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John von Neumann

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Contents

Abstract. **This article is dedicated to John von Neumann, one of the most outstanding scholars of the 20th century. His life was short but bright, and his contribution to almost all branches of mathematics, as well as to physics, economics, biology, and astronomy was enormous. He constructed some of the first computers and he was among the key persons in the American atomic project. Development of his ideas will continue to play a vital part in various areas of pure and applied mathematics.**

Most mathematicians
prove what they can,
von Neumann proves what he wants.
A widespread opinion

Alles Vergängliche
Ist nur ein Gleichnis;
Das Unzulängliche
Hier wird's Ereignis...
Johann Wolfgang Goethe. *Faust* [1]

1. Introduction

The 20th century has found its place in history as a period of the extraordinary development of the physical and mathematical sciences. It is in the 20th century that two fundamental theories — the theory of relativity and quantum mechanics — evolved; they proved to be determinant not only for the physics of the 20th century but, in essence, also for the further development of civilization. First-rate results in mathematics paralleled outstanding achievements in physics. Physics is guided by a mathematical language to express its laws; however, the interaction between mathematics and physics is a complex process far from being unambiguous.

John von Neumann
(28.12.1903 – 8.02.1957)

[1] All things transitory
But as symbols are sent;
Earth's insufficiency
Here grows to event... (*Translated by Bayard Taylor*)

¶ The author is also known by the name M I Monastyrsky. The name used here is a transliteration under the BSI/ANSI scheme adopted by this journal.

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Each of these sciences exercises its specific approach to the problem, and differing interests and styles hamper one from being equally successful in both disciplines. For this reason, scientists who have produced major results in both mathematics and physics are very rare.

John von Neumann holds one of the most honored places among those outstanding scholars of the 20th century whose achievements have received recognition from both scientific communities. Even a simple enumeration of his scientific results is impressive for the extent of their scope and extraordinary productivity.

However, the life of von Neumann is interesting and instructive in a much broader context.

His participation in two extremely important technological projects of the 20th century — development of nuclear weapons and creation of the principles of modern computers — gives grounds to rate von Neumann an influential figure of the past century, who is virtually unknown to the general public. The attitude of his contemporaries to von Neumann was far from unambiguous. Although he was recognized as an outstanding mathematician immediately after the publication of his earliest studies, his conservative views and active anticommunism did not command sympathy in the university community, where "leftist" ideology was widely accepted. The history of the past fifty years has demonstrated that, in this area, von Neumann was also more sagacious than many of his colleagues.

John von Neumann passed away at a relatively young age, and some of the people who knew him are still alive.

Interest in von Neumann's personality is ever-growing. A witness to this fact is the recent appearance of a number of books, including the most complete biography of von Neumann, written by Macrae [1]. In his book, Nicholas Vonneuman [2] describes in detail the years of his brother's childhood and youth. Some other books gathering interesting facts on von Neumann's life are also available [3 – 5]. However, the writings of von Neumann himself are the most important material for reflecting on his creation. Almost all his works were published in a collection [6]. These six *in folio* volumes give a quite adequate idea of the admirable results produced by von Neumann and of his fantastic capability for work. The range of his interests was enormously wide — from set theory to the structure of the human brain and climatological problems. However, the best monument to a scholar is the work of the followers who develop the scholar's ideas. Not only have the studies of von Neumann given us clues for solving some particular, challenging problems but they also have determined the avenue of development of many scientific disciplines up to the present day. A comprehensive analysis of von Neumann's studies and the assessment of their role in modern science would require the work of a team of experts. Such an analysis has partially been done [7, 8]. Of particular interest is the special issue of the *Bulletin of the American Mathematical Society* that appeared shortly after von Neumann's death and that collected articles of famous mathematicians who knew him well [7]. Various aspects of von Neumann's mathematical activity have been analyzed in the proceedings of a number of conferences [8, 9].

Obviously, my short article cannot be a substitute for this vast literature. I will only try to tell the reader about von Neumann without repeating the content of the available books and I will dwell on some particular features of von Neumann's life that seem interesting to our contemporaries and that have received relatively little attention in the already published materials.

2. Life history of János – Johann – John von Neumann

On December 28, 1903, in Budapest, a boy was born into the family of the banker Max Neumann and received the name of János. This family was well-to-do. On the side of the mother, Margaret Kann, the Neumanns descended from an ancient family of Jewish bankers. In 1913, the father of János received hereditary nobility on merit from the Emperor Franz Joseph (which was reflected by the prefix *von*).

Hungary, then part of the Austro – Hungarian Empire, was highly autonomous and was among the most democratic and prosperous countries in Europe.[2] The yearly gain in its gross national product was 6 – 10% from 1867 to the beginning of World War I. Hungary was famous for its illustrious musicians, poets, and actors. The status of scientists was very high. It is during this period that the University of Budapest had an excellent professoriate. School education resided at a very high level.

Three schools were regarded as the best ones in Budapest and in all of Hungary — the Lutheran Gymnasium, the Minta Gymnasium, and the Real Gymnasium. It is sufficient to mention only a few names of outstanding scientists, graduates of these schools. These included T von Kármán and E Teller (they graduated from the Minta Gymnasium, which was founded by the father of von Kármán), L Szilard (Real Gymnasium), and D Gabor, E Wigner, and M Polanyi (Lutheran Gymnasium).

János entered the Lutheran Gymnasium at the age of ten, but he had received a good home education before. The Lutheran Gymnasium was famous for excellent teaching in classical subjects, including the Greek and Latin languages; natural sciences were also taught at a very high level. The teacher of mathematics Laszlo Ratz was especially noted; he comprehended very soon that the abilities of his pupil were extraordinary. In essence, von Neumann had an individual learning program. The university teachers J Kürschák, G Szegö, and M Fekete gave him lessons. The leading Budapest mathematician L Fejér carefully supervised him. By the time he graduated, the pupil was a formed professional mathematician. His first study, on the distribution of zeroes of Chebyshev polynomials, co-authored by M Fekete,[3] was published in 1922, when János was 18 years old. Clearly, János aimed to enter the faculty of mathematics; however, his father believed that mathematics could not guarantee a stable income and insisted on choosing a more 'practical' profession. A compromise was reached in a nontrivial way. János entered two universities simultaneously — the famous Eidgenössische Technische Hochschule (ETH) in Zürich, as

[2] According to the Austro – Hungarian Treaty of 1867, Hungary was a completely independent state, apart from the spheres of finance, army, and diplomatic representatives. Among other things, this resulted in the passing of a bill on the complete equality of Jews in their rights. This drastically distinguished Hungary from all other countries of Eastern Europe. During that period, Hungary (along with the United States) attracted numerous emigrants from many countries, especially from Russia and Romania.

[3] M Fekete (1886 – 1957) was a professor at the University of Budapest. He emigrated from Hungary in the 1930s and died in Israel the very year his illustrious student did. Fekete himself was a well-known mathematical analyst; he devoted his entire scientific life to the theory of approximation of functions by polynomials with integer coefficients.

a chemical engineer, and the mathematical faculty of the University of Budapest. Studies at the University of Budapest did not occupy much time. As a rule, János visited the university at the end of the semester to pass the examinations. Chemistry took much more time. Nevertheless, he had two diplomas by 1925, five published works (mainly on the foundation of set theory), and fairly good contacts with well-known German mathematicians.

It was also clear to him that he could no longer stay in Hungary.

What occurred in Hungary after World War I? The defeat and the disintegration of the Austro–Hungarian Empire led to the Communist Revolution in March 1919. The Hungarian followers of Lenin remained in power for only four months, but they left behind painful memories. In his book [4], Pais notes the strong anticommunism practized by von Neumann and Wigner, which stemmed from the impressions of their youth. The leaders of the Hungarian Communist Party participated in the Civil War in Russia (in particular, the head of the Hungarian Republic Béla Kun was 'famed' for shooting captive officers of the Wrangel army in the Crimea in 1920), so that it is beyond question that they also committed follies in their own country during those four months. However, as often happens, the pendulum of history swung in the opposite direction, and the extreme fascist, Admiral Horthy, came to power after the communists. A definitely anti-Semitic, fascist dictatorship was established in Hungary. In particular, a five-percent limit (similar to that accepted in tsarist Russia) was introduced at the universities. However, in contrast to Russia, where Jews were discriminated against by religious criteria, the discrimination in Hungary was purely racial. Since, in addition, Hungary had sustained substantial territorial losses (of two thirds of its area and the same fraction of its population, according to the Treaty of Trianon, 1920), a few years changed it from a prosperous country into the gloomy boondocks of Europe.

That is really a very instructive story. The well-known quote of Pascal comes to mind: "Cutting off 300 prominent heads in France is sufficient to change it into a country of idiots". This quantitative estimate does not vary considerably from country to country, and this statement remains valid even nowadays.

We should, however, give Horthy's regime its due: the borders were not closed, and everybody wishing to leave Hungary could do so.

Throughout the year 1926, János, who had become Johann von Neumann, stayed in Göttingen as a Rockefeller fellow. Here, he met outstanding mathematicians and physicists, first of all, David Hilbert, and became his nontenured assistant. This was a fantastic time. It is in Göttingen, in Max Born's school, that quantum mechanics was created, and von Neumann actively participated in developing it. A year later, after having moved to Berlin, he continued close collaboration with mathematicians and physicists from Göttingen.

However, excellent mathematicians also worked at the University of Berlin. Von Neumann was especially influenced by Erhard Schmidt, a prominent specialist in functional analysis.

Over a short, two-year period which was, however, fantastic in terms of productivity, von Neumann published 20 papers or so. These were purely mathematical works, including his dissertation (Habilitationschrift) "Die Axiomatisierung der Mengenlehre" [10], as well as studies in quantum mechanics and, finally, the famous paper "Zur Theorie der Gesellschaftspiele" [11] that opened up a new mathematical discipline — games theory.

In 1929, he got a position at the University of Hamburg, again as privatdocent. However, already in 1930 he was invited to Princeton University as a visiting professor for one year and he gladly accepted.

Besides his interest in America, at least two things compelled him to take this step. The first reason was related to the difficult economic state of Germany and the growing activity of Nazism. Von Neumann had seen Horthy's regime and Béla Kun's Bolshevist Revolution in Hungary, and it was more clear to him than to many of his contemporaries what the victory of the Nazis might result in. The second reason was more practical. Von Neumann was not content with the privatdocent position, which was not promising in terms of high salaries and, moreover, was very 'unstable'. On the other hand, professorship guaranteed a highly honorary and beneficial position in Germany, but obtaining it was not realistic. As von Neumann said, "every year there were 40 candidates for three professor positions, so that the mathematical expectation exceeded ten years".

Thus, he moved to Princeton in 1930, where he spent a year as a visiting professor and then got a permanent position. Von Neumann, who became John (permanently this time), worked at Princeton University until 1933, when the Institute for Advanced Study was founded [12]. A Einstein and O Veblen became the first professors of this institute. Von Neumann shared the honor of being in the first group with J W Alexander, H Weyl, and M Morse.

The merits of the 30-year-old scholar were thus deservedly recognized. He already had a number of first-rate achievements by that time. First and foremost, besides the already mentioned work on the foundation of set theory, a cycle of studies on the mathematical foundations of quantum mechanics should be noted; they were summarized in the fundamental monograph *Mathematische Grundlagen der Quantenmechanik* [13].

Von Neumann's studies during this period can be broken down into the following thematic groups:

(1) quantum mechanics;
(2) ergodic theory;
(3) spectral theory of linear operators;
(4) game theory.

(Below, we will analyze von Neumann's scientific achievements in greater detail.)

It is beyond doubt that von Neumann was the most illustrious mathematician of his generation in America.

Von Neumann spent the remainder of his life working at the Institute for Advanced Study. These activities paralleled many other responsibilities; in particular, he frequently consulted for various commissions and military services. Nevertheless, whatever von Neumann was engaged in, including the strenuous work on the atomic bomb (1943–1945), he always eventually returned to Princeton, to his own institute and home, where he had created an inimitable atmosphere of a European lifestyle.

It is in Princeton that he carried out his remarkable studies on factor theory and the theory of topological groups, wrote the book *Theory of Games and Economic Behavior* [16] co-authored by his friend, the economist O Morgenstern, and constructed the famous 'Johniac' computer.

However, von Neumann did not feel comfortable within the narrow bounds of purely academic scholarly activities. At the very beginning of his work at Princeton, he established close collaboration with various organizations developing applications. In particular, he was a consultant to a number of important military (mainly artillery) projects. Von Neumann's mercurial, well-organized mind made him unrivalled in diverse discussions. For the military community, he was an authority beyond exception. He was said to be worth a whole division. Of particular importance were his merits in the development of atomic weapons — the area that required knowledge of not only nuclear physics but also fluid mechanics (at the early stages of this work, nuclear physicists were out of touch with it). Von Neumann, who became interested in the problem of turbulence even in the mid-1930s, was an extremely valuable acquisition to the project.

The work on the Manhattan Project has been described in an immense number of books and articles, so I will not add anything new. [4]

It seems important to note only one circumstance that resulted in the creation of such a powerful weapon, which remains a factor in politics in the modern world even now. A small number of first-rate scientists, mainly European expatriates, opposed Hitler's tyranny. Were it not for Hitler, there would not have been an atomic bomb for a long time.

However, after having bred this monster, they could not stop. At this point, we are treading on the infirm ground of speculation. Somehow or other, von Neumann found himself among those scientists who decided to go through thick and thin even after Hitler was brought down. Stalin became their bitter enemy. Von Neumann was one of those scientists who hated Stalin. It is interesting that he did not adopt this attitude at once. Even in 1936, in particular, he realized that a war in Europe was possible and, having an objective view of the capabilities of France, he believed that only the Soviet Union would be able to stop Hitler.

Von Neumann visited the Soviet Union in September 1935 to take part in the famous Moscow Topological Conference. For almost 30 years, this meeting, with its brilliant attendance, organized by P S Aleksandrov became the last major international mathematical (and not only mathematical) conference held in the Soviet Union. The list of Soviet and especially foreign participants is extremely impressive: virtually all famous topologists of that time (J Alexander, S Lefschetz, H Hopf) as well as young mathematicians who became famous later (A Weil, H Whitney, G de Rham, W Hurewicz) attended this meeting. Von Neumann gave a talk on the uniqueness of the Haar measure. This study was later published in the *Matematicheskiĭ sbornik* (*Mathematical Collection*). His other long paper on algebraic constructions in quantum mechanics was also published there. One more von Neumann paper published in the Russian journal *Trudy Tomskogo universiteta* (*Proceedings of Tomsk University*) was devoted to studying the metric properties of the space of infinite-dimensional matrices. The somewhat baffling choice of place for publishing this work has a curious relationship to the questions under discussion and can be accounted for as follows.

In the early 1930s, when far-sighted people recognized the danger of Hitler's coming to power, many of them — first of all, scientists — began considering emigration from Germany. Moving to the Soviet Union, which they regarded as a very progressive country with a great future, became one of the attractive aims. Moreover, many scientists shared left, pro-communist views. A number of widely known German mathematicians of Jewish extraction immigrated to the Soviet Union. It is sufficient to mention S Cohn-Vossen, a prominent geometer, Hilbert's co-author in writing the book *Geometry and the Imagination*; the topologist, Leningrad University professor G M Müntz; A I Plesner, the author of remarkable studies in functional analysis, in particular, the first review article published in *Uspekhi matematicheskikh nauk* (*Advances in Mathematical Sciences*), and F I Frankl, the well-known specialist in gas dynamics.

A small group of prominent mathematicians, including Fritz Noether — the brother of the famous Emmy Noether — and Stefan Bergman came to Tomsk University. This university, which had already been rated among the best in Russia, became an important center of mathematical research. In particular, the publication of the journal, in which the studies of widely known mathematicians and other interesting papers appeared, was indicative of the activity and recognition of the university. Unfortunately, this period of bloom proved to be short. Fritz Noether, for whom an important notion of the modern theory of partial differential equations — the Noetherity of an operator — is named, was arrested in 1937 and shot in 1941 in the 'famous' Orel prison. Stefan Bergman managed, after a number of incidents, to escape from the Soviet Union through Tbilisi. Subsequently, he became a professor at Stanford University, where he worked together with such outstanding mathematicians as Gabor Szegö (one of von Neumann's first teachers), György Pólya, D Spencer, and many others, including the well-known Russian mathematician Ya V Uspensky, who held the rank of academician and emigrated during the Soviet period. Undoubtedly, von Neumann was aware of the sad fate of the German emigrants in the Soviet Union.

The years of the war were an extremely fruitful period in von Neumann's scientific career. They were, however, mainly devoted to applied problems. Scientifically, his participation in the Manhattan Project included the development of numerical techniques for solving gas dynamic equations. In particular, his and Richtmyer's work on the description of shock waves using the equations of fluid dynamics, which included the development of finite-difference methods of solving such equations, became a classic and opened a new avenue of research; these studies continue up to now. Although von Neumann was greatly engaged in applied problems, he found time to write, in cooperation with his close friend Oskar Morgenstern, the monograph *Theory of Games and Economic Behavior* [16]. This book develops the ideas of his much earlier study [11]. In the opinion of a number of outstanding modern economists, its material constitutes the basis of all mathematical methods used in economics. Without a doubt, if von Neumann had been alive in 1969, he would have won a Nobel Prize in Economic Sciences.

Some remarkable features were present in von Neumann's postwar life.

In science, he was mainly absorbed in the development of computers. He managed to persuade the management of the Institute for Advanced Study to launch, in Princeton, the construction of a new computer of the highest performance. It is worth noting that these ideas did not suit some professors,

[4] Let us only mention that von Neumann and Klaus Fuchs put forward a principle for constructing the hydrogen bomb — radiative implosion. A patent for this discovery was issued in 1946, and it has not been declassified up to now [17].
Since Fuchs was the chief Soviet spy in atomic project, the developers of the Soviet hydrogen bomb were evidently aware of this result.

who believed that their solitary, quiet life would end with the arrival of engineers, computer specialists, and other similar public — an inevitable accompaniment of such affairs.

It is known that, when this project was being discussed by the tenured professors of the Institute for Advanced Study, von Neumann claimed that the computer would speed up calculations by a factor of several hundred; this prompted the famous number theorist Carl Ludwig Siegel to object that he did not need this at all: if knowing some logarithm was necessary, he could calculate it in his mind, even without a table.

Nevertheless, von Neumann succeeded in implementing this project. It was important that the Director of the Institute, Frank Aydelotte, and Robert Oppenheimer, who became the next director in 1947, supported this undertaking.

Another area in which von Neumann managed to make fundamental discoveries was the theory of automata. His ultimate aim was to comprehend and, possibly, to master simulating the action of the brain. Regrettably, his untimely death prevented von Neumann from achieving a deep insight into this subject. The fundamental question he raised was as follows. Let a set of structural blocks of an automaton be specified, each block (or element) operating with a certain probability of its trouble (failure). Is it possible to construct an arbitrarily large and complex automatic machine whose probability of trouble (failure) could be controlled, i.e., could be made arbitrarily small or at least smaller than a given value? In other words, as von Neumann himself said, can a reliable machine be made of unreliable elements? The analogy between the operation of such an automaton and that of the brain is quite clear. It is well known that, in some cases, the human brain can continue functioning as a whole entity even if some areas of its cortex are substantially damaged. Von Neumann's approach was remarkable: previously, the development of computers was based on assigning a decisive role to each element, so that the failure of any block stopped the operation of the entire machine. Von Neumann's extraordinary ability to formalize any problem and impart a distinct mathematical meaning to it also revealed itself in this area of his activity. In essence, he developed a formal theory of automata by elaborating a system of techniques which still retain their fundamental value [18]. One more of von Neumann's interesting ideas concerning automata is the creation of self-reproducing automata or even automata with progressively increasing complexity [19]. Von Neumann outlined his idea of the evolution of automata at a symposium in Pasadena in 1948 and developed it in an incomplete manuscript. His considerations on the theory of self-reproducing automata remain highly important even now. For example, fighting computer viruses raises problems very close to von Neumann's theory.

After the war, in parallel with his purely scientific work, von Neumann took progressively larger part in the action of high governmental institutions. His career in public service culminated when he was appointed to the Atomic Energy Commission — the highest governmental body planning all activities associated with atomic policy in the US. This commission consisted of only five people who were nominated by the President and confirmed by the Congress. Von Neumann's speeches were always distinguished by an exclusive clarity of formulation.

Admiral Strauss, who headed the commission in those years, noted that there was no need to continue the discussion after a speech by von Neumann.

Von Neumann also encountered then serious moral problems. One of the most widely known actions that forced all prominent atomic scientists to take a position was the famous action against Robert Oppenheimer: the scientific head of the Manhattan Project was accused of anti-American activities. This was a fairly intricate action, and we will not discuss it. However, if we read von Neumann's testimony at the Un-American Activities Committee and keep in mind his substantial dissent from Oppenheimer's opinion of both the possibilities of creating the hydrogen bomb and political issues, we can qualify his answers as quite worthy. He signed (although not immediately) a letter by a group of professors in defense of Oppenheimer. He was among those who rejected any attempts to accuse Oppenheimer of spying or tactically sabotaging the creation of the hydrogen bomb. At the same time, von Neumann himself advocated the development of the military atomic industry and adhered to extremist views in discussing US atomic strategy. "It is known" that he even considered a preventive nuclear attack on the Soviet Union. From the present-day standpoint, these views seem monstrous; however, we should keep in mind the atmosphere of the late 1940s and early 1950s. This was the period of the death-throes of Stalinist regime. Von Neumann, as a politically active person who was also versed in history, clearly understood what the policy of concessions to Stalin could result in. The Korean War, which began in 1950, led the world to the verge of a direct armed conflict between the US and USSR. We recall here these well-known facts only to illustrate how it may be difficult to reach global political decisions. On the other hand, to adequately assess any declaration, one must consider its background and the evolution of thinking of the particular person. Let us give only two examples.

The first example refers to the famous mathematician and philosopher, a votary of peace, Bertrand Russell, who founded the Pugwash Movement at the end of his life. He also spoke of a first strike on the USSR at nearly the same time as von Neumann did. Several years later, he became a supporter of disarmament, and when someone reminded him of his previous statements, Russell completely denied them, so that an old newspaper with these statements had to be shown to him. After changing his views, he simply forgot the old ones.

We borrow another example from the life of Andreĭ D Sakharov [20]. During the same years, he was absorbed in considering a plan for sweeping away the US through an attack with an underwater-based torpedo equipped with a hydrogen bomb. It was meant to surreptitiously approach the US coast and launch the torpedo. Another, no less exotic plan involved simultaneously exploding several depth-charges and producing artificial tsunamis. It is remarkable that an admiral with whom Sakharov shared this plan rejected Sakharov's idea quite nobly and said that the navy does not wage war against noncombatants. Perhaps this reply stimulated Sakharov's evolution toward the person of truly democratic convictions he later became.

Let us, however, return to von Neumann.

His veritably fantastic capacity for work faltered in 1955, and shortly thereafter he was diagnosed as having cancer. The subsequent two years were spent in fighting against the severe disease. It is not unlikely that one of the factors that triggered the cancer was von Neumann's participation in testing the hydrogen bomb at Bikini in 1954, where the precautions taken by the people involved were absolutely insufficient.

Von Neumann continued working until nearly the last days of his life, and he died on February 8, 1957, in a governmental hospital in Washington. One of the most brilliant minds of the 20th century passed away; however, his works survived, and this is a happy feature of a scientist's life.

It is hopeless to try to describe, even in outline, the entire spectrum of von Neumann's studies, to say nothing of assessing their influence. His works penetrated the entire building of modern mathematics, and many his ideas still remain to be developed. Let me have a look at some of them — naturally, in view of my own interests and capabilities.

3. Von Neumann's scientific achievements

In 1954, von Neumann filled out a questionnaire of the US National Academy of Sciences to which he was elected in 1937. When answering the question on which of his scientific results he regarded as the most important, he noted three cycles of studies. Among them, the foundations of quantum mechanics were mentioned first; investigations in the ergodic theory and the theory of operators were considered the two other principal achievements. All these results, most interesting to physicists, are in inherent unity. It is worth dwelling on them particularly.

3.1 Studies in quantum mechanics

The work on the mathematical foundations of quantum mechanics, which was done by von Neumann during his stay in Germany, mainly in Göttingen and Berlin, are wonderful for their depth, vigor, and insight. This cycle of studies was completed when the author had not yet reached 30. It was summarized in the monograph *Mathematische Grundlagen der Quantenmechanik*, published in 1932 and remaining seminal in this area even today.

A fundamental result of paramount importance obtained by von Neumann is the proof of the theorem on the impossibility of introducing 'hidden parameters' in quantum mechanics. This statement forms a reliable groundwork for the probabilistic interpretation of quantum mechanics and demonstrates that nondeterministic elements cannot be eliminated in the measuring process. As is well known, the problem of quantum-mechanical determinism prompted high-pitched debates immediately after quantum mechanics was developed. Einstein, Bohr, Heisenberg, and other founding fathers of modern physics participated in these discussions; they stimulated a comprehensive investigation of the fundamental issues of quantum mechanics, including the development of the theory of quantum measurements, which is well underway at present. Nevertheless, von Neumann's basic result — the impossibility of evolving a completely deterministic quantum mechanics — remains inviolable.

Another result, which was obtained by von Neumann in 1927 and which is important to statistical quantum physics, is the introduction of the density matrix — a key tool for the description of a system of particles rather than a single particle. Von Neumann gave a highly general formulation to the density-matrix approach, which immediately came into use in the quantum theory of measuring. However, he shares his merit in this area with L D Landau, who introduced the density matrix shortly (several months) before, but for a special case, in the problem of damping in wave mechanics [21]. Von Neumann cited this remarkable study of the 19-year-old Landau. In the quantum-theory studies of the 1920s, which inspire our admiration for the outstanding results, two facts attract our particular attention. First, we note the very rapid publications and the high awareness of the works of colleagues the acting personages had; second, the correct citation. We should not forget that most results that are now considered classical were obtained by very young people — in many cases, in their first works.

Among the studies that definitely have physical applications, we should mention the cycle of papers (1927 – 1928) co-authored by von Neumann's childhood friend E Wigner [14]. In these works, methods of the theory of (symmetric and orthogonal) group representations were applied for the first time to classifying the spectra of multilevel atoms. As Wigner remembered later, these papers were written directly by him, but von Neumann gave him invaluable help noting studies by Frobenius and Schur that contain the needed mathematical apparatus. Any specialist can undoubtedly value such hints in similar general problems.

Other than particular results, the systematic treatment of the foundations of quantum mechanics in terms of Hilbert spaces constitutes another, no less important fundamental work by von Neumann.

Virtually the same approach is still used in all modern textbooks on quantum mechanics. The technique of Hilbert spaces, besides having purely methodical merits, also led von Neumann to a comprehensive analysis of the operators acting in Hilbert spaces.

Investigation of linear unbounded operators, numerous examples of which are supplied by quantum mechanics, became a subject of von Neumann's longstanding studies. His achievements in this area were also outstanding.

It is worth mentioning that he was extremely fortunate in these activities. While his quantum-mechanical studies of the 1920s were stimulated by contacts with physicists and mathematicians at Göttingen, he could discuss the mathematical problems of operator theory when cooperating in Berlin with a prominent expert in this field, Erhard Schmidt. [5] Professor Schmidt was well acquainted with von Neumann since he was a member of the jury that awarded a doctorate (Habilitation) to von Neumann for his dissertation in the axiomatics of set theory. Von Neumann did not forget his old teacher and paid tribute to Schmidt in his last paper on operator theory published in *Festschrift*, 1954, when Schmidt's 75th birthday was being celebrated.

3.2 Operator theory

The year 1929 was especially fruitful in von Neumann's scientific career. He published ten papers that year, only two of which (on more special physical applications) were co-authored by someone else — his friend E Wigner. Two papers were dedicated to the theory of operators; they have had an extremely profound impact on all subsequent developments in this branch of mathematics.

In the first study, "Allgemeine Eigenwerttheorie Hermitiescher Funktionaloperatoren", he constructed a spectral decomposition for Hermitian and more general normal operators. He managed to prove a theorem on spectral decomposition for unbounded operators, including opera-

[5] E Schmidt was among those few German authorities on mathematics staying in Hitlerite Germany, who behaved decently. For this reason, many of his 'non-Aryan' colleagues among the authors of *Festschrift* had good memories of him. His life was destined to be a long one: he survived von Neumann by two years.

tors with continuous spectra. These results were a substantial advance compared to the classical results obtained by his predecessors Hilbert, Schmidt, E Hellinger, H Hahn, and T Carleman, who mainly considered only bounded operators. Somewhat later, he and M H Stone independently developed a theory of operational calculus for such operators. It is remarkable that von Neumann suggested a fundamental new idea. The analysis of the properties of a particular operator can be substantially simplified if one considers a whole family of analogous operators possessing an additional algebraic structure.

In the paper "Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren", he introduced a ring of operators that came later to be known as the von Neumann algebra.

A *von Neumann algebra* (W *algebra*) is a ring (or, according to the modern terminology, algebra) of operators that includes, along with the operator itself, its adjoint one and all limiting operators in the so-called weak operator topology.

The basic result presented in that paper is the so-called double-commutant theorem that determines the conditions under which the second commutant of a W algebra coincides with this algebra itself.

However, the most important result of this work is certainly the introduction of W algebras.

In 1943, I M Gel'fand and M A Naĭmark introduced a wider class of operator algebras — the so-called C^* algebras which include the W algebras. The importance of the notion of operator W algebras (other than the purely mathematical interest in them) is the fact that they naturally appear in the theory of group representations and therefore have natural physical applications. Another natural class, a C^* algebra, is an algebra of observables in the quantum field theory and quantum statistical physics.

In view of the rich inherent problems and possible applications of the theory of operator algebras, this field of research remains topical even now.

However, von Neumann himself, partly together with his co-worker F Murray, found quite a new field of research related to operator algebras. He revisited these problems later, during his Princeton period. In the cycle of four papers entitled "Operator rings. I–IV" and several related ones, von Neumann and Murray scrutinized a special class of operator algebras that was denoted by von Neumann as 'factors'.

Factors are the rings of operators W whose center, i.e., $W \cap W'$, consists of scalar operators $\{\lambda E\}$ only. What is the ring of factors composed of?

In a finite-dimensional case, the Schur lemma [22] can easily be used to demonstrate that the matrix algebras Mat_n operating in R^n space are such rings. The corresponding invariant classifying these rings is dim (R^n), the dimension of R^n space.

However, this is far from the case for the infinite dimension. Besides the factors of the class of full matrix algebras Mat_n, known as *type*-I *factors*, several other classes of factors exist, whose properties are completely different.

The invariant that classifies these classes is the dimension of the factor, dim W. This fairly crude invariant makes it possible to divide the factors into four classes. The dimension of the factor has a number of peculiar properties. In contrast to the case of type-I factors, dim W can assume any real values or belong to a certain real interval — for example (at the proper normalization), the interval (0, 1). Such factors are called *class*-II_∞ and *class*-II_1 *factors*, respectively. There exist *class*-III *factors*, whose dim W takes a value of 0 or ∞. In any class of factors that have the same dimension, the infinity of pairwise nonisomorphic factors exist.

The introduction and classification of factors can definitely be characterized as an outstanding mathematical discovery. To achieve it, von Neumann needed not only a consummate management of the techniques of operators and Hilbert spaces but also a deep comprehension of set theory.

As for the cycle of von Neumann's studies in factor theory, it could hardly be said when factors would have been discovered if he had not done so.

Not only were these studies aimed at solving some challenging problems, but, as with everything done by von Neumann, they also contained more extensive plans.

Von Neumann expected that his theory would be applicable, apart from group representations, to quantum physics. In these investigations, von Neumann was well ahead of his time. Interest in factor theory revived in the late 1960s. By that time, fundamental results in factor theory were obtained, series of nonisomorphic type-III factors were constructed, and many problems formulated by von Neumann were solved [15]. However, no very bright applications were known in other areas of mathematics and physics. A breakthrough occurred in 1984, when the young mathematician Vaughan Jones found an elegant application of factor theory to the theory of knots. Using type-II_1 factors, he constructed a new invariant that discriminates nonhomeomorphous knots [23]. Furthermore, interesting links to Temperley–Lieb statistical lattice models were revealed.

This confirms the well-known empirical rule: deep and elegant mathematical theories are never wasted. As von Neumann himself stated, "Modern mathematics can be applied after all. It is not clear *a priori*, is it, that could be so" [7].

3.3 Ergodic theory

In 1929, von Neumann turned, in parallel with the theory of operators, to another very interesting problem — analysis of the statistical properties of a macroscopic system from the standpoint of quantum mechanics. His considerations resulted in the proof of the ergodic theorem for a quantum-mechanical system. Since this study is widely known (and published in a Russian translation as an appendix to book [13]), we will, without dwelling on it, pass to another interesting subject which is related to the beginning of von Neumann's American period of life. We will discuss the proof of the ergodic theorem in classical mechanics — a problem that absorbed such scholars as L Boltzmann, J W Gibbs, and some others. Although many first-rate scientists put in considerable effort, no approaches to solving this problem were implemented. An unexpected success was achieved in 1931; it was related to a crucial remark made by the American mathematician B O Koopman. A Weil wrote that he had also noted this fact; however, only Koopman's remark was published. Let us recall the result produced by Koopman.

Consider a function $f(x)$, $x \in X$, and a transformation T: $f(x) \to f(Tx) = g(x)$. This transformation induces the operator U: $f \to g$ on the space of functions. If T is a measure-conserving transformation, the operator U is isometric. If the reversibility condition is additionally imposed on the transformation T, the operator U will be unitary.

The classical ergodic hypothesis can be formulated for discrete transformations T^n as the following statement: the ratio of the 'time' spent by the iterations of a point, $x \to T^k x$, in E to the total number of k values considered (i.e., $k \leqslant n$) has a limit as $n \to \infty$. Here, the very existence of a limit in any sense is far from obvious. If f is the characteristic function of the set E, our hypothesis can be rewritten as the statement that the series

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

converges in the mean.

Von Neumann reformulated the condition of convergence in terms of unitary operators and proved the following theorem (we present it here in a somewhat more general form, after the prominent Hungarian mathematician F Riesz).

Von Neumann's statistical ergodic theorem. *Let U be an isometric operator in a complex Hilbert space H, and let P be a projector onto a subspace invariant with respect to U; then*

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j f \to Pf \text{ for any vector } f \in H.$$

Von Neumann's original proof is given in note [24]. Riesz's formulation and proof are presented in the remarkable book *Lectures on Ergodic Theory* by von Neumann's disciple P Halmos [25].

Von Neumann's theorem refers to convergence in mean (in terms of the norm in L_2). However, as von Neumann himself proved in note [26], this statement is sufficient to prove the ergodic hypothesis. This theorem was proved by von Neumann in October 1931 but published only in January 1932 in *Proceedings of National Academy of Sciences*. However, as early as in December 1931, the outstanding American mathematician G D Birkhoff published two notes that contained the proof of a stronger result — the individual ergodic theorem that states a point-by-point convergence (almost everywhere) of time averages to a spatially averaged function. This result was stronger, and the proof based on delicate combinatoric consideration was more complex. Thus, most of the fame went to Birkhoff. However, a detail poorly known, especially to the contemporary readership, and remarkable in the context of scientific ethics is present in this history. Its traces can be found among minor notes that accompany both Birkhoff's and von Neumann's papers; they say little to a reader who does not know the origin of the history. To understand what happened, we should remember that Birkhoff was then the most competent and influential mathematical authority in America. A word from him was sufficient to obtain or fail to obtain a professorate at a university. At the same time, von Neumann was, although a genius, only a young man newly appointed to his position at Princeton University. After proving his theorem in October 1931, he told Birkhoff of it. Birkhoff immediately assessed the importance of von Neumann's result and started proving a strengthened version of von Neumann's theorem. He was well prepared for this activity because he had worked in a closely related area — the theory of dynamical systems — for a few dozen years. After very hard work for a month, he proved the individual ergodic theorem. Then, he did something very improper. This can easily be noted by comparing the submission dates of two his papers. The first one was received on November 22, and the second on December 1; both appeared as soon as in the December issue of *Proceedings*. Von Neumann's note, not yet published, as Birkhoff mentioned, appeared only a month later, early in 1932. It was regarded by many people as a weaker version of Bikhoff's work. This history did not remain unnoted, so that Birkhoff was even forced to publish another note (co-authored by Koopman) under the expressive title "On the history of the proof of the ergodic theorem" [27].

Birkhoff's unfair action consisted in the fact that he delayed von Neumann's paper until the publication of his own. Any specialist can understand the importance of a breakthrough in a problem that has seemed unassailable.

Although von Neumann usually took the priority question fairly coolly, this history — in view of the importance of the result — upset him very much, and he still remembered it many years later. In his studies in ergodic theory, focused on the spectral properties of dynamical systems, von Neumann revealed a remarkable phenomenon: systems that possess different sets of states can have the same spectrum. Among such systems are, in particular, dynamical systems constructed in the form of shifts of Bernoulli sequences with varying numbers of states.

The question naturally arose as to whether such dynamical systems were isomorphic — in particular, if shifts in the binary or ternary Bernoulli systems are considered. Von Neumann expected a negative answer. This challenging problem was solved by A N Kolmogorov only after von Neumann's death. In 1958, Kolmogorov constructed a new invariant of a dynamical system — entropy — and computed it for Bernoulli systems with different numbers of states to show that the entropy is different for them. In particular, it is log2 for shifts on binary sequences, and log3 on ternary sequences.

Kolmogorov's entropy opened new horizons in the theory of dynamical systems, which is among the most actively developing branches of modern mathematics.

It is no accident that Kolmogorov's name appears in our article on von Neumann. These two outstanding mathematicians of the 20th century were of the same age; they knew well and appreciated each other's studies. They did not directly compete in solving particular problems but, in essence, they often continued each other's work. Now, when Kolmogorov's diaries and correspondence have partly been published [28], we know how carefully he kept watch over von Neumann's work and even likened himself (under cover) to von Neumann. Even in fields far from mathematics, the interests and tastes of von Neumann and Kolmogorov proved to be similar. In particular, both knew Goethe's *Faust* thoroughly and liked it.

Similarly, von Neumann was well up on Kolmogorov's works (especially, prewar ones). In particular, he prepared an excellent review of studies in turbulence, in which he paid tribute to Kolmogorov's results [26]. As is known, Kolmogorov's classical studies on turbulence published before the World War II were poorly known in the West and were subsequently rediscovered by the outstanding scientists L Onsager, W Heisenberg, and C von Weizsäcker.

The last, very symbolic meeting between Kolmogorov and von Neumann took place at an International Mathematical Congress in Amsterdam in 1954, where von Neumann delivered the opening lecture "Unresolved problems of mathematics", and Kolmogorov gave the closing lecture "General theory of dynamical systems and classical mechanics".

In the context of von Neumann's contribution to the science of the 20th century, we would like to note some particular features of his creativity that distinguish him even against the background of the many outstanding mathematicians the 20th century was so rich in.

The first thing that distinguishes von Neumann is the comprehension of mathematics as one whole. Among the scientists of this class, we can remember D Hilbert, H Poincaré, H Weyl, and, out of those who are alive, I M Gel'fand.

Second, there are many excellent mathematicians who can solve a difficult but strictly formulated problem.

There are mathematicians who formulate and solve problems that, in view of their contemporaries, do not deserve attention. Their importance becomes clear only many years later. Such mathematicians are very few in number.

Last but not least, there are mathematicians who regard mathematics as part of all sciences; they find and solve problems that are interesting and important to various branches of knowledge.

Thus, if we consider the intersection of all the 'four sets', we will find that only some individual mathematicians belong to it; von Neumann would undoubtedly be among them.

4. Comments on the literature

In addition to the books and papers cited in the body of the text, we included some other books and reviews in the reference list; the reader can use them to gain an idea of the state of the art in the corresponding areas [29 – 31]. A Russian translation of a book by von Neumann's close friend, the prominent mathematician S Ulam [3], appeared recently. Unfortunately, the translation is highly careless and contains anecdotal mistakes. A brochure by Danilov [32] published in 1981 gives an insight into von Neumann's activities. It also contains a list of von Neumann's works translated into Russian. Its factual material is based on Ulam's book whose distribution over the USSR was, however, prohibited in that period. Only the special depository of the Library for Social Sciences of the USSR Academy of Sciences had a copy of this book. For this reason, not even Danilov could give a bibliographic reference. In 1987, two volumes of von Neumann's selected works were published in the series *Klassiki Nauki* (*Classics of Science*) edited by Kolmogorov and Sinaĭ [33]; they included his famous studies in factor theory and ergodic theory. A bibliography of von Neumann's works was also presented. Von Neumann's studies were supplemented with comments written by distinguished mathematicians, which outlined the state of the art in the corresponding areas. There is no need to mention that substantial progress has been achieved and outstanding results have been produced during the past 20 years.

John von Neumann has left fertile ground for further research by his followers.

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References

1. Macrae N *John von Neumann* (New York: Pantheon Books, 1992)
2. Vonneuman N *J von Neumann as Seen by His Brother* (Private Printing: Meadow Book, 1987)
3. Ulam S M *Adventures of a Mathematician* (New York: Scribner, 1976) [Translated into Russian (Izhevsk: RKhD, 2001)]
4. Pais A *The Genius of Science*: *A Portrait Gallery* (Oxford: Oxford Univ. Press, 2000) [Translated into Russian (Moscow: Inst. Komp'yuternykh Issledovaniĭ, 2002)]
5. Poundstone W *Prisoner's Dilemma* (New York: Doubleday, 1992)
6. von Neumann J *Collected Works* Vols 1 – 6 (Ed. A H Taub) (New York: Pergamon Press, 1961 – 1963)
7. "John von Neumann, 1903 – 1957" *Bull. AMS* **64** (3, Pt. 2) (1958)
8. Glimm J, Impagliazzo J, Singer I (Eds) *The Legacy of John von Neumann* (Proc. of Symp. in Pure Math., Vol. 50) (Providence, RI: American Math. Soc., 1990)
9. *Von Neumann Centennial Conf.*: *Linear Operators and Foundations of Quantum Mechanics, Budapest, Hungary, 15 – 20 October 2003*
10. Von Neumann J "Die Axiomatizirung der Mengenlehre" *Math. Z.* **27** 669 (1928)
11. Von Neumann J "Zur Theorie der Gesellschaftspiele" *Math. Ann.* **100** 295 (1928)
12. Monastyrskiĭ M I "IAS i IHES — istoriya dvukh institutov" ('IAS and IHES: The history of two institutes'), in *Istoriko-Matematicheskie Issledovaniya* (*Vtoraya Ser., Vyp. 6*) (Moscow: Yanus-K, 2001) p. 73
13. Von Neumann J *Mathematische Grundlagen der Quantenmechanik* (Berlin: J. Springer, 1932) [Translated into English: *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton Univ. Press, 1955); Translated into Russian (Moscow: Nauka, 1964)]
14. Von Neumann J, Wigner E *Z. Phys.* **49** 73 (1928); **51** 844 (1928)
15. Kadison R V, Ringrose J R *Fundamentals of the Theory of Operator Algebras* (New York: Academic Press, 1983 – 1992)
16. Von Neumann J, Morgenstern O *Theory of Games and Economic Behavior* (Princeton: Princeton Univ. Press, 1944) [Translated into Russian (Moscow: Nauka, 1970)]
17. Rhodes R *Dark Sun*: *the Making of the Hydrogen Bomb* (New York: Simon & Schuster, 1995)
18. von Neumann J *The Computer and the Brain* (New Haven, Conn.: Yale Univ. Press, 1958) [Translated into Russian: *Kiberneticheskiĭ Sbornik* (1) 11 (1960)]
19. von Neumann J *Theory of Self-Reproducing Automata* (Urbana: Univ. of Illinois Press, 1966) [Translated into Russian (Moscow: Mir, 1971)]
20. Sakharov A D *Vospominaniya* (Memoirs) Vol. 1 (Moscow: Prava Cheloveka, 1996)
21. Landau L *Z. Phys.* **45** 430 (1927) [Translated into Russian: *Sobranie Trudov* (Collected Works) Vol. 1 (Moscow: Nauka, 1969) p. 19]
22. Weyl H *The Classical Groups*; *their Invariants and Representations* (Princeton, NJ: Princeton Univ. Press, 1939) [Translated into Russian (Moscow: IL, 1947)]
23. Jones V, in *Proc. of the Intern. Congress of Mathematicians, Kyoto, 1990* (Tokyo, 1991) [Translated into Russian (Moscow: Mir, 1996) p. 167]
24. von Neumann J *Proc. Natl. Acad. Sci. USA* **18** 70 (1932)
25. Halmos P R *Lectures on Ergodic Theory* (New York: Chelsea Publ. Co., 1956) [Translated into Russian (Moscow: IL, 1959)]
26. von Neumann J *Proc. Natl. Acad. Sci. USA* **18** 263 (1932)
27. Birkhoff G D, Koopman B O *Proc. Natl. Acad. Sci. USA* **18** 279 (1932)
28. Shiryaev A N (Ed.) *Kolmogorov* Book 3 *Zvukov Serdtsa Tikhoe Ekho... Iz Dnevnikov* (Soft echo of the heartbeat... From the diaries) (Moscow: Fizmatlit, 2003)
29. Holevo A S *Statistical Structure of Quantum Theory* (Berlin: Springer-Verlag, 2001) [Translated into Russian (Moscow-Izhevsk: Inst. Komp'yuternykh Issledovaniĭ, 2003)]
30. Zureck W H *Rev. Mod. Phys.* **75** 715 (2003)
31. Voiculescu D, in *Proc. of the Intern. Congress of Mathematicians Aug. 3 – 11, 1994, Zürich, Switzerland* (Ed. S D Chatterji) (Basel: Birkhäuser Verlag, 1995) [Translated into Russian (Moscow: Mir, 1999) p. 323]

32. Danilov Yu A *John von Neumann* (Moscow: Znanie, 1981)
33. Von Neumann J *Izbrannye Trudy po Funktsional'nomu Analizu* (Selected Papers on Functional Analysis) Vols 1, 2 (Moscow: Nauka, 1987)