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What is mathematical physics?

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Matematicheskaya Fizika (Mathematical Physics). Encyclopedia (Ed. L D Faddeev) (Moscow: Bol'shaya Rossiĭskaya Entsiklopediya, 1998) 692 pp. ISBN 5-85270-304-4.

I have learned a great deal from the one-volume encyclopedia *Mathematical Physic*s published by the 'Bol'shaya Rossiĭskaya Entsiklopediya' publishing house.

On p. 237, *a vector tangent to a manifold* is defined thus:

"it is *a linear functional* (*operator*) which associates every differentiable function f with a vector $v(f)$ satisfying the conditions

$$v(f+h) = v(f) + v(h)\,, \quad v(cf) = c \cdot v(f)\,,$$

$$v(fh) = f \cdot v(h) + h \cdot v(f)\,."$$

The editors should have to oppose confusing the operators and functionals, even though they may be unaware that the case in point is actually neither an operator nor a functional and that the author calls the $v(f)$ *vector* what is simply a *number* in reality (to correct the wrong definition now). It seems likely that no one had read this definition before me. Actually, tangent vector to a manifold is the velocity vector of motion along the manifold.

I have gained the distinct impression that this novel ignorant definition of a tangent vector is not an accidental misprint, for in the subsequent pages I have encountered similar absurdities in many encyclopedia articles (and even in the majority of the articles which I managed to read).

The following definition is on page 264:

"*Quaternion* is an element of a set $\mathbb{H}$ of combinations (not defined in the article) 1, i, j, and k, which satisfy the conditions

$$ji = -ji = k."$$

To an ordinary reader like me it is evident that the k vector, which is opposite itself here, is zero. Hamilton apprehended k differently when he invented it. Unlike Hamilton, who described the rotations of a three-dimensional space, the author of the encyclopedia article considers quaternions to be formal symbols, and these patiently tolerate any axioms.

Many other definitions in the book are no better, either. On p. 328, *unitarity* is defined by the formula

$$\sum_k u_{ik} u^*_{ik} = \delta_{ij}\,.$$

The reader is not informed of the significance of either the symbol $*$ or the subscript j, which makes its appearance from somewhere on the right-hand side.

The reader unfamiliar with unitarity will remain so upon reading the 'definition' in this encyclopedia. I hope that real readers are in fact familiar with the true definition.

However, when simple ideas (like preservation of a Hermitian scalar product by an operator) are replaced with complex formulas, misprints are impermissible, because they become insurmountable obstacles to the reader of a formalized text.

The missing quantors[1] ("for any i" in the preceding formula) give rise to the same difficulties for the reader.

It is stated on p. 679 that "*entropy is referred to be the total differential of the state function*". It astounded me (for I considered, after Gibbs, entropy itself to be the state function rather than the differential), but the new encyclopedia introduces new terminology.

Linear inhomogeneous differential operators are termed *nonlinear* in this encyclopedia. On p. 193, for instance, the operator which maps the function u onto the function $\mathrm{d}u/\mathrm{d}x + 1$ is considered to be nonlinear: it is stated there that linear differential expressions are linear in the derivatives of the u function, while *all other* expressions *are spoken of as nonlinear*. This new principle makes it possible to achieve tremendous and rapid advance in nonlinear mathematics and physics (it would suffice simply to employ the formula of perturbation superposition).

Some articles in the encyclopedia discuss important facts in such a way that it is impossible to understand anything. For instance, on p. 336: "Penrose explicitly described geometric structures on T_0, *nonintersecting* l_∞". The author forgot to mention what does not intersect l_∞, and for this reason alone the sentence is incomprehensible (neither T_0 nor l_∞ were defined or mentioned earlier). This situation reinforced the impression I formed that no one had read the encyclopedia before me.

Not far from the place described above, the author mentions a mysterious group of "*SL* projective transformations". I have always thought of *SL*-involved transformations as being linear and not projective, carefully making a distinction between the *SL* and *PSL* groups (following

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[1] Quantors are the expressions "there exists" and "for any". For instance, the statement "$f(x)$ is positive" (without quantors) is an inexact abbreviation of one of the exact statements "there exists x such that $f(x) > 0$" and "$f(x) > 0$ is positive for any x", and without quantors there is no way of telling which of the two is meant. – (*Editor's note*.)

Bourbaki's example), but the *new* encyclopedia accustoms me to *new* terminology (illogical and confusing).

The article about the *rotations of a solid* (on p. 593) informs the reader that 'the central problem of the famous work of S V Kovalevskaya *was posed and solved by Poincaré*''. It is a pity the author did not name the date: did Poincaré do this prior to Kovalevskaya or after her? In any case, Poincaré said nothing of his work on this subject when he awarded a prize to Kovalevskaya for her discovery.

As a matter of fact, the problem was posed by K Weierstrass, who suggested that his learner Kovalevskaya should prove (by Poincaré's method) the *nonexistence* of new cases of integrability of the equations of motion of a heavy rigid body. Kovalevskaya refuted her teacher's idea by discovering a new case of integrability (precisely where Poincaré's method fails). In recent years, several Moscow mathematicians confirmed the nonexistence of new cases (besides Kovalevskaya's) of integrability following Weierstrass's idea.

Prior to that, Poincaré had proven (in a remarkable paper on celestial mechanics) the 'nonintegrable nature' of the three-body problem (in a work fallaciously awarded a prize, the prize money for which he spent to replace the fallacious paper in the *Acta Matematika* journal with the revised version, which later grew into the *New Methods of Celestial Mechanics*).

All this area (including the discussion of the influence of Poincaré's work on Kovalevskaya's discovery) was recently studied and developed by several Moscow mathematicians, but the appropriate reference to the current status of the problem is missing from the paper in the encyclopedia.

By the way, *Lindshtedt's method*, which was elaborated by Poincaré, is referred to as Linshtein's method (on p. 327 and p. 626). Lindshtedt was neither a Muscovite nor a mathematician. The authors of the encyclopedia nonetheless discriminate against this remarkable Swedish astronomer (for reasons unknown to me), who had anticipated the Bogoliubov averaging method by many decades. Poincaré transferred Lindshtedt's theory from the general case to the case of Hamilton equations, while Bogoliubov transferred the Poincaré theory from the case of Hamilton equations to the general case.

Unexpected innovations like those listed above are not infrequently encountered in the encyclopedia, but some of the most important subjects of mathematical physics have been omitted; the simple contents of several basic facts is at best replaced with long lists of their consequences and identities based on the omitted theories (like the above-described attempt to discuss the unitarity of a matrix without mentioning the conservation of Hermitian structure by the operator).

On p. 223, the concept of a *linear operator* is introduced for beginners with the following words: ''the matrix K_{ij} maps a vector x_j into the vector $y_i = K_{ij}x_j$''. This mysterious phrase will certainly do no harm to those who are already familiar with operator matrices. But no one will be able to *understand* what is termed the operator matrix, unless he knows it beforehand. The author of the article did not mention that he made use of the Einstein convention on summation and is supposedly unaware that y_i is *not vector* but a *number* (a *component of a vector*).

I will mention only a few examples of the *cardinal concepts of mathematical physics* omitted in the encyclopedia (instead, it is overfilled with minor details pertaining to less significant subjects; however, really important black holes *turned out to be the titles of seven different articles*).

In the section devoted to partial differential equations (on p. 193), *characteristics* (and with them the *wave – corpuscle dualism* based on them) were completely forgotten. In this case, also lacking is the reference to the article ''Contact geometry'' from the same encyclopedia in which these important objects of ray and wavefront geometry were (duly) described.

The authors also forgot about the *vector-potential of a magnetic field* and about its experimental examination in the (also forgotten) *Aharonov – Bohm effect*, which is of paramount significance for the understanding of the philosophic foundations of quantum mechanical principles. Instead of this, the authors of the article write that ''*the vector-potential of a magnetic field is an unobservable mathematical abstraction*'', which is, as is well known, wrong and contradicts the experiments mentioned above. This outdated delusion of the authors of the encyclopedia underlines the absolute need for a discussion of the Aharonov – Bohm effect in the *Mathematical Physics* encyclopedia.

A strange gap in the encyclopedia is the total absence of the *variational theory of eigenvalues*, including the basic *Rayleigh – Fisher – Courant inequalities* which describe the behavior of the eigenfrequencies of oscillating systems with an increase in their rigidity and under the imposition of constraints.

Also absent is the *Wigner – von Neumann theorem on the repulsion of eigenvalues* (as is the entire important *theory of the eigenvalue distribution for random matrices*).

Instead of this, the reader will encounter (on p. 552) the incorrect statement that ''*a complex $n \times n$ matrix always possesses n complex eigenvectors*'' (in this case, for some reason the author additionally requires that the matrix determinant should be nonzero).

Suchlike ignorance would not allow a student to graduate from a second-year university course, and it is impermissible in encyclopedic articles. It demonstrates the *absolute need for including in the Mathematical Physics encyclopedia an article about Jordan normal matrix forms* (which were mentioned without definition in the article about Hamiltonian systems).

The fundamental *Sturm and oscillatory proprietes theories* for the solutions of differential equations are completely absent in the encyclopedia, although *real mathematical physics is impossible without these theories*.

I have the impression that the authors purposely avoid describing principal objects and theories, compensating this by abundant quoting of the less significant unoriginal following works.

The authors of the articles made wide use of the terms they have not defined (for instance, 'C^* algebras' or 'deviators of tensors') but unfortunately forgot to define the most necessary objects as well, for instance, *Young diagrams* (which are absent, as is the *theory of symmetrical group representations*, depending on them).

The *intertwining number* of two group representations is defined (on p. 567) as the dimensionality of the space of intertwining operators from one representation to the other, but the definition of these intertwining operators remains an enigma.

Neither the *refractive index*, the *Snell law of refraction*, nor the *optical Fermat variational* principle (which undoubtedly deserve three separate articles in the *Mathematical Physics* encyclopedia) — none of these three basic subjects are presently discussed in the encyclopedia. Also absent is the *wave transformation theory* (which explains the formation of waves of a new type on the emergence of a multiple root of the dispersion relation and is required in plasma physics as well as in seismology and geophysics).

The *Gibbs phenomenon* was forgotten, together with its tomographic applications (where this phenomenon accounts for the remarkable artifacts which display on computer tomographic sections of a body the nonexistent lines attributable to the poor convergence of computer-used Fourier series at the boundary contours of bone images). When describing 'vector analysis', the encyclopedia's authors forgot to mention its *principal homotopy formula*

$$L = id + di\,,$$

which expresses the Lie derivative L, also referred to as the fisherman's derivative [2]. This derivative is much more important than the 'covariant derivative' but was also forgotten in the encyclopedia.

Among the main concepts forgotten are *homologies* and *homotopies, manifold orientations* and *characteristic classes of bundle*s, *the Möbius band*, and many others without which modern mathematical physics is incomprehensible.

Chern numbers were correctly used in the physical paper about 'topological charges', but their definition did not find its way into the encyclopedia, nor did the definitions of *knots*, *linking coefficients, cobordisms*, etc., which are also basic to modern mathematical physics.

The absence of all these fundamental ideas and concepts may leave readers with the false impression that the level of mathematical physics in Russia is as poor as that represented in the encyclopedia. Fortunately, so far this is not the case.

A N Kolmogorov learned mathematical analysis by reading articles about it published in the Brokhaus – Efron Encyclopedia. No genius would have been able to comprehend mathematical physics by reading the *Mathematical Physics* Encyclopedia under review here.

[2] Lie's derivative (vector-field-directed derivative) is the time derivative of the tensor field carried by the phase flow of this field past an immobile observer ('fisherman').