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Nonperturbative QCD and supersymmetric QCD

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Contents

Abstract. Nonperturbative phenomena in quantum chromodynamics (QCD) and, in particular, confinement are reviewed. As an example of the treatment, an exact solution of the $\mathcal{N} = 2$ supersymmetric Yang – Mills gauge theory is presented. Prospects for application of the duality idea in QCD are discussed.

> "Thank God for creating the world with anything significant being simple, and anything complicated having no significance"
> *Grigory Skovoroda, XVIII century peripatetic Ukrainian philosopher*

1. Introduction

I belong to the generation of ITEP theorists who never met Isaac Yakovlevich Pomeranchuk, but have heard his closest friends retell numerous jokes, utterances, or cock-and-bull stories about him or liked by him. Many theorists of the older generation claim that the above quotation from Grigory Skovoroda[1] was one of the most often repeated by Pomeranchuk. It complied with his ideas of physics, of the relationship between complexity and simplicity in science.

[1] Quoted arbitrarily, reproduced from narratives by the staff of the ITEP theoretical division.

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In the present review, nonperturbative phenomena are discussed. The importance of this subject is clear, at the very least, from the fact that our flesh and the mass of numerous objects surrounding us owe their origin by 99% to nonperturbative phenomena. Somehow, contrary to the words of Skovoroda, nonperturbative phenomena represent an extremely complex subject. The advance toward understanding it is very slow (somewhat like in the case of turbulence). However, an enormous amount of work has been performed during the past 25 years, and the contours of simplicity have started to show through. The picture is not complete, yet, but certain fragments are quite visible. Some of these fragments are dealt below with.

2. Nonperturbative dynamics

2.1 The Higgs phase

Our world is described by the Standard Model with the gauge group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Interactions are mediated by the respective vector gauge bosons: photons, the W- and Z-bosons, and gluons. Gravity is also described by a gauge theory — GRT (general relativity theory). Its mediator is a massless boson of spin 2 — the graviton. Matter consists of quarks and leptons. The Standard Model is arranged in such a way that neither quarks nor leptons can be assigned mass 'by hand'. The mechanical (current) mass of quarks and leptons originates from interaction with the scalar Higgs field $\phi(x)$.

According to the existing dogma, the Higgs field undergoes condensation in space, so that the vacuum expectation value of the field differs from zero: $\langle 0|\phi(x)|0\rangle = \eta \neq 0$. This is a well understood nonperturbative phenomenon — the Higgs effect. It is said that gauge theories are in the Higgs phase. The nonzero Higgs field vacuum expectation value leads to nonzero masses of the gauge W- and Z-bosons, quarks, and leptons. It appears that the Higgs field is a fundamental field

of the Standard Model. All attempts at constructing the field $\phi(x)$ as a composite field (like the condensate in the Bardeen – Cooper – Schrieffer theory of superconductivity) led to contradictions with experimental data. The boson corresponding to the Higgs field has not yet been found experimentally. Its revelation is the main task of experimental physics of the next decade.

Nuclei, protons, and neutrons are composed of light u- and d-quarks. The mechanical (current) mass of the three quarks in a nucleon amounts to about 10 MeV, which is to be compared with the nucleon mass on the order of 1000 MeV. Thus, the Higgs effect (which is well understood) explains 1% of the mass of matter surrounding us. The remaining 99% are related to a much less understood phenomenon — confinement.

2.2 The confinement phase

The fundamental fields of the $\mathrm{SU}(3)_\mathrm{c}$ theory of strong interactions (chromodynamics) are eight gluons A^a_μ $(a = 1, \ldots, 8)$ and quark triplets q^i_f $(i = 1, 2, 3,\ f = 1, \ldots, N_f)$. Neither gluons nor quarks are observed in the form of free states. At small distances quarks are clearly seen (in experiments on deep-inelastic scattering). However, at large distances only colorless hadrons [$\mathrm{SU}(3)_\mathrm{c}$ singlets] are observed. This phenomenon has been termed color confinement.

A qualitative explanation of confinement resides in that the force lines of the chromoelectric field between two static color charges squeeze to a force tube of constant thickness, so that all the field energy is concentrated within this tube and is, therefore, proportional to its length: $V(r) \simeq \sigma r$. Such a potential corresponds to a constant interaction force between two quarks, so they can never become free from this interaction, i.e., they never leave each other. The potential interaction energy of light quarks is concentrated in the energy of the tube, and 99% of the nucleon mass consists of the chromoelectric field energy inside the tube.

2.3 The Meissner effect

A similar phenomenon takes place in the theory of superconductivity. In superconductors, the charged condensate ϕ^{--} (Cooper pairs) differs from zero at temperatures below the phase transition temperature: $\langle\phi^{--}\rangle \neq 0$. As a result, a magnetic field cannot penetrate deep into the superconductor (the Higgs effect in $3D$ U(1) theory). Therefore, if a magnetic charge g (monopole) and an antimonopole $\tilde{g}$ are inserted into a superconductor, then (to minimize the free energy) the magnetic force lines form a tube — the Abrikosov tube — so that the interaction energy between the monopole and antimonopole is proportional to the distance between them.

2.4 The dual Meissner effect

The analogy to confinement was immediately noticed by 't Hooft [1], Mandelstam [2], and Gribov [3]. Confinement looks like the dual Meissner effect when all electric (gauge) charges q have to be replaced by magnetic charges g (and vice versa), and the electric field E is replaced by the magnetic field B (and vice versa). Thus, the qualitative picture of confinement is that monopoles condensate in a vacuum, and the charges do not escape.

To answer the questions of what kind of monopoles in $\mathrm{SU}(3)_\mathrm{c}$ chromodynamics are, why they condensate, and so on, is quite difficult. It is possible to receive direct answers to these questions with the help of supercomputers. Gauge theories in continuous spacetime are replaced by gauge theories on a discrete lattice of finite length. Computers simulate feasible fluctuations within such a field theory. If the sizes of significant fluctuations turn out to exceed the size of the discrete lattice, then one can assume discretization not to be important and computer simulation not to be far from real field theory in real spacetime. The success achieved in this direction has been great (a review of this work is presented in the article by Bornyakov et al. [4]).

It is possible, however, to understand some properties of nonperturbative physics without turning to supercomputers. Here, it is necessary to leave the straight road toward the goal and to move along detour paths. The main tools along these paths are the concepts of analyticity and duality.

A good example of 'detour maneuvering' in physics is represented by the work of Gribov, Ioffe, and Pomeranchuk [5] on the $\mathrm{e^+e^-}$ annihilation into hadrons, in which analyticity was applied for relating the behavior of the cross section ($\mathrm{e^+e^-} \to$ hadrons) and the properties of electromagnetic current commutators at small distances.

Similar approach in QCD resulted in the creation of the method of QCD sum rules, which has permitted the calculation of hadron parameters in terms of the parameters of a nonperturbative QCD vacuum.

2.5 Nonperturbative vacuum in QCD

The main elements of the method of QCD sum rules comprise asymptotic freedom, analyticity (holomorphism), and duality. The first success achieved by this method consisted of the construction of the dispersion theory of charmonium in 1977 [6]. We shall deal with this example is greater detail.

Of the total electromagnetic current, we single out the part related to the heavy c-quark, $J^\mathrm{c}_\mu = \bar{c}\gamma_\mu c$, and consider the polarization operator corresponding to this current:

$$\Pi^\mathrm{c}_{\mu\nu} = \mathrm{i}\int \mathrm{d}^4x\, \exp(\mathrm{i}qx)\langle 0|T\{J^\mathrm{c}_\mu(x)\, J^\mathrm{c}_\nu(0)\}|0\rangle = (g_{\mu\nu}q^2 - q_\mu q_\nu)\,\Pi^\mathrm{c}(q^2)\,. \tag{1}$$

It is readily shown that $\Pi^\mathrm{c}(q^2)$ is an analytical function of the variable q^2 with singularities at $q^2 > 0$. The discontinuity of $\Pi^\mathrm{c}(q^2)$ is related to the production of resonances J/ψ, ψ', and ψ'', as well as of D-meson pairs and so forth.

With the help of analyticity (i.e., dispersion relations) one can calculate function $\Pi^\mathrm{c}(q^2)$ throughout the entire q^2-plane in terms of masses and widths of resonances, and of parameters of the continuous spectrum. On the other hand, owing to asymptotic freedom, it is also possible to calculate the same function $\Pi^\mathrm{c}(q^2)$ far from singularities in perturbation theory, dealing with quarks and gluons.

In the case of heavy c-quarks, the point $q^2 = 0$ resides sufficiently far from the threshold, and comparison of the two different calculations of $\Pi^\mathrm{c}(q^2)$ in the vicinity of $q^2 = 0$ leads to the QCD sum rules:

$$\int \mathrm{d}q^2\, \frac{1}{(q^2)^n}\, R^\mathrm{pt}_\mathrm{c} = \int \mathrm{d}q^2\, \frac{1}{(q^2)^n}\, R^\mathrm{exp}_\mathrm{c}\,, \tag{2}$$

where R^pt_c and $R^\mathrm{exp}_\mathrm{c}$ are the cross sections (normalized) of $\mathrm{e^+e^-}$ annihilation, taken from perturbation theory and experiments, respectively.

This is an example of duality. One and the same physical reality can be described either in terms of interacting quarks and gluons, or in terms of hadrons. If relation (2) were valid

for all n values, then we would have pointwise (local) duality: $R^{\rm pt} = R^{\rm exp}$, which certainly does not conform to reality. But, if relation (2) holds valid for several nearest n values, then integral duality takes place, where the averaged physical cross sections differ insignificantly from the averaged theoretical-perturbative cross sections.

However (and this was a completely unexpected present from nature), if real masses and widths are substituted into relation (2), then the lowest J/ψ resonance already turns out to be dominant in the sum rules for the first few n values. As to theoretical computations, they are represented as an expansion in terms of the small parameter $\alpha_s(m_c)$ — the running constant of strong interactions at the scale of m_c. Therefore, in the left-hand side of Eqn (2) one can deal with only the first term of the expansion, if one is not too interested in an accuracy exceeding 10%. As a result, it is possible to 'calculate' the parameters of the J/ψ resonance within perturbation theory. These calculations are in quite good agreement with experimental data. Such was the theory of charmonium in 1977 [6].

The theory described above does have one essential defect. Within the theory itself it is impossible to find any indications as to when the duality relation (2) stops being valid. Indeed, as the number n increases, the contribution from the J/ψ resonance saturates the dispersion integral with a higher and higher accuracy. On the other hand, neither does the accuracy of theoretical-perturbative calculations deteriorate as the number n increases. Therefore, the impression is created that relation (2) holds valid at all n values, which is, naturally, wrong. No local duality exists, the narrow J/ψ resonance is in no way similar to the production cross section of $c\bar{c}$-quarks in perturbation theory, and equality (2) must break down at large n values.

A way to resolve the paradox was proposed in Refs [7, 8]. The idea is that, besides perturbation theory, nonperturbative fluctuations are also important. A QCD vacuum contains gluon condensate $\langle \alpha_s G^a_{\mu\nu} G^a_{\mu\nu} \rangle \neq 0$, where $G^a_{\mu\nu}$ is the gluon field tensor, and α_s is the strong interaction coupling constant. It is precisely the interaction of quarks with the vacuum condensate that is responsible for violation of the duality relation (2). At small n, this interaction is small, as compared with perturbation theory, and relation (2) works well. However, starting from a certain $n_{\rm cr}$, the interaction with nonperturbative condensates increases drastically, and duality relation (2) must be renounced.

The mechanism of duality violation is especially transparent in the case of polarization operators with light quarks. In this case, far from physical singularities, the duality relation has the form

$$\Pi^{\rm exp}(q^2) \approx \Pi^{\rm pt} + \frac{c}{q^4} \langle \alpha_s G^a_{\mu\nu} G^a_{\mu\nu} \rangle + \ldots \tag{3}$$

Here, function $\Pi(q^2)$ is normalized so as to be dimensionless, the coefficient c is determined by the concrete form of light quark currents, and the suspension points correspond to the contribution from operators of higher dimension.

The perturbative contribution of $\Pi^{\rm pt}(q^2)$ depends on slowly varying functions such as $\ln q^2$, i.e., the function $\Pi^{\rm pt}$ is nearly a constant throughout the entire range of q^2 values. At large q^2 values, the nonperturbative contribution behaves like an expansion in terms of small power corrections to the main perturbative term. However, when q^2 decreases, the power terms 'burst' and start to dominate in the right-hand side of formula (3). The scale separating large and small q^2 values is obviously determined by the condensate value. The transition is made very rapidly — it is exponential. This is what the modified duality looks like.

As was already mentioned, nature is such that power corrections permit us (in a number of cases) to go so close to the physical region that resonances start to dominate in the dispersion integrals. Here, the corrections to the perturbative computations still remain small. As a result, it is possible to calculate the parameters of the resonances. This is precisely what the QCD sum rules are. Thus were the masses and widths of light mesons calculated [9], as were the masses and magnetic moments of nucleons [10], and many other things. The number of articles on the QCD sum rules amounts to hundreds.

3. Nonperturbative operator expansion

Operator expansion outside the framework of perturbation theory [11] represents the theoretical justification for the sum rules. Such an expansion is very similar to the procedure proposed by Wilson for constructing the effective action. All theoretical-field fluctuations are assumed to be separable into short-wave and long-wave fluctuations. The QCD asymptotic freedom guarantees that the short-wave fluctuations can be described with sufficient accuracy within perturbation theory and that their contribution can be calculated explicitly. As to the long-wave fluctuations, their computation is difficult, since they correspond to the strong coupling regime. However, the behavior of such fluctuations is formally described by effective action with an infinite series expansion in local operators.

For the T-product of currents $J(x)$ at small distances, the procedure reduces to the relation

$$\mathrm{i} \int \mathrm{d}^4 x \, \exp(\mathrm{i}qx) \, T\{J(x), J(0)\} \underset{q^2 \to \infty}{\approx} \sum_n c_n(\mu, q^2) \, \frac{O_n(\mu)}{[q^2]^{\mathrm{Dim}\, O_n/2+d}} \, . \tag{4}$$

Here, the parameter μ separates small distances from large ones. The coefficients $c_n(\mu, q^2)$ explicitly take into account the contribution of small distances, i.e., of perturbation theory, of small-sized instantons, and so forth. They are similar to the running coupling constants in front of the new operators in the effective action. Operators $O_n(\mu)$ are similar to the new terms of the effective interaction for long-wave fluctuations in the Wilson effective action. The powers in $[q^2]^{-(\mathrm{Dim}\, O_n/2+d)}$ are singled out so as to make the coefficients $c_n(\mu, q^2)$ dimensionless. Their dependence upon q^2 is very weak (via $\ln q^2$, etc.).

In the case of processes such as annihilation ($e^+e^- \to$ hadrons or $Z \to$ hadrons), one should consider the vacuum expectation of the T-product of the appropriate currents. Precisely the vacuum expectations of operators O_n are the condensates $\langle \alpha_s G^2 \rangle$, $\langle \bar{q}q \rangle, \ldots$ All the ignorance concerning strong interactions is encoded in the form of a series of unknown condensates. Given a certain level of ingenuity, this turns out to be sufficient to calculate certain parameters of certain hadrons.

4. Hidden scale in QCD

It is, however, possible to do without ingenuity and to obtain important information about strong interactions for nothing.

Indeed, the interaction of various currents with *vacuum fields* is not universal. A ρ-meson, say, may be produced from a vacuum by the current $j_\mu = \bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d$. Perturbation theory is described by a loop (the polarization operator). Interaction with gluon fields is also described by a loop and is proportional to $(\alpha_s/16\pi)G^a_{\mu\nu}G^a_{\mu\nu}$.

If one now turns to gluon currents, for example, to the scalar current $J = \alpha_s G^a_{\mu\nu}G^a_{\mu\nu}$, then perturbation theory is again described by a loop, while the interaction with vacuum fields is enhanced: it proceeds in the tree approximation. Thus, the gluon corrections to gluon currents turn out to be enhanced with respect to the gluon corrections to the quark currents by factors on the order of $2\pi/\alpha_s$. Therefore, the scales of duality violation (which also means the mass scale of resonances) for ordinary mesons and glueballs are essentially different.

A confirmation of this observation can be found in exact low-energy theorems. By introducing new objects — condensates — into field theory, one can expect new relations to arise for these objects. One such theorem assumes the following form [12]:

$$\mathrm{i}\int \mathrm{d}^4x \left\langle T\left\{\frac{3\alpha_s}{4\pi}\,G^2(x),\,\frac{3\alpha_s}{4\pi}\,G^2(0)\right\}\right\rangle = \frac{18}{b}\left\langle\frac{\alpha_s}{\pi}\,G^2\right\rangle. \quad (5)$$

In perturbation theory, the left-hand side of relation (5) starts with terms on the order of α_s^2, however, the right-hand side is proportional to α_s (actually, to $\alpha_s^{(0)}$, since $\alpha_s G^2$ is a renormalization invariant).

Substituting concrete numbers, one readily verifies that, while the mass scale for the 'old' hadrons is $\Lambda^2 \approx 1\ \mathrm{GeV}^2$, the mass scale for glueballs in certain channels amounts to $\Lambda^2 \approx 16\ \mathrm{GeV}^2$ [12]. This prediction stills awaits confirmation either from direct experiments or from computer simulations. (It must be noted that a large scale of duality violation does not always mean that the lowest resonance has a large mass. The distance between resonances must be large.)

5. Preliminary results

All these exercises with QCD may provide several lessons.

(1) A 'physical' vacuum is not like emptiness. It is filled up with nonperturbative condensates. The interaction of quarks and gluons with vacuum fields turns out to be decisive in the formation of colorless hadrons.

(2) There exist numerous exact theorems for nonperturbative condensates. This subject has not yet been exhausted. The most recent theorem was revealed at the beginning of 2003 [13].

(3) There exists a natural restriction for the applicability of QCD sum rules. The procedure for dividing large and small distances seems trivial and gives rise to no unpleasant obstacles. However, in spite of asymptotic freedom, the knowledge of small distances happens to be insufficient. Besides perturbation theory, it is necessary to take into account small-sized instantons [11]. These fluctuations are well known, and they can be taken into account. There also exist less known objects. Thus, for instance, the divergence of perturbation theory series (ultraviolet and infrared renomalons) lead to power corrections to the coefficient functions, which confuses the analysis of power corrections (see the works of V I Zakharov, starting from the middle of the 1990s).

(4) QCD seems to be too complicated a theory for the theoretical evaluation of condensates and of other vacuum structures without turning to supercomputers. More simple, but still interesting, theories must be sought.

6. Supersymmetric QCD

Such a possibility is presented by supersymmetric theories. Below follows the outline of a concise introduction to supersymmetry.

Supersymmetry (SUSY) is the symmetry between the bosons and fermions:

$$\begin{aligned}
&|\mathrm{boson}\rangle \overset{\mathrm{SUSY}}{\leftrightarrow} |\mathrm{fermion}\rangle\,,\\
&E^n_{\mathrm{B}} \leftrightarrow E^n_{\mathrm{F}}\,,\\
&\mathrm{gluon\ g} \leftrightarrow \mathrm{gluino\ \tilde{g}}\,,\\
&\mathrm{quark\ q} \leftrightarrow \mathrm{squark\ \tilde{q}}\,.
\end{aligned} \quad (6)$$

In addition to spacetime x_μ, *superspace* contains odd dimensions described by Grassmann (odd) coordinates θ_α, $\bar{\theta}_{\dot{\alpha}}$ $(\alpha, \dot{\alpha} = 1, 2;\ \{\theta_\alpha\theta_\beta\} = 0)$.

Supercalculus is quite simple, here are all the rules:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\,\theta = 1\,,\quad \frac{\mathrm{d}}{\mathrm{d}\theta}\,1 = 0 \qquad \text{(differentiation)},$$

$$\int \mathrm{d}\theta\cdot\theta = 1\,,\quad \int \mathrm{d}\theta\cdot 1 = 0 \quad \text{(integration)}.$$

Since the variables θ_α anticommute, all *superfields* have finite expansions in θ. The scalar chiral superfield has the form

$$\phi(x,\theta) = \phi(x) + \theta_\alpha\psi^\alpha(x) + \theta_\alpha\theta^\alpha F(x)\,.$$

It contains two fermion fields $\psi^\alpha(x)$, the scalar field $\phi(x)$, and an additional scalar field $F(x)$. Thus, the number of fermion degrees of freedom is always equal to the number of boson degrees of freedom.

Nonrenormalization theorems represent a key property of supersymmetric theories. The creators of supersymmetry Gol'fand and Likhtman already noted [14] that a significant reduction of divergences takes place in such theories.

Let us 'prove' the theorem that the energy of a vacuum within supersymmetry is equal to zero.

Proof 1. The numbers of boson and fermion degrees of freedom coincide, boson and fermion masses are equal to each other. The boson and fermion loops are present with opposite signs, so corresponding loops cancel each other out exactly.

Proof 2. Due to supersymmetry, the energy of vacuum boson modes, E^{B}_n, is precisely equal to the energy E^{F}_n of fermion modes:

$$E_{\mathrm{vac}} = \sum_n E^{\mathrm{B}}_n - \sum_n E^{\mathrm{F}}_n = \sum_n (E^{\mathrm{B}}_n - E^{\mathrm{F}}_n) \equiv 0\,. \quad (7)$$

And, finally, we give the superfield proof.

Proof 3. The energy of the vacuum is equal to the integral over superspace of the vacuum energy density $\varepsilon_{\mathrm{vac}}$. By virtue of the homogeneity of space, the vacuum energy density $\varepsilon_{\mathrm{vac}}$ cannot be dependent upon coordinates x_μ and, consequently, coordinates θ_α, $\bar{\theta}_\alpha$. Therefore, one finds that

$$E_{\mathrm{vac}} = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\varepsilon_{\mathrm{vac}} = 0 \quad (8)$$

due to the rules for integrating Grassmann variables: $\int d\theta = 0$.

All this reasoning provides hope that supersymmetric theories can be resolved exactly, which, doubtless, is a very attractive feature of the theory. On the other hand, no exact supersymmetry is observed in our world. For this, there is no need to perform special experiments. No one has ever seen a scalar electron of mass 0.5 MeV. For this reason, theorists made the assumption that supersymmetry exists (only at small distances) at high energies, while at low energies it is violated, so that the masses of the electron and scalar electron differ from each other. But the high-energy contributions of e and ẽ cancel out. During the past decade, enormous effort has been made in searching for supersymmetric particles. The bounds imposed on their mass exceed a hundred GeV. No traces of supersymmetry exist yet. And the main argument in favor of SUSY is its beauty.

In one of his works, N Seiberg wrote: "It would be a pity if nature did not take advantage of such an excellent idea like SUSY". However, as N Bohr wrote on another occasion, "it is not up to us to instruct God on what to use in creating the world".

Let us now go back to supersymmetric theories.

7. Gluino condensate

Let us start with SU(2) supergluodynamics. It contains three gluons A^a_μ (massless, transverse) and three gluinos λ^a_α (two spin orientations). The gluino field λ^a_α is the lower component of the chiral superfield W^a_α. The vacuum averages of the lower components of chiral fields can be readily shown to be independent of coordinates. Thus, in particular, the vacuum correlator

$$\Pi(x,y) = \langle 0 | T\{\lambda^2(x), \lambda^2(y)\} | 0 \rangle \equiv \text{const}\,. \tag{9}$$

Due to the asymptotic freedom at small distances $|x-y| \ll 1/\Lambda$, where Λ is a parameter determining the run of the coupling constant, it is possible to calculate the contribution to the vacuum correlator $\Pi(x,y)$. The contribution of perturbation theory is zero, since correlator (9) emits two fermions at point x and emits (but does not absorb) two fermions at point y. No such Feynman diagrams exist. On the other hand, the SU(2) instanton contains precisely four zero fermion modes and gives a nonzero contribution to relationship (9).

Let us now consider large distances: $|x-y| \gg 1/\Lambda$. In the limit of infinite distances, one should expect the decay of correlations:

$$\Pi(x,y) \underset{|x-y|\to\infty}{\Longrightarrow} \langle \lambda^2 \rangle^2\,. \tag{10}$$

Thus, for SU(2) one obtains [15]

$$\langle 0 | \mathrm{Tr}\,\lambda^2 | 0 \rangle^2 = \frac{2^{10}\pi^4}{5} \exp\left(-\frac{8\pi}{g_0^2}\right) \frac{M^6}{g_0^4} = \frac{144}{5}\,\Lambda^6\,. \tag{11}$$

Similar calculations for SU(N_c) yield

$$\langle \lambda^a_\alpha \lambda^a_\alpha \rangle = c\Lambda^3 \exp\left(\frac{2\pi i}{N_c}\,k\right).$$

The coefficient c is calculated exactly, and the number $k = 0, 1, \ldots, N_c - 1$ 'counts' the various vacuum states.

Instantons lie in one of the SU(2) subgroups of the O(4) = SU(2) × SU(2) group, and precisely one half of the supersymmetry is not violated by the external instanton field. This half of supersymmetry is readily shown to be sufficient to demonstrate the absence of perturbative corrections to expression (11), just as there are no corrections to the vacuum energy [15]. Consequently, instanton calculation of the gluon condensate is a precise calculation.

In relation (11), we first expressed the gluino condensate in terms of the cut-off parameter M^2 and the bare coupling constant g_0, and then in terms of the physical parameter Λ.

Clearly, the physical quantity $\langle \lambda^2 \rangle$ must not depend on the cut-off procedure. Hence, one immediately obtains the 'exact' β-function (NSVZ [16]):

$$\beta(\alpha_s) = -3N_c\,\frac{\alpha_s^2}{2\pi}\,\frac{1}{1 - N_c\alpha_s/2\pi}\,. \tag{12}$$

Such was the state of the theory in the years 1983 – 1984. Ten years later a new revolution took place in supersymmetric field theories: in a certain sense a breakthrough into the strong coupling region was achieved.

8. Strong coupling regime

The revolution in supersymmetric field theories is based on two constituent parts: holomorphy and electric – magnetic duality.

Holomorphy asserts that the superpotential W_{eff} depends holomorphically on all chiral fields, as well as on the coupling constants (i.e., it depends on the chiral fields, but not on the complex conjugate fields).

Electric – magnetic duality signifies that one and the same physics can be described within either the *electric* or *magnetic* formulation. A well-known example is the Zwanziger formalism for describing the electrodynamics of electric and magnetic charges (monopoles).

8.1 The power of holomorphy

The principle of holomorphy, known since the 1980s, started to be really and fully applied during the revolution of the 1990s [17]. We shall show how it works, taking advantage of the Wess – Zumino model. The superpotential $W(\phi)$ takes the form

$$W(\phi) = m\phi^2 + g\phi^3\,. \tag{13}$$

If the mass m and charge g are dealt with like external fields, then the superpotential $W(\phi)$ is symmetric relative to the two U(1) symmetries:

$$m \to \exp(i\alpha)\,m\,, \qquad g \to \exp(i\beta)\,g\,,$$

with charges

	$U(1)$ charge	$U(1)_R$ charge
ϕ	1	1
θ	0	1
m	−2	0
g	−3	−1

Interaction does not violate the symmetries, therefore, the effective action must also satisfy these symmetries. There

exists only one combination of fields and constants that meets such requirements:

$$W_{\text{eff}} = m\phi^2 f\left[\frac{g\phi}{m}\right],$$

where f is an arbitrary function.

Let us, however, consider the limiting case: the coupling constant $g \to 0$, while $g\phi/m$ is an arbitrary number.

In the limit $g \to 0$, the effective action coincides with the tree action:

$$W_{\text{eff}} \underset{g\to 0}{\Rightarrow} W_{\text{tree}}\,.$$

Hence, we immediately obtain

$$f[t] = f_{\text{tree}}(t) = 1 + t\,, \qquad t = \frac{g\phi}{m}$$

for all values of t. Therefore, the following relationships hold true:

$$W_{\text{eff}} \equiv m\phi^2 + g\phi^3 = W_{\text{tree}}\,, \tag{14}$$

i.e., the superpotential does not undergo renormalization. The proof was obtained beyond the framework of perturbation theory. It looks somewhat simplified, but, apparently, contains no shortcomings.

8.2 $\mathcal{N} = 2$ SUSY gluodynamics

Let us describe the 'exact solution' obtained by Seiberg and Witten in 1994 in work [18], which marked the beginning of the revolution of the 1990s.

Consider $\mathcal{N} = 2$ SU(2) supergluodynamics. In terms of $\mathcal{N} = 1$ superfields, the theory contains the chiral gauge supermultiplet $W^a_\alpha = (\lambda^a, A^a_\mu)$ and the chiral multiplet $\Phi^a = (\phi^a, \psi^a)$, where $a = 1, 2, 3$. Thus, there exist three vector fields, six spinor fields, and three complex scalar fields.

The interaction Lagrangian has the form

$$\mathcal{L} = -\frac{\mathrm{i}}{16\pi^2}\int \mathrm{d}^2\theta\, \tau W^2 + \frac{1}{g^2}\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \Phi^+ e^V \Phi\,, \tag{15}$$

where

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2}$$

determines the charge and θ-term in the gluon interaction Lagrangian.

The scalar field ϕ^a condensates (the interaction potential contains flat directions in $\phi^a\phi^a$):

$$\langle \phi^a \phi^a \rangle = u \neq 0\,,$$

and SU(2) symmetry is violated down to U(1). Massless states arise: the photon A^3_μ, photino (λ^3, ψ^3), and massless scalar ϕ^3. In addition, massive $\mathrm{W}^\pm$-bosons, a massive Dirac spinor $\psi^\pm$, and massive scalar field $\phi^\pm$ exist. This is not the end of the story: the theory contains massive 't Hooft – Polyakov monopoles that represent one of the components of the respective $\mathcal{N} = 2$ supermultiplet. Thus, besides an 'empty' monopole, there is a monopole with an occupied fermion mode and a monopole with two occupied fermion modes.

At low energies, it is possible to integrate out the degrees of freedom corresponding to heavy W-bosons, monopoles, and their superpartners. There remain the massless photon, the photino, and the neutral boson. At a classical level, these light degrees of freedom do not interact and the theory seems trivial and uninteresting. However, taking into account quantum mechanics leads to interaction. The exact Seiberg – Witten solution pertains precisely to this 'noninteracting' system.

The effective action for the light $\mathcal{N} = 2$ supermultiplet appears as follows:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\,\mathrm{Im}\left\{\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \Phi^+ \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} + \frac{1}{2}\int \mathrm{d}^2\theta\, \frac{\partial^2 \mathcal{F}}{\partial \Phi^2}\, W^2\right\}. \tag{16}$$

Here, the function $\mathcal{F}(\phi)$ is holomorphic. The form of action (16) is fully determined by $\mathcal{N} = 2$ supersymmetry. The 'charge' in the effective action is the effective 'charge', i.e., one finds

$$\tau(\phi) = \frac{\partial^2 \mathcal{F}}{\partial \phi^2} = \frac{1}{2\pi}\,\theta(\phi) + \frac{4\pi\mathrm{i}}{g^2(\phi)}\,.$$

At a 'classical' level, the masses of heavy particles can be written down in a symmetric form

$$m = \sqrt{2}\,|aQ_{\mathrm{e}} + a_D Q_{\mathrm{m}}|\,, \tag{17}$$

where Q_{e} and Q_{m} are the electric and magnetic charges, respectively, and the functions

$$a = \sqrt{u}\,, \qquad a_D = \frac{4\pi\mathrm{i}}{g^2}\, a = \tau a$$

are holomorphic functions of the variable $u = \langle \phi^a \phi^a \rangle$.

At large values of the variable u, the effective charge tends toward zero and perturbative calculations are reliable:

$$\begin{aligned} &u = a^2\,, \\ &a_D(u) = \frac{\partial \mathcal{F}}{\partial a} = \frac{\mathrm{i}}{\pi}\, a\left(\ln\frac{a^2}{\Lambda^2} + 1\right). \end{aligned} \tag{18}$$

The point $u = \infty$ is a singular point of functions (18), namely, the branch point. When the detour around a point $u = \infty$ is performed, functions a_D and a transform into each other:

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_\infty \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Matrix M_∞ is termed a monodromy matrix. The form of matrix M_∞ follows from the explicit form (18) of the quasiclassical functions $a(u)$ and $a_D(u)$. Clearly, the branch point cannot be alone: the cut having started somewhere must also terminate somewhere. This means that functions $a(u)$ and $a_D(u)$ must possess additional singularities.

Simple arguments can be made in favor of one additional singularity leading to a contradiction. (We shall not dwell upon a detailed demonstration. As a hint we can say that from the definition of function $\tau(\phi)$ it is clear that the function $\mathrm{Im}\,\tau(\phi)$ is always positive and increases at large ϕ. On the other hand, $\mathrm{Im}\,\tau(\phi)$ is a harmonic function and cannot have a

minimum.) Let there exist two additional singularities. The effective action of SU(2)-gauge theory is Z_2-symmetric, i.e., the theory is symmetric under the substitution $u \to -u$. Therefore, the singularity points reside at the symmetric points u_0 and $-u_0$.

Assume the singularities in S_{eff} to be related to the appearance of massless states. In Ref. [18], the singularity u_0 was considered to correspond to a massless monopole with charges $(Q_{\text{e}}, Q_{\text{m}}) = (0, 1)$, so that

$$a_D(u_0) = 0\,, \qquad a(u_0) \neq 0\,.$$

This seems to be insufficient for calculation of the monodromy in the vicinity of point u_0. However (and this is a very bold step), one can switch from the electric description of the remaining U(1)-theory to the dual (magnetic) description: $\phi \to \phi_D$, $W \to W_D$. In these variables, the interaction of monopoles looks like QED electrodynamics with charge unity. QED is an infrared-free theory, therefore, perturbative calculations in the vicinity of $a_D = 0$ are reliable. The monodromy matrix M_{u_0} is found in a manner similar to M_∞.

The monodromy near point $-u_0$ can be reconstructed on the basis of a detour around points ∞ and u_0, being equivalent to a detour around point $-u_0$ in the opposite direction, i.e., one has

$$M_\infty M_{u_0} M_{-u_0} = 1\,.$$

Thus, duality permits us to fix the monodromy matrices.

The Riemann–Gilbert theorem is known from mathematics. Consider functions $F_i(z)$ to transform as follows when a detour around z_0 is performed:

$$F_i\big[(z - z_0)\exp(2\pi\mathrm{i})\big] = M_{ij}(z_0)\,F_j(z)\,.$$

If $\{z_0\}$ and $\{M_{ij}(z_0)\}$ are known, then functions $F_j(z)$ can be reconstructed in one and only one way. In our case, the monodromies are determined, therefore, Gilbert and Riemann guarantee that $m_W(u)$, $m_m(u)$, and $a_D(u)$ can be calculated in terms of u_0.

The problem seems complicated. In Ref. [18], excellent mathematics are presented, permitting us to find the exact solution. However, the solution of an even more general problem has been known since the 1930s, when problems concerning the motion of particles in periodic potentials in solid-state physics were being resolved. In the case of a shift by a period, monodromy matrices arose for two independent solutions of the Schrödinger equation. And all (at least, many) interesting and solvable problems were solved 70 years ago.

For information, we give the exact solution corresponding to our problem:

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \mathrm{d}x\,\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,,$$

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u} \mathrm{d}x\,\frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,.$$

Here, $u_0 = 1$ (i.e., the variable u is measured in units of u_0).

One can verify that the 'exact' charge $\tau(u) = \mathrm{d}a_D/\mathrm{d}a$ differs from the 'exact' perturbative charge, found in Section 7, by nonperturbative power terms. It has been demonstrated explicitly that the respective first terms of this power expansion correspond to one-, two-, ... instanton contributions to the effective charge [19]. This assertion is most likely correct for all terms of the power series.

Similar solutions have also been obtained for $\mathcal{N} = 2$ theory with an arbitrary gauge group and for SU(N_{c})-gauge theory with matter [20].

8.3 Condensation of monopoles

It is interesting to break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. To this end, it is sufficient to add a mass term for matter to the action:

$$\Delta W = m\,\mathrm{Tr}\,\phi^2 = mu\,.$$

If the mass is small, one could expect that variation of the effective action can be taken into account by applying perturbation theory.

In terms of the dual description, the effective theory in the vicinity of point $u = u_0$ can be represented in the form

$$W_{\text{eff}} = \sqrt{2}\,a_D\tilde{M}M + mu(a_D)\,,$$

where M and $\tilde{M}$ are the chiral fields for the monopole and antimonopole, respectively. The first term corresponds to the mass term of the monopole, and the second corresponds to the mass term of matter. The extremality condition of W_{eff} in variable a_D, M, and $\tilde{M}$ is written down as

$$\sqrt{2}\,M\tilde{M} + mu' = 0\,,$$

$$a_D M = a\tilde{M} = 0\,.$$

The solution of the set of equations has the form

$$\langle M\rangle = \langle\tilde{M}\rangle = \sqrt{-\frac{m}{\sqrt{2}}\,u'(0)} \neq 0\,.$$

Thus, we have obtained the Higgs phase of the magnetic U(1) theory.

For a complex value of u_0, the monopole field undergoes condensation. Recall what we started from. For $u = 0$ we had, at the Lagrangian level, massless gluons and gluinos and a massive (with mass m) matter field. In the exact solution, the point $u = 0$ is not singled out in any way, and no massless charged states exist. Thus, confinement is realized in the initial theory. The initial hypothesis for monopoles condensating under confinement has been confirmed, in a certain sense. If $u = u_0$ has a complex value, the scalar component of the chiral monopole field has a nonzero vacuum expectation value.

8.4 Recent progress

At the beginning of 2003, exact solutions were found for $\mathcal{N} = 1$ supersymmetric SU(N_{c})-gauge theory with matter in the adjoint representation and with arbitrary interaction [21], as well as with matter in the adjoint and fundamental representations [22]. Apparently, in this direction it turns out to be possible to obtain certain exact results for nonsupersymmetric theories [23].

9. Conclusion

In conclusion, it is to be emphasized that:

- the idea of a relationship between confinement and monopole condensation has been confirmed in supersymmetric theories;

• the origin of confinement still remains unclear within QCD;

• the progress achieved in understanding the issue is, nevertheless, remarkable;

• the time scale for progress in the theory is on the order of 8 – 10 years, and this gives rise to hope.

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References

1. 't Hooft G, in *High Energy Physics*: *Proc. of the EPS Intern. Conf., Palermo, Italy, 23 – 28 June 1975* (Intern. Physics Series, issue 6, Ed. A Zichichi) (Bologna: Compositori, 1976)
2. Mandelstam S *Phys. Rep.* **23** 245 (1976); *Phys. Rev. D* **19** 2391 (1979)
3. Gribov V N, private communications
4. Bornyakov V G et al. *Usp. Fiz. Nauk* **174** 19 (2004) [*Phys. Usp.* **47** 17 (2004)]
5. Gribov V N, Ioffe B L, Pomeranchuk I Ya *Yad. Fiz.* **6** 587 (1967) [*Sov. J. Nucl. Phys.* **6** 427 (1967)]
6. Novikov V A et al. *Phys. Rev. Lett.* **38** 626 (1977); *Phys. Rep.* **41** 1 (1978)
7. Vaĭnshteĭn A I et al. *Yad. Fiz.* **28** 465 (1978) [*Sov. J. Nucl. Phys.* **28** 237 (1978)]
8. Vaĭnshteĭn A I, Zakharov V I, Shifman M A *Pis'ma Zh. Eksp. Teor. Fiz.* **27** 60 (1978) [*JETP Lett.* **27** 55 (1978)]
9. Shifman M A, Vainshtein A I, Zakharov V I *Nucl. Phys. B* **147** 385 (1979)
10. Ioffe B L *Nucl. Phys. B* **188** 317 (1981); Ioffe B L, Smilga A V *Pis'ma Zh. Eksp. Teor. Fiz.* **37** 250 (1983) [*JETP Lett.* **37** 298 (1983)]
11. Novikov V A et al. *Nucl. Phys. B* **174** 378 (1980); **249** 445 (1985)
12. Novikov V A et al. *Nucl. Phys. B* **191** 301 (1981)
13. Vainshtein A *Phys. Lett. B* **569** 187 (2003); hep-ph/0212231
14. Gol'fand Yu A, Likhtman E P *Pis'ma Zh. Eksp. Teor. Fiz.* **13** 452 (1971) [*JETP Lett.* **13** 323 (1971)]
15. Novikov V A et al. *Nucl. Phys. B* **229** 407 (1983)
16. Novikov V A et al. *Nucl. Phys. B* **229** 381 (1983)
17. Seiberg N *Int. J. Mod. Phys. A* **16** 4365 (2001); hep-th/9506077
18. Seiberg N, Witten E *Nucl. Phys. B* **426** 19 (1994)
19. Dorey N, Khoze V V, Mattis M P *Phys. Rev. D* **54** 2921, 7832 (1996)
20. Argyres P C, Plesser M R, Seiberg N *Nucl. Phys. B* **471** 159 (1996)
21. Cachazo F et al. *JHEP* **0212** 071 (2002)
22. Cachazo F, Seiberg N, Witten E *JHEP* **0304** 018 (2003); hep-th/0303207
23. Armoni A, Shifman M, Veneziano G *Nucl. Phys. B* **667** 170 (2003); hep-th/0302163