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Topological phase in classical mechanics

G B Malykin, S A Kharlamov

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Contents

Abstract. **The historical development of the concept of the topological phase in classical mechanics from the mid-19th century to the present is discussed. There are three stages to be recognized in this period. The first, the mid-19th century stage, is concerned with studying the kinematics of rigid body rotation and includes such milestone developments as the Euler theorem on finite rotation of rigids, Gauss formula for the sum excess of the angles of a spherical polygon, Rodrigues's proof of the noncommutativity property of two finite rotations, and, finally, Hamilton's *Lectures on Quaternions* where the solid angle theorem is formulated and proved. The experimental discovery of the nonholonomic error of gyroscopes and its exhaustive explanation by A Yu Ishlinskiĭ represent the second stage. The third stage, which started in the 1980s, has witnessed the rediscovery of the nonholonomic effect in the framework of Hamiltonian formalism and is dominated by the study of how the topological phase — or an additional angle — forms in a mechanical system being treated in action – angle variables.**

G B Malykin Institute of Applied Physics, Russian Academy of Sciences, ul. Ul'yanova 46, 603950 Nizhniĭ Novgorod, Russian Federation
Tel. (7-8312) 16 43 70
E-mail: malykin@mail.nnov.ru
S A Kharlamov V I Kuznetsov Research Institute of Applied Mechanics, Aviamotornaya ul. 55, 111123 Moscow, Russian Federation
Tel. (7-095) 273 33 00
E-mail: niipm-gyro@dol.ru; sakharlam@mtu-net.ru

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1. Introduction

About 20 years ago M Berry considered a number of examples from quantum mechanics in his well-known paper [1] and raised the question of the occurrence of a topological phase describing the Schrödinger wave function evolution for a time-dependent Hamiltonian. This topological (geometric) phase was named the *Berry phase* by B Simon in the paper [2] issued just before paper [1] as an announcement of Berry's publication. The Berry phase represents an additional phase incursion experienced by any quantum-mechanical system due to its spatial evolution. The phase incursion is defined by topological properties of the evolution operator in Hilbert space rather than the local quantum dynamics.

For two decades, a great number of papers devoted to this problem were published. The concept of topological phase was extended considerably and was applied in different areas of physics as shown in Refs [3 – 9] where the topological phase is often called the Berry phase. In Ref. [10], Berry introduced the concept of topological phase into optics for a light beam propagating along a nonplanar trajectory. However, soon after the publication of work [10] it became clear that the topological phase in optics was formulated long before this publication. For a beam propagating along the nonplanar trajectory, this concept was introduced by S M Rytov in Refs [11, 12] published in 1938 and 1940, respectively, as well as by V V Vladimirskiĭ in Ref. [13] published in 1941. For a beam propagating along a straight line with varying polarization state, the concept was formulated by S Pancharatnam in 1956 [14, 15]. These priorities were noted by Berry, who recognized the role of Pancharatnam's papers [14, 15] in Ref. [16], as well as the contribution of Rytov [11] and Vladimirskiĭ [13] works in Ref. [4]. Thus, the question of priority in the discovery of the topological phase in optics is now completely answered. Recently, the terms *Rytov effect, Rytov – Vladimirskiĭ phase*, and *Pancharatnam phase* have become soundly established. A totally different situation is observed in classical mechanics. Here, there is a widely held opinion that the topological phase in mechanics was introduced by J Hannay in Ref. [17] published in 1985. However, this opinion is only partially correct. In this way the present paper is aimed at the successive and systematic presentation of papers devoted to the topological phase in classical mechanics since the 19th century.

2. W R Hamilton's (1805 – 1865) work

Sir William Rowan Hamilton had formulated and proven a remarkable theorem in the first edition of his famous treatise *Lectures on Quaternions* [18] published in 1853. This theorem, which is now named the solid angle theorem in classical mechanics, was given on pages 339 and 340 as Article 355. Its formulation is:

"355. ... And on *physical* or rather *geometrical* side, so far as regards the general theory of *compositions of rotations*, we arrive (in the plan of recent articles) at this remarkable theorem, that the *infinitely many infinitesimal and conical* ROTATIONS (*themselves* now, and not their halves) of the PERIMETER of ANY *closed figure on a sphere, compound themselves into a* SINGLE *resultant and finite rotation, represented by the* TOTAL AREA *of figure*; it being still understood that elements of this area may become negative."

This citation reproducing its typographical typesetting shows the author's mastery to make the formulation clear and easy to understand.

The proof of this theorem followed from the total theory of quaternion multiplication that was presented in Lecture VI occupying 140 pages of the treatise. This ambiguity may be the reason the theorem is little known. Later, the theorem was mentioned in the book on mechanics written by Horace Lamb [19], where a brief story of the theorem formulation and proof was given.

The sequence of events leading to the theorem's deduction and proof was as follows. To understand and prove that a rigid body rotates about any of its axes describing a closed nonintersecting curve on the unit sphere, it is necessary to resort to two fundamental results. First, one should be familiar with the Leonhard Euler (1707 – 1783) theorem [20] that any given initial orientation of a rigid body can be transferred into any given final orientation by a rotation about a fixed axis. Second, one should be able to apply the Karl Friedrich Gauss (1777 – 1855) formula [21] that defines the sum of the angles of a spherical polygon. The first problem solved was the problem of the summation of two finite rotations. The credit for its solution belongs to Olinde Rodrigues (1794 – 1854) who demonstrated in 1840 that two finite rotations of a rigid body are noncommutative [22]. The merit of Hamilton is that he applied quaternions to solve the same problem in 1853.

According to H Lamb [19], M J Donkin (professor of astronomy at Oxford from 1842 to 1869) solved the problem of the summation of two finite rotations in 1851 and presented the following simple solution [23]. Let the point of intersection of the body's axis with the unit sphere describe an arc of the great circle AB as the result of the first rotation, and an arc BC as the result of the second rotation. Donkin proved that the finite rotation equal to two previous ones takes place about the axis which is perpendicular to the plane going through the fixed point O and the mid-points X and Y of arcs AB and BC. Moreover, the angle of rotation is equal to the arc of the great circle connecting the points X and Y.

The second problem leading to the solid angle theorem reduces to the problem of summation of three finite rotations. Hamilton considered such a sequence of three rotations of a rigid body that the point of intersection of the body-fixed axis and the unit sphere describes a closed spherical triangle. He proved that, as a result of these rotations, the body is turned about this axis and the angle of this turn is equal to the spherical excess of the sum of the angles of a spherical triangle composed of arcs of the unit-sphere great circles.

According to the Gauss theorem [21], the spherical excess is equal to the area of the spherical triangle divided by the area of sphere, or the solid angle under which the triangle is seen from the center of the sphere. It should be noted that the Gauss theorem is valid for a spherical polygon with an arbitrary number of sides. Undoubtedly, Hamilton knew the Gauss theorem since he arrived at the final formulation of his theorem taking the limit when the number of spherical polygon sides infinitely increases.

It is interesting to note that, as mentioned by Lamb [19], Hankel [1] produced in 1847 the simple proof of the theorem that is equivalent to the solid angle theorem. He demonstrated that if the point returns to its starting position after three successive rotations on the sphere, then the angle of equivalent resulting rotation is equal to the spherical excess of the spherical triangle formed by three arcs corresponding to the rotations under consideration. The information presented by Lamb [19] does not allow one to conclude undoubtedly if Hankel gave this theorem either pure geometrical or combined geometrical and kinematical meaning. Therefore, until the Hankel work is found one cannot definitely state that Hankel anticipated Hamilton. To make the final decision it is necessary to elucidate if Hankel had drawn the physical conclusions from his mathematical results, i.e., considered the corresponding features of rigid body kinematics.

However, in succeeding years of the 20th century, the solid angle theorem had been forgotten. Its fate was to be 'rediscovered' several times in at least three different areas of science (optics, quantum mechanics, and classical mechanics). These different areas are connected by another discovery of Hamilton [24], i.e., the analogy between optics and mechanics, made in 1828 – 1837.

3. The Felix Klein (1849 – 1925) problem

The essence of the analogy discovered by Hamilton is that the transformation group of the conservative mechanical system motion and the group of light propagation in the wave theory of Christian Huygens (1629 – 1695) are both tangent or canonical transformation groups [25]. Augusten Cauchy (1789 – 1857), trying to develop the Hamilton analogy between optics and mechanics, found this analogy in the vibrations of an elastic medium rather than the dynamics of mechanical systems. Thus, Cauchy excluded the further development of the analogy between analytical dynamics

[1] Unfortunately, Lamb gave neither the reference to the corresponding publication, nor Hankel's initials. Therefore, it is not clear who was mentioned, Wilhelm Hankel (1814 – 1899), German physicist, professor at Leipzig University, or his son Herman Hankel (1839 – 1873), well-known mathematician, professor at Tubingen University.

and the post-Huygensian theories of light. Felix Klein (1849 – 1925) called attention to this fact in Ref. [26]. The next stimulus to developing the optical – mechanical analogy in the scope of analytical dynamics was given by N G Chetaev (1902 – 1959) in Refs [27, 28], where this problem was named the Klein problem. As a result of the Klein problem solution in these works, the fundamental relation between the mechanics of conservative systems and geometrical optics was discovered. Thus, it is instructive to consider the works on the geometrical phase in optics once more.

4. S M Rytov's (1908 – 1996) and V V Vladimirskiĭ's (born in 1915) works

Let a polarized light beam propagate along a nonplanar trajectory. A natural moving trihedron or Frenet trihedron [2] is connected to this trajectory in such a manner that the unit vectors constituting this trihedron are tangent, normal, and binormal to the curvilinear beam trajectory. In 1938 – 1940, Rytov showed (see, Refs [11, 12]) that the plane of beam polarization rotates with respect to the Frenet trihedron. Later on, this result was developed by Vladimirskiĭ in a graduate thesis written under the tutorship of Rytov. In 1941, Vladimirskiĭ [13] found that after the tangent returns to its initial state at some point of the trajectory, the plane of beam polarization will differ, in general, from its starting position. This phenomenon is not observed if the beam trajectory is a planar curve and beam polarization is constant. Moreover, if the Frenet trihedron returns to its original position after any cyclic spatial evolution then the plane of polarization will be twisted by a certain angle relative to its initial position. The value of the angle is equal to the area of the unit-sphere surface which is bounded by the closed curve inscribed on the unit sphere by one of the trihedron's unit vectors (say, the unit vector of tangent) in the course of evolution. The analogy between the rotation of the plane of polarization of light and the rotation of a rigid body was revealed and studied by I S Emel'yanova [29 – 31].

5. W Pauli's (1900 – 1958) and A D Galanin's (1916 – 2000) works

The relation between the Dirac wave equation for electron and geometrical optics was investigated by Wolfgang Pauli in his paper [32] published in 1932. As a starting point of investigation he accepted the analogy between classical mechanics and geometrical optics, according to which the light beams in geometrical optics, corresponding to any quantum-mechanical problem, and the trajectories in the respective classical mechanical counterpart completely coincide. Relying on this analogy, Pauli expended the solution of the Schrödinger equation as a power series in the Planck constant ratio to an imaginary unit and applied this approach to the relativistic Dirac wave equation for elementary particles (electron, proton) in a given external electromagnetic field. Neglecting the spin of the electron, he showed that the classical trajectories which can be obtained by the passage to the limit, similar to the passage from wave optics to a geometrical one, completely coincided with the trajectories of a pointwise charged electron in relativistic mechanics. Considering the effect of the spin of the electron, Pauli assumed that it is manifested in the higher approximations. He wrote: "...*Concerning the effects related to the spin and their influence on the distributions of density and current, it should be noted that these effects, combined with diffraction effects, will have an affect at first on the amplitudes ... of the next approximation, but we will not concentrate on their calculations in detail*".

The properties of the spin of an electron and meson were studied by A D Galanin [33], a post-graduate tutored by I E Tamm in 1942. The starting point of his approach to the problem was that the wave equation can be transformed into the equation of geometrical approximation by expending the solution into the power series of a small parameter. This approach was applied to the Maxwell, Schrödinger, Dirac, and Proca equations. Here, as the zeroth approximation one obtains the equations of geometrical optics (in the case of Maxwell equations) or the Hamilton – Jacobi equations (in the case of classical mechanics). In the first approximation, the equations of energy and charge conservation are obtained. However, if one invokes the results of Rytov [11], showing that the Maxwell equations give the variation of polarization state of the wave field along the beam trajectory in the first approximation, and applies them to the Dirac equation, then, as the first approximation, one finds the variation of spin orientation of an electron moving along the classical path. Implementing this program, Galanin derived the equations describing the spin precession of an electron and meson and obtained the formula for the Thomas precession [34, 35] as the ultimate result. Summarizing the study carried out, Galanin wrote: "*... in the case of small velocities, the electron and meson manifest their gyroscopic properties as classical spinning particles*".

6. A Yu Ishlinskiĭ's (1913 – 2003) work

In 1944, A Yu Ishlinskiĭ published an important paper [36] which remained relatively unknown for a long time. This paper brought to light the effect of the combined rolling and pitching motion of a ship. In Ref. [36], it was shown that in the general case where rolling and pitching are out of phase, any axis fixed in the ship performs a conical motion that leads to the variation of ship attitude and, as a consequence, its course.

In 1949, the solid angle theorem got an unexpected experimental verification during the tests of a single-axis gyrostabilizer drift under stand angular oscillations, carried out by engineer M L Effa [3]. The test program specified the measurements of gyrostabilizer drift under the effect of angular oscillations about two mutually orthogonal axes with ratio of frequencies 2:1. In this case, the end of the stabilization axis of the tested instrument described a figure eight on the spherical surface. Thus, by virtue of the Hamilton theorem, such an excitation should not lead to instrument drift. The drifts observed were relatively small and may be generated by imperfections in the testing instrument. However, in one of test runs, the frequencies of two stand angular oscillations were equal to each other. Since the oscillations had a certain phase shift, the stabilization axis described an

[2] The *Frenet trihedron* is a natural moving trihedron of the spatial curve suggested by Jacque Frederique Frenet (1816 – 1900). The theory of such trihedrons was developed by Gaston Darboux (1842 – 1917), therefore the moving trihedron is sometimes called a *Darboux trihedron*.

[3] Mark Leopol'dovich Effa (1925 – 1994) was a Doctor of technical sciences, Lenin prize winner, engineer, and scientist, who created and developed the gyroscopes for missiles, rockets, and spacecraft.

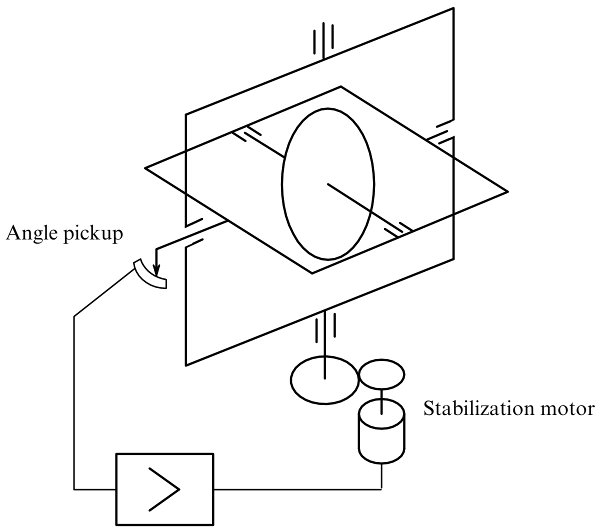

Figure 1. Schematic of a single-axis gyrostabilizer. The rotation of the outer gimbal leads to the precession of the gyroscope (represented by the inner gimbal and rotor) about the inner gimbal axis. The angle of precessional rotation is measured by an angle pickup, the output signal of which is amplified by the feedback amplifier and transformed into current applied to a stabilization motor. The motor compensates all the disturbing torques about the outer gimbal axis. Therefore, the angular rate of the outer gimbal is reduced to zero.

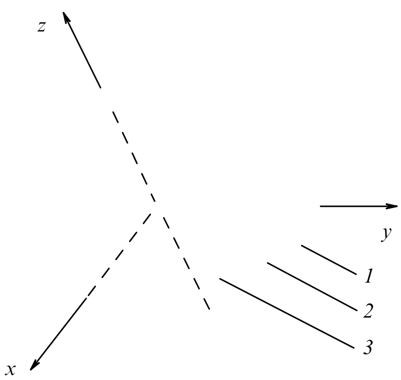

Figure 2. Gyroscope in cardan suspension, suggested by L Foucault. The instrument consists of the rotor (*3*), inner gimbal (*2*), and outer gimbal (*1*). Coordinate system *xyz* is attached to the inner gimbal of the suspension.

elliptic cone and the observed gyrostabilizer drifts increased by two orders of magnitude. The results were registered by the test engineer and presented to A Yu Ishlinskiĭ who gave them a short, but exhaustive explanation. [4]

The scheme of the single-axis gyrostabilizer tested is illustrated in Fig. 1. It is based on the gyroscope in a cardan suspension (see Fig. 2) that had been suggested by Jean Bernard Leon Foucault (1819 – 1868) to the Paris Academy of Sciences on 27 September 1852 as an instrument for the experimental confirmation of the Earth's rotation [40]. It is known that the experiment carried out by L Foucault was unsuccessful due to instrument imperfections. The single-axis gyrostabilizer is an instrument in which the drawbacks that prevented the success of Foucault's experiment are eliminated.

If a rotor of the gyroscope shown in Fig. 2 is brought into rotation with high angular velocity and all the system is free to move then one can see immediately that the Foucault experiment was unsuccessful due to the effect of friction in the rotor and gimbal bearings. To compensate the effect of rotor bearing friction, the stator of the electric drive motor is mounted on the inner gimbal. The motor should provide a constant angular velocity Ω of rotor rotation with respect to the stator or inner gimbal of cardan suspension. As a result, the rotor acquires an angular moment H equal to the production of the rotor polar moment of inertia J and angular velocity Ω. The spin of elementary particles represents a microscopic analogue of the gyroscope angular moment.

The equations of gyroscope rotation as a rigid body having constant spin maintained by the electric drive motor are written down as

$$
\begin{aligned}
&A\,\frac{\mathrm{d}\omega_x}{\mathrm{d}t} + (C-B)\,\omega_y\omega_z + \omega_y H = M_x\,,\\
&B\,\frac{\mathrm{d}\omega_y}{\mathrm{d}t} + (A-C)\,\omega_z\omega_x - \omega_x H = M_y\,,\\
&C\,\frac{\mathrm{d}\omega_z}{\mathrm{d}t} + (B-A)\,\omega_x\omega_y = M_z\,,
\end{aligned}
\tag{1}
$$

where A, B, and C are the moments of inertia of the gyroscope about axes of the coordinate system xyz shown in Fig. 2; ω_x, ω_y, and ω_z are the projections of the gyroscope angular velocity vector, and M_x, M_y, and M_z are the projections of the resultant torque acting on the gyroscope onto the same axes. In the derivation of equations (1) it was assumed that axes x, y, and z are principal central axes of gyroscope inertia (the validity of this assumption is one of the concerns in the practice of gyro design and development). Equations (1) differ from the classical Euler equations [20]

$$
\begin{aligned}
&A\,\frac{\mathrm{d}\omega_x}{\mathrm{d}t} + (C-B)\,\omega_y\omega_z = M_x\,,\\
&B\,\frac{\mathrm{d}\omega_y}{\mathrm{d}t} + (A-C)\,\omega_z\omega_x = M_y\,,\\
&C\,\frac{\mathrm{d}\omega_z}{\mathrm{d}t} + (B-A)\,\omega_x\omega_y = M_z
\end{aligned}
\tag{2}
$$

by terms proportional to the gyroscope spin H. Equations (1) were deduced by Russian scientist B V Bulgakov (1900 – 1952) in Ref. [41], while the idea of such forms of equations was put forth by A N Krylov (1863 – 1945) and Yu A Krutkov (1890 – 1952) in their monograph [42].

The gyroscope supported by bearings of its inner axis y inside the outer gimbal (see Fig. 1) composes a single-gyro frame with this gimbal. Such a structure represents the basis for the design of gyroscopic instruments like a free gyro and single-axis gyrostabilizer. The instruments serve as a reference for sensing the rotation of rockets, aircraft, ships, and other vehicles moving in space. Since the angular rates of vehicles are usually low, the terms of the Eulerian part in equations (1) are much less than the terms proportional to the spin of the gyroscope and are, in general, negligible. As a result, one

[4] This analytical result was published in a book [37] issued in 1952. This book had a limited number of copies and was available only to a small group of specialists. However, A Yu Ishlinskiĭ included the explanation of the effect in the course of lectures *Theory of Gyroscopes* which was attended by one of authors (S A Kharlamov) at Moscow State University in 1956. Only in 1963 did the second enlarged edition of the book [38] become widely known. Later, this effect was included in a textbook [39].

obtains the following equations

$$\omega_y H = M_x, \quad -\omega_x H = M_y \tag{3}$$

governing the precession of the gyroscope. As one can see from Fig. 1, the torques acting on the gyroscope about axis z are balanced by the torques of reaction forces in the bearings of the inner and outer gimbals.

Let the single-gyro frame be installed on a moving vehicle in such a manner that its outer axis is horizontal, while the inner axis is vertical. Moreover, all the spurious torques about these axes, like the bearing friction torques, are made as low as possible. In this case, from equations (1) follow that

$$\omega_x = 0\,, \qquad \omega_y = 0\,,$$

i.e., axes x and y are nonrotating. Thus, the instrument can be used for producing signals about vehicle rotation and generating commands to the vehicle attitude control. [5]

If one tries to turn the outer gimbal of the Foucault gyroscope with a fast spinning rotor, then, according to equations (3), the rotor will precess, tending to align its angular moment vector with the outer gimbal axis. This effect is used to compensate the friction torque in the bearings of the outer gimbal axis. To this aim, a stabilization motor is mounted as shown in Fig. 1. The motor generates a torque that balances the friction torque and other disturbing torques. The balancing torque is produced by a command signal proportional to the rotor precession angle which is measured by an angle pickup on the inner gimbal axis. [6] For the single-axis gyrostabilizer, equations (3) governing its precessional motion take the form

$$\omega_y H = M_x^{(1)} + M_x^{(2)}\,, \quad -\omega_x H = M_y\,, \tag{4}$$

where $M_x^{(1)}$ is a total disturbing torque, and $M_x^{(2)}$ is the stabilizing torque, while the torque M_y is usually considered as negligible for modern high-precision instruments. The sum of the torques in the first equation is equal to zero by virtue of the single-axis gyrostabilizer operation principle. The second equation shows that the angular velocity of outer gimbal rotation is equal to zero. This angular velocity is written down as follows

$$\omega_x = \frac{d\alpha}{dt} + \frac{d\psi}{dt}\sin\theta\,, \tag{5}$$

where α is the angle of outer gimbal rotation about its axis, and ψ and θ are the angles of instrument base rotation during the motion of the end C of a unit vector directed along the outer gimbal axis on the unit sphere (see Fig. 3, where angle α is not shown for the sake of simplicity). Rewriting formula (5) in the differential form and performing the integration, one can find that the variation of the outer gimbal rotation angle due to motion of point C along the closed loop (see Fig. 3) is

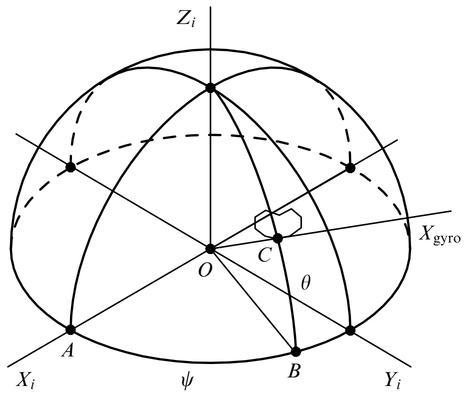

Figure 3. Unit sphere attached to the inertial coordinate system $X_i Y_i Z_i$. Axis $X_{\rm gyro}$ of the gyrostabilizer shown in Fig. 1 describes a closed curve on the sphere. Orientation of this axis is governed by the angles $AOB = \psi$ and $BOC = \theta$. Point C is the trace of the gyrostabilizer axis on the sphere.

defined by the relation

$$\alpha - \alpha_0 = -\oint_L \sin\theta \, d\psi\,. \tag{6}$$

Transforming this contour integral into a double surface integral by using the Stokes formula, one obtains

$$\alpha - \alpha_0 = \iint_S \cos\theta \, d\psi \, d\theta\,. \tag{7}$$

The integrand is the element dS of a unit sphere area. Therefore, one arrives at

$$\alpha - \alpha_0 = S\,. \tag{8}$$

By definition, the surface S on the unit sphere is a measure of the solid angle formed by a closed conical surface. Thus, the rotation angle of the single-axis gyrostabilizer about its outer axis is equal to the solid angle described by this axis due to the cyclic motion of the instrument base. [7] As mentioned above, the important implication of this result was published by A Yu Ishlinskiĭ in Ref. [36] in 1944. Based on the analysis of the geometry of bicardan suspension [44] by using the direction cosine matrices, he revealed the effect of combined rolling and pitching motion of a ship about an arbitrarily directed axis on the variation of course indication. The main contribution is the demonstration of the important influence of nonlinear (quadratic) effects of small angular oscillations on the attitude kinematics of rigid bodies.

As shown by Ishlinskiĭ in his monographs [37, 38], the effect considered above is intimately connected with a parallel transfer of a vector in Riemann geometry [45, 46]. Thus, if a vector is subjected to a parallel transfer along the closed contour on the sphere surface then the vector will be turned by an angle after returning to the original point. The angle of turn is equal to the solid angle formed by the contour. This phenomenon is sometimes called *anholonomy* (see, e.g., Refs [4, 6, 47]). In the general case of the parallel transfer of a vector (or an axis fixed in a rigid body), the final vector orientation is related to its initial orientation and every point

[5] Similar gyroscopic instruments were applied for attitude control of the German A-4 rocket developed in the 1940s. They could not 'see' the Earth's rotation, but could guide a rocket on the prescribed trajectory. In the mean time, due to many engineering innovations, instruments of this type were improved and attained such a level of perfection that they could 'see' the Earth's rotation excellently.

[6] There are engineering methods to reduce the effect of the friction in the bearings of the inner gimbal axis, but they are beyond the scope of the problems under consideration.

[7] In Ref. [43], A Yu Ishlinskiĭ noted that he derived formula (8) in 1943. This formula was first published in the book [37] (second edition [38]).

of its path, along which the vector is transferred, by a nonholonomic (nonintegrable) relation. It should be noted that V V Vladimirskiĭ applied the formalism of the parallel transfer of a vector in the paper [13].

7. G Heinrich's work

In 1950, Austrian researcher G Heinrich published paper [48] that is close conceptually to A Yu Ishlinskiĭ's work. He considered the conical motion of a ship, occurring due to a phase shift between rolling and pitching motions, and found the effect of this motion on the course indication of single-rotor gyrocompasses. The gyrocompass studied represents in essence the gyroscope in cardan suspension shown in Fig. 2, the outer axis of which is vertical with the angular momentum vector of the gyroscope held in the horizontal plane by a control system with a tilt indicator installed on the inner gimbal. It was shown that gyrocompasses also have errors generated by the conical oscillations of their bases, but in contrast to the single-axis gyrostabilizer's unbounded angles of rotation, these errors are limited. The reason is that the gyrocompass has a directional torque aligning its rotor spin axis with the direction to North and preventing unlimited rotation. The outcomes of the work [48] were presented in the book of Kurt Magnus [49], where the results of the papers of H L Price [50, 51] were also reviewed. In Refs [50, 51], the errors of a directional gyro were studied. These errors were generated by an aircraft bank in which the aircraft axis perpendicular to the fuselage axis and the axis of the wings described a cone in space.

8. L E Goodman and A R Robinson's work

In 1958, L E Goodman and A R Robinson published a paper [52] where they reopen all the above-described results of Ishlinskiĭ in the form of a generalized kinematical theorem of finite rotations. The generalization was really reduced to the consideration of the conical motion effect on the rotating rigid body of the gimbal in the gyroscopic instrument. It was shown that the continuous conical motion of the rigid body rotation axis results in an increment of the angular velocity of rotation. This paper became widely known and generated the boom of studies on gyroscopic instruments. In a short time period, a large number of theoretical and experimental papers were devoted to the effect of angular vibrations on the drifts of gyroscopes and errors of gyroscopic instruments.

9. J Hannay's work

The works of J Hannay [17, 53] deserve a more detailed consideration in order to estimate his contribution to the development of modern representations of topological phase in classical mechanics. In Ref. [17], he aimed to extend the quantum-mechanical results of Berry [1] to the semiclassical case where the Planck constant h tends to zero. To this end, Hannay considered an integrable mechanical system with one degree of freedom, the Hamiltonian of which depends on the parameters. The mechanical system is integrable if the parameters are constant, i.e., in any point of the parameter space. As canonical variables of this mechanical system, the 'action – angle' variables are selected and the 'action' variable yields the first integral for the system under consideration.

Starting from the presentations of the adiabatic invariants, taken from the book by V I Arnold [54], Hannay studied the system behavior in the case where the parameters of the Hamiltonian are slowly varied along a closed trajectory in the parameter space. In this case, the action variable represents the adiabatic invariant. However, long before J Hannay and V I Arnold, the adiabatic invariants of mechanical systems depending on the parameters were studied by A A Andronov, L I Mandel'shtam, and M A Leontovich in a paper [55]. Unfortunately, this publication, which contains clear definitions of the stationary and temporary adiabatic invariants, was missed by the next generation of scientists.

Hannay's main contribution is the demonstration that the slow variation of the Hamiltonian parameters along closed contour in the parameter space leads to an additional increment of the angle variable that depends on the closed contour. It is this increment that was declared by M Berry and J Hannay [52] as the topological phase in classical mechanics. Due to the high scientific authority of Berry, this declaration has become the widespread opinion, and not only among Western scientists.

Besides theoretical considerations, Hannay gave three examples intended to support the significance of the author's conclusions. As the first example, the frictionless motion of a bead along a closed wire loop was considered. The effect studied is manifested in this system if the wire loop is rotated about the axis perpendicular to its plane. It was shown that the variation of the mean angular velocity of the bead position vector is equal to the angular velocity of the loop. This result is trivial and may be easily obtained by an application of the angular momentum conservation theorem of mechanics.

In the second example, a harmonic oscillator was discussed. In this case the Hamiltonian was taken in the most complicated form. For this oscillator, the action variable, which is equal to the ratio of total oscillator energy to its natural frequency, is the adiabatic invariant. This fact was already shown by Soviet physicists in 1928 [55]. That the angle variable may have an additional increment under the variations of oscillator parameters one can find from the formulas given in a paper [55].

As the third example, it was stated that the angle of rotation of a spinning top varied due to the conical motion of its spin axis. This paragraph of paper [17] summarized the theoretical findings of L E Goodman and A R Robinson's work [52] published in 1958. No arguments are given relating these findings to Hannay's theories.

10. Development, generalization, and application of the solid angle theorem

10.1 Further application of the solid angle theorem in classical mechanics

In the last years of the 20th century, work related to the topological phase in classical mechanics was continued. In Refs [56 – 58], Yu K Zhbanov and V F Zhuravlev refined the formulation of the solid angle theorem (in particular, they considered the accumulation of this angle during the conical motion of a rigid body axis if the cone is not closed), simplified the proof, and made it more obvious. The relation of the solid angle theorem to the noncommutativity of rigid body finite rotation was studied. The most complete representation of rotation noncommutativity and its role in navigation was presented by N I Krobka [59].

In Ref. [60], S E Perelyaev studied the existence of an effect similar to the Ishlinskiĭ effect in multidimensional spaces. He showed that this effect is peculiar only to three-dimensional space and is not observed in four-dimensional space. The author postulated that the effect is produced due to the coincidence of the space dimension and the number of parameters representing the SO(3) rotational group. In Ref. [61], A V Krutov investigated in detail the geometrical aspects of rigid body rotation in space.

Among the papers devoted to the problem under consideration, paper [62] published by Indian scientists (R Simon, N Mukunda, E C G Sudarshan) is of special interest. Its authors generalized the Hamilton theory of rotations [18] by using the group theory. They showed that the rotations defined by Hamilton as oriented great circle arcs on the unit sphere are equivalent to the elements of the SU(2) group. By analogy, they introduced oriented arcs on the unit hyperboloid of one sheet and demonstrated their equivalence to the elements of the Sp(2, R) group.

As for further development of Berry's and Hannay's studies, significant progress was attained by Y Aharonov and J Anandan [63]. They generalized the concept of the Berry phase by eliminating the adiabatic approximation. However, this generalization comes at a fairly high cost, since quantum states do not always return to the initial state after the cyclic evolution of the Hamiltonian. Therefore, the geometrical phase cannot be determined in some cases by the method of Ref. [63]. The Hannay angle is the classical analogue of the geometrical phase. It was studied by Dae-Yup Song [64] for a harmonic oscillator with a time-periodic Hamiltonian. The author considered the existence of the Hannay angle in the phase state of the oscillator, the canonical structure of which is such that the action variable is the exact (but not adiabatic) invariant of the Hamilton system.

10.2 Further application of the solid angle theorem in polarization optics and development of the analogy between optics and mechanics

The results obtained by S M Rytov and V V Vladimirskiĭ [11–13] in geometrical optics were advanced further in polarization optics. In 1956, S Pancharatnam [14, 15] showed that a light beam propagating along a plane trajectory (in particular, a straight line) acquired a phase incursion if the beam polarization state is subjected to cyclic evolution, so that the point on the Poincaré sphere [9] representing this state traces a closed curve. In this case the phase increment is equal to the solid angle under which the closed curve is seen from the center of the sphere. In Ref. [65], G B Malykin and Yu I Neĭmark showed that the Pancharatnam phase of a light beam passed through a unimode fiber lightguide (UFL) has a nonholonomic relation with the azimuthal angles of linear birefringence axes at lightguide input and output. This fact is a direct implication of the analogy between optics and mechanics. The applications of the Pancharatnam phase were reviewed in Refs [3–9].

Recently, Malykin showed in papers [66, 67] that the well-known phenomenon of polarizational nonreciprocity of a fiber ring interferometer (FRI) [68–70], leading to a counterpropagating wave phase difference which is not related with the interferometer rotation, can also be represented as a nonreciprocal geometrical (topological) phase. As shown in Refs [66, 67] the counterpropagating wave phase difference in the FRI generated by the polarizational nonreciprocity [8] is equal to a solid angle which is based on a spherical triangle given by three points of the Poincaré sphere. These points correspond to the light polarization states on the FRL input and the two outputs.

Notice that in spite of the fact that both the nonreciprocal geometrical phase of the UFL and the Pancharatnam phase are defined in terms of the solid angle on the Poincaré sphere, they cannot be joined to each other. The first phase is defined only for two counterpropagating waves in the ring, while the second one may exist for a single wave. The first one is determined only by the polarization state at the ring input and two outputs, while the second one depends on the evolution of the polarization state along the entire beam path. Therefore, in contrast to the Pancharatnam phase, the nonreciprocal geometrical phase of the UFL has a holonomic association with the beam polarization state at the UFL input and both outputs.

Another principal difference between the Rytov–Vladimirskiĭ phase, on the one hand, and the Pancharatnam phase, on the other hand, should be noted. In the first case, one deals with a solid angle in real space, while, in the second case, one considers a solid angle on the Poincaré sphere, i.e., in the space of Stokes vectors [9, 71].

The effect treated by Ishlinskiĭ in Refs [37, 38] leads to the generation of errors of mechanical gyroscopes, which serve as an indication of orientation according to their operation principle based on the inertial properties of rigid bodies. The nonreciprocal geometric phase of the UFL results in the generation of errors in fiber optical gyroscopes which are the angular velocity sensors [70], and their operation principle is based on the utilization of the Sagnac effect (effect of the special theory of relativity) [70, 72–75]. Thus, the solid angle theorem found application for the evaluation of errors of both mechanical and optical gyroscopes. This is another manifestation of the deep analogy between optics and mechanics.

11. Solid angle theorem in the special relativity theory

A rather unexpected and, from the physical viewpoint, very interesting implication of the solid angle theorem was found in the special theory of relativity. In Refs [77–79], it was shown that a relation similar to that predicted by the classical theorem also exists in the special theory of relativity: the angle of rigid body rotation in its motion along a plane trajectory, caused by the Thomas precession (relativistic kinematic effect) [34, 35, 80], is equal to the solid angle observed in the immovable frame of reference, which is traced by the body-fixed axis due to the variation of body image rotation in the immovable frame of reference. The last phenomenon is called a relativistic aberration [81–84]. It is caused by the Lorentzian shrinkage of length and propagation time lag of light emitted by different regions of the body. Thus, the Thomas precession can be considered the relativistic analogue of the Ishlinskiĭ effect or solid angle theorem.

Notice that even in the simplest case of circular motion, it is a very difficult problem to obtain an analytical relation between angular velocities of the Thomas precession and orbital motion. As shown in Ref. [85], different authors

[8] This difference was computed separately for the waves traveling along 'slow' and 'fast' birefringence axes of the FRL.

obtained different relations for this case. There are about ten known relations that significantly differ from one another. Certain relations can be found in a number of versions which have different signs and coefficients. The method suggested by one of the authors of this paper (G B Malykin) for the preparation of the Thomas precession formula is based on the application of the solid angle theorem [77 – 79]. It allows one to obtain the correct result by using simple trigonometric manipulations, being much less complicated than other known methods. Moreover, it enables one to clearly illustrate the physical sense of the Thomas precession.

12. Conclusions

There have been three waves of interest in or three stages of study of the topological phase in mechanics. In the first stage, Sir William Rowan Hamilton elaborated the theory of rigid body rotations, based on the findings of his predecessors as an object of application of his theory of quaternions. In 1853, he formulated and proved the theorem that is now called the *solid angle theorem*. The theorem was included by Horace Lamb in his classical treatise on mechanics in 1929, but later was forgotten for two decades.

In the second stage, at the beginning of the 1940s, the theorem was formulated and proved by A Yu Ishlinskiĭ, posed in a more general way and in connection with gyroscopic instruments. At the end of the 1940s, the theorem found experimental confirmation as a result of testing the single-axis gyrostabilizer on a rotationally oscillating stand. In 1956, Ishlinskiĭ already included the theorem in the course *Theory of Gyroscopes* where he demonstrated the nonholonomic nature of this effect. In 1958, L E Goodman and A R Robinson formulated and proved the theorem in parallel and independently in connection with the effect of finite rotation on the errors of gyroscopic sensing elements. These publications stimulated a large number of publications devoted to the effect of angular vibrations on the accuracy of gyroscopic instruments.

In the third stage, J Hannay extended the concept of the topological phase that was introduced by M Berry in quantum mechanics to classical Hamilton mechanical systems. Here, it was shown that the angle variable of an integrable mechanical system acquires an additional increment in the course of adiabatic and cyclic variation of the Hamiltonian in the parameter space. This increment, called the Hannay angle, is in essence the angle found by Hamilton for a rigid body, and Ishlinskiĭ for gyroscopic instruments in real three-dimensional space, but observed in the phase space of an integrable Hamilton mechanical system. The necessary condition of the existence of the Hannay angle is the adiabatic nature of the canonical variable 'action' or its exact invariance.

It should be emphasized that, besides its use in classical mechanics, the concept of topological phase is also employed in optics and quantum mechanics. Remarkably, the Russian scientists S M Rytov, V V Vladimirskiĭ, A D Galanin, and A Yu Ishlinskiĭ obtained the first results in solving the problem of topological phase in optics, the physics of elementary particles, and mechanics during the short (1938 – 1952), but very dramatic period of history in Russia. It is also necessary to note that the manifestations of topological phase in optics and mechanics are related by the fundamental analogy between these disciplines that was developed in the works of N G Chetaev.

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