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Dirac in 20th century physics: a centenary assessment

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Abstract. **Current views on Dirac's creative heritage and on his role in the formation and development of quantum physics and in shaping the physical picture of the world are discussed. Dirac's fundamental ideas in later life (1948 – 1984) and their current development are given considerable attention.**

1. Introduction

In August of 2002, the scientific community commemorated the birth centenary of P A M Dirac — one of the most original thinkers in 20th century physics. Our paper is concerned with the traits of his creative activity and his impact on the development of contemporary natural science.

Reminiscing about Dirac, A Salam, a Nobel Prize Laureate in Physics 1979, emphasized (see Ref. [1], p. 84) that "*Paul Adrien Maurice Dirac was undoubtedly one of the greatest physicists of this or any century. In three decisive years – 1925, 1926 and 1927 — with three papers, he laid the foundation, first of quantum physics, second of the quantum theory of fields, and third, of the theory of elementary particles ... No man except Einstein has had such a decisive influence in so short a time on the cause of physics in this century*".

Assessing the comparative role of personalities and separate accomplishments in the history of humankind's spiritual culture, the more so for a period of centuries and millennia (as done in Salam's statement), is an extremely difficult task. Even restricting oneself only to the scope of physics, one cannot but admit that it is conventional to equally admire the exquisite effects of experimenters — the authors of so-called *experimentum crucis* — and the brilliant insights of theorists — the authors of basic theories, which put the comprehension of a whole class of natural phenomena in order. What criteria should be applied to evaluate accomplishments so different in nature is a separate question.

Of course, it is possible to scrupulously count the number of references to the papers of one scientist or another in the publications of other authors and evaluate the so-called citation index. This relatively formal approach is the simplest to realize because there is no need to analyze the contents of the papers and their real value, and it is possible to resort to only the services of statisticians in lieu of the expensive services of analysts. Another way is to question specialists and experts engaged in a given or related field of knowledge, which is taken advantage of in one form or another when allotting grants, awarding prizes, etc. At best these methods allow us to determine the circle of best-known or most frequently cited scientists, but no more than that.

There exists a more objective criterion, namely, the assessment of accomplishments, which is only applicable, truth to tell, to acknowledged classics of science: to judge the contribution of a scientist by the number of 'nominal' results — principles, effects, phenomena, formulas, and equations bearing his name. Should this criterion be applied, Dirac would be among the indisputable leaders in 20th century physics: the Dirac equation, the Dirac transformation theory, the Dirac field, Dirac matrices, the Dirac delta function, Dirac brackets, the Dirac theory of holes, the Dirac interaction representation, the Dirac quantization rule, the Dirac monopole, Fermi – Dirac statistics, Dirac conjugation, the Dirac propagator, Dirac mechanics — this is by no means the complete list of appellations and terms that have firmly entered modern textbooks and monographs.

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Getting somewhat ahead of our presentation, by and large it is valid to say that the words and notions introduced by Dirac form the basis of the language in which quantum physics expresses itself. Examples are an observable, a state, commutation relations, the Dirac unit of action (the Serbian letter $\hbar$), 'bra' and 'ket' vectors, c- and q-numbers to denote respectively classical and quantum quantities. The bracket notation for matrix elements, the particle creation and annihilation operators, and even the functional integral have also been inherited by modern physics from Dirac. (Some details of Dirac's 'quantum word creation' are given in Appendix 7.2.)

Reverting to the above criteria, we note that even the highest of levels specified is 'somewhat tight' for the evaluation of the creative work of scientists who are famous for more than just separate key ideas or accomplishments. The case in point is personalities who have actually changed our notions of the major aufbau principles of the surrounding world. It is generally recognized that among the creators of this rank are I Newton, who laid the foundations of the classical physical picture of the world (PPW), and A Einstein, who brilliantly completed its creation and paved the way for the nonclassical PPW. According to the aforementioned viewpoint of Salam, which is also shared by other famous scientists (see, for instance, Refs [1–6]), Dirac is among the personalities of precisely this scale.

Given below in support of this opinion, which is not universally accepted, are the arguments and facts which are testimony to Dirac's fundamental role in the formation of contemporary PPW. That is why we will not enumerate the well-known facts of Dirac's biography, which have repeatedly been published (in particular, by *Phys. – Usp.* [7]) and, if need be, can be found in the collected books [1–4, 8] as well as in autobiographic articles of Dirac himself [9]. To start with, we invoke an ancient truth that "everything is apprehended by comparison" and somewhat develop the above judgement of Salam, who ranked Dirac with Einstein. This comparison is instructive, the more so as the destinies and the features of creative activity of these two physicists have much in common.

2. Strokes on the canvas: Dirac and Einstein

Dirac and Einstein are similar primarily because of their profound and highly original thinking. In fact, neither one had tutors or numerous pupils and they seldom needed references to anybody else's works, while the papers each of them co-authored can be counted on one hand. Their deepest ideas were elaborated in practically complete solitude, either at the writing-desk of a patent clerk (Einstein) or during walks through the local environs (Dirac). Solitariness and isolation (in thought, creative activity, and everyday life) are the most distinctive features of both Dirac and Einstein, inherent in them until their very last years. [1]

Another distinguishing feature of Dirac's and Einstein's style of scientific creative work is simplicity, which is made a principle and yet has nothing in common with elementariness. Dirac remarked in the last page of the third edition (1947) of his *The Principles of Quantum Mechanics* [10] that "... *we should require of a satisfactory theory that its equations have a simple solution for any simple physical problem* ...". Einstein echoes him in his "Autobiographishes" [11]: "*The eminent heuristic significance of the general principles of relativity lies in the fact that it leads us to the search for those systems of equations which are in their general covariant formulation the simplest ones possible*...". Multipage computations and 'tedious' proofs are not found in their works, and the results and formulas they arrived at comply with the highest standards of 'mathematical beauty'. The amazing elegance and masterly ease with which logically harmonious theories emerged in the works published by Dirac and Einstein may be compared only with Mozart's style in music or with drawings made by Picasso and Dali.

It would be quite reasonable to suspect that there is some mystery behind all this... And it turned out that such was indeed the case! Each of them made use of his own 'know-how', which had long remained 'concealed from the uninitiated'. Einstein's magic wand of sorts was the preference he showed for 'the theory of principle', with thermodynamics being its embodiment for him. In his seventieth year, Einstein wrote in the above-mentioned "Autobiographishes": "*Reflections of this type made it clear to me as long ago as shortly after 1900, i.e., shortly after Planck's trailblazing work, that neither mechanics nor thermodynamics could* (*except in limiting cases*) *claim exact validity. By and by I despaired of the possibility of discovering the true laws by means of constructive efforts based on known facts. The longer and the more despairingly I tried, the more I came to the conviction that only the discovery of a universal formal principle could lead us to assured results. The example I saw before me was*

[1] We note that everything listed here pertains equally to Newton. However, a comparative analysis of the life and activities of the three greatest physicists would lead us far beyond the scope of this paper.

thermodynamics. The general principle was there given in the theorem: the laws of nature are such that it is impossible to construct a perpetuum mobile (*of the first and second kind*)...''. A detailed analysis of the thermodynamic origins in Einstein's thought can be found in Klein's article ''Thermodynamics in Einstein's thought'' [12] (see also Ref. [13]), where he showed that Einstein was directly or indirectly guided by the thermodynamic view of the world even when constructing theories outwardly remote from thermodynamics.

Dirac also had a 'secret' of his own, which emerged under the following circumstances. On receiving a bachelor's degree from Bristol University in 1921, youthful Paul made an attempt to continue education in Cambridge University. However, he, a recent immigrant, was refused a stipend and returned to Bristol where he was granted permission to attend lectures unofficially at the Mathematical Department, without payment for the education. But, as the saying goes, every dark cloud has a silver lining. The strongest impression of this period was produced by the lectures of the mathematician P Fraser, who managed to inculcate in his pupils the apprehension of the beauty of mathematical constructions and simultaneously a demand for the rigor of mathematical arguments. The mathematical beauty of physical laws, not without Frasers's influence, became for Dirac the intuitive criterion of correctness of physical theories.

It was Fraser who acquainted Dirac with projective geometry. ''*I was strongly impressed by its mathematical beauty*,'' — Dirac wrote later on. — ''*It seems to me that for the most part physicists know little of projective geometry, and I would say that this is a gap in their education. Projective geometry always operates on a plane space, but it is a powerful tool for its investigation, which equips us with methods, e.g., the method of unique correspondence, that yield results as if by magic... I have always invoked projective geometry considerations in my work... Projective geometry has been an extraordinarily useful research apparatus, but I have written nothing about it. It seems to me that I have not even mentioned it in my papers* (*though I am not quite sure of it*) [2], *for I realized that the majority of physicists are hardly familiar with it. On arriving at some result I would translate it into the analytical language and transform my arguments to equations. Any physicist could understand such an argument...*'' (see Ref. [9], p. 12). In this case, the story recurs (the other way round, though). In the 17th century, Newton, on obtaining the majority of his results with the aid of the methods of analysis he himself had elaborated, would convert them to geometrical language, in which his celebrated *Philosophiae Naturalis Principia Mathematica* were written, with the same purpose: to make the presentation clear to the majority of contemporary physicists.

While a post-graduate student at Cambridge, Dirac used to attend tea parties in Prof. G Baker's house (Fraser was also his student at his time), each of which ended with some communication on the results obtained employing projective geometry methods. After one of these tea sessions, a novice — Paul Dirac — had the courage to read a communication on a new method of solving projective geometry problems. That was the first lecture in his life [17]. The problems of the special relativity theory, which captivated Dirac early in his scientific career (the then obtained results were set forth in the second paper [18] of Dirac's publication list), were easily and simply solved in terms of projective geometry. [3]

From projective geometry Dirac derived not only the idea of spinors (homogeneous coordinates for isotropic lines), but he also transferred the Poncelet principle of duality to quantum mechanics, introducing not only the vectors of state, but dual vectors as well. It is noteworthy that, despite the fact that Dirac revealed his 'secret' which allowed him to arrive at outstanding results, the physical community practically refused to be interested in this information. In any case, projective geometry and Lobachevskian geometry have never been included in the list of obligatory courses of physics departments, and papers 'restoring' Dirac's original train of thought in terms of projective geometry are not found in the scientific literature. That is why Appendix 7.1 to our paper outlines briefly the simplest ideas of projective geometry.

It seems likely that the proximity of life and creative styles of Dirac and Einstein was by no means accidental. Worthy of note is their certain solitude in their families during childhood, as well as the oppressive feeling of being everlasting foreigners in society, which accompanied them throughout their lives, no matter where they were or in what capacity they worked. One cannot help noting a chain of astonishing coincidences: both received only basic technical education, both failed to get a job in their profession upon graduating with a higher education, and no one supported them during their first years of scientific research. They were compelled to live, devoting themselves to self-education, in small towns away from scientific centers. These circumstances undoubtedly slightly delayed the emergence of their first scientific papers. At the same time, they might have been the reason that the subsequent papers (only 1.5 years later!) of the young researchers fell right away into the category of unique works.

Indeed, Einstein and Dirac equally displayed an extraordinarily bright outburst of intellect, which embodied an original and many-sided blossoming at a relatively early age. [4] The ideas formulated in their early works exerted an immediate and sometimes decisive influence on their contemporaries and provided the basis for radically new physical theories. At a very early age they were elected respectively to the Prussian Academy of Sciences and the London Royal Society. At as early an age as thirty, they joined the world scientific elite as the main speakers at the First (Einstein, 1911) and Seventh (Dirac, 1933) Solvay Congresses. Naturally, both of them were Nobel Prize Laureates (Dirac, along with W Heisenberg, becoming a Laureate extremely early — at the age of 31). But Einstein and Dirac obviously stand out for the scale of their accomplishments even among Nobel Laureates.

These successes were great enough to go to their heads, but this did not happen. The great respect which Einstein and Dirac won from their contemporaries and progenies was based not only on the admiration for their scientific genius.

[2] Dirac's doubts are fully justified. Not only did he mention projective geometry methods, but he also made direct recourse to them, in particular, in papers which were mathematical in nature [14–16].

[3] Much has been written, in particular at a popular level in the book [19], about the relation between space geometry of the special relativity theory and Lobachevsky 'imaginary geometry' (which in turn is intimately related to projective geometry).

[4] Such examples are frequent in mathematics, for instance, E Galois, N H Abel, N N Bogolyubov, in music — W Mozart, in theoretical physics mention can be made of I Newton, J C Maxwell, L D Landau, Ya B Zel'dovich, R Feynman.

Both great physicists were distinguished by high human qualities, among which modesty is of special note. It manifested itself, of course, both in everyday life and in relations with other people. As regards science, they would never accentuate their role and, moreover, sometimes publicly underestimated their accomplishments [5]: no struggle for priority, all conceivable respect to the contributions to science made by their predecessors and contemporaries. Suffice it to refer to how Dirac throughout his life used to give Heisenberg his due for the initial idea.

Finally, we cannot help mentioning yet another feature which draws the scientific destinies of Dirac and Einstein closer. Having become classics of natural science relatively early in life, both of them experienced long periods of ideological solitude and even oblivion. The most active part of the physical community prematurely assumed that they no longer mattered, considering them has-beens. Many of their ideas advanced during the several last decades of their lives were underestimated by their contemporaries and have not been fully appreciated even to the present day. It is pertinent to note that the first comprehensive collections of the works of these outstanding physicists were issued not by the academic publishing houses in Germany, Great Britain, or the USA. The world's first four-volume collection of Einstein's works was published in the USSR in 1965 – 1967, while the publication of Dirac's collected scientific works is also now for the first time being undertaken in Russia [20].

It seems likely that the immanent properties of human consciousness require a significant historical distance to apprehend the true contribution of one personality or another. (Suffice it to remember what place Newton occupied in physics in the view of the scientific circles in the middle of the 18th century.) It therefore comes as no surprise that Dirac's true role in physics is gaining recognition in a gradual manner. We hope that our paper will convince the reader that Dirac was not only one of Newton's most deserving successors as a Lucasian Professor in Mathematics at Cambridge, but also continued the cause of constructing a proper physical picture of the world, pioneered by Newton.

3. Founders of quantum mechanics: Heisenberg – Dirac – Schrödinger

The advent of quantum mechanics is one of the greatest events in the history of civilization. To reveal the true contribution to the common cause from each of the heroes of this epoch is therefore an important task not only for science historians. Of course, we are not dealing with priority matters, all the more since a man like Dirac attached no significance to them.

It is well known that quantum mechanics was for the most part the fruit of creative activity of very young physicists. In this connection it deserved the name 'Knabenphysik'(boys' physics) from W Pauli. (Indeed, Pauli himself was born in 1900, Heisenberg in 1901, and Dirac and P Jordan in 1902.) It therefore makes sense to compare the conditions in which these talented youths were educated and became scientists. It must be said that it is one thing to grow up in continuous communication with coryphaei — with A Sommerfeld (Münich), M Born (Göttingen), N Bohr (Copenhagen), and P Ehrenfest (Leiden) — which actually took place in the scientific lives of Pauli, Heisenberg, and Jordan. And it is quite another matter to be, like Dirac, a research student of the famous Cavendish Laboratory in Cambridge, in which, however, there were no prominent scientists engaged directly in the problems of atomic physics. [6]

The first idea which initiated the origin of 'new' quantum mechanics was undeniably stated by Heisenberg in the summer of 1925 [21]. However, the noncommutativity of dynamic variables, which came to light in Heisenberg's matrix mechanics, depressed primarily the author himself, who regarded it as a substantial fallacy of his theory. This judgement was initially shared by Born and Jordan, who became engaged in its development along with Heisenberg. On R Fowler's advice, Dirac took up the same work in September 1925. With the boldness of thought inherent in him and the knowledge of Hamiltonian dynamics, Dirac came to consider the noncommutativity of canonically conjugate variables as Heisenberg's main contribution to the construction of quantum dynamics. Upon familiarizing himself with the proof of Heisenberg's first paper, he prepared on his own a fundamental article "The fundamental equations of quantum mechanics" [22] by 7 November 1925, which saw light on 1 December 1925.

Interestingly, it is in this work that the modern form was imparted to the Heisenberg equations

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = [\hat{x}, \widehat{H}]\,,$$

this being done for an arbitrary observable x, an arbitrary Hamiltonian H, and any operator representation. This is precisely the equation form which has entered all textbooks and monographs on quantum mechanics. [7]

In point of fact, Dirac's paper turned out to be the second publication on quantum mechanics, for the well-known paper by Born and Jordan [23] (although it was submitted on 27 September 1925) was published somewhat later and had not been accessible to Dirac beforehand. Born, Heisenberg, and Jordan — the authors of the celebrated 'paper of three' [24], which proved to be the fourth paper on this topic submitted to publication — in its preparation had a copy of Dirac's paper [22] given by the author himself. Heisenberg's friendly letter of 20 November 1925 to Dirac runs as follows: "*I have read your excellent work with the keenest interest. All your results are undoubtedly correct, with the understanding, of course, that one has faith in the new theory... I hope you will not be grieved about the fact that a part of your results was obtained in our institute some time ago... In your results you have advanced much further, and this is especially true of the general definition of differentiation and the relation between quantum conditions and the Poisson brackets*". And that is indeed the

[5] When Salam asked Dirac what he regarded as his most significant contribution to physics, the answer astounded him — the Poisson bracket. "*But with characteristic modesty, he added after a pause that for a long time he felt ecstatic and pleased, till he found essentially the same remark made by Hamilton as a footnote in one of his papers written in the last century*" (see Ref. [1], p. 84).

[6] Dirac took his first journey to the 'continent' in September 1926, when his principal results in quantum mechanics had already been obtained and published in seven most important papers.

[7] Here, historical analogies suggest themselves again: it is well known that the modern form of Newtonian laws, in particular the second law, was first imparted by L Euler, while the Maxwell equations acquired their modern form of writing in H Hertz's works. It only remains to remark: while Euler did this within 70 years after Newton, and Hertz within 20 years, for Dirac it took only two (!!) months.

case, because Refs [23, 24] are only concerned with the equations for coordinate and momentum operators and only in the energy representation, though for a broader class of Hamiltonians in comparison with the pioneering paper by Heisenberg [21].

Furthermore, Dirac published in 1926 a series of papers on quantum mechanics [25], including "On quantum algebra" and "On the theory of quantum mechanics". Based on these papers, he prepared by May of 1926 a Ph.D. thesis "Quantum mechanics". In the history of physics this was the first purely 'quantum' thesis; four years later it formed the basis of his fundamental monograph *The Principles of Quantum Mechanics* (first edition — 1930) [26].

In his approach to the construction of quantum mechanics, Dirac proceeded from the Hamiltonian form of analytical dynamics. This enabled him not only to introduce, in the most natural way, the idea of noncommutativity of dynamic variables into the mathematical apparatus of the new science, but also to organically incorporate the qualitatively new concept of a *quantum state* — the basic concept of wave mechanics proposed by E Schrödinger at the end of January and published on 13 March 1926. The theory of transformations elaborated primarily by Dirac allowed its author to convincingly demonstrate the equivalence of the approaches of Heisenberg (matrices), Schrödinger (wave functions), and the most general one, belonging to Dirac himself (q numbers).

A remark must be made concerning the heroic period of quantum physics elaboration (1925–1934). Among theoretical physicists were supporters of either the Heisenberg–Born–Jordan matrix mechanics, or the de Broglie–Schrödinger wave mechanics. The standpoint of Dirac, whose works were oriented from the outset to the formation of quantum mechanics proper, clearly stood out against this background. In support of this statement we adduce the fact that in his 28 papers, written on the subject during that period, the term 'wave mechanics' is encountered in the title of only one paper, while 'matrix mechanics' is not used at all.

Therefore, we have every reason to believe that the 'new' quantum mechanics is the common creation of Heisenberg, Dirac, and Schrödinger, wherein the basic ideas which allowed for unifying different approaches and representing quantum mechanics as a qualitatively new science are due to Dirac. In this respect, the part played by Dirac is quite comparable to Einstein's role in the development of relativity theory, which also unified the contributions of three authors — H Lorentz, H Poincaré, and Einstein himself. In this case, the Nobel Committee made an adequate assessment of the contributions of each of the founders of quantum mechanics and awarded Nobel Prizes in Physics for its creation to Heisenberg (1932) and Dirac and Schrödinger (1933), with the Prizes presented (it so happened) simultaneously to all three of them in December 1933.

In this connection we allow ourselves only a few remarks. Firstly, the universally accepted statistical interpretation of quantum mechanics, which can be traced back to Einstein's ideas from his radiation theory, is commonly related only to Born's name. The latter did introduce it when discussing the interpretation of microparticle scattering in three-dimensional configuration space. Similar ideas were simultaneously and independently put forward by Dirac in his work "The physical interpretation of quantum dynamics" [27], with the only difference being they were formulated not for the wave function in ordinary space, but for the probability amplitude of any process (not only scattering) in an arbitrary Hilbert space of states.

Secondly, away back in autumn 1926 Dirac discussed the problem of simultaneous measurability of the coordinate and momentum of a microparticle, coming close to the formulation of the *uncertainty relation*. In his famous 1927 paper on uncertainty relations, Heisenberg directly pointed out that its source was the Dirac theory of transformations.

Thirdly, it is traditionally believed that the creation of quantum statistical mechanics is primarily related to the name of J von Neumann. Indeed, the original idea of the density matrix was advanced by L Landau and von Neumann in 1927. However, it is not generally known that this idea was realized in Dirac's works done during 1929–1931 [28] and in his monograph [26], wherein the principles of quantum statistical mechanics were developed even before von Neumann's well-known monograph saw light in 1932 [29].

Fourthly, it was Dirac [30] who first came to consider the scattering theory as a description of the transition between single-particle 'in' and 'out' states in the momentum representation with fixed values of momentum, spin, polarization type, etc. His approach, unlike the initial Born collision theory, has proved to be equally applicable in nonrelativistic and relativistic domains for any microparticles undergoing scattering, including photons, and for any targets. In fact, this work of Dirac contained the initial elements of S-matrix theory, whose development is associated with the names of Heisenberg, E Stückelberg, and Bogolyubov.

And finally, fifthly, Dirac made a substantial contribution to the progress of approximate techniques of quantum-mechanical calculations. Following Schrödinger, who worked out the perturbation theory for stationary states, he developed a version of this theory for unsteady states. Also, Dirac significantly improved the techniques for calculating multielectron systems. In particular, while the wave function of an electron system in the initial Hartree–Fock method is expressed as the product of two determinants, in Dirac's paper [31], where the spin variables are not separated out from the wave functions of individual electrons right from the start, it is expressed in terms of a single determinant, which significantly simplifies calculations. In Ref. [32], he introduced a correction to the theory of a Thomas–Fermi atom to allow for the electron exchange interaction, which significantly improved the accuracy of this computing method. Dirac expounded all the above-listed methods in a supplement to the first Russian edition (1932) of his monograph [33].

Dirac titled his main work on quantum mechanics — the monograph *The Principles of Quantum Mechanics* — in the spirit of Newton. This work, which ran into four revised editions during his lifetime, by its contents is the best exposition of the elements of quantum mechanics and has assumed its rightful place in the treasury of physical classics, along with Newton's *Philosophiae Naturalis Principia Mathematica*, Maxwell's *A Treatise on Electricity and Magnetism*, and J Gibbs's *Elementary Principles in Statistical Mechanics*. The book was written in a new quantum language elaborated by Dirac, which was initially disapproved of some physicists. Even Heisenberg wrote in his review of the German translation of the book that "... *Dirac supposedly conceives quantum mechanics, particularly its physical content, more 'symbolically' than is required*" [5].

Like Newton, Dirac began the exposition of quantum mechanics with basic definitions and axioms. [8] He considered in detail the distinctions between the classical and quantum approaches to the description of physical phenomena and the ensuing profound changes in the opinion of physicists on the mathematical foundations of their science. Dirac wrote, "*With the recognition that there is no logical reason why Newtonian and other classical principles should be valid outside the domains in which they have been experimentally verified has come the realization that departures from these principles are indeed necessary. Such departures find their expression through the introduction of new mathematical formalisms, new schemes of axioms and rules of manipulation, into the methods of mathematical physics*" (see Ref. [20], Vol. 1, p. 28).

The advantages of Dirac's approach to the exposition of the elements of quantum mechanics eventually received general acceptance. Interestingly, Einstein, who had never perceived the quantum theory as the unified scientific theory of the microscopic world and persistently sought contradictions in formulations and interpretations of quantum laws, would permanently carry precisely *The Principles of Quantum Mechanics* of Dirac, as attested to by witnesses. D D Ivanenko wrote in the foreword to the first Russian edition of *The Principles* ... (at that time translated as *The Elements* ...): "*Among all the books issued, Dirac's The Elements ... stands out primarily for its exceptional integrity and breadth of scope... Compared with other books on this subject in our field, one can say with some exaggeration that, alongside The Elements ..., Sommerfeld's supplementary volume Wellenmechanischer Ergangzungsband presents itself like collected solutions of a number of particular problems; de Broglie's Introduction a l'Etude de la Mecanique Ondulatoire is merely an introduction concerned primarily with the passage from classical to quantum mechanics; Elementare Quantenmechanik by Born and Jordan is an exposition of an intentionally limited part of the material...* (*the Schrödinger equation is absent in the book*)*, and, lastly, Frenkel's Einführung in die Wellenmechanik, the book most intelligible to the reader, is devoid, like all the above, of only one thing — the exposition of the system of quantum mechanics. It is precisely the exposition of the system that is afforded by Dirac's book, this being done in the most superior way, which is free from any provincialism, i.e., employing a restricted method, posing problems close to the author, etc.*" (see Ref. [20], Vol. I, p. 13).

4. Dirac's role in the elaboration of quantum field theory and the theory of elementary particles

That which was done by Dirac to lay the foundations of quantum mechanics alone would suffice to rank him among the 'immortals'. Meanwhile, at virtually the same period (1927 – 1934) Dirac was laying the foundations of two more exceptionally fruitful approaches to the study of the microworld — the quantum field theory and the theory of elementary particles. The former resulted from giving deep thought to Schrödinger's wave mechanics. According to his own reminiscences, Dirac asked himself the question: "*What if we take the Schrödinger wave equation and try to apply the quantization procedure to the wave function itself? It has always been assumed that the wave function is expressed in terms of ordinary numbers, i.e. c-numbers. The question now arises: what if they are transformed to q-numbers? ... Here is how the method known as the second quantization emerged*" [9]. [9]

4.1 Dirac as the founder of quantum field theory

It is generally recognized that the first work on quantum field theory was Dirac's paper "The quantum theory of emission and absorption of radiation" [34]. In this paper, for the first time the method of secondary quantization was proposed, the quantization of electromagnetic field was performed, and the coefficients entering Einstein's radiation theory were consistently calculated in the framework of the quantum theory. As a result of further development of the ideas outlined in this work, the arsenal of physicists was enriched with a qualitatively new object — *quantum field*, which allowed for the elimination of the contradictions between the corpuscular and wave interpretations of electromagnetic radiation.

Dirac's fundamental role in the elaboration of quantum field theory has been comprehensively investigated for a long time (see, for instance, articles by R Jost [6], V Weisskopf [35], and J Mehra [7]). For this reason we will not delve deeply into this topic, but will restrict ourselves to only a short summary of the most thorough, in our opinion, paper by B V Medvedev and D V Shirkov "P A M Dirac and formation of the basic notions of quantum field theory" [36]. The authors of the paper note that the theory of quantum fields has assumed different aspects more than once. In this case, "... *not only the details, but also, in a certain sense, the basic concepts*" of the theory experienced significant changes. This process is most naturally subdivided into the following three stages.

In the first stage (1927 – 1948), which may be referred to as the theory formation stage, the main effort was directed toward extending the methods of quantum mechanics to relativistic systems with an infinite number of degrees of freedom, i.e., to field systems. It was Dirac who contrived and proposed employing the majority of the technical means required for the solution of this problem. Apart from the general theory of transformations from one representation to another, which was proposed in Ref. [27], in the same paper Dirac introduced the first generalized function, the δ-function (present-day quantum field theory is unthinkable without employing generalized functions), as well as the rules for manipulating these functions. Subsequently proposed was the method of secondary quantization [34] and the so-called 'many-time formalism' [37] — the main working tool in relativistic quantum calculations right up to the emergence of the explicitly covariant formulation of quantum electrodynamics due to S Tomonaga, J Schwinger, R Feynman, and F Dyson.

"*However, the main obstacles to the transfer of the methods of quantum mechanics to field systems were not the technical problems,*" *the authors of the summarized paper* [36] *noted,* "*but supposedly the necessity to overcome the psychological*

[8] As stated by H Rechenberg [17], in doing this "... *he closely followed Baker's example, especially his book entitled The Principles of Geometry. From this book, Dirac practically copied the necessary statements in about the same order the mathematician had written them down. Also, as regards the geometric interpretation of the formalism, in two places he used Baker's scheme. On the one hand, he concluded from this book that it was possible to construct a mathematically consistent theory with noncommuting variables, and, on the other hand, he derived the geometric interpretation of what he named 'q-numbers'...*". Therefore, projective geometry has played its part in the creation of the masterpiece of the world's scientific literature.

[9] The term 'secondary quantization' itself was presumably proposed by V A Fock.

barrier of contraposing two forms of matter — particles and fields — which were perceived from the classical standpoint as absolutely different essences''. In fact, Dirac obviated the problem of wave–corpuscle dualism even in Ref. [34], wherein he established that ''... *the Hamiltonian which describes the interaction of the atom and the electromagnetic waves can be made identical with the Hamiltonian for the problem of the interaction of the atom with an assembly of particles moving with the velocity of light and satisfying the Einstein–Bose statistics...*''. The same paper first saw the emergence of a quantized electromagnetic field which satisfied the equations of classical electrodynamics but whose values were quantum-mechanical operators acting on the Schrödinger wave function; in this case, this wave function is often referred to as the state amplitude. The development of this central idea, in which the majority of the contrivers of quantum mechanics took an active part, was detailed in Ref. [36, Sections 2–5], which permits us to pass on to the results of the first stage at one.

Summarizing the activities of a large group of theorists (including Heisenberg, Pauli, Jordan, Fock, E Fermi, O Klein, E Wigner, and others), Medvedev and Shirkov concluded that ''... *These 15–20 years were actually a time of the agonizing development of a fundamental new paradigm (and of becoming accustomed to it) in which classical particles and fields come to have completely equal rights as two different manifestations of a single unitary object: a quantized field. The new understanding of a basic organizational mechanism of nature was developed by various people in small pieces, which only gradually combined to form a unified picture*'' [36].

It may be pertinent to note that this 'painful process' was brought to logical completion only 65 years later in Shirkov's work [38]. He noted, in particular, that the term 'quantized field', which was actively employed at the formation stage of quantum field theory, from the outset assumes the prime nature of the classical field and the secondary nature of the quantum one. But this reflects only the historical sequence of the origin of these terms since, as is well known, the quantum picture is more adequate to the physical reality and the classical picture is merely some approximation to it. It was therefore proposed to replace 'the historical ordering' of terms with the logical ordering and consider just the quantum fields as the prime essence. If this field is transformed according to Fermi–Dirac approach, in the classics it corresponds to the concept of a point particle. And if it is transformed according to Bose–Einstein approach, it corresponds to the concept of a classical relativistic field. In this case, once again there prevails a principle referred to as 'the Ockham razor': ''*essences should not be needlessly multiplied*''. To take the place of both the fields and particles of classical physics, a universal essence comes up — a quantum field, which boils down to primary matter constituents as well as quanta which transfer the interaction between the present-day prime elements.

In fact, quantum field theory almost entirely assumed its present-day aspect during the second stage, which can be dated to 1949–1964. The main problem of this stage was 'combatting divergences'; their inevitable emergence was first pointed out presumably by Ehrenfest almost immediately after the publication of Dirac's paper [34]. Ehrenfest noted that invoking the notion of a point electron would inevitably lead to its infinite intrinsic energy. Five years later, in Ref. [39] Dirac distinctly formulated the causes of this phenomenon, which was inherited from the classical problem of the electron interaction with the radiation field: ''*The classical equations which deal with this problem are of two kinds,* (*i*) *those that determine the field produced by the electron* (*which field is just the difference of the ingoing and outgoing fields*) *in terms of the variables describing the motion of the electron, and* (*ii*) *those that determine the motion of the electron. Equations* (*i*) *are quite definite and unambiguous, but not so equations* (*ii*). *The latter express the acceleration of the electron in terms of field quantities at the point where the electron is situated and these field quantities in the complete classical picture are infinite and undefined*''.

A year later, in his Solvay report [40], Dirac actually came up with the seed idea of charge renormalization. He stated that external charges should polarize the vacuum in his theory, with the effect that ''... *the electric charges which are normally observable for the electron, the proton, and other electrified particles are not the charges which are actually carried by these particles and which figure in the fundamental equations; they are instead smaller*''. He carried out calculations of this new physical effect, which reduced to a logarithmically diverging integral whose cut-off at momenta on the order of 100 ms (which corresponds to the classical electron radius) yielded a 'radiative correction' to the electron charge, which reduced it by about a factor of 1/137. Yet another year later, Weisskopf [41] also arrived at a similar result; he showed that the intrinsic electron energy with the inclusion of the Dirac vacuum diverges logarithmically, so that its addition to the 'mechanical' mass remains small even when the cut-off is effected at the Schwarzschild radius.

As a result, the development of these initial attempts 'to combat divergences' took two paths. On the one hand, Stückelberg [42] and H Kramers [43] formulated the central idea of the renormalization method: the final values for observables can be obtained, for instance, by appropriate subtraction of an infinite magnitude (of some characteristic) for a free electron from the similar infinite magnitude for a bound electron. This approach makes it possible to retain the deep-rooted notions of particles as points of geometrical space and of the local nature of quantum field theory. These ideas were brilliantly realized by Schwinger, Feynman, and Dyson in the late 1940s with a record accuracy of agreement between theoretical predictions and experiments. However, the unconventional technique of quantum–field calculations called for a sufficiently rigorous mathematical substantiation.

And such substantiation of renormalization technique did appear as a result of a thorough analysis of the mathematical nature of quantum–field infinities, which was reliant on the Sobolev–Schwartz theory of generalized functions. It transpired that the divergences (from the viewpoint of this theory) are a manifestation of the uncertainty in the operation of multiplication of the propagators of point particles (which are the generalized functions) in the event of coincidence of their spatio-temporal arguments. N N Bogolyubov and his pupils (O S Parasyuk, D V Shirkov, and others) [44–47] elaborated the R-operation technique: extension of the definition of the products of causal propagators in such a way as to ensure the finiteness of resultant expressions in all orders of the perturbation theory. In this way there came into existence the notion of renormalizable and nonrenormalizable models of quantum field theory, which became one more criterion for the selection of models rich in content. The modern treatment of the renormalizability concept was given by Shirkov in Ref. [48].

The ultimate embodiment of renormalization ideology and simultaneously the central result of the second stage of development of quantum field theory is the advent of renormalization group approach whose foundations were laid in Refs [49 – 51]. [10] The renormalization group method for the first time made it possible to go beyond the framework of weak coupling approximation and to obtain, on this basis, record-accurate data in the calculation of higher-order radiative corrections. However, the authors of Ref. [36] noted: "*As a result of all these studies, the outlook for the future prospects of renormalizable quantum field theories seemed a bit gloomy. It appeared that the qualitative diversity of renormalizable quantum field theories was negligible: for any renormalizable model, the only possible effects of interaction — for small coupling constants and moderate energies — were unobservable changes in the constants of free particles ... The existing theory — again, regardless of the specific model — was inapplicable to large coupling constants or asymptotically high energies. Quantum electrodynamics remained the only* (*although brilliant*) *application to the real world, which met these requirements*".

Now is as good a time as any to recall another line of 'combatting divergences', which Dirac chose for himself, working actually in complete 'solitude'. Having generated the initial idea of charge renormalization, he practically abandoned the further development of these ideas. In addition, more than once he argued against the development of QFT along these lines (see, for instance, Ref. [9]). Dirac would persistently seek the solution of the resultant problems by way of abandoning the notion of an electron as a point object. In particular, his quest resulted in the emergence of theories with indefinite metrics, one of the versions of which was first proposed in his Bakerian lecture "The physical interpretation of the quantum mechanics" [52]. Such theories later found numerous applications.

It is well known that Dirac did not achieve much success in quantum electrodynamics by following this path, but the original ideas and approaches suggested by Dirac became (in the majority of cases) the 'seeds' of the third stage of developing the quantum field theory, which will be discussed at length in Section 5.

4.2 The Dirac equation and principles of elementary particle theory

Dirac's next basic result is his celebrated relativistic equation of an electron, which has not revealed all its properties to physicists nor to mathematicians. This is how Weisskopf, one of the first CERN directors, assessed this event in his semi-autobiographic article "Growing up with field theory" [35]: "*In 1928, Dirac published two papers dedicated to the new relativistic equation for the electron. This was his third outstanding contribution to the foundations of modern physics* (*the first contribution was the new formulation of quantum mechanics – 'The Transformation Theory...', and the second one was the theory of radiation ...*)". Apart from satisfying the principles of relativism and probabilistic interpretation of quantum mechanics, it contained information about the half-integer spin of an electron and its magnetic moment, and also provided a gauge invariant description of the electron interaction with electromagnetic field.

[10] An intelligible exposition of this approach is contained in Ref. [36, Section 8].

True, in this case an electron acquired a new degree of freedom — it could move into states with negative energy. This appeared to be so odd that one might as well abandon the results obtained. We are reminded that quantum mechanics in fact had inherited the problem of negative energies from the special relativity. According to the formula for relativistic energy $E = c\sqrt{m^2c^2 + p^2}$, which contains a square root, it can assume both positive and negative values. In other words, the particle energy can assume formally any value in the range between mc^2 and infinity, as well as from $-mc^2$ to minus infinity. In the classical theory, where particle trajectories are continuous, problems do not arise, for a particle cannot pass into a negative-energy state. In the quantum theory, the probability of such a transition is nonzero, so that the particle can change the sign of its energy in a stepwise manner, without going through the intermediate states.

The paradoxicality of the ensuing conclusions did not frighten Dirac. He chose another way — he believed in the reality of negative-energy states and, taking advantage of the Pauli exclusion principle, filled all unreal states with real electrons. Dirac termed the collection of these states a 'sea' or an 'ocean', which "*is occupied with electrons without the restriction for a negative energy and therefore there is nothing like a bottom in this electron ocean*" [9]. Dirac believed that electrons with a negative energy are not observed, because they make up a continuous invisible background against which all world events take place. However, when a high-energy photon finds itself in the 'Dirac electron sea', under certain conditions it can knock out one of the countless 'sea' electrons. The empty place, a 'hole', will behave like a quasi-particle with a positive charge. [11]

The situation changed when Dirac took the next step by assuming that the 'holes' in the electron sea should be treated not as quasi-particles, but as real positively charged particles which would be experimentally observable, in principle, as free objects. We are reminded that only electrons, protons, and photons were known from experiment late in the 1920s, so that even atomic nuclei were assumed to be collections of tightly coupled electrons and protons. It is proceeding from precisely the available opportunities that Dirac initially selected a proton as a candidate for a 'hole'. As a result, the 'elementary particle physics' known by that time would have actually been described with a single equation — everything would be simple and beautiful.

We emphasize that the proposed theory of 'holes' was not taken seriously by the majority of physicists and, whenever considered by individual theorists, the aim was primarily to disprove it. Dirac himself was not discouraged by these circumstances, and he continued to elaborate the theory under the title 'the theory of electrons and protons', assuming that the glaring difference in the masses of electrons and protons would later be possible to explain by the special features of interaction in the electron sea. In particular, as early as 1930 he calculated the annihilation cross section for electrons and 'holes', obtaining by so doing (as it turned out later) the correct cross section for the annihilation of electrons and ... the then unknown positrons.

[11] We emphasize that Dirac interpreted the vacancies among the occupied states of this type as 'holes' almost right away. He proceeded from the scheme of occupation of some atomic electron shells and their restructuring at molecular formation, which was employed in the theories of multielectron atoms and chemical valence, as well as in the description of the origin of X-ray atomic spectra.

In the May of 1931, in the paper "Quantized singularities in the electromagnetic field" [53] Dirac clearly pointed out for the first time that the combined employment of the principles of the quantum theory and the relativity theory requires that to each charged particle there corresponds its own oppositely charged antiparticle with the same mass. That is why the role of 'holes' with respect to electrons should be played by qualitatively new objects — antielectrons, which were termed positrons before long. Simultaneously, Dirac stated that there are also bound to exist the antipodes of protons — antiprotons. Slightly more than a year went by when an American physicist C Anderson announced on August 2, 1932 (not long before Dirac's birthday) the discovery of the positron in cosmic rays. (The antiproton was obtained at an accelerator in 1955, and the antineutron in 1956.)

The above events call for several comments, primarily concerning the role of R Oppenheimer's well-known letter [54] in the establishment of the positron concept. This letter contains a preliminary estimate of the cross section for electron – positron annihilation as a process which follows from previously advanced Dirac's theory. Since the resultant estimate did not correspond to the observed stability of these particles, Oppenheimer suggested that (i) the holes in the electron background should not be identified with protons; (ii) electrons and protons should be treated as absolutely independent particles; (iii) all negative-energy electron states should be completely filled with electrons to eliminate holes, and (iv) in order to compensate for the infinite negative charge of the electron background, a similar background with an infinite positive charge should be introduced, filling completely, i.e., without any holes, with protons the negative-energy levels of the similar proton background.

Therefore, according to the idea of Oppenheimer, both for electrons and protons, the holes in the corresponding backgrounds are lacking and cannot be produced in principle. That is why the processes of annihilation or production of massive particles should not take place at all. The only unconventional positively charged particles whose existence could be hypothesized on the basis of Oppenheimer's suggestions were protons in negative-energy states, but not positrons; far from it!

We next note that up to the present day it is possible to encounter the following assertion in the scientific literature: to discover positrons required cosmic photons with energies of more than 1 MeV. The collisions of the latter with nuclei made it possible to observe electron – positron pairs whose components were deflected differently by a magnetic field. In reality, this requirement was not necessary at all: even five years prior to Anderson's experiments, events were known which now are referred to as positive β decay of nuclei. [12] In these events, positrons emerged one at a time and with any arbitrarily low energy. However, observers interpreted their 'incorrect' deflection in the magnetic field as the backward (i.e., towards the source) motion of electrons.

We would also like to emphasize that the positrons themselves were not the point. The basic idea advanced by Dirac in these papers, which now is frequently overlooked, was the possibility of principle to produce and destruct particles of any mass on keeping the corresponding conservation laws. Of course, the theoretical possibility of the interconversion of kinetic energy and rest energy follows from the special relativity, and the majority of physicists agreed with it by the late 1920s. However, this did not in the least imply that the number and sort of particles could vary in elementary processes. The long-standing resistance to the recognition of a photon as one of elementary particles was supposedly due to this circumstance, for photons had the capacity to be radiated and absorbed. In the long run, an exception was made for massless photons. At the same time, the only corroboration of energy interconversion processes for nonzero-mass objects was the occurrence of radioactivity and the simplest nuclear reactions, which were commonly treated by analogy with molecular dissociation and chemical reactions. Even β decay was initially interpreted by analogy with the ionization of atoms. To put it another way, the number and sorts of nonzero-mass particles were always assumed to be the same at the onset and the end of any process, and only a relatively small energy redistribution was dealt with when the same particles moved from a bound state to the free state and back.

Having postulated the possibility of the production and annihilation of electron – positron pairs (and the production and annihilation operators themselves appeared even in Dirac's pioneering work on quantum theory in 1925 [22]), Dirac predicted for the first time the interconversion of elementary particles of any mass, including the processes wherein the rest energy of the initial particles was completely converted to the kinetic energy of the final particles. The success of this prediction subsequently had an enormous impact on changing world outlook (Weltanschauung) of the scientific community as a whole, for the implications of the special relativity enriched with the quantum theory were brought to their logical conclusion.

Finally, we are reminded that the existence of antiprotons predicted by Dirac, which now appears to be almost trivial, was disapproved by many physicists even after the discovery of positrons. The point is that anomalous magnetic moments were discovered in protons and neutrons by that time, and the question of whether the Dirac equation could be applied for their description proved to be an open question (with all the ensuing consequences) [13].

But Dirac was not confused by these doubts. His Nobel lecture [56] concluded with a new prevision: "*If we accept the view of complete symmetry between positive and negative electric charges so far as concerns the fundamental laws of Nature, we must regard it rather as an accident that the Earth* (*and presumably the whole solar system*) *contains a preponderance of negative electrons and positive protons. It is quite possible that for some of the stars it is the other way about, these stars being built up mainly of positrons and negative protons. In fact, there may be half the stars of each kind. The two kinds of stars would both show exactly the same spectra, and there would be no way of distinguishing them by present astronomical methods*". While the discovery of 'antistars' has never been reported, there is significant progress in the cause of experimental discovery of antimatter pioneered by Anderson 70 years ago. In August of 2002, in fact on the centenary of Dirac's birth, the international team of the 'ATHENA' project in CERN for the first time produced tens of thousands of antihydrogen atoms in one experiment, i.e., an almost macroscopic dose of antimatter. In principle, the door was thereby opened to the production of antimolecules and later 'antiliquids', 'anticrystals', etc.

[12] The classical source on this problem and its history is monograph [55].

[13] According to present-day notions, the Dirac equation underlies the description only of truly basic structureless fermions — leptons and quarks.

It is pertinent to add a few words about the fundamental significance of these Dirac's ideas. Of course, since the 1940s no one can be surprised by discoveries of theoretically predicted particles (from the Yukawa meson to the *t*-quark). However, Dirac's theory was more than the first successful prediction in this series. In his report presented at the XIVth International Conference on Cosmic Rays in 1975 Heisenberg emphasized: "*Of significance was by no means the discovery of yet another previously unknown particle; of significance was the discovery of a new symmetry, the particles-antiparticles conjugacy intimately related to the Lorentz group of special relativity theory as well as to the conversion of the kinetic energy of colliding particles to the rest energy of new particles and back*" [57]. Elaborating on this idea, I Yu Kobzarev noted that "... *the new symmetry of nature discovered by Dirac has proved to be significant not only for fermions. Its intimate connection to the relativistic invariance was subsequently embodied in the celebrated CPT theorem which presently underlies the theory of elementary particles. This symmetry was experimentally borne out by the discovery, for practically every particle, of its associated antiparticle different from it*" [58].

The subsequent destiny of Dirac's idea of the 'sea' of negative-energy electrons turned out to be quite extraordinary. It underwent a qualitative evolution in quantum electrodynamics itself and, more broadly, in quantum field theory. A radically new notion was eventually introduced — the physical vacuum, qualitatively different from the classical notion about 'void'. The vacuum is filled with virtual pairs of electrons and positrons, virtual photons, as well as virtual pairs and basic quanta of other types. The last exert an effect on the properties of real objects, which shows up in the renormalization of charge and mass as well as in polarization effects, which were also considered by Dirac for the first time [59], and so forth. But even today, despite all the modifications, the initial idea of the Dirac's negative-energy sea exhibits amazing vitality: it is employed to advantage, for instance, in the interpretation of 'anomalies' in the quantum field theory [60].

However, the theory of holes-quasiparticles advanced most significantly and found numerous applications outside quantum field theory proper. It underlies the band theory of electronic spectra in semiconductors and is employed in the theories of multielectron atoms and chemical valence, the nuclear shell model, the theory of supercharged nuclei [61], and, lastly, the theory of superconductivity. In fact, Dirac's notions of 'holes'-quasiparticles have proved to be extremely fruitful in all physical systems whose energy spectra possess a gap or a Fermi sphere.

It is pertinent to note that no one had anticipated so quick an experimental corroboration of the existence of antiparticles predicted by Dirac. For just on the eve of this event many famous theorists (L D Landau, V A Fock, N Bohr, and several others), to put it mildly, could hardly believe so crazy a hypothesis. Even Pauli, although a witty and slightly adventurous person who had just advanced a hypothesis (true, a speculative one) for the existence of the neutrino, cast strong doubt on Dirac's predictions in a famous review paper on quantum mechanics [62]: "*In this theory, the laws of nature are precisely symmetric about electrons and antielectrons, and it seems unsatisfactory for this reason alone... We do not think this way out should be considered in earnest*". However, Dirac's brilliant intuition and his conviction that pretty mathematical results are efficient in physics won out this time, too. This was a triumph. In this connection Weisskopf emphasized in the above-mentioned paper [35] that "*The theoretical predictions concerning new basic processes and the new properties of matter had been made before any experimental indications were made on that score. On the contrary, all previous experience contradicted the symmetry between positive and negative electric charge*".

The discovery of a positron as the confirmation of the existence of basic antimatter constituents produced an impression on the broad public, comparable only with the confirmation of the results on the general theory of relativity in the observations of light ray deflection in the solar gravitational field in 1919. Dirac, like Einstein in his time, instantly became a world celebrity, but this had no effect on his mode of life and style of scientific work. Meanwhile, everybody was expecting him to report equally quick and sensational results. However, such results were not to be. Regular routine scientific work was underway, which was oriented, as we now understand, to a distant perspective and therefore remained outside the scope of current attention (and sometimes of understanding) of colleagues. Furthermore, the Second World War broke out and after it the acute period of the cold war. The physics society's interest in Dirac's creative work began to gradually fade away.

5. Basic ideas of the 'later' Dirac

Since 1934, i.e., after laying the foundations for three basic theories, Dirac lived and went on working for 50 long years. It is inconceivable that a person of his intelligence and the power of engrossing in scientific work would rest on his laurels at the age of 32 and not make significant contributions to science any more. This viewpoint is nevertheless rather popular, largely due to some stereotypes created by famous Dirac biographers (see, for instance, Mehra's article [7] as well as Ref. [63]).

Thus, the article by R Dalitz [63], a famous theoretical physicist, which opened a collection of memories of Dirac published by his friends and colleagues, gave the list of 24 most significant (in the view of the author) Dirac's papers. The last paper in the list is dated 1948, and the 1934 – 1948 period is represented by only six papers. Therefore, strange as it may seem, the majority (150!!) of Dirac's papers which saw light after 1934 remained outside the field of view of Dalitz, who was seemingly treating Dirac's name and his heritage with benevolence and distinction. [14]

Meanwhile, these works, which have not engaged the attention of biographers, contain several basic ideas, each of which deserves at least a thorough paper, if not a separate monograph. In the subsequent discussion we therefore restrict ourselves to only a summary outline of consolidated series of his papers whose ideas (in our opinion) have either proved to be fruitful or contain incompletely revealed potential for the development of modern theoretical and mathematical physics.

[14] It is pertinent to establish the consistent conservatism of Dalitz's standpoint. Almost 10 years later, in 1995, as an editor he prepared the publication, which was unique in many respects, of Dirac's selected papers [64], again including only the papers published before 1949.

5.1 Classical Hamiltonian dynamics with constraints — Dirac mechanics — and quantization of gauge fields

It is likely that the series of papers on generalized Hamiltonian dynamics [65] (see also lectures [66]) constitutes Dirac's greatest contribution to theoretical physics in the 1950s–1980s. In fact, this is the next stage in the development of analytical dynamics after Hamilton himself, and for this reason the title 'Dirac mechanics' [67, 68] is increasingly often employed in the modern literature, side by side with Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics.

Dirac's works on the generalization of Hamiltonian dynamics made their appearance at the time when quantum field theory was going through its most difficult period. After the stunning successes of quantum electrodynamics related to the names of Tomonaga, Schwinger, Feynman, and Dyson, there set in the 'Time of Troubles' of heavily dispiriting failures at meson theories of nuclear forces, where the renormalization procedures, which had shown themselves to be advantageous in electrodynamics, would not do any good. The so-called 'zero charge problem' nonplussed the eminent theorists in all its magnitude. Their opinion was most clearly formulated by Landau: "*... the Hamiltonian method for strong interactions has become obsolete and should be buried, naturally, and rendered homage it has deserved*" [69]. On these grounds they attempted to 'discard' the whole quantum field theory as 'being out of date' and replace it with semiphenomenological approaches like the analytical theory of S-matrix, reggistics, current algebra, etc. [15] True, this viewpoint was by no means unanimously shared. An intensive search for new approaches and generalizations was underway, and an increasingly more powerful mathematical apparatus was invoked for the development of the quantum-field approach.

In his lectures [66] Dirac explained in detail why the development of the apparatus of relativistic quantum field theory called first and foremost for the extension of the capabilities of classical Hamiltonian dynamics and why on this path it is necessary to successively go through all the stages from the relativistic-invariant action principle to the Hamiltonian and only then to the quantum theory. As a preliminary, he elucidated the way in the situation when the conventional transfer from the Lagrangian $L(q,\dot{q})$ to the Hamiltonian $H(p,q)$ is impossible, i.e., when the conventional definition of the generalized momentum $p_i = \partial L/\partial \dot{q}_i$ is unsolvable for some set of generalized velocities $\dot{q}_i$. For systems with a finite number of degrees of freedom, this situation occurs when the rank of a Hessian $\partial^2 L/(\partial \dot{q}_i \partial \dot{q}_j)$ is smaller than the number of degrees of freedom. The corresponding Lagrangians are termed singular or special. In going over to systems with an infinite number of degrees of freedom (condensed media, field systems), the problem persists and is even aggravated. In real situations, the latter takes place for the majority of modern models in particle physics, such as the gauge Yang–Mills model, the supersymmetric generalizations of Yang–Mills fields, supergravity, superstring, membrane, and bag models, etc., in which the fields have one geometrical significance or another.

As is well known, the theories of non-Abelian gauge fields (or Yang–Mills fields) occupy a special place in the contemporary notions of the nature of fundamental interactions. First of all, based on the principle of gauge invariance, physicists had at their disposal a simple and efficient algorithm of constructing 'dynamics from symmetries'. Simple and elegant, yet amazingly informative, Yang–Mills Lagrangians came to replace the immense expressions for the Lagrangians of the meson theories of the late 1940s. In any case, the Standard Model, which represents our present-day understanding of the physics of elementary particles and fields, was constructed on the basis of such theories. However, we are reminded that the Yang–Mills fields were perceived by theorists, for more than ten years after their introduction, as an elegant but useless construction, which was, at most, of academic interest. The reason lay, in particular, with the massless gauge vector bosons predicted by the theory, which had never manifested themselves in experiments (for more details, see, for instance, Refs [71, 72]).

We note that Dirac 'betrayed', when solving this range of problems, his traditional 'emploi' of a researcher personally developing his ideas to all conceivable logical consequences and played the part of a 'playmaker' rather than the main 'goal-scorer'. The generalization of Hamiltonian formalism proposed by Dirac relies on reducing the initial phase space by imposing first- and second-class constraints corresponding to the system — the *Dirac reduction* — making it possible to find the modified Poisson bracket — the *Dirac bracket* — and construct the corresponding Hamiltonian formalism. Even in the first paper (1950) of the series [65] he also proposed the scheme of operator quantization of the systems with constraints (as a matter of fact, Dirac was developing his approach for precisely this purpose). However, in the application of this scheme to the gravitational field [73] the problems emerged with multiplier ordering and relativistic covariance, among others. [16] Feynman's attempt (1963) to carry out the quantization of Yang–Mills fields by employing the methods which had proven advantageous in quantum electrodynamics also encountered certain contradictions (the violation of unitarity condition was discovered).

The further narration of the creation of the quantum theory of gauge fields would be a digression from our main subject. Omitting the intermediate stages, we therefore point out straight away that the method of continual integration developed by Feynman (1948) has eventually proved to be the most adequate apparatus for the quantization of gauge fields. The starting point for Feynman was Dirac's idea, which was proposed in Ref. [74] as far back as 1933, that the temporal evolution of a quantum system on a finite time interval can be represented as a composition of a large number of evolutions over short time intervals. Relying on his theory of transformations developed earlier, Dirac showed that the final transform function appears in this case in the form of a multiple integral of the product of a large number of 'elementary' transform functions taken over the possible values of dynamic variables at intermediate points in time. Most significantly, Dirac suggested that the wave function transformations should be determined employing the exponent of the classical action of the system. This idea was further refined in an infrequently cited paper [75]. The development

[15] The author's preface to book [70] says: "*...The general level of the book assumes familiarity by the reader with the principles of nonrelativistic quantum mechanics* (*including scattering theory*) *as well as with the Lorentz group. No background in quantum field theory is required. Indeed, as pointed out in the preface to my 1961 lecture notes, lengthy experience with Lagrangian field theory appears to constitute a disadvantage when attempting to learn S-matrix theory*". No comment is necessary!

[16] It is well known that these problems were solved at a later time, but the construction problem of the quantum theory of gravitation still remains unsolved owing to definite nonrenormalizability.

and formalization of these ideas led to Feynman integrals, which are referred to as path integrals in the quantum mechanics of systems with a finite number of degrees of freedom, and as functional integrals in the quantum field theory (for more details, see Ref. [36]).

The first issue of the journal *Teoreticheskaya i Matematicheskaya Fizika* (Theoretical and Mathematical Physics) saw light in 1969; it opened with L D Faddeev's paper entitled "Integral Feĭnmana dlya singulyarnykh lagranzhianov" ("The Feynman integral for singular Lagrangians") [76]. The paper gave the general recipe for the quantization of systems with constraints within the formalism of a continual integral, which has gained general acceptance and is reproduced in practically all guides and textbooks on the quantum theory of gauge fields up to the present time. From the very title of the work it follows that doing this required accomplishing, at the very least, the synthesis of two of Dirac's ideas mentioned above: the generalized Hamiltonian formalism, and the continual integral.

However, the task was not limited to the synthesis alone. It took a certain development of the Dirac scheme to carry out gauge group reduction, since, owing to the gauge invariance of the theory, the principal objects in it are not the potentials A_μ but their equivalence classes (orbits). Next obtained was an explicit expression of the Feynman measure for Dirac's generalized Hamiltonian dynamics. It was found that the requisite reduction is most naturally realized employing precisely the generalized Feynman integral. As a result of this and other accomplishments, which we do not mention here and which the reader can familiarize himself utilizing monograph [72], the gauge field theories have occupied a fitting position in particle physics, and the dynamics of systems with constraints has become an actively advancing independent direction (see, for instance, Refs [67, 77]).

5.2 The Dirac monopole and topological ideas in physics

Another fruitful direction in modern theoretical physics, which is also closely related to Dirac's name, is the problem of a solitary magnetic charge (monopole). It reduces to the question: why are magnetic field sources similar to electric charges absent in nature? For otherwise, electric and magnetic fields enter the Maxwell equations quite symmetrically. This brings up the natural question: why did nature require so evident an asymmetry as regards the sources of electric and magnetic fields?

Speaking at the symposium held at Loyola University (USA) and dedicated to his 80th birthday, Dirac explained his interest in the problem in the following way: "*Another example of pretty mathematics led to the idea of the magnetic monopole. When I did this work I was hoping to find some explanation of the fine-structure constant $\hbar c/e^2$. But this failed. The mathematics led inexorably to the monopole. From the theoretical point of view one would think that monopoles should exist, because of the prettiness of the mathematics*" [78].

After a thorough analysis of the known facts on the fundamental unobservability of the phases of the wave functions in quantum mechanics, which, in addition, are defined correct to 2π and become nonintegrable in the presence, for instance, of an electromagnetic field, Dirac showed in 1931 in Ref. [53] that the hypothesis of the existence of solitary magnetic monopoles with a charge μ is not at variance with the principles of quantum mechanics, provided that $e\mu = 2\pi\hbar cn$, where n is an integer. Therefore, if the monopoles were discovered, the above formula, termed the *Dirac quantization condition*, would be an explanation of the quantized nature of the electric charges of the known particles. "*Under these circumstances one would be surprised if Nature had made no use of it*", Dirac noted at the end of the paper [53].

In a series of papers [79], Schwinger generalized the Dirac quantization conditions to the interaction of two particles each of which possesses both electric and magnetic charges:

$$(e_1\mu_1 - e_2\mu_2) = 2\pi\hbar cn\,,$$

which he termed *dions*. In this case, when such a dion is produced from two bosons with nonzero total electric and magnetic charges, the resultant bound state should obey the Fermi – Dirac statistics, i.e., there occurs the so-called *Fermi – Bose transmutation*. Currently, such transmutations are actively being investigated in the framework of *supersymmetric theories*.

True, the Dirac monopole proved to be a highly exotic (according to the notions of those days) solution containing a chain of singularities — the *Dirac string* — which is unobservable with the fulfilment of quantization conditions. In the view of M Atiyah [80], Dirac's work was in point of fact the first application of topological ideas in quantum physics. In this connection he wrote that "*... topology around the monopole* (*a 3-dimensional version of the winding numbers in a plane*) *would affect the wave function of the particle, and this in turn would lead to the quantization of its electric charge. Thus, the discreteness of charge is directly related to the discreteness of topological 'winding numbers'...*" In a paper dated 1948 [81], Dirac developed the general theory of interaction between charges and magnetic poles (positive and negative) and, in particular, endeavored to explain the inseparability of magnetic poles by the fact that they are connected by the Dirac string (the so-called monopole confinement). This idea was subsequently harnessed many times in different versions of string models of baryon, in which quarks were placed in lieu of monopoles at the ends of strings (see, for instance, Ref. [82]).

The idea of the Dirac monopole received the most interesting development in the grand unified theory. In 1974, A M Polyakov and G 't Hooft found a soliton-type solution with a unit magnetic charge (topological in nature) in one of the versions of electroweak theory — the Georgi – Glashow model. Unlike the Dirac monopole, the 't Hooft – Polyakov monopole is finite in dimensions and possesses finite values of energy, momentum, etc. What is most important, the magnetic charge of these monopoles should be topologically nontrivial, and their mass should be 10^6 times the proton mass. The monopoles predicted by the grand unified theories should be still more massive. Their mass should be 10^{16} times the proton mass. It is evident that the energy of not only the most modern accelerators, but also of the highest-energy cosmic rays, would be too small to give birth to this 'mammoth of the microworld'. However, early in the universe's evolution, when energy was abundant, monopoles could well have been produced that survive to the present day. That is why the quest for the monopoles does not cease in circumterrestrial space and near space.

One of the possible ways of detecting monopoles was derived 'with a pen and a sheet of paper' by V A Rubakov in 1981 and somewhat later by C Callan (the Callan – Rubakov effect, or the monopole catalysis) [83]. They discovered that a proton in the presence of a monopole should instantly decay into a positron and mesons. The monopole itself remains safe and sound in the process (by the law of magnetic charge conservation) and further capable of destroying the ambient material. The monopole trace in the material would therefore be accompanied by an easily detectable chain of 'proton catastrophes'. Neither the idea of the Dirac monopole, nor the idea of the 't Hooft–Polyakov monopole has been directly borne out in experiment. Despite this fact, they have lent impetus to the development of new directions [17] not only in physics, but in mathematics as well, have impelled physicists to master the unconventional mathematical apparatus of algebraic topology, and have simultaneously generated considerable interest among pure mathematicians in physical problems (see, for instance, Ref. [85]).

It is noteworthy that Dirac introduced, in the style inherent only in him, a new mathematical object to describe dynamics in the monopole field — *a many-valued functional*. Investigating its properties called for a substantial development of variational methods carried out by S P Novikov [86]. Prior to Dirac, the employment of topology was at the periphery of physicists' attention. Having introduced the idea of a monopole and its attendant topological singularity, Dirac pioneered the penetration of the elements of topology and the corresponding language in physics. These have found numerous applications in the present-day versions of elementary particle physics, in the physics of condensed media, and in cosmology, particularly in the development scenarios of the early universe. Therefore, even though magnetic monopoles have not been discovered experimentally, their numerous 'twins' (skyrmions, thorons, holons, etc.) have occupied a fitting place in theoretical physics (see, for instance, Refs [87, 88]).

5.3 Dirac's ideas in the realm of gravitation and cosmology

Speaking on the occasion of the centennial anniversary of Einstein's birth in 1979, Dirac briefly outlined his hypothesis of large numbers advanced back in 1937–1938 [89]. Under this hypothesis, all very large numbers composed of various physical and astronomical constants are not in fact fixed but are related by simple laws to the epoch — the time elapsed from the instant of the universe's creation. [18] The stated hypothesis allows an unambiguous choice among three possible evolution scenarios of our universe. Should this hypothesis prove to be true, this would manifest itself in a reduction of the gravitational constant, in a variation of interplanetary distances, etc.

Dirac developed these ideas for almost half a century, although they found a relatively narrow response among the scientific community. In recent years, the situation has taken a turn for the better as regards these ideas. Firstly, Dirac's hypothesis for the existence of two time scales — gravitational and atomic (electromagnetic) — may be realized in modern supergravitation approaches, where the number of dimensions increases not only with reference to spatial variables, but with reference to temporal variables as well. It also correlates with the modern ideas [90] according to which the gravitational and electromagnetic interactions are realized in spaces of different dimensionality.

Secondly, the idea of the time decrease of the gravitational constant and its attendant weakening of the gravitational interaction between visible and 'dark' matter may prove to be verisimilar. The point is that the latest discoveries of observational astronomy are indicative of the significant part played in the universe by so-called 'vacuum matter', or 'quintessence', as a fundamentally new material object. In this connection, efforts could well be made to ascribe the effective, after Dirac, time decrease of the intensity of gravity to the time increase of the role of peculiar 'antigravity'. Dirac's idea itself, which consists in the possibility to relate the big numbers known in physics to the age of the universe, has never been disproved. However, all this is still beyond the range of the experimental capabilities of contemporary physics.

5.4 Dirac's work on mathematical physics

Apart from the above-listed ideas, the work carried out by Dirac during the last 50 years of his life contains a lot of other discoveries and findings. Of these we point out only the most striking ones (in light of modern views). Having actually pioneered the development of the theory of renormalizations, later Dirac would repeatedly characterize this approach merely as a temporarily inevitable approach, bearing in mind the necessity of eliminating divergences. He spent a lot of time and mounted a serious effort to construct a quantum field theory with renormalizations, but without divergences. On the one hand, it is conceivable that these efforts were spent in vain, for the modern renormalization procedure reliant on the Bogolyubov R-operation is mathematically irreproachable. However, the very idea of constructing a truly finite quantum field theory is nowadays being realized in the so-called supersymmetric models which exhibit the remarkable property of cancellation of ultraviolet divergences in all orders of the perturbation theory (for more details, see Ref. [91]). Singletons, which have recently come under intensive investigation in conformal field theories, also rely on the conformal group representation proposed by Dirac in 1936 [15].

By and large, Dirac's works concerned with the problems of group representation theory deserve special consideration. Investigating the Lorentz group representations in Ref. [92], Dirac observed: "*The finite representations of this group, i.e. those whose matrices have a finite number of rows and columns, are all well known, and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications*". In this paper he proposed a new method of studying such representations, which leads to a new variety of tensor quantities in spacetime with an infinite number of components and a positive definite square of their length. He termed them *expansors*. Not only did Dirac determine the properties of expansors, but he also applied them for the description of a 4-dimensional harmonic oscillator, as well as for a particle with a spin, deriving in doing so several amazing consequences. Nevertheless, this work has not come, according to D P Zhelobenko, to the attention of experts in this field.

In Ref. [16], Dirac took advantage of projective geometry techniques to construct the quaternion representation of the Lorentz group, making it possible not to restrict oneself (as is done in the majority of textbooks) to the Lorentz transforma-

[17] For instance, research into topological and geometrical phases in quantum theory and optics (the Berry, Vladimirskiĭ, Anandan, etc. phases). For more details, see Ref. [84].

[18] It is not difficult to trace the connection between this idea of Dirac and the ancient dream of the philosophers of the Pythagorean school: to relate the basic laws of nature to the properties of integer numbers.

tions along one axis, but to comprehensively study the relativistic particle kinematics in the case of arbitrary motion of the frame of reference. To the best of our knowledge, this work has also remained unnoticed.

In the physics of pre-Planck distances, rather many recent papers have been devoted to the study of the properties of membranes (two-dimensional generalizations of a string) and p-branes (its p-dimensional generalizations). Curiously, in Refs [93, 94] Dirac first introduced membrane-like objects and wrote for them the relativistic-invariant action (which is frequently referred to in the literature as the Nambu – Goto action) with the aim of explaining experimental data on muons. This is one more testimony in favor of the opinion that Dirac may also be regarded as one of the trailblazers of the rapidly advancing string theory and its various modifications.

In principle, practically all of Dirac's work can be regarded as particular realizations of a new powerful method which emerged in the course of the mutual progress of physics and mathematics toward unification. Dirac expounded this method in detail in Ref. [95]: "*The method of advance is to begin with the selection of a branch of mathematics which in your opinion can serve as a basis for the new theory. In doing this you should be guided in great part by the considerations of mathematical beauty. It is also likely that preference should be given to the branch of mathematics which relies on an interesting transformation group, since transformations play a great role in a modern physical theory; both the relativistic and quantum theories supposedly suggest that the significance of transformations is more fundamental than the significance of equations. On selecting the branch of mathematics, there is good reason to elaborate it in the corresponding directions, simultaneously bearing in mind how it can lend itself to a natural physical interpretation*". It may be said without gross exaggeration that Dirac's method today has been adopted by the majority of theoretical physicists. By the way, the issues of the interrelation between physics and mathematics were of concern to him throughout his life, he would readily discuss this subject, and he digressed to discuss it in his works dedicated to absolutely different problems (see, for instance, Ref. [96]).

Of course, this list of the fundamental ideas of the 'later' Dirac can be continued. However, based even on the foregoing one can arrive at a definite conclusion: the creative heritage of this genius of 20th century physics harbors a wealth of potential heretofore unknown and yet untapped.

6. Dirac and the present-day physical picture of the world

In summary, we would like to emphasize that Dirac's contribution to the progress of civilization is not limited to the above-listed fundamental theoretical discoveries. As evidenced by the course of time, his work has led to qualitative changes in our notions of nature as a whole, which is commonly referred to as the physical picture of the world. From the modern viewpoint, the main components of the PPW are, on the one hand, the abstract images of material objects and, on the other hand, the conceptual apparatus invoked to describe the most important properties of these objects. Dirac's ideas have led to significant additions and radical changes of both PPW components.

We are reminded that the main models of objects in physics for the first 150 years after Newton were massive material points (corpuscles) or their associations (solids, ideal liquids), with central forces acting instantly between them and all this taking place in an absolutely empty space for an absolutely continuous flow of time. In this case, the conceptual apparatus reduced only to the characteristics of material objects. In general terms, such was the first PPW. M Faraday and Maxwell supplemented this picture with fields and electromagnetic waves seemingly alien to it, and Lorentz was the first to guess that both the field and substance are the forms of matter, although qualitatively different. As is well known, the construction of the classical PPW version was completed by Einstein, whose relativity theory removed evident contradictions between the mechanical and field notions of the surrounding world; however, in this case our notions of the geometry of the universe changed significantly.

Proceeding from relativistic and quantum principles, Dirac in his turn showed that, along with conventional matter, there is also bound to exist its antipode — 'antimatter'. It may be said without exaggeration that Dirac actually discovered a 'second' nature for us by doubling the number of material objects amenable to observation and study. And Weisskopf's observation is absolutely correct [35] that "... *these predictions rank with the greatest achievements of natural science*".

From these predictions of Dirac there also followed the possibility of interconversion, including the creation and destruction, of nuclei and elementary particles, including those which are not observed under ordinary terrestrial conditions. Studying these processes in space and in terrestrial conditions has opened up the way to the cognition of the early stages of the evolution of the universe.

Furthermore, Dirac laid the foundations of quantum field theory which has elicited the qualitative unity of matter at the microlevel. According to modern views, the notion of the quantized field, which he introduced, is the most basic and universal form of describing matter, which underlies all its observable (both wave and corpuscular) manifestations. Finally, the qualitatively new conception of the physical vacuum, which is being actively developed in the modern models of quantum theory and cosmology scenarios, emerged under the impact of Dirac's work.

No less significant is Dirac's contribution to the second PPW component — the conceptual apparatus of physics. We dwell only on the most significant contribution, on the introduction of two fundamentally new notions in locution — observables and states, which pertain to two qualitatively different aspects of the physical reality — the object as such, and its macroenvironment. The natural development of this idea is the modern notion that all physical objects exist not by themselves, but as if in a 'fur coat', experiencing an uncontrollable quantum action (on a Planck constant scale) from macrosurroundings which may also include the means of observation. In this connection, the independent characteristics of both the object itself and its state, determined by the uncontrollable action of the environment, turn out to be equally the subject of the physical theory.

Dirac's viewpoint of principle concerning the role of macrosurroundings in the formation of the state of a microsystem was reflected in his discussion with Heisenberg at the Fifth Solvay Congress (1927) in connection with Bohr's report "Quantum postulate". Dirac spoke positively in the sense that the reduction of a wave packet takes place because "... *The Nature chooses and decides in favor of a specific state*

ψ_n *with a probability* $|C_n|^2$*. This choice cannot be rejected, and it determines the subsequent evolution of the state*'' (see Ref. [20], Vol. II, p. 206). At the same time, Heisenberg insisted that ''... *it is our observations that give us the reduction to the eigenfunction*'', obviously overestimating at that moment the part played by the subjective factor.

There is another question: to what extent should environmental action be taken into account in the description of macro- and microobjects? For the dynamics (but by no means for thermodynamics!) of macroobjects, the existence of a 'fur coat' does not ordinarily play a significant part, so that for them there is good reason to restrict ourselves to only one class of characteristics — the observables. However, we have a completely different situation with microobjects. The concept of a quantum state acquires an independent role, with the result that the number of characteristics describing the physical reality in the microscopic world is actually doubled. Moreover, underestimating the role of one or another characteristic leads to paradoxes of the Einstein–Podolsky–Rozen type. Furthermore, attempts to give an interpretation of quantum phenomena on the basis of our usual, 'obvious' notions are nothing more nor less than a veiled hope for the existence in nature of the so-called 'hidden parameters'... That is why the results of the well-known experiments on the verification of Bell inequalities can be regarded as the confirmation of the correctness of Dirac's approach to the description of quantum realities and, first and foremost, of the idea of the integrity of quantum states.

To appreciate the extraordinariness of Dirac's innovation specified above, we revert to the formation period of quantum mechanics. Prevailing at that time was a tradition which can be traced back to Newton: to reduce the description of the natural objects to the study of their physical characteristics by themselves. In this case, it went without saying that these characteristics were undoubtedly observable. In other words, unobservable quantities introduced into physics on the basis of some speculative considerations had, according to this tradition, to be eliminated in the construction of any theory.

Many physicists believed that Einstein, too, was among the adherents of this tradition. In any case, he was presumed to proceed from such considerations when constructing the relativity theory. In particular, Heisenberg also adhered to this tradition and initially considered the observability principle as the basis for the quantum theory he was constructing. That is why, according to his own recollections [97], he was hoping for mutual understanding and support of his views when he informed Einstein of his initial premise during their conversation in 1926. However, a kind of discomfiture was in store for him, for Einstein spoke on this subject quite definitely: ''*Theory alone decides on what precisely can be observed*''. It should be said straight away that this statement significantly extends the scope of notions on observability and is at variance with the usual principles of classical science.

It is likely that Heisenberg's excessive concern with the observability problem in its simplified interpretation was actually a manifestation of the rudiments of classical thinking, which were not so easy to abandon. In the years when the 'new' quantum mechanics was under construction, in fact, there existed no other way of thinking apart from the classical one and Heisenberg was by no means alone in this respect. For instance, Fock, following Heisenberg, at that time spoke of quantum mechanics as of ''*a relativity theory with respect to means of observation*'', which could be adopted merely as a useful metaphor. Bohr also paid certain tribute to classical views in his initial statements concerning the principle of complementarity.

Dirac's standpoint was radically different: even in his first paper on quantum mechanics he managed 'to hold himself aloof' from too straightforward a classical view of nature and began formulating the quantum language of its description. Eventually, he showed that, along with the characteristics of objects by themselves known from classical physics and being as if on the face of phenomena, there exists the second independent set of characteristics — the characteristics of object states theretofore concealed from the attention of researchers, much like the opposite side of the Moon. In fact, this has led to the doubling of the number of characteristics employed in the conceptual apparatus of physics, this being true, as it has turned out, of not only quantum physics.

As emphasized by Faddeev [98], in the modern view ''... *the main notions participating in the formulation of a physical theory are observables and states...*'' He next showed in what sense the existing physical theories — classical and quantum mechanics, nonrelativistic and relativistic dynamics — can be considered as different realizations of the corresponding algebraic structures, the quantum-to-classical mechanics transfer and the relativistic-to-nonrelativistic dynamics transfer being regarded in this case as the deformations of these structures in the parameters $\hbar$ and $1/c^2$, respectively. Based on this general scheme, Faddeev observed that ''*From the standpoint of modern mathematics, the two principal revolutions in physics and natural science in general are deformations of unstable structures into the stable ones. From this viewpoint fashionable talks about the change of paradigms are losing their luster, to say the least*''. In this case, a similar scheme could have been revealed even in the 19th century; quantum mechanics and the relativity theory could have been arrived at simply by searching for other realizations of these general schemes. But ''... *the scheme itself appeared only after the discovery of quantum mechanics in the description of its general structure. Here, the part of fundamental importance was played by P Dirac. Only then was it recognized that classical mechanics is another realization of the same scheme*''.

This implies that the conceptual apparatus elaborated by Dirac makes it possible to adequately formulate not only the nonclassical PPW version, but also the classical one, which traces its origin to Newton. More recently, it was found that the conceptual apparatus elaborated by Dirac is applicable not only to mechanics. Today it has proven to be efficient in classical and statistical thermodynamics, including the theories of fluctuations [99, 100] and Brownian motion [101, 102, 109].

Therefore, there are strong grounds to believe that Dirac's works have led to qualitative changes in the Weltanschauung of the scientific community, completing the epoch of transfer from the classical view to the quantum view and, what is more, to the nonclassical view of nature initiated by Planck [103, 104, 110]. To put it another way, the radical change of the contents of both PPW components is Dirac's contribution of paramount importance to the cognitive activity of humanity as a whole. Before our very eyes the PPW is progressively acquiring the form of an adequate basic model of nature, which embodies in indissoluble unity the ideas of Newton, Einstein, and Dirac.

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7. Appendices

7.1 Projective geometry: elementary concepts

Projective geometry (see, for instance, Ref. [105]) originated from the teaching of perspective of the Renaissance; well-known painters indulged in it prior to others — Albrecht Dürer and Leonardo da Vinci — and da Vinci's canvas 'the Last Supper' is considered to be the canon for that stage of development of the future branch of mathematics. As a mathematical discipline in its own right, this science took shape (by concurrence of circumstances) in Russia, in the town of Saratov, which was the residence of Jean-Victor Poncelet (a captive lieutenant of Napoleon's army) from March of 1813 through June of 1814. He took advantage of the 'spare time' to make notes of his future *Traité des Propriétés Projectives des Figures* (Treatise on the Projective Properties of Figures) later published in Paris in 1822. That year is considered to be the birthday of this mathematical discipline, although several of its assertions (theorems) were formulated and proved even in the 17th century by G Désargues and B Pascal.

If, in lieu of Cartesian coordinates (x, y) of some point in a plane, one introduces *homogeneous* coordinates $(x_1 : x_2 : x_3)$ related to the Cartesian ones as $x = x_1/x_3$; $y = x_2/x_3$, it is easily seen that the homogeneous coordinates of an arbitrary point in a plane cannot simultaneously all vanish and are defined correct to a constant factor, for the triplets (x_1, x_2, x_3) and $(\lambda x_1, \lambda x_2, \lambda x_3)$ define the Cartesian coordinates of the common point (hence there appears the designation adopted for them). The name of the coordinates is related to the fact that the equation of any straight line is written in these coordinates in a homogeneous form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\,. \tag{7.1}$$

Second-order curves are also represented in a similar homogeneous form

$$a_{ij}\, x_i\, x_j = 0; \quad i, j = 1, 2, 3\,. \tag{7.2}$$

From Eqn (7.1) it follows that the equalities $x_1 = 0, x_2 = 0$ define, respectively, the Y- and X-axes in the plane, while the equality $x_3 = 0$ is the equation for an *ideal* (infinitely far) straight line, which is the locus of *ideal* points of the plane. In the ideal line there intersect any two parallel straight lines, for instance, the straight lines $x_2 = kx_1 + b_1$; $x_2 = kx_1 + b_2$ intersect at a point $(1 : k : 0)$, and so forth. A straight line supplemented with an ideal point is termed a *projective straight line* and is designated as RP^1, while a plane complemented with an ideal straight line is termed a *projective plane* RP^2. These are the simplest objects of projective geometry that allow a natural generalization to higher dimensionalities.

Projective geometry contains a wealth of amazing facts, which are quite unusual to a person with conventional (Euclidean-geometrical) thinking. In particular, from the equation for the best-known second-order curve [like Eqn (7.2)] — a circumference

$$x_1^2 + x_2^2 + a_0 x_3^2 + 2a_1 x_1 x_3 + 2a_2 x_2 x_3 = 0 \tag{7.3}$$

it follows that any circumference passes through two ideal imaginary points $(1 : i : 0)$ and $(1 : -i : 0)$, which are referred to as the *cyclic* points of the plane. The straight line which is defined by formula (7.1) and passes through any of the cyclic points is remarkable in that the length of any of its segments is equal to zero, while such straight lines themselves are termed *isotropic*. In this case, exactly two such isotropic straight lines pass through any point of the plane. This is but one step to spinors, which were discovered by the French geometrician E J Cartan in 1913, and introduced into physics by Dirac (see, for instance, Ref. [106]).

The second remarkable statement of projective geometry is the *principle of duality* (Poncelet): to any proposition with participation of the terms 'point' and 'straight line' there corresponds a dual proposition, which results from the first one by a simple permutation of these terms (in the case of projective space, 'plane' is added to these terms). For instance, the equation of a straight line (7.1), which is symmetric in form about a and x, for fixed x and variable a defines a set of straight lines passing through a point x, i.e., is the equation of a point.

In the general case, projective geometry studies the properties of figures which remain invariable under *projective transformations* of the form

$$x' = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}\,, \qquad y' = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}\,, \tag{7.4}$$

which define a one-to-one projective plane mapping onto itself. For spaces of higher dimensionality, projective transformations are obtained by a simple generalization of formulas (7.4) and in every case make up a *projective group*, which comprises as special cases the similarity group and the affine transformation group. On this basis, the mathematician Arthur Cayley in 1859 even enunciated a principle: *projective geometry is the entire geometry*!, which later proved to be only conventionally true. [19]

7.2 Dirac's 'quantum dictionary'

Dirac's ingenious way of thinking manifested itself even in his mode of inventing the terminology of quantum theory. He possessed an amazingly capacious spatial thinking, which enabled him to easily operate not only onto real bodies, but on abstract physical notions as well. That is why operating in the spirit of projective geometry it is as if he aspired to project many-dimensional physical abstractions onto the 'plane of thinking' of ordinary researchers. Having found that the conventional notions of vectors in finite-dimensional spaces are insufficient for describing the states of quantum-mechanical systems, he came up with the idea of generalizing these notions and going over to vectors in infinite-dimensional spaces (two years later, the mathematician J von Neumann

[19] On this occasion, see V G Boltyanskiĭ's notes to F Klein's lectures [107].

'recognized' these vectors as the elements of Hilbert spaces and gave a rigorous exposition of Dirac's apparatus in his monograph [29]). In his *The Principles of Quantum Mechanics* Dirac introduced this innovation as a quite natural one and continued: "*It is desirable to have a special name for describing the vectors which are connected with the states of a system in quantum mechanics, whether they are in a space of a finite or an infinite number of dimensions. We shall call them ket vectors, or simply kets, and denote a general one of them by a special symbol* $|\rangle$. *If we want to specify a particular one of them by a label, A say, we insert it in the middle, thus* $|A\rangle$. *The suitability of this notation will become clear as the scheme is developed*" (see Ref. [20], Vol. I, p. 29).

The term 'ket' is the second part of the word 'bracket'. For the vector conjugate to $|A\rangle$, Dirac introduced a 'bra' vector, which is the first part of the same word, and designated it by $\langle B|$. "*A scalar product* $\langle B|A\rangle$ *now appears as a complete bracket expression, and a bra vector* $\langle B|$ *or a ket vector* $|A\rangle$ *as an incomplete bracket expression. We have the rules that any complete bracket expression denotes a number and any incomplete bracket expression denotes a vector, of the bra or ket kind according to whether it contains the first or second part of the brackets* ..." (this is how simply and naturally introduced is the characteristic termed by modern physicists, after Feynman, as the $|A\rangle$-to-$|B\rangle$ transition probability amplitude).

In his *The Principles...*, Dirac made extensive use of the so-called δ-function, which he had introduced in Ref. [34] back in 1927, and which he needed "*to get a precise notation for dealing with ... infinities*". He considered the δ-function as "*a function of the real variable x which vanishes everywhere except inside a small domain, of length* ϵ *say, surrounding the origin* $x = 0$, *and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations...*".

Even someone who was not a mathematician understood that it was a 'trick' rather than a rigorous definition. But this did not confuse Dirac, who treated the δ-function without any respect — differentiated, integrated, multiplied by other functions, etc. The mathematicians of that time perceived Dirac's actions simply as a play on formulas. Those who harnessed the δ-function in their calculations preferred to conceal it in their publications and provided 'conventional' proofs of the theorems obtained with its aid. But then the mathematicians S L Sobolev and L Schwartz in their works elaborated the theory of generalized functions, the Dirac δ-function being their special case. All the results obtained by Dirac without substantiation thereby acquired 'legitimate status'.

"*I encountered the notation problem in connection with a Poisson bracket*", Dirac remembered. "*I borrowed all the information about it from Whittaker's book Analytical Dynamics, where parentheses were used for Poisson brackets, and square brackets were used for Lagrange brackets. The quantum theory does not employ the Lagrange brackets, it makes use of only the Poisson bracket. That it why Whittaker's designations seemed inconvenient to me. They suggest the idea of a scalar product known from the vector analysis. However, the scalar product is symmetric about permutation of the two terms involved, while the Poisson bracket is antisymmetric about their permutation. That is why I boldly took advantage of the other designation of the bracket... Since then, everybody does so. It turned out that the quantity antisymmetric about permutation of the two terms involved is quite convenient to designate by square brackets*" [9] [20].

When the equality $uv = vu$ is fulfilled, mathematicians-algebraists say that u is 'permutable' with v. The word 'permutability' seemed somewhat inappropriate to Dirac, since physicists, on the subject of permutations, commonly imply that rearranged are several quantities rather than two, as in our case. That is why Dirac introduced the word 'commute' (from the Latin *commutare* — 'change'). "*I do not think that mathematicians had used it before me*", he wrote. — "*I declared: when* $uv = vu$, *u and v commute with each other. Since then, this term has also come into use*".

Another typical example of Dirac's word creation is the introduction of c- and q-numbers. "*The situation was that I had to deal with new, quantum variables, which appeared quite mysterious to me, and therefore I invented a new word for them. I called them q-numbers to distinguish them from ordinary variables, which figured in mathematics and which I termed c-numbers... I next undertook to construct the theory of q-numbers; c-numbers can be treated simply as the special case of q-numbers which have the property that they commute with any quantities... I had no idea of the origin of q-numbers and believed that the Heisenberg matrices provided an example of q-numbers, but it might well turn out that q-numbers had a more general significance... I continued to elaborate the theory, and in doing this I was free to make any assumptions I needed, provided that they did not give rise to immediate contradictions. I was not going to find out the mathematical nature of q-numbers, nor did I intend to elucidate the accuracy of calculations with them*" [9].

Following Dirac's example, physicists would resort to such terms as 'fermions' (for particles with a half-integer spin) and 'bosons' (for particles with an integer-valued spin). He proposed their use in his lectures on elementary particles and their interactions, which were given in Princeton actually a year before the discovery of charged π mesons by S Powell, G Occhialini et al. in 1947. All massive particles known at that time possessed only half-integer spins, but Dirac had no doubt of the verity of Yukawa's hypothesis and believed that the discovery of mesons was only a matter of time.

Thus there gradually formed the vocabulary of terms that came to be 'spoken' by the new science – quantum physics. As justly observed by B V Medvedev in the introductory article to the collected works of Dirac [108]: "*Not only did Dirac turn quantum mechanics from a set of recipes for the solution of particular problems to a consistent and logically closed theory, but he also devised the language — of notions, terms, and symbols — in which we express ourselves in any division of the quantum theory. It can be said without gross exaggeration that in the event we are — like in a children's game — suddenly forbidden to use this language we would find ourselves in the situation of the builders of the tower of Babel*".

[20] We note that the Poisson brackets (the Poisson structures) play about the same part in classical Hamiltonian mechanics as the vector product in the vector algebra of Euclidean space, with the difference that the brackets should be nondegenerate. A more general notion of the Poisson structure that needs not necessarily obey the nondegeneracy requirement originated in the works of the Norwegian mathematician S Lie on the theory of continuous groups, which was elaborated for the integration of the systems of first-order partial linear differential equations. The interest in these works of Lie was rekindled due to Dirac and his work on the generalization of Hamiltonian mechanics (see Section 5.1)

References

1. Taylor J G (Ed.) *Tributes to Paul Dirac* (Bristol: A. Hilger, 1987)
2. Salam A, Wigner E P (Eds) *Aspects of Quantum Theory* (Cambridge: Univ. Press, 1972)
3. Mehra J (Ed.) *The Physicist's Conception of Nature*: *Symp. on the Development of the Physicist's Conception of Nature in the 20th Century, Italy, 1972. Dedicated to P A M Dirac on the Occasion of His 70th Birthday* (Dordrecht: Reidel, 1973)
4. Kursunoglu B N, Wigner E P (Eds) *Reminiscences About a Great Physicist*: *Paul Adrien Maurice Dirac* (Cambridge: Cambridge Univ. Press, 1987)
5. Heisenberg W "Buchbesprechung: P A M Dirac 'The Principles of Quantum Mechanics' " *Metallwirtschaft* **9** 988 (1930)
6. Jost R "Foundations of quantum field theory", in Ref. [2] p. 61
7. Smorodinskiĭ Ya A *Usp. Fiz. Nauk* **148** 527 (1986) [*Sov. Phys. Usp.* **29** 281 (1986)]; see also Ref. [20] Vol. I, p. 613; Mehra J *Usp. Fiz. Nauk* **153** 135 (1987); see also Ref. [20], Vol. II, p. 779
8. Medvedev B V (Ed.) *P A M Dirak i Fizika XX Veka* (P A M Dirac and 20th Century Physics) Collected Articles (Moscow: Nauka, 1990)
9. Dirac P A M *Vospominaniya o Neobychaĭnoĭ Epokhe* (Recollections of the Extraordinary Epoch) Collected Articles (Ed. Ya A Smorodinskiĭ) (Moscow: Nauka, 1990); see also Ref. [20] Vol. I, p. 576
10. Dirac P A M *The Principles of Quantum Mechanics* 3rd ed. (Oxford: Clarendon Press, 1947)
11. Einstein A "Autobiographishes", in *Albert Einstein, Philosopher – Scientist* (Ed. P A Schilpp) (New York: Tydor, 1949) [Translated into Russian: *Sobranie Nauchnykh Trudov* Vols I – IV (Ser. "Klassiki Nauki", Eds I E Tamm, Ya A Smorodinskiĭ, B G Kuznetsov) (Moscow: Nauka, 1965 – 1967)]
12. Klein M J "Thermodynamics in Einstein's thought" *Science* **157** 509 (1967) [Translated into Russian: in *Eĭnshteĭnovskiĭ Sbornik 1978 – 1979* (Moscow: Nauka, 1983) p. 150]
13. Mehra J "Einstein and the foundation of statistical mechanics" *Physica A* **79** 447 (1975)
14. Dirac P A M "Homogeneous variables in classical dynamics" *Proc. Cambr. Philos. Soc.* **29** 389 (1933) [See the translation into Russian in Ref. [20] Vol. II, p. 461]
15. Dirac P A M "Wave equations in conformal space" *Ann. Math.* **37** 429 (1936) [See the translation into Russian in Ref. [20] Vol. II, p. 485]
16. Dirac P A M "Applications of quaternions to Lorentz transformations" *Proc. R. Irish Acad. A* **50** 261 (1945) [See the translation into Russian in Ref. [20] Vol. II, p. 553]
17. Rechenberg H "Osnovaniya geometrii i kvantovaya algebra" (The foundations of geometry and quantum algebra); see Ref. [8] p. 15
18. Dirac P A M "Note on the relativity dynamics of a particle" *Philos. Mag.* **47** 1158 (1924) [See the translation into Russian in Ref. [20] Vol. II, p. 16]
19. Dubrovskiĭ V N, Smorodinskiĭ Ya A, Surkov E L *Relyativistskiĭ Mir* (Relativistic World) (Bibliotechka 'Kvant', Vyp. 34) (Moscow: Nauka, 1984)
20. Dirac P A M *Sobranie Nauchnykh Trudov* T. I *Kvantovaya Teoriya* (*Monografii, Lektsii*) T. II *Kvantovaya Teoriya* (*Nauchnye Stat'i*: *1924 – 1947*) T. III *Kvantovaya Teoriya* (*Nauchnye Stat'i*: *1948–1984*) T. IV *Gravitatsiya i Kosmologiya. Vospominaniya i Razmyshleniya.* (*Lektsii, Nauchnye Stat'i*: *1937 – 1984*) (Collected Scientific Works Vol. I Quantum Theory (Monographs, Lectures) Vol. II Quantum Theory (Scientific Papers: 1924 – 1947) Vol. III Quantum Theory (Scientific Papers: 1948–1984) Vol. IV Gravitation and Cosmology. Memoirs and Thoughts (Lectures, Scientific Papers: 1937–1984)) (Ser. 'Klassiki Nauki', Ed. A D Sukhanov) (Moscow: Fizmatlit, 2002 – 2004)
21. Heisenberg W *Z. Phys.* **33** 879 (1925) [Translated into Russian: *Usp. Fiz. Nauk* **122** 574 (1977)]
22. Dirac P A M *Proc. R. Soc. London Ser. A* **109** 642 (1925); [Translated into Russian: *Usp. Fiz. Nauk* **122** 611 (1977), see also Ref. [20] Vol. II, p. 59]
23. Born M, Jordan P *Z. Phys.* **34** 858 (1925) [Translated into Russian: *Usp. Fiz. Nauk* **122** 586 (1977)]
24. Born M, Heisenberg W, Jordan P *Z. Phys.* **35** 557 (1926) [Translated into Russian: Heisenberg W *Izbrannye Trudy* (Selected Works) (Moscow: URSS, 2001) p. 127]
25. Dirac P A M *Proc. R. Soc. London Ser. A* **110** 561 (1926); **111** 281, 405; **112** 661 (1926); *Proc. Cambridge Philos. Soc.* **23** 412, 500 (1926) [See the translation into Russian in Ref. [20] Vol. II, pp. 72, 91, 117, 147, 140, 163]
26. Dirac P A M *The Principles of Quantum Mechanics* (Oxford: The Clarendon Press, 1930)
27. Dirac P A M "The physical interpretation of quantum dynamics" *Proc. R. Soc. London Ser. A* **113** 621 (1927) [See the translation into Russian in Ref. [20] Vol. II, p. 171]
28. Dirac P A M "The basis of statistical quantum mechanics" *Proc. Cambridge Philos. Soc.* **25** 62 (1929); "Note on the interpretation of the density matrix in the many electron problem" *Proc. Cambridge Philos. Soc.* **27** 240 (1931) [See the translation into Russian in Ref. [20] Vol. II, pp. 208, 279]
29. von Neumann J *Mathematische Grundlagen der Quantenmechanik* (Berlin: J. Springer, 1932) [Translated into Russian (Moscow: Nauka, 1964)]
30. Dirac P A M "Über die Quantenmechanik der Stossvorgänge" *Z. Phys.* **44** 585 (1927) [See the translation into Russian in Ref. [20] Vol. II, p. 191]
31. Dirac P A M "Quantum mechanics of many-electron systems" *Proc. R. Soc. London Ser. A* **123** 714 (1929) [See the translation into Russian in Ref. [20] Vol. II, p. 213]
32. Dirac P A M "Note on the exchange phenomena in the Thomas atom" *Proc. Cambridge Philos. Soc.* **26** 376 (1930)[See the translation into Russian in Ref. [20] Vol. II, p. 268]
33. Dirac P A M "Priblizhennye metody (Dop avt. k rus. per.)" ("Approximate methods (Author's supplement to the Russian edition"), in Dirac P A M *Osnovy Kvantovoĭ Mekhaniki* (The Elements of Quantum Mechanics) (Moscow-Leningrad: Gostekhizdat, 1932); see also Ref. [20] Vol. I, p. 303
34. Dirac P A M "The quantum theory of emission and absorption of radiation" *Proc. R. Soc. London Ser. A* **114** 243 (1927) [See the translation into Russian in Ref. [20] Vol. II, p. 285]
35. Weisskopf V F "Growing up with field theory", Preprint MIT (1980) [Translated into Russian: *Usp. Fiz. Nauk* **138** 455 (1982)]
36. Medvedev B V, Shirkov D V *Usp. Fiz. Nauk* **153** 59 (1987) [*Sov. Phys. Usp.* **30** (9) 791 (1987)]; see also Ref. [20] Vol. I, p. 626
37. Dirac P A M, Fock V A, Podolsky B "On quantum electrodynamics" *Phys. Z. Sowjetunion* **2** 468 (1932) [See the translation into Russian in Ref. [20] Vol. II, p. 409]
38. Shirkov D V "Quantum field — the only form of matter", Preprint MPI-Ph/92-54 (1992)
39. Dirac P A M "Relativistic quantum mechanics" *Proc. R. Soc. London Ser. A* **136** 453 (1932) [See the translation into Russian in Ref. [20] Vol. II, p. 399]
40. Dirac P A M "Theorie du positron", in *Septiéme Conseil de Physique Solvay*: *Structure et Propertiétés des Noyaux Atomiques*, *22 – 29 October 1933* (Paris: Gautier Villars, 1934) p. 203 [See the translation into Russian in Ref. [20] Vol. II, p. 431]
41. Weisskopf V *Z. Phys.* **89** 27; **90** 817 (1934)
42. Stückelberg E G G *Ann. Phys.* (Leipzig) **21** 367 (1935); *Helv. Phys. Acta* **9** 255 (1938)
43. Kramers H A *Nuovo Cimento* **15** 108 (1938)
44. Bogolyubov N N, Parasyuk O S *Dokl. Akad. Nauk SSSR* **100** 25 (1955)
45. Bogolyubov N N, Shirkov D V *Usp. Fiz. Nauk* **55** 149 (1953); **57** 2 (1955)
46. Bogolyubov N N, Parasyuk O S *Izv. Akad. Nauk SSSR, Ser. Matem.* **20** 585 (1956)
47. Parasyuk O S *Izv. Akad. Nauk SSSR, Ser. Matem.* **20** 843 (1956)
48. Shirkov D V *Ann. Phys.* (Leipzig) **47** 230 (1990)
49. Stückelberg E G G, Petermann A *Helv. Phys. Acta* **26** 499 (1953)
50. Gell-Mann M, Low F E *Phys. Rev.* **95** 1300 (1954)
51. Bogolyubov N N, Shirkov D V *Dokl. Akad. Nauk SSSR* **103** 203 (1955)
52. Dirac P A M "The physical interpretation of the quantum mechanics" (Bakerian Lecture 1941) *Proc. R. Soc. London Ser. A* **180** 1 (1942) [See the translation into Russian in Ref. [20] Vol. II, p. 587]
53. Dirac P A M "Quantized singularities in the electromagnetic field" *Proc. R. Soc. London Ser. A* **133** 60 (1931) [See the translation into Russian in Ref. [20] Vol. II, p. 388]

54. Oppenheimer J R "On the theory of electrons and protons" *Phys. Rev.* **35** 562–563 (1930)
55. Rutherford E, Chadwick J *Ellis Radiation from Radioactive Substances* (Cambridge: Cambridge Univ. Press, 1930)
56. Dirac P A M "Theory of Electrons and Positrons", in *Nobel Lectures — Physics 1932–1941* (Amsterdam: Elsevier, 1965) p. 320 [See the translation into Russian in Ref. [20] Vol. I, p. 381]
57. Heisenberg W "Cosmic radiation and fundamental problems in physics" *Naturwissenschaften* **63** 63 (1976)
58. Kobzarev I Yu "K istorii positrona" (On the history of the positron), see Ref. [8] p. 21
59. Dirac P A M "Discussion of the infinite distributions of electrons in the theory of the positron" *Proc. Cambr. Philos. Soc.* **30** 150 (1934) [See the translation into Russian in Ref. [20] Vol. II, c. 441]
60. Jackiw R "Effects of Dirac's negative energy sea in quantum numbers" *Helv. Phys. Acta* **59** 835 (1986)
61. Popov V S "Kvantovaya elektrodinamika sverkhsil'nykh poleĭ" (Quantum electrodynamics of superstrong fields), in *Sovremennaya Teoriya Elementarnykh Chastits* (Modern Theory of Elementary Particles) (Moscow: Nauka, 1984) p. 127
62. Pauli W *Die allgemeinen Prinzipien der Wellenmechanik* (Handbuch der Physik, 2 Aufl., Bd. 24) (Berlin: Springer, 1933) s. 82 [Translated into Russian (Moscow-Leningrad: Gostekhizdat, 1947)]
63. Dalitz R H "Biographical sketch of the life of Professor P A M Dirac, OM, FRS", in Ref. [1]
64. Dirac P A M *The Collected Works of P.A.M. Dirac, 1924–1948* (Ed. R H Dalitz) (Cambridge: Cambridge Univ. Press, 1995)
65. Dirac P A M "Generalized Hamiltonian dynamics" *Can. J. Math.* **2** 129 (1950); "The Hamiltonian form of field dynamics" *Can. J. Math.* **3** 1 (1951); "Generalized Hamilton dynamics" *Proc. R. Soc. London Ser. A* **246** 326 (1958) [See the translation into Russian in Ref. [108], as well as in Ref. [20] Vol. III]
66. Dirac P A M *Lectures on Quantum Mechanics* (New York: Belfer Graduate School Science, Yeshiva Univ., 1964) [Translated into Russian (Moscow: Mir, 1968); see also Ref. [20] Vol. I, p. 386]
67. Arnol'd V I, Kozlov V V, Neĭshtadt A I "Matematicheskie aspekty klassicheskoĭ i nebesnoĭ mekhaniki" (Mathematical aspects of classical and celestial mechanics), in *Itogi Nauki i Tekhniki. Ser. Sovremennye Problemy Matematiki. Fundamental'nye Napravleniya* Vol. 3 (Moscow: Izd. VINITI, 1985)
68. Borisov A V, Mamaev I S *Puassonovy Struktury i Algebry Li v Gamil'tonovoĭ Mekhanike* (Poisson Structures and Lie Algebras in Hamiltonian Mechanics) (Izhevsk: Udmurtskiĭ Universitet, 1999)
69. Landau L D, in *Teoreticheskaya Fizika v 20 Veke* (Theoretical Physics in the 20th Century) Dedicated to the memory of W Pauli (1960); Landau L D *Sobranie Trudov* (Collected Works) Vol. 2 (Moscow: Nauka, 1969) p. 421
70. Chew G F *The Analytic S-Matrix* (New York: W.A. Benjamin, 1966) [Translated into Russian (Moscow: Mir, 1968)]
71. Okun' L B *Vvedenie v Kalibrovochnye Teorii: Lektsii, Prochitannye na Shkole OIYaI–TsERN* (Moscow: Izd. MIFI, 1983)
72. Slavnov A A, Faddeev L D *Vvedenie v Kvantovuyu Teoriyu Kalibrovochnykh Poleĭ* (Introduction to the Quantum Theory of Gauge Fields) 2nd ed. (Moscow: Nauka, 1987)
73. Dirac P A M "The Lagrangian in quantum mechanics" *Phys. Z. Sowjetunion* **3** 64 (1933) [See the translation into Russian in Ref. [20] Vol. III]
74. Dirac P A M "The Lagrangian in quantum mechanics" *Phys. Z. Sowjetunion* **3** 64 (1993) [See the translation into Russian in Ref. [20] Vol. II, p. 573]
75. Dirac P A M "On the analogy between classical and quantum mechanics" *Rev. Mod. Phys.* **17** 195 (1945) [See the translation into Russian in Ref. [20] Vol. II, p. 626]
76. Faddeev L D *Teor. Matem. Fiz.* **1** 3 (1969)
77. Gitman D M, Tyutin I V *Kanonicheskoe Kvantovanie Poleĭ so Svyazyami* (Canonical Quantization of Fields with Constraints) (Moscow: Nauka, 1986)
78. Dirac P A M "Pretty mathematics" *Int. J. Theor. Phys.* **21** 603 (1982) [See the translation into Russian in Ref. [20] Vol. III]
79. Schwinger J *Phys. Rev.* **144** 1087; **151** 1048, 1055 (1966); **173** 1536 (1968); *Science* **165** 757 (1969)
80. Atiyah M, in *Quantum Physics and the Topology of Knots: Proc. XIth Intern. Congress of Mathematical Physics* (Ed. E Lieb) (Boston: International Press, 1995)
81. Dirac P A M "The theory of magnetic poles" *Phys. Rev.* **74** 817 (1948) [See the translation into Russian in Ref. [20] Vol. III]
82. Marinov M S *Usp. Fiz. Nauk* **121** 377 (1977) [*Sov. Phys. Usp.* **20** 179 (1977)]
83. Rubakov V A *Pis'ma Zh. Eksp. Teor. Fiz.* **33** 658 (1981) [*JETP Lett.* **33** 644 (1981)]; *Nucl. Phys. B* **203** 311 (1982); Callan C G (Jr) *Phys. Rev. D* **25** 2141; **26** 2058 (1982)
84. Markovski B, Vinitsky S I (Eds) *Topological Phases in Quantum Theory* (Singapore: World Scientific, 1989)
85. Monastyrskiĭ M I, Sergeev A G (Eds) *Monopoli: Topologicheskie i Variatsionnye Metody* (Monopoles: Topological and Variational Methods) Collected Papers (Moscow: Mir, 1989)
86. Novikov S P "Gamil'tonov formalizm i mnogoznachnyĭ analog teorii Morsa" (Hamiltonian formalism and the many-valued analog of the Mors theory) *Usp. Matem. Nauk* **37** 3 (1982)
87. Rybakov Yu P, Sanyuk V I *Mnogomernye Solitony. Vvedenie v Teoriyu i Prilozheniya* (Many-Dimensional Solitons. Introduction to the Theory and Applications) (Moscow: Izd. RUDN, 2001)
88. Monastyrskiĭ M I *Topologiya Kalibrovochnykh Poleĭ i Kondensirovannykh Sred* (Topology of Gauge Fields and Condensed Media) (Moscow: PAIMS, 1995)
89. Dirac P A M "A new basis for cosmology" *Proc. R. Soc. London Ser. A* **165** 60 (1938) [See the translation into Russian in Ref. [20] Vol. IV]
90. Rubakov V A *Usp. Fiz. Nauk* **171** 913 (2001) [*Phys. Usp.* **44** 871 (2001)]
91. Salam A "Dirac and Finite Field Theory", see Ref. [1] p. 84
92. Dirac P A M "Unitary representations of the Lorentz group" *Proc. Roy. Soc. London Ser. A* **183** 284 (1945) [See the translation into Russian in Ref. [20] Vol. II, p. 537]
93. Dirac P A M "An extensible model of the electron" *Proc. R. Soc. London Ser. A* **268** 57 (1962) [See the translation into Russian in Ref. [20] Vol. III]
94. Dirac P A M "Particles of finite size in the gravitation field" *Proc. R. Soc. London Ser. A* **270** 354 (1962) [See the translation into Russian in Ref. [20] Vol. III]
95. Dirac P A M "The relations between mathematics and physics" *Proc. R. Soc. Edinburgh A* **59** 122 (1938–1939) [See the translation into Russian in Ref. [108], as well as in Ref. [20] Vol. IV]
96. Vizgin Vl P "P.A.M. Dirac on the interrelation between physics and mathematics", see Ref. [8] p. 95
97. Heisenberg W "Kvantovaya mekhanika i beseda s Eĭnshteĭnom" (Quantum mechanics and the conversation with Einstein) *Priroda* (5) 87 (1972)
98. Faddeev L D "Matematicheskiĭ vzglyad na evolyutsiyu fiziki" (Mathematical view on the evolution of physics) *Priroda* (5) 11 (1989)
99. Uffink J, van Lith J "Thermodynamic uncertainty relations" *Found. Phys.* **28** 323 (1998)
100. Rudoĭ Yu G, Sukhanov A D "Termodinamicheskie fluktuatsii v podkhodakh Gibbsa i Eĭnshteĭna" (Thermodynamic fluctuations within the Gibbs and Einstein approaches) *Usp. Fiz. Nauk* **170** 1265 (2000) [*Phys. Usp.* **43** 1169 (2000)]
101. Guth E "Brownian motion and indeterminacy relations" *Adv. Chem. Phys.* **15** 363 (1969)
102. Oksak A I, Sukhanov A D "Predstavlenie kvantovogo brounovskogo dvizheniya v metode kollektivnoĭ koordinaty" (Representation of quantum Brownian motion in the method of collective coordinate) *Teor. Matem. Fiz.* **136** 115 (2003) [*Theor. Math. Phys.* **136** 994 (2003)]
103. Sukhanov A D "K stoletiyu neklassicheskoĭ fiziki" (To the centenary of nonclassical physics), in *100 Let Kvantovoĭ Teorii. Istoriya. Fizika. Filosofiya: Trudy Mezhdunarodnoĭ Konferentsii* (100 Years of the Quantum Theory. History. Physics. Philosophy: International Conference Proceedings) (Editor-in-chief E A Mamchur) (Moscow: NIA–Priroda, 2002) p. 39
104. Sukhanov A D "Novyĭ podkhod k sootnosheniyu neopredelennosteĭ energiya–vremya" (A new approach to the energy–time uncertainty relation) *Fiz. Elem. Chastits At. Yadra* **32** 1177 (2001) [*Phys. Part. Nucl.* **32** 619 (2001)]
105. Glagolev N A *Proektivnaya Geometriya* (Projective Geometry) (Moscow-Leningrad: Gostekhizdat, 1963); Busemann H, Kelly P J

Projective Geometry and Projective Metrics (New York: Academic Press, 1953) [Translated into Russian (Moscow: IL, 1957)]
106. Cartan E *The Theory of Spinors* (Cambridge, Mass.: MIT Press, 1966) [Translated into Russian (Moscow: IL, 1947)]
107. Klein F *Elementarmathematik vom höheren Standpunkte* Bd. 2 *Geometrie* (Berlin: J. Springer, 1924) [Translated into English: *Elementary Mathematics from an Advanced Standpoint*: *Geometry* (New York: The Macmillan Co., 1939); Translated into Russian (Moscow: Nauka, 1987)]
108. Dirac P A M *K Sozdaniyu Kvantovoĭ Teorii Polya*: *Osnovnye Stat'i 1925 – 1958 Godov* (On the Creation of Quantum Field Theory: The Main Papers of 1925 – 1958) (Ed. B V Medvedev) (Moscow: Fizmatlit, 1990)
109. Sukhanov A D "Obobshchennye sootnosheniya neopredelennosteĭ v kvantovoĭ mekhanike i teorii brounovskogo dvizheniya" (Generalized uncertainty relations in quantum mechanics and the theory of Brownian motion) *Teor. Matem. Fiz.* (2003) (in press)
110. Sukhanov A D "Sootnosheniya neopredelennosteĭ Shredingera i fizicheskie osobennosti korrelirovanno-kogerentnykh sostoyaniĭ" (Schrödinger uncertainty relations and physical features of correlated-coherent states) *Teor. Matem. Fiz.* **132** 449 (2002) [*Theor. Math. Phys.* **132** 1277 (2002)]