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REVIEWS OF TOPICAL PROBLEMS

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Lattice SU(2) theory projected on scalar particles

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Contents

Abstract. Lattice measurements provide a unique possibility to directly study the anatomy of vacuum fluctuations, that is, their action and entropy. In this review, we discuss properties of vacuum fluctuations that are naturally called magnetic monopoles, or scalar particles. Magnetic monopoles are defined on the lattice as closed trajectories. One of the basic observations is that the length of these trajectories is measured in physical units (fermi) and does not depend on the lattice spacing *a*. Their thickness, on the other hand, determined in terms of the distribution of the non-Abelian action, is of order of the resolution *a*. Moreover, these infinitely thin — within presently available resolution — trajectories are unified into infinitely thin surfaces.

1. Introduction

This year, we remember Academician I Ya Pomeranchuk (1913 – 1966) on what would have been his ninetieth birthday. I first saw Isaak Yakovlevich more than 40 years ago, at his lecture at the Moscow Institute of Physics and Engineering. Although I was later a party to many physics discussions in his presence, I cannot claim to have understood much during these discussions. However, Isaak Yakovlevich had an enormous emotional impact on the physicists around and I remember the emotional atmosphere of those years very well. The pain and feeling of tragedy caused by his untimely death do not lessen with the passing of time.

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The content of this review is not so easy to outline. If we were to say that the review is devoted to magnetic monopoles, this would sound attractive but would not actually be very meaningful: the theme is too broad. If, on the other hand, we say that we interpret the results of recent lattice measurements performed mostly by physicists and students from the Institute of Theoretical and Experimental Physics (ITEP), we could scare the reader off with too narrow a topic.

The actual scope of the review is a kind of compromise between these two extremes: the measurements discussed are indeed very recent, but the physical problems touched are quite simple and fundamental.

The style of presentation assumes that the review is self-contained and can be read without addressing other sources. This impression should be true for the reader familiar with the basics of monopole physics and of lattice formulation of field theories. We hope that in any case, the general picture and the logic of presentation are readily understandable. In the list of references, we give a few reviews that could be consulted to refresh or gain knowledge of the background, such as generalities of lattice gauge theories. To a great extent, the review is based on the original papers [1 – 6].

2. Topological defects in lattice gauge theories

2.1 Two scales: *a* and $\Lambda_{\rm QCD}^{-1}$

To understand the material presented below, it is crucial to realize that observables on a lattice can depend on two distinct parameters, a and $\Lambda_{\rm QCD}^{-1}$. The lattice spacing a is assumed to be small, $a \to 0$. It serves as an ultraviolet cut off. The characteristic hadronic scale[1] $\Lambda_{\rm QCD}^{-1}$ naturally arises in description of the low-energy or infrared physics. The value of a can be varied on the lattice; a small dimensionless parameter is then given by the product $a\Lambda_{\rm QCD}$.

[1] By $\Lambda_{\rm QCD}$, one usually understands the scale at which the running coupling is of the order of unity. Instead of $\Lambda_{\rm QCD}$, we could have chosen any other physical unit, for example, ${\rm fm}^{-1}$.

Various vacuum fluctuations are sensitive to a and $\Lambda_{\rm QCD}^{-1}$ in different ways. The best known example of vacuum fluctuations seems to be instantons (see, e.g., Ref. [7] for a review). Their classical action is equal to

$$S_{\rm cl} = \frac{8\pi^2}{g^2}\,, \tag{1}$$

where g is the SU(2) coupling constant. The probability of finding an instanton of the size $\rho_{\rm inst}$ is proportional to

$$W \propto \exp\left(-\frac{8\pi^2}{g^2(\rho_{\rm inst})}\right), \tag{2}$$

where we have accounted for the running of the coupling. It is obvious that probability (2) is growing with $\rho_{\rm inst}$ and the characteristic size of instantons is stabilized at

$$\rho_{\rm inst} \sim \Lambda_{\rm QCD}^{-1}\,. \tag{3}$$

In other words, instantons represent the vacuum fluctuations in the infrared limit.

Another well-known example of vacuum fluctuation is given by zero-point fluctuations. The corresponding density of the vacuum energy is approximately equal to

$$\epsilon \approx \frac{1}{2}\sum_{\omega}\hbar\omega \approx \frac{\rm const}{a^4} \tag{4}$$

and is strongly divergent in the ultraviolet range.

Later, we discuss vacuum fluctuations where the two scales, a and $\Lambda_{\rm QCD}^{-1}$, coexist. This coexistence, or fine tuning, of the two scales distinguishes these fluctuations from the well-known examples mentioned above.

2.2 Topological defects

A well-known example of topological solitons is instantons. The topological charge is defined as

$$Q_{\rm top} = \frac{g^2}{32\pi^2}\int G^a_{\mu\nu} G^a_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,{\rm d}^4 r\,, \tag{5}$$

where $G^a_{\mu\nu}$ is the non-Abelian field strength tensor. Instantons are natural topological defects in the case of the SU(2) gauge theory because there exists a nontrivial lower limit on the action associated with a nonzero topological charge (5),

$$S_{\rm cl} \geqslant |Q_{\rm top}|\,\frac{8\pi^2}{g^2}\,. \tag{6}$$

If we consider the U(1) subgroup of SU(2), then magnetic monopoles become natural topological excitations (see, e.g., Refs [8, 9] for reviews). In four-dimensional space, $d = 4$, monopoles are represented by closed trajectories and are characterized by their length $\mathcal{L}$. The corresponding lower limit for the action is given by

$$S_{\rm mon} \geqslant {\rm const}\cdot\frac{\mathcal{L}}{a}\,|Q_{\rm M}|^2\,. \tag{7}$$

Because of the Dirac quantization condition, the magnetic charge $Q_{\rm M}$ is inversely proportional to the electric charge, $|Q_{\rm M}| \propto 1/e$, and the overall constant depends on details of regularization.

Relation (7) can be readily understood. Indeed, the topological invariant is given by the magnetic flux

$$Q_{\rm M} = \frac{1}{4\pi}\int \mathbf{H}\,{\rm d}\mathbf{s}\,, \tag{8}$$

where $\mathbf{H}$ is the radial magnetic field of the monopole and $\int {\rm d}\mathbf{s}$ is an integral over the surface of a sphere. Because of the Bianchi identities for the electromagnetic field strength tensor, $\partial_\mu \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$, the flux is independent of the radius of the sphere.

The mass of the monopole is given by

$$M_{\rm mon} = \frac{1}{8\pi}\int_a^\infty \mathbf{H}^2\,{\rm d}^3 r \sim \frac{1}{e^2}\int_a^\infty \frac{1}{r^4}\,{\rm d}^3 r \sim {\rm const}\cdot\frac{1}{e^2 a}\,, \tag{9}$$

where we have to introduce an ultraviolet cut off, a, because of the divergence of the integral at small distances. For a given flux, the minimum of mass (9) is reached on the spherically symmetric magnetic field.

For Z_2 gauge theories, the natural topological excitations are closed surfaces (see [10] for a review and references). Within the scope of the present review, we cannot discuss the Z_2 gauge theories in detail and just mention the results. The action related to topological defects is given by

$$S_{\rm vortex} = {\rm const}\cdot\frac{A}{a^2}\,, \tag{10}$$

where A is the surface area.

In concluding this section, we emphasize that only the bound in (6) is valid in the SU(2) theory. As far as the bounds in (7) and (10) are concerned, the full SU(2) theory does not involve any lower bounds for the action related to monopole trajectories and Z_2-surfaces. Relations (7) and (10) are only valid if the invariance is restricted to the corresponding subgroups of the SU(2) theory, namely U(1) and Z_2.

2.3 Energy – entropy balance

We now discuss monopoles in the Abelian case in more detail. The first glance at the lower bound in the corresponding action (7) seems to convince us that the monopoles cannot play any dynamical role in the continuum limit, $a \to 0$. Indeed, the monopole mass tends to infinity in this limit and the factor $\exp(-S_{\rm mon})$ seems to entirely suppress the monopoles.

But we have not yet taken the entropy into account. For point-like monopoles, the entropy factor is readily calculable. Indeed, the entropy is now the number of trajectories of the same length $\mathcal{L}$. Without the lattice regularization, the task of evaluating the number of trajectories $N_{\mathcal{L}}$ would puzzle any theorist. On the lattice, on the other hand, the calculation is quite straightforward. Indeed, the monopoles occupy the centers of cubes. In $d = 4$, any cube has 8 neighboring cubes. At each step, the trajectory can therefore be continued in eight various directions. The number of steps is $\mathcal{L}/a$. Therefore,

$$\tilde{N}_{\mathcal{L}} = 8^{\mathcal{L}/a}\,. \tag{11}$$

Two remarks concerning Eqn (11) are now in order. First, we used the approximation of point-like monopoles. In fact, the monopole field depends on the distance to the center: $|\mathbf{H}| \sim 1/er^2$. The approximation of point-like monopoles is justified by the ultraviolet divergence in the mass [see Eqn (9)]. Second, Eqn (11) actually refers to neutral point-like particles. Conservation of the magnetic charge implies that the trajectories cannot have open ends and are closed. Moreover, if the same piece of a trajectory is covered in both possible directions, this piece is not included in the trajectory at all. In the language of field theory, this statement corresponds to cancellation of charges between a monopole and an anti-monopole.

Instead of Eqn (11), the relation

$$N_{\mathcal{L}} = 7^{\mathcal{L}/a} \tag{12}$$

must be used for the number of monopole trajectories. Upon calculating entropy (12), we can find the probability of observing a monopole trajectory of length $\mathcal{L}$,

$$W(\mathcal{L}) = \frac{c_1}{\mathcal{L}^3} \exp\left[\left(-\frac{c_2}{e^2} + \ln 7\right)\frac{\mathcal{L}}{a}\right], \tag{13}$$

where $c_{1,2}$ are constants. The factor $1/\mathcal{L}^3$ actually arises because the trajectories are closed. The arguments presented above [see Eqns (9) and (12)] suffice only to derive the exponential factor in Eqn (13). Derivation of the factor $1/\mathcal{L}^3$ can be found, for example, in Ref. [11].

It is crucial for us that the constant c_2 in Eqn (13) is known explicitly for the lattice regularization (although not analytically but only numerically). We can then find the value of the electric charge

$$e^2_{\text{cr}} \equiv \frac{\ln 7}{c_2} \tag{14}$$

such that the exponential suppression vanishes for any length $\mathcal{L}$ of the trajectory. This is the point of phase transition to the monopole condensation.

The main result in this section is that for $a \to 0$, only an infinitely narrow band of values e^2 is physical,

$$e^2_{\text{cr}} - \text{const} \cdot a \leqslant e^2_{\text{phys}} \lesssim e^2_{\text{cr}} + \text{const} \cdot a\,. \tag{15}$$

Indeed, if we overstep the upper limit in (15), then the monopoles are too copious: they occupy a finite part of the four-dimensional lattice volume V_4. If e^2 is below the lower limit in (15), the monopoles are practically absent. Choosing e^2 within the limits (15) can be called fine tuning of the parameters.

2.4 Why monopoles at all?

The reader following our presentation should start wondering at some point why we consider the monopoles at all. Indeed, everything that we have had to say so far about the monopoles in non-Abelian theories [SU(2), for definiteness], seems to testify against the possible dynamical role of the monopoles.

We summarize these arguments again:

• monopoles are not natural topological excitations in the case of SU(2). In other words, there is no lower limit on the non-Abelian action in SU(2), which means, in fact, that any topological definition of the monopoles in SU(2) can be satisfied on gauge copies of the trivial field $A_\mu \equiv 0$;

• even in the U(1) case, where monopoles can be defined consistently as topological excitations, one has to choose (figuratively speaking, 'by hand') the value of the constant e^2 to be very close to a certain fixed value e^2_{cr} [see Eqn (15)].

It is worth mentioning, therefore, that the original interest in lattice monopoles stems from a clear physical idea. This is the so-called dual-superconductor model for confinement (see, e.g., Ref. [9] for a review). To clarify this model, we recall that the magnetic field does not penetrate the bulk of the standard superconductor. But if an external field is applied, the superconductor is destroyed and the condensate of the Cooper pairs disappears in the region of the strong field.

Quantitatively, the picture is realized as the Abrikosov vortex, which is a solution of coupled classical equations for a charged scalar and electromagnetic fields in the presence of two magnetic poles placed into a superconductor. (The magnetic poles can be visualized as the end points of a long solenoid.) The magnetic field streams into a tube connecting the poles, and the energy grows linearly with the distance between the poles.

We now imagine that there is a condensation of a magnetically charged particles in the vacuum state of non-Abelian theories, $\langle 0|\varphi_{\text{M}}|0\rangle \neq 0$. Then, if external heavy quarks are introduced into the vacuum, their (color) electric field also streams into a tube and there arises a linear potential between the quark – antiquark pair, which corresponds to the quark confinement.

Our interest in monopoles is focused on determining the properties that they have as objects of an ('effective') field theory. What is specific for the topic is that because of the difficulties outlined above, we cannot start with a microscopic theory of monopoles, but must turn to their phenomenology, which is surprisingly rich and calls for theoretical understanding.

In other words, the theoretical pessimism (see the beginning of the section) yields to the physical idea of the QCD vacuum as a dual superconductor. This is the outcome of the phenomenological studies. Creation of an adequate theory is still ahead of us.

3. Lattice data on magnetic monopoles and vortices

3.1 Maximal Abelian projection

The starting point of the phenomenology of monopoles is their definition. The idea of this definition is simple: because monopoles are natural topological defects in the Abelian case, one should substitute, or project, the original configuration of SU(2) fields on the closest Abelian field configuration and then define the monopoles 'inside' this Abelian configuration. We still have to explain what is understood by the 'closest' Abelian configuration.

We begin with a simple analogy. Imagine that two jets of particles are produced in a central collision (Fig. 1a). We define the 'closest' collinear configuration of the particle momenta in two steps. First, using the rotational invariance, we choose an axis, or unit vector $\mathbf{e}$, such that the sum of the moduli of projections of the momenta on the axis is

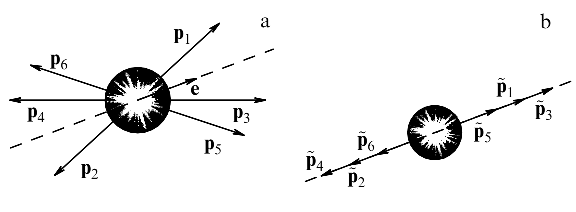

Figure 1. Particle momenta $\mathbf{p}_i$ and the choice of the axis $\mathbf{e}$ by maximizing the sum of momenta projections on the axis (a); the corresponding collinear momenta closest to the original ones (b).

maximal,

$$\max \sum_i |\mathbf{p}_i \mathbf{e}| .$$

Second, we replace the momenta with their projections on the axis,

$$\mathbf{p}_i \to \tilde{\mathbf{p}}_i \equiv \mathbf{e} |\mathbf{p}_i \mathbf{e}| \operatorname{sign}(\mathbf{p}_i \mathbf{e}) .$$

The momenta thus projected can be called the collinear configuration of the momenta closest to the original one (Fig. 1b).

In the case of a gauge theory, our basic object is the vector potential A^a_μ, where a is the color index, $a = 1, 2, 3$ for the SU(2) group. We consider A^3_μ as the electromagnetic field and $A^{1,2}_\mu$ as the field of charged gluons. As the first step — analogous to the example given above — we use gauge invariance to find the maximal projection of A^3_μ,

$$\max \sum_{i,\mu} |A^3_\mu(x_i)|^2 ,$$

where x_i are positions of the lattice sites. At the next step, we set $A^{1,2}_\mu \equiv 0$ in the chosen gauge.

As a result, we replace the original field configuration of $A^{1,2,3}_\mu$ by the 'closest' Abelian field configuration $\bar{A}^3_\mu$. Our magnetic monopoles are then nothing else but Dirac monopoles in terms of the projected fields $\bar{A}^3_\mu$. The Dirac monopoles correspond to singular fields, and the corresponding monopole current can be determined in terms of violations of the Bianchi identities,

$$\partial_\mu \epsilon_{\mu\nu\rho\sigma} \partial_\rho \bar{A}_\sigma \equiv j_\nu . \tag{16}$$

More precisely, we use a lattice analog of Eqn (16), and therefore all singularities are resolved in terms of the lattice spacing.

The result of the above procedure is given by a set of monopole trajectories. First, a representative set of non-Abelian field configurations is generated (see, e.g., Ref. [12] for a review of the lattice formulation of gauge theories). At this step, only the original non-Abelian Lagrangian of the SU(2) theory is used. Then each configuration is replaced by its maximal Abelian projection, within which the monopole trajectories are determined. These trajectories are the starting point of our analysis. The procedure can be iterated for various values of the lattice spacing a.

3.2 Monopole clusters

The entire network of monopole trajectories, defined for each field configuration, decays into clusters. It is important to distinguish between an infinite, or percolating, cluster and finite clusters. In realistic measurements, the percolating cluster is understood as the one stretching from one boundary to another boundary of the lattice volume V_4 (where the subscript '4' indicates the space – time dimension). Clearly, the length of the percolating cluster is proportional to V_4,

$$\mathcal{L}_{\text{perc}} \equiv \rho_{\text{perc}} V_4 , \tag{17}$$

where ρ_{perc} is the density of monopoles. The percolating cluster exists for each field configuration and is in a single copy (see [13] and references therein).

For finite clusters, their most important characteristic is the distribution in length, $N(L)$. Experimentally,

$$N(L) \sim \frac{1}{L^3} . \tag{18}$$

Thus, finite clusters are dominated by short ones with the length of the size of several lattice spacings. We can introduce the corresponding density of finite-length monopoles,

$$\mathcal{L}_{\text{uv}} \equiv \rho_{\text{uv}} V_4 ,$$

where $\mathcal{L}_{\text{uv}}$ is the total length of finite clusters and the subscript uv indicates that 'ultraviolet' (short) clusters dominate.

So far, we have mainly discussed definitions. A remarkable observation is that the monopole densities ρ_{perc} and ρ_{uv} satisfy simple scaling relations.[2] The monopole density ρ_{perc} is independent of the lattice spacing a,

$$\rho_{\text{perc}} \approx 0.62 \sigma^{3/2}_{\text{SU}(2)} \approx c_{\text{perc}} \Lambda^3_{\text{QCD}} . \tag{19}$$

Here, $\sigma_{\text{SU}(2)}$ is the string tension for the SU(2) gluodynamics and c_{perc} is a constant. As for ρ_{uv}, the density of finite-size clusters diverges linearly as $a \to 0$,

$$\rho_{\text{uv}} \approx \frac{c_{\text{uv}}}{a} \Lambda^2_{\text{QCD}} , \tag{20}$$

where c_{uv} (like c_{perc} above) is a constant.

What is most striking about relations (19) and (20) is that they are formulated entirely in terms of a and Λ_{QCD}, which are perfectly gauge invariant. On the other hand, the definition of the monopoles uses projection on a particular U(1) subgroup of the original SU(2). Therefore, relations (19) and (20) indicate that the projection is only a means to detect objects that have an SU(2)-invariant meaning.

3.3 Fine tuning of the parameters

Relation (19) implies that the probability of finding a monopole belonging to the percolating cluster in a particular lattice cube is given by

$$W_{\text{mon}} \approx \exp\left(-\frac{9(4\pi)^2}{44} \frac{1}{g^2(a)}\right) \propto (a\Lambda_{\text{QCD}})^3 , \tag{21}$$

where $g^2(a)$ is the running coupling normalized at the lattice spacing a and the coefficient in front of $1/g^2(a)$ is determined in terms of the β-function.

Comparing (21) and (2), we note that — as distinguished from the instanton case — the probability W_{mon} of finding a monopole explicitly depends on the lattice spacing a. One could think that this distinction is rooted in mere definitions. By construction, the monopole trajectory is infinitely thin. If we identify an instanton with its center, then the probability of finding an instanton would look similar to (21).

The basic difference between instantons and monopoles is revealed through measurements of the action associated with the monopoles. The corresponding data are reproduced in Fig. 2 [1]. The data are crucial for the whole framework presented here, and we describe them in some detail.

The procedure for measuring the action is as follows. First, using the maximal Abelian projection for each original non-Abelian configuration, one determines the monopole

[2] We here quote the latest data [14]; further references can be found therein.

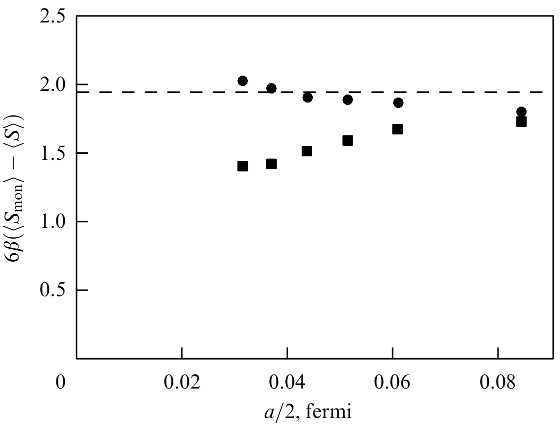

Figure 2. The non-Abelian action associated with the lattice monopoles belonging to the percolating cluster (squares) and averaged over all the monopoles (circles). The dashed line corresponds to the monopole action 'ln 7'$\times(L/a)$. The data are from Ref. [1].

positions. The monopoles occupy the centers of lattice cubes. Then the non-Abelian action is measured on the plaquettes belonging to the monopole cubes (see, e.g., Ref. [12]). Next, one averages the monopole action over all (or only percolating) monopoles and subtracts the plaquette average over the entire the lattice from this average. This difference is plotted in Fig. 2 as a function of the lattice spacing.

An important point is that the difference is plotted in the so-called lattice units of the action $S_{\rm lat}$. In the continuum limit, this unit itself is ultraviolet divergent,

$$S_{\rm plaq} \sim \frac{1}{a^4} = {\rm const} \cdot S_{\rm lat} \,. \tag{22}$$

In other words, the scale of the lattice unit of the action corresponds to the contribution of zero-point fluctuations [see (4)].

Finally, the field-theory mass of the monopole can be estimated as

$$M(a) \sim a^3 \left(\langle S^{\rm mon}_{\rm plaq} \rangle - \langle S^{\rm lat}_{\rm plaq} \rangle \right) . \tag{23}$$

If we compare (23) with the corresponding continuum-limit expression (9), then the factor a^3 in (23) represents the volume (or $\int {\rm d}^3 r$), while the action on the plaquettes corresponds to $\mathbf{H}^2$. However, the field value is not set in advance now, but directly measured on the lattice.

In Fig. 2, the data for the percolating monopoles are separated from the whole of the monopoles. We see that the action for the percolating monopoles is indeed lower than average. This agrees with the idea that monopoles with lower action are condensed.

But the most important point for us is that the monopole action is close to a constant in 'lattice units'. In other words, at the lattices now available, we have the estimate

$$M_{\rm mon}(a) \sim \frac{1}{a} \,,$$

the same as for point-like and Dirac monopoles! As we discussed in detail above, such monopoles can survive on the lattice only because of fine tuning, ensuring that the suppression due to the mass divergence is almost exactly canceled by the enhancement due to large entropy.

Comparison with (21) allows us to estimate the precision with which cancellation occurs,

$$M_{\rm mon}(a) - \frac{\text{`}\ln 7\text{'}}{a} \sim \frac{\rm const}{g^2(a)} \,, \tag{24}$$

where the 'ln 7' term represents the entropy in (12). We put ln 7 in quotation marks because Eqn (12) neglects the effect of neighbors and is therefore an approximation. Also, in Eqn (23) we took only plaquettes closest to the centers of the monopole cubes into account, which introduces an error as well. The quantity 'ln 7' is also given in Fig. 2 for the sake of orientation. We see that 'ln 7'$/a$ is indeed close to the experimental value of the monopole mass (as a function of a).

To summarize, there is evidence that the monopole action and entropy indeed satisfy a relation similar to Eqn (15) that we derived for a U(1)-symmetric Lagrangian. But in the U(1) case, relation (15) can be satisfied only at the expense of artificially adjusting the value of the coupling e^2. In the non-Abelian case, the coupling g^2 cannot be fixed at all because the coupling depends on the scale. Nevertheless, Eqn (24) is satisfied 'automatically', without any artificial adjustment.

3.4 Thin and heavy surfaces

We have already mentioned that in the case of the Z_2 gauge group, natural topological excitations are given by closed surfaces. In the case of the SU(2) gauge theory, one can also define closed surfaces using projection of the original non-Abelian fields on the closest Z_2-configuration of the fields. It turns out that surfaces defined in this way also possess remarkable properties in terms of the original SU(2) theory. In this section, we briefly summarize the findings. Due to a lack of space, our presentation might sometimes be sketchy (many details are clarified in review [10]).

At the first step, the gauge invariance is used to ensure that the SU(2) fields generated on the lattice by the standard non-Abelian action are as close as possible to the matrices $\pm I$. The projection on the Z_2 theory actually took two steps. The maximal Abelian projection was constructed first (see above). At the second step, the remaining U(1) freedom was used to make the Abelian fields as close as possible to the Z_2 fields. Finally, the Abelian fields are replaced by the closest Z_2 fields. The closed surfaces are the unification of all the negative plaquettes constructed on the projected Z_2 fields.

The two-dimensional surfaces defined in this way are infinitely thin by construction. This does not automatically mean, however, that there are some physical infinitely thin objects behind them.

To clarify the nature of the surfaces, the non-Abelian action associated with the surfaces and their total area were measured as functions of the lattice spacing a [3]. We recall that similar measurements in the case of monopoles revealed a remarkably simple picture: the non-Abelian action associated with the monopoles is approximately constant in the lattice units [see Fig. 2 and Eqn (24)], while the total length of the percolating trajectories is approximately a constant in the physical units (measured in fm).

The story repeated itself in the case of surfaces as well! More specifically, the action density per unit area is quadratically divergent in the ultraviolet limit,

$$S_{\rm vortex} \approx 0.5\, \frac{A}{a^2} \,, \tag{25}$$

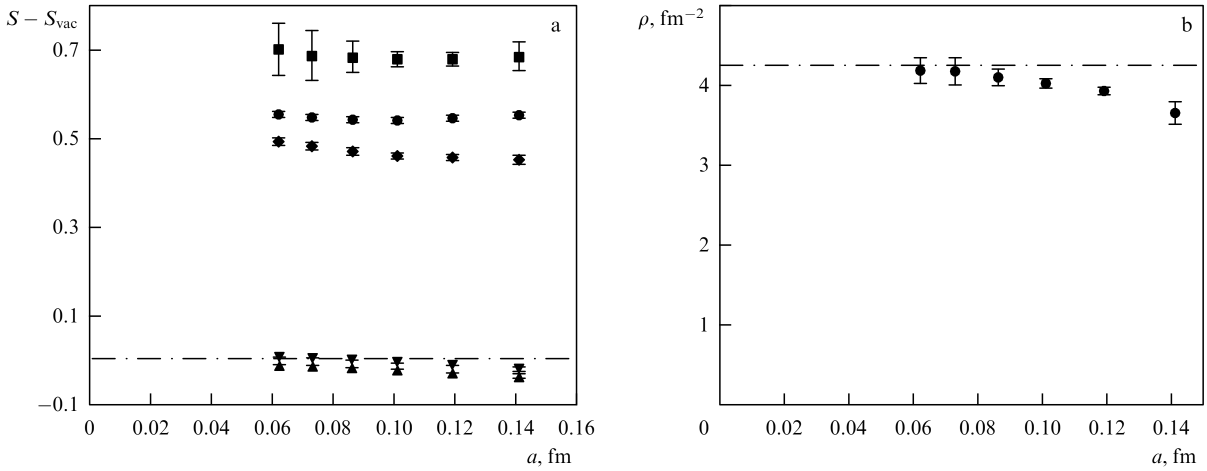

Figure 3. (a) The non-Abelian action associated with the vortices: averaged over the entire area of the vortices (circles); separately for the plaquettes that belong both to the vortices and monopoles (squares); separately for the plaquettes that belong to the vortices but not to the monopoles (diamonds); for the plaquettes neighboring the vortex (triangles). Geometrically, there are two different types of neighboring plaquettes and the action was measured separately for each of them. (b) Average area of the vortices in the unit lattice volume. The plots are borrowed from [3].

where A is the area of the surface. The corresponding data are reproduced in Fig. 3a.

As regards the total area, it grows proportionally to the lattice volume and is approximately constant in the physical units,

$$A_{\rm vortex} \approx 4({\rm fm})^{-2} V_4 \,. \qquad (26)$$

The corresponding data are reproduced in Fig. 3b.

The ultraviolet divergence of the density of the action in the limit $a \to 0$ [see (25)] implies strong suppression of the probability of finding such a vortex by the action factor,

$$\exp(-S) \propto \exp\left(-{\rm const} \cdot \frac{A}{a^2}\right).$$

On the other hand, the observation of the scaling of the total area of the vortices [see Eqn (26)] implies that the suppression due to the action is canceled by the enhancement due to the entropy.

Theoretically, it is known that the entropy of a surface can indeed grow as $\exp({\rm const} \cdot A/a^2)$. The mechanism of the entropy growth is similar to the monopole (trajectories) case. Namely, the surface on the lattice consists of pieces of size a^2 and neighboring plaquettes can be mutually oriented in various ways. Unfortunately, so far the constant in the exponential has not been evaluated. Thus, we cannot establish by straightforward calculation that action (25) corresponds to the fine tuning of the entropy and action. But we can claim that in the experiment, the surfaces are indeed very 'crumpled,' which means that the plaquettes belonging to the surface change their directions randomly at each step, and therefore their entropy is certainly exponentially growing with their total area A.

To summarize, two-dimensional surfaces are present in the vacuum state of the lattice SU(2) theory. The thickness of the surfaces is determined in terms of an excess of non-Abelian action and is less than the resolution a at presently available lattices. The area of the surfaces, on the other hand, scales in physical units. In the opinion of the author of this review, discovery of such vortices is one of the most remarkable observations made in lattice simulations.

4. Interpretation and implications

4.1 The nature of thin vortices

The discovery of infinitely thin and heavy vortices is very recent and there exist no detailed papers on their interpretation. In this section, we nevertheless outline a simple picture that can serve the purpose of orientation.

It seems reasonable to discuss the lattice monopoles in terms of the Dirac monopoles. This appears to be a reasonable approximation because monopoles are indeed associated with singular fields [see (24)]. Moreover, it is convenient to use the gauge where all the monopole fields are along the third direction in the color space. The potential corresponding to a Dirac monopole is then of the form

$$A_r^3 = A_\theta^3 = 0 \,, \quad A_\varphi^3 = \frac{Q_{\rm M}}{4\pi} \frac{1+\cos\theta}{r \sin\theta} \,, \qquad (27)$$

where r, θ, and φ are the spherical coordinates and $Q_{\rm M}$ is the magnetic charge, in our case $Q_{\rm M} = 4\pi/g$.

We note that potential (27) is singular at $\theta = 0$. The singularity corresponds, of course, to the Dirac string. In making lattice estimates, it is natural to replace

$$r \sin\theta \to a \qquad (28)$$

to regularize integration at the singularity.

We now consider the non-Abelian action

$$S = \frac{1}{4} (G_{\mu\nu}^a)^2 \,, \qquad (29)$$

where $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g(a)\,\epsilon^{abc} A_\mu^b A_\nu^c$. Characteristic perturbative fields on the lattice are then of the order

$$A_\mu^a \sim \frac{1}{a} \,, \qquad (30)$$

where $1/a$ corresponds to the ultraviolet divergence discussed in Section 2.1.

If the coupling $g(a)$ is small, the Abelian part of the field strength tensor, $S_{Ab} \sim (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2$, dominates the action (29). We must now take the effect of the Dirac string into account. A property of lattice regularization is that the action associated with the Dirac string itself vanishes. But there remains the interference term

$$S_{\rm int} \sim g^2(a)\, A^+_\mu A^-_\nu A^3_\mu A^3_\nu \sim \frac{1}{a^4}\,. \tag{31}$$

The constant g^2 is here canceled because the string potential contains $1/g$ [see Eqn (27)], and we estimated the charged fields $A^\pm_\mu$ as zero-point fluctuations, $A^\pm \sim 1/a$.

Thus, the ultraviolet divergence in the action associated with the surfaces is ascribed in the model under consideration to the interaction of charged gluons with Dirac strings that span the surfaces populated with monopoles. It is worth emphasizing that the interpretation in terms of the 'Dirac string' is actually gauge-dependent. But we can also formulate some gauge-independent predictions of the model.

4.2 Association of the monopoles with the vortices

One of apparent consequences of the model is the prediction that the surface is populated with monopoles. Indeed, the surfaces are now swept by the Dirac strings and the strings end up at the monopoles. This prediction agrees very well with the data [10, 3]. Namely, 90–97% of all the monopole trajectories belong simultaneously to the vortices and, vice versa, about $1/3$ of all the plaquettes belonging to the vortices belong to the cubes containing monopoles.

Another prediction of the model considered is that the distribution of the non-Abelian action of the monopoles is not spherically symmetrical. Indeed, the Dirac strings lying on the surfaces are attached to the monopoles and make a contribution to the non-Abelian action. This lack of spherical symmetry is also known from the measurements. Also, it is naturally explained that the surfaces are thin. Indeed, they have the thickness of the Dirac strings.

On the other hand, it would of course be very premature to claim that our model is the correct one. Indeed, all the consequences discussed above are only qualitative; we have not made quantitative predictions.

4.3 Standard problem of the Standard Model

We return to relations (13) and (14), which in the U(1) case allow us to find the critical value of the electric charge that corresponds to the onset of monopole condensation. We now consider these relations from a somewhat different standpoint. The properties of monopoles are determined by a single parameter, $M(a)$, and therefore the monopole action is given by $S_{\rm mon} = M(a)\mathcal{L}$, where $\mathcal{L}$ is the length of the trajectory. But this is exactly the classical action of a particle of mass $M(a)$! The onset of monopole condensation corresponds to a vanishing mass of the scalar field, $m^2 = 0$.

At first glance, we are contradicting ourselves. On one hand, according to Eqn (14) in the continuum limit, $a \to 0$, the condensation begins when $M(a) \to \infty$. On the other hand, we know that condensation begins when the mass squared vanishes. The resolution of the paradox is that $M(a)$ is to be identified with the bare mass, while the physical, or renormalized, mass differs from it [11].

We formulate this important statement in more detail. We define a particle propagator *a la* Feynman as a path integral,

$$\tilde{D}(x, x') = \sum_{\rm path} \exp\left(-S_{\rm cl}(x, x')\right), \tag{32}$$

where $S_{\rm cl}(x, x')$ is the classical action corresponding to a particular path,

$$S_{\rm cl} = M(a)\,\mathcal{L}(x, x'), \tag{33}$$

where $\mathcal{L}(x, x')$ is the length of the path connecting the points x, x' (in Euclidean space–time). It then turns out that the propagator $\tilde{D}$ in (32) is proportional to the standard propagator of a particle with the mass $m_{\rm prop}$ given by

$$m^2_{\rm prop} = \frac{8}{a}\left(M(a) - \frac{\text{`}\ln 7\text{'}}{a}\right). \tag{34}$$

We emphasize that Eqn (34) is a general field-theory relation in no way specific to the monopoles. Moreover, knowing relation (14), we can recover Eqn (34), up to an overall constant. Indeed, the vanishing of the right-hand side of Eqn (33) signifies a phase transition to monopole condensation. In the language of field theory, this is the transition to the tachyon mass of the scalar field, or $m^2_{\rm prop} = 0$. Moreover, because of the Lorentz invariance, we should have a relation for mass squared, $m^2_{\rm prop}$, not mass itself, and this consideration fixes the power of a in Eqn (34).

What is most remarkable about Eqn (34) is that it looks very similar to the standard relation for the Higgs mass in the Standard Model,

$$m^2_{\rm Higgs} = M^2_{\rm rad} - M^2_0\,. \tag{35}$$

Here, $M^2_{\rm rad}$ is the radiative correction to the mass, which diverges quadratically at large virtual momenta and M^2_0 is the so-called counterterm. Comparison of (35) and (34) shows that the standard expression for the radiative correction, $M^2_{\rm rad}$, is indeed the same as for $M_{\rm mon}(a)/a$. This is not surprising because both describe the same effect. On the other hand, the counter term M^2_0 is replaced in Eqn (34) by $8\text{`}\ln 7\text{'}/a^2$. In other words, introduction of the lattice regularization allows us to fix the counterterm in our case.

Thus, the fine tuning of the coupling discussed in Section 3.3 in fact provides an example of solving the 'standard problem' of the Standard Model: how to keep the Higgs mass much less than the absolute value of any of the two terms in the right-hand side of Eqn (35).

Because this conclusion is important for us, we rephrase it in somewhat different terms. We have started with the SU(2) gauge theory. We next defined an algorithm to identify trajectories of magnetic monopoles. The non-Abelian action associated with the monopoles then turned out to be divergent in the ultraviolet (at the lattices now available), with the same divergence as for point-like particles. Monopoles have spin zero, and we can therefore say that we have constructed a projection of the SU(2) theory on point-like scalar particles.

In doing so, we also inherited the main problem of the theory of scalar charged particles, the quadratic divergence in the mass. However, the lattice data indicate strongly that this problem is somehow fixed within the resultant effective theory of scalar particles. Indeed, the scaling properties like (19) unequivocally indicate that the infrared scale $\Lambda_{\rm QCD}$ coexists with the lattice spacing scale a, which characterizes the size of the scalar particles.

It would be instructive to elucidate the mechanism of the fine tuning that is at work in the effective theory of scalar particles. At this moment, we can only suggest some preliminary considerations concerning this question.

4.4 Solution to the problem of quadratic divergence

To understand how the problem of quadratic divergence is solved, we must translate the data on the monopole densities [see (19), (20)] into the more familiar language of field theory. We now explain how this is to be done.

We start with the classical action (34). It is quite obvious that the average length of the trajectory is given by the derivative of the partition function Z,

$$\langle \mathcal{L} \rangle = \frac{\partial}{\partial M} \ln Z \,. \tag{36}$$

Using (34), we can replace the differentiation with respect to the bare mass M with the differentiation with respect to the propagating (field-theory) mass m_{prop},

$$\langle \mathcal{L} \rangle = \frac{8}{a} \frac{\partial}{\partial m^2_{\text{prop}}} \ln Z \,. \tag{37}$$

The derivative of the partition function with respect to m^2_{prop} is related to the vacuum expectation value $\langle |\varphi|^2 \rangle$,

$$\langle |\varphi|^2 \rangle = \frac{\partial}{\partial m^2_{\text{prop}}} \ln Z \,, \tag{38}$$

where φ is the complex scalar field describing the monopoles. Indeed, the standard Lagrangian of the scalar field depends on m^2_{prop} through the term $m^2_{\text{prop}} |\varphi|^2$.

Finally, the average length of the monopole trajectories reduces, in a purely phenomenological way, to the monopole density,

$$\langle |\varphi|^2 \rangle = \frac{a}{8} (\rho_{\text{perc}} + \rho_{\text{uv}}) \propto c_{\text{perc}} a \Lambda^3_{\text{QCD}} + c_{\text{uv}} \Lambda^2_{\text{QCD}} \,, \tag{39}$$

where $c_{\text{perc, uv}}$ are constants entering the expressions for monopole densities (19) and (20). The data on the monopole densities imply that the vacuum expectation value $\langle |\varphi|^2 \rangle$ does not have any quadratic divergence.

Of course, the following question immediately arises: how can one explain the behavior $\rho_{\text{uv}} \propto 1/a$ of the density? The answer is that such a dependence of the density of short clusters on a is directly related to the fact that the monopoles actually 'live' on surfaces of dimension $d = 2$, not in the whole space $d = 4$. Indeed, this is obvious already on dimensional grounds.

Thus, the solution to the standard problem of the Standard Model is that the monopoles are associated with surfaces. As far as we can judge, this is a novel solution of the fine tuning problem. At this moment, it is difficult to say whether such a solution is universal.

5. Conclusion

We have briefly reviewed the lattice data that indicate a novel picture of the vacuum in the Yang – Mills theories in Euclidean space – time. Namely, there is evidence that there exist thin lines (monopole trajectories) and thin surfaces (the so-called vortices). The thickness is defined in terms of the distribution of the non-Abelian action and is less than the experimental resolution, that is, the lattice spacing a. In its turn, the minimum accessible value of a is approximately $a \approx (3 \text{ GeV})^{-1}$. As regards the average length of the trajectories and the area of the vortices, they are independent of the lattice spacing a (with high accuracy) and are measured in fm and fm^2 respectively. This coexistence of the two different scales, $\Lambda^{-1}_{\text{QCD}}$ and a, in vacuum fluctuations can be called fine tuning.

Moreover, we have argued that the fine tuning understood in this way is actually identical with the phenomenon first discussed in relation with the Standard Model, or more precisely, with Higgs particles. In the case of the Standard Model, the problem is how the mass of the scalar particle can be much smaller than its inverse radius. In the case of the monopoles, the solution to this standard problem of the theory of scalar particles is that the monopoles are actually associated with surfaces (vortices), rather than with the entire four-dimensional space.

From the theoretical standpoint, the main conclusion is that there is evidence of the existence of objects whose description takes us beyond field theory, namely infinitely thin surfaces. The theory of such objects is an open field.

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