

REVIEWS OF TOPICAL PROBLEMS

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Self-similar anomalous diffusion and Lévy-stable laws

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Contents

Abstract. **Stochastic principles for constructing the process of anomalous diffusion are considered, and corresponding models of random processes are reviewed. The self-similarity and the independent-increments principles are used to extend the notion of diffusion process to the class of Lévy-stable processes. Replacing the independent-increments principle with the renewal principle allows us to take the next step in generalizing the notion of diffusion, which results in fractional-order partial space–time differential equations of diffusion. Fundamental solutions to these equations are represented in terms of stable laws, and their relationship to the fractality and memory of the medium is discussed. A new class of distributions, called fractional stable distributions, is introduced.**

1. Introduction

When embarking on laboratory training in general physics during our juvenile years, all of us were introduced to the theory of errors. At the very least, this required comprehension of the fact that the result of any measurement X of a quantity a under study is, according to Zaĭdel's [1] vivid expression, *burdened by a random error* Δ:

$$X = a + \Delta\,.$$

As a rule, it cannot be completely eliminated but can be reduced by repeating the experiment and computing the arithmetical (sample) mean of the measured quantities $X_1, \ldots, X_n$:

$$\frac{1}{n}\sum_{j=1}^{n} X_j = a + \Delta_n\,.$$

Here, a is still the true value of the physical quantity (we assume the systematic error to be eliminated), and Δ_n is the random error depending on the number of repetitive measurements n. For large n, this error exhibits a normal (Gaussian) distribution.

Two years later, when studying the probability theory, we learned that the error should have such a distribution by virtue of the *central limit theorem.*

The central limit theorem (CLT). *Let the random variables X_j be independent and let them have the same distribution with a mean $\langle x\rangle = a$ and a variance $\sigma^2 < \infty$. Then*

$$\frac{\sum_{j=1}^{n} X_j - na}{\sqrt{n}\,\sigma} \overset{\mathrm{d}}{\sim} G\,,$$

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where G is a random variable with a standard, normal (Gaussian) density of distribution

$$p_{\mathrm{G}}(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right),$$

and the symbol $\stackrel{\mathrm{d}}{\sim}$ *denotes the asymptotic* (*for* $n\to\infty$) *identity of the distributions of the random variables that appear on both sides of this sign*.

We gained an idea of *random processes and relaxation* phenomena watching the quiver of the pointer of a measuring instrument due to the passage of a tram beyond the window of the laboratory (the quiver damped rapidly); the behavior of particles suspended in a fluid and viewed through a microscope provided an example of persistent random walk — *Brownian motion*, or *diffusion*. Our course in molecular physics acquainted us with the equation of diffusion, which has the following form in the one-dimensional case:

$$\frac{\partial p(x,t)}{\partial t}=D\,\frac{\partial^2 p(x,t)}{\partial x^2},$$

where $p(x,t)$ is the distribution density for the coordinate x of a diffusing particle at time t, and D is the diffusion coefficient. If the particle was located at the origin $x=0$ at the initial time $t=0$,

$$p(x,0)=\delta(x),$$

the solution to this equation has the form

$$p(x,t)=(Dt)^{-1/2}g\big(x(Dt)^{-1/2}\big),$$

where

$$g(x)=\frac{1}{2\sqrt{\pi}}\exp\left(-\frac{x^2}{4}\right)$$

is a Gaussian distribution with a double variance. We note that the equation of diffusion in this case can also be represented as

$$\frac{\partial p(x,t)}{\partial t}=D\,\frac{\partial^2 p(x,t)}{\partial x^2}+\delta(x)\,\delta(t),$$

where a zero initial condition is assumed.

These formulas form a basis for the theory of linear, normal diffusion. The equation of diffusion can most simply be derived from the *continuity equation*

$$\frac{\partial p(x,t)}{\partial t}=-\frac{\partial j(x,t)}{\partial x}$$

and *Fick's law*

$$j(x,t)=-D\,\frac{\partial p(x,t)}{\partial x}.$$

Various physical, but not only physical, problems lead to the diffusion model: the same equations of diffusion describe the behavior of neutrons in a nuclear reactor, of stock market prices, and of pollen particles suspended in a fluid. Many famous physicists, such as Einstein, Smoluchowski, Langevin, Fokker, Planck, Ornstein, Uhlenbeck, Chandrasekhar, etc., contributed to the study of Brownian movement. A senior reader may be aware of the pioneering studies in this area from a collection of papers by Einstein and Smoluchowski [2]. In particular, Smoluchowski mentioned the theory of Brownian movement to be developed by Einstein, Smoluchowski, and Langevin [2, p. 319]. The studies by Einstein, Langevin, and Smoluchowski [3–5] were dated 1905, 1908, and 1912, respectively. However, the diffusion equation appeared for the first time in a dissertation by Louis Bachelier, Poincaré's student, in 1900. The dissertation was entitled "The Theory of Speculations" and analyzed the random evolution of market prices.

The fact that phenomena completely different in nature can be described by the same equations suggests unambiguously that *the issue is not in a particular mechanism of the phenomenon but rather in a certain quality common to the entire class of similar phenomena*. Describing this quality in terms of axioms or definitions makes it possible to purge the pattern of phenomena from details weakly affecting the process and to study the resulting model in a pure form. This is a task for mathematicians, and it is their efforts that have developed the theory of random processes, which was first applied to Brownian motion. A very important contribution to the mathematical theory of Brownian movement was made by Wiener, who proved that the trajectories of a Brownian process are almost everywhere continuous but nowhere differentiable. For this reason, Brownian motion is sometimes called a Wiener process. Besides Wiener, such famous theorists as Bernoulli, Doob, Kac, Feller, Bernstein, Lévy, Kolmogorov, Stratonovich, etc. developed mathematical aspects of Brownian motion.

I would like to generalize here the notion of diffusion using the theory of random processes as a background, in such a manner that the processes of anomalous diffusion, which have received so much attention in recent years (see, e.g., Refs [6–12]), appear or, even better to say, emerge against this background in a most natural way. These processes differ from normal diffusion in that the width of a diffusion packet grows according to the law t^H, where $H\neq 1/2$ ($H<1/2$ in the case of *subdiffusion* and $H>1/2$ in the case of *superdiffusion*). As this takes place, the diffusion packet may not preserve its normal shape, and additional information on the process is necessary to analyze this shape. Such information can either be extracted from the specific physical situation or be introduced externally as a certain principle obeyed, in particular, by normal diffusion — Brownian motion. The *self-similarity principle*, according to which

$$p(x,t)=t^{-H}q(xt^{-H}),$$

can be used as such a principle; here, H is not necessarily $1/2$, and $q(x)$ is not necessarily a normal distribution.

We present here a survey of basic ideas underlying this approach and the most important results obtained.

2. Anomalous diffusion as a random process

2.1 Wiener processes

To better comprehend the relationship between diffusion and other random processes, we recall some definitions [13]. We restrict ourselves to considering one-dimensional motion, so that the coordinate of a randomly walking particle $X(t)$, $t\geqslant 0$ will be real. In this case, the notion of a random process coincides with the notion of a random function [14, p. 589].

2.1.1 *The random process* (*random function*) $\{X(t)\}$ is a collection of random variables $X(t)$ defined on the same probability space and corresponding to all possible t. The random variable $X(t)$ is called the *coordinate* of the random process at time t, and the random realization $X(\cdot)$ is referred to as the *trajectory* of the random process.

According to the tradition commonly accepted in the physical literature, we will characterize the random process $\{X(t)\}$ by a set of finite-dimensional densities $p_n(x_n, t_n; \ldots; x_1, t_1)$, $t_n > t_{n-1} > \ldots > t_1$, assuming their existence.

2.1.2 The random process $\{X(t)\}$ is called *Gaussian* if, for any $n \geqslant 1$ and $t_1, \ldots, t_n$, the density $p_n(x_n, t_n; \ldots; x_1, t_1)$ is an n-dimensional normal (Gaussian) distribution. Since the normal distribution is uniquely determined by its first and second moments, two functions are sufficient to specify a Gaussian process — the mathematical expectation $m(t) = \langle X(t) \rangle$ and the correlation function $B(t,u) = \langle [X(t) - m(t)][X(u) - m(u)] \rangle$.

2.1.3 The random process $\{X(t)\}$ is called *Markovian* if, for any $n \geqslant 1$ and $t_1 < t_2 < \ldots < t_n < t$, the conditional density $p(x,t|x_n,t_n;\ldots;x_1,t_1)$ depends solely on the last coordinate:

$$p(x,t|x_n,t_n;\ldots;x_1,t_1) = p(x,t|x_n,t_n)\,.$$

The interpretation of the Markovian properties of the process assumes that the future is independent of the past provided the present is known. In other words, according to Lévy, the past affects the future only through the present; he also emphasizes an analogy with the Huygens principle: "it can be said that this is the *Huygens principle in calculating probabilities*" [15]. In a sense, the Markovian properties can be considered a stochastic generalization of the dynamical principle carried by the equations of mechanics: the future of a mechanical system is completely determined by its current phase coordinates and does not depend on the past.

The conditional density $p(x,t|y,u)$ is designated as $p(y,u \to x,t)$ and referred to as the *transition density*. It obeys the Kolmogorov–Chapman equation, which is a reformulation of the above definition:

$$p(y,u \to x,t) = \int p(y,u \to z,v)\,p(z,v \to x,t)\,\mathrm{d}z\,,$$
$$u < v < t\,.$$

We note that there are two 'intermediate' arguments (z and v) here, but the integration with respect to one of them (z) also removes the dependence on the other (v). The transition density combined with the one-dimensional density $p(x,0)$ specified at the initial time $t=0$ completely determines the Markovian process for $t > 0$.

2.1.4 The random process $\{X(t)\}$ is called a *process with independent increments* if, for any $n \geqslant 1$ and $0 < t_1 < t_2 < \ldots < t_n < t$, the random variables $X(0), X(t_1) - X(0), \ldots, X(t_n) - X(t_{n-1})$ are mutually independent. Lévy terms such processes *additive* [15]. They obviously belong to the class of Markovian processes, with

$$p(y,u \to x,t) = p(0,u \to x-y,t)\,.$$

2.1.5 A process with independent increments is called *homogeneous* if the distributions of the differences $X(t+\tau) - X(t)$ are t-independent:

$$p(y,u \to x,t) = p(0,0 \to x-y,t-u) \equiv p(x-y,t-u)\,.$$

Lévy terms such processes *linear* and notes that "processes differing from Brownian ones" are present among them [15].

2.1.6 A homogeneous process is referred to as a *stochastically* (*weakly*) *continuous* one if

$$\int_{|x|>\varepsilon} p(x,\tau)\,\mathrm{d}x \to 0$$

for $\tau \downarrow 0$ and any $\varepsilon > 0$.

We list here the most commonly accepted definitions of the diffusion process (Brownian motion).

2.1.7 *Brownian motion* $\{X(t)\}$ can be defined as a homogeneous process with a transition density $p(x,t)$ that is a fundamental solution to a parabolic differential equation [16, p. 319].

2.1.8 *Brownian motion*, or a *Wiener–Bachelier process*, is a process with independent increments that starts at the origin and has a normal distribution with zero mathematical expectation and a variance proportional to t. Wiener and Lévy have shown that the trajectories of such a process are continuous with a probability of unity and that this property distinguishes this process among the wider class of infinitely divisible processes [17, Ch. VI, § 4].

2.1.9 A *Wiener process* (*Brownian motion*) is a homogeneous process with independent increments and continuous trajectories [18], i.e.,

(a) for $0 < t_1 < t_2 < \ldots$, the random variables $X(t_1)$, $X(t_2) - X(t_1)$, ... are independent;

(b) $X(t+\tau) - X(t)$ is t-independent;

(c) $\lim \mathsf{P}\,(|X(t+\tau) - X(t)| \geqslant \varepsilon)/\tau = 0$ for $\tau \to 0$ and for all $\varepsilon > 0$ (the *Lindeberg condition*).

It can easily be seen that, according to property (a), a Wiener process is Markovian: if the increment $X(t_n) - X(t_{n-1})$ does not depend on other increments $X(t_{n-1}) - X(t_{n-2}), \ldots, X(t_1) - X(0)$, it also does not depend on the values $X(t_{n-2}), \ldots, X(0)$ themselves; therefore, $X(t_n)$ at a given $X(t_{n-1})$ is also independent of all preceding values $X(t_{n-2}), \ldots, X(0)$. Property (b) ensures the homogeneity of the process and, for a uniform discretization, the identity of the distributions of increments. Thus, two properties — (a) and (b) — lead us to the CLT. The last remaining detail is provided by property (c) — the continuity of trajectories. This property can, however, be replaced by specifying a normal distribution of trajectories [19, p. 17], by specifying a linear law of the increase of the variance with time, or simply by the requirement that the variance be finite. Its linear time dependence results from the independence of increments, and the normal law follows from the CLT. Thus, the definition 2.1.9 can be rewritten in the following equivalent form:

2.1.10 A Wiener process (W *process*) is a homogeneous process with a finite variance.

2.2 Self-similarity and stability

There is, however, an important feature that distinguishes a simple summation of independent finite-variance terms, for which the normal distribution appears as an asymptotic law (at large numbers of terms), from a Wiener process with a normal distribution valid for any finite time. Brownian motion is self-similar and, therefore, it does not discriminate between long and short times.

2.2.1 *Self-similarity* (*scale invariance, scaling*) is a particular type of symmetry that makes it possible to compensate a rescaling of some variables by a corresponding rescaling of others [20, pp. 8, 351].

In our case, the self-similarity of the distribution density $p(x,t)$ can be expressed as follows:

$$p(x,t) = t^{-H} p(xt^{-H}, 1)\,.$$

In terms of the random variable, this property assumes the form

$$X(t) \stackrel{\mathrm{d}}{=} t^H X(1)\,,$$

or

$$X(at) \stackrel{\mathrm{d}}{=} a^H X(t)\,,$$

for $a > 0$ and any fixed t. Generally, a random process $\{X(t)\}$ is called *self-similar* if the distributions of all its finite-dimensional vectors are self-similar:

$$\big(X(at_1), \ldots, X(at_n)\big) \stackrel{\mathrm{d}}{=} \big(a^H X(t_1), \ldots, a^H X(t_n)\big)\,.$$

For the self-similarity of a homogeneous Markovian process, however, the self-similarity of the one-dimensional distribution is sufficient.

2.2.2 We now define an L *process* as a homogeneous process with a self-similar one-dimensional distribution

$$p(x,t) = t^{-1/\alpha} g^{(\alpha)}(xt^{-1/\alpha})\,,$$

where $\alpha = 1/H$ and $g^{(\alpha)}(x)$ is the density of the distribution, not yet known. Thus, we can pass from a W process (Section 2.1.10) to an L process (Section 2.2.2) by replacing the finite-variance condition with the scaling requirement.

Obviously, such a replacement cannot narrow the class of processes considered, since a W process satisfies this condition and thus remains in the L class with $\alpha = 2$. Can any new processes appear in this case? The answer depends on the existence of non-Gaussian L processes satisfying the scaling condition.

Let us consider two instants of time, t and $t+\tau$. The random coordinates of the L process at these times are related as follows:

$$X(t+\tau) = X(t) + X(\tau)\,.$$

If $X(0) = 0$, the random variables $X(t)$ and $X(\tau)$ are the increments of the process over the nonintersecting intervals $(0,t)$ and $(t, t+\tau)$, thus being independent. The distribution density for their sum is the convolution of the densities for the two summands:

$$p(x, t+\tau) = p(x,t) * p(x,\tau) \equiv \int_{-\infty}^{\infty} p(x-x', t)\, p(x', \tau)\,\mathrm{d}x'\,.$$

It is convenient to pass from the densities to the characteristic functions

$$\tilde{p}(k,t) = \big\langle \exp\{\mathrm{i}kX(t)\}\big\rangle = \int_{-\infty}^{\infty} \exp(\mathrm{i}kx)\, p(x,t)\,\mathrm{d}x\,,$$

for which the convolution reduces to the multiplication

$$\tilde{p}(k, t+\tau) = \tilde{p}(k,t)\,\tilde{p}(k,\tau)\,,$$

and the scaling condition assumes the form

$$\tilde{p}(k,t) = \tilde{g}^{(\alpha)}(kt^{1/\alpha})\,.$$

2.2.3 These relations yield the functional equation

$$\tilde{g}^{(\alpha)}\big(k(t+\tau)^{1/\alpha}\big) = \tilde{g}^{(\alpha)}(kt^{1/\alpha})\,\tilde{g}^{(\alpha)}(k\tau^{1/\alpha})\,,$$

which specifies the class of *strictly stable laws*, which, for brevity, we will simply call stable laws. The characteristic functions $\tilde{g}^{(\alpha)}(k)$ can be expressed in terms of elementary functions and written in several forms. We present here the two most widely used, forms A and C, according to the classification proposed by Zolotarev [21]:

$$\tilde{g}^{(\alpha,\beta)}(k) = \begin{cases} \exp\left\{-|k|^{\alpha}\left[1 - \mathrm{i}\beta \tan\left(\dfrac{\alpha\pi}{2}\right)\operatorname{sign} k\right]\right\}, & \alpha \neq 1\,, \\ \exp\{-|k|\}\,, & \alpha = 1\,, \end{cases} \tag{A}$$

$$\tilde{g}(k;\alpha,\theta) = \exp\left\{-|k|^{\alpha} \exp\left[-\mathrm{i}\left(\frac{\theta\alpha\pi}{2}\right)\operatorname{sign} k\right]\right\}. \tag{C}$$

Here, $\alpha \in (0,2]$ is the characteristic exponent of the stable law, and $\beta \in [-1,1]$ and $\theta \in [-\theta_\alpha, \theta_\alpha]$, $\theta_\alpha = \min\{1, 2/\alpha - 1\}$ are the parameters of asymmetry. We will denote the stable random variables specified by these characteristic functions as $S^{(\alpha,\beta)}$ and $S(\alpha,\theta)$, respectively.

Now we put aside the discussion of stable distributions and return to L processes.

2.3 Equations for L processes

We recall that we have introduced an L process as a homogeneous self-similar law. Its random value at time t is

$$X^{(\alpha,\beta)}(t) \stackrel{\mathrm{d}}{=} t^{1/\alpha} S^{(\alpha,\beta)}\,.$$

Prokhorov and Rozanov [16] describe the family of such laws as *stable processes*, whereas Samorodnitzky and Taqqu [22] suggest reserving this name for the processes whose finite-dimensional distributions are multidimensional stable ones (see Section 3.2.16) and using the following terminology for the processes we consider.

2.3.1 The random process $X(t)$ is called a (*standard*) *α-stable Lévy motion* with parameters $0 < \alpha \leqslant 2$ and $-1 \leqslant \beta \leqslant 1$ if

(1) $X(0) = 0$ almost certainly;

(2) $X(t)$ is a process with independent increments;

(3) $X(t+\tau) - X(t) \stackrel{\mathrm{d}}{=} \tau^{1/\alpha} S^{(\alpha,\beta)}$ at any t and τ.

For brevity, this process will be referred to here as an $\mathrm{L}^{(\alpha,\beta)}$ *process*, while a Wiener process will be denoted as an $\mathrm{L}^{(2,0)}$ process.

To clarify the difference between the trajectories of a W process and an L process, we consider the behavior of the function

$$Q(\tau,\Delta) \equiv \frac{\mathsf{P}\big(|X(t+\tau) - X(t)| \geqslant \Delta\big)}{\tau}$$

as $\tau \to 0$. For a W process,

$$Q_{\mathrm{W}}(\tau,\Delta) = \frac{1}{\sqrt{\pi}\,\tau^{3/2}} \int_{\Delta}^{\infty} \exp\left(-\frac{x^2}{4\tau}\right)\mathrm{d}x = \frac{1}{\sqrt{\pi}\,\tau} \int_{\Delta/\sqrt{\tau}}^{\infty} \exp\left(-\frac{z^2}{4}\right)\mathrm{d}z\,.$$

Upon resolving the indeterminacy according to L'Hospital's rule, we obtain

$$\lim_{\tau \to 0} Q_{\mathrm{W}}(\tau, \Delta) = 0\,,$$

which reflects the continuity of the trajectories of the Wiener process. On the other hand, for an L process with $\alpha < 2$, we have

$$Q_{\mathrm{L}}(\tau, \Delta) = \frac{\mathsf{P}\big(|S^{(\alpha,\beta)}| \geqslant \Delta\tau^{-1/\alpha}\big)}{\tau}\,.$$

It is known that

$$\mathsf{P}\big(|S^{(\alpha,\beta)}| \geqslant x\big) \propto x^{-\alpha}\,, \qquad x \to \infty\,,$$

so that

$$\lim_{\tau \to 0} Q_{\mathrm{L}}(\tau, \Delta) = \mathrm{const}\cdot\Delta^{-\alpha} > 0\,, \qquad \tau \to 0\,.$$

The trajectories of an L process *are not continuous*, and this is their basic distinction. The amplitude of the jumps (discontinuities) is α-dependent: the smaller the α, the larger the jumps. Thus, among L processes, only $\mathrm{L}^{(2,0)}$ has continuous trajectories.

The width of the diffusion packet at $\alpha < 2$ grows with time more rapidly than $\sqrt{t}$ does; specifically, it is proportional to $t^{1/\alpha}$. Its shape is described by a stable distribution with an exponent equal to α. The variance is infinite, but nothing can prevent us from using any other measure of width (e.g., the full width at half-maximum or the width of the interval containing a fixed probability). We thus arrive at the *stochastic model of superdiffusion*, which was considered in detail by Zolotarev et al. [23] and Uchaikin [24].

The trajectories of L processes for different α and β are illustrated in Fig. 1, which presents the results of Monte Carlo simulations.

By analogy with normal diffusion, we introduce a positive constant, the 'diffusion coefficient' D, via the relationship $X^{(\alpha,\beta)}(t) \stackrel{\mathrm{d}}{=} (Dt)^{1/\alpha} S^{(\alpha,\beta)}$.

2.3.2 *The characteristic function of the state of the L process* $X^{(\alpha,\beta)}(t)$ *at time* t has the form ($\alpha \neq 1$)

$$\tilde{p}^{(\alpha,\beta)}(k,t) = \big\langle \exp\big\{\mathrm{i}kX^{(\alpha,\beta)}(t)\big\}\big\rangle = \exp\left\{-Dt|k|^{\alpha}\left[1 - \mathrm{i}\beta\tan\left(\frac{\alpha\pi}{2}\right)\operatorname{sign}k\right]\right\}.$$

From here, the equation governing this function follows immediately:

$$\frac{\partial\tilde{p}^{(\alpha,\beta)}(k,t)}{\partial t} = -|k|^{\alpha}D\left[1 - \mathrm{i}\beta\tan\left(\frac{\alpha\pi}{2}\right)\operatorname{sign}k\right]\tilde{p}^{(\alpha,\beta)}(k,t)\,.$$

The inversion of this equation (inverse Fourier transform) can be obtained using formulas for the inverse Feller potential (see Ref. [25]) and leads to the equation that describes the evolution of the spatial distribution [24],

$$\frac{\partial p^{(\alpha,\beta)}(x,t)}{\partial t} = DK^{(\alpha,\beta)}\big(x;\, p^{(\alpha,\beta)}(\,\cdot\,,t)\big)\,,$$

where the right-hand side can be written in either of two equivalent forms:

$$K^{(\alpha,\beta)}\big(x;\, p^{(\alpha,\beta)}(\,\cdot\,,t)\big) = \frac{A}{\Gamma(-\alpha)}\int_{-\infty}^{\infty}\frac{1 + \beta\operatorname{sign}(x-\xi)}{|x-\xi|^{1+\alpha}} \times \big[p^{(\alpha,\beta)}(x,t) - p^{(\alpha,\beta)}(\xi,t)\big]\,\mathrm{d}\xi\,,$$

or

$$K^{(\alpha,\beta)}\big(x;\, p^{(\alpha,\beta)}(\,\cdot\,,t)\big) = \frac{A}{\Gamma(-\alpha)}\int_0^{\infty}\big[2p^{(\alpha,\beta)}(x,t) - (1+\beta)\,p^{(\alpha,\beta)}(x-\xi,t) - (1-\beta)\,p^{(\alpha,\beta)}(x+\xi,t)\big]\,\xi^{-1-\alpha}\,\mathrm{d}\xi\,,$$

where

$$A = 1 + \beta^2\tan\left(\frac{\alpha\pi}{2}\right).$$

These expressions involve fractional derivatives with respect to x. In particular, for the symmetric case ($\beta = 0$),

$$K^{(\alpha,0)}\big(x;\, p^{(\alpha,0)}(\,\cdot\,,t)\big) = -D^{\alpha}p^{(\alpha,\beta)}(x,t)\,,$$

where D^{α} denotes the Riesz derivative of order α with respect to x, while for a one-side process ($\alpha < 1$, $\beta = 1$), we have

$$K^{(\alpha,1)}\big(x;\, p^{(\alpha,1)}(\,\cdot\,,t)\big) = -\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]^{-1} D_{+}^{\alpha}\,p^{(\alpha,1)}(x,t)\,,$$

where D_{+}^{α} is the Marchaud fractional-derivative operator (see Ref. [25]).

2.4 Generalized Fokker – Planck – Kolmogorov equation

Let us present two other definitions of diffusion processes.

2.4.1 By a *diffusion process* $\{X(t)\}$ whose phase space is a real axis, we mean a continuous Markovian process with a transition density $p(x, t \to x+\xi, t+\tau)$ satisfying the following conditions at any $\varepsilon > 0$ [16, p. 335]: for $\tau \downarrow 0$,

$$\frac{1}{\tau}\int_{|\xi|>\varepsilon} p(x, t \to x+\xi, t+\tau)\,\mathrm{d}\xi \to 0\,,$$

$$\frac{1}{\tau}\int_{|\xi|\leqslant\varepsilon} p(x, t \to x+\xi, t+\tau)\,\xi\,\mathrm{d}\xi \to \mu(x,t)\,,$$

$$\frac{1}{\tau}\int_{|\xi|\leqslant\varepsilon} p(x, t \to x+\xi, t+\tau)\,\xi^2\,\mathrm{d}\xi \to \sigma^2(x,t)\,.$$

The coefficient $\mu(x,t)$ characterizes the systematic (average) displacement in the OX direction, which is called the drift.

2.4.2 A *diffusion process* can be defined as a Markovian process with continuous realizations, whose transition density $p \equiv p(y, u \to x, t)$ obeys either the *forward Kolmogorov equation*

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\big[\mu(x,t)\,p\big] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\big[\sigma^2(x,t)\,p\big]$$

or the *backward* (*adjoint*) one,

$$-\frac{\partial p}{\partial u} = \mu(y,u)\,\frac{\partial p}{\partial y} + \frac{1}{2}\,\sigma^2(y,u)\,\frac{\partial^2 p}{\partial y^2}$$

[the first of them is also called the *Fokker – Planck – Kolmogorov* (*FPK*) *equation*] with the initial condition $p = \delta(x-y)$, for $t \downarrow u$ and $u \uparrow t$, respectively [14, p. 170].

The process specified by the last two definitions is no longer homogeneous, since the coefficients μ and σ depend on time and coordinates. Physically, it represents diffusion in a nonuniform and nonstationary medium. Clearly, if μ and σ are sufficiently smooth functions, this process coincides locally (over short distances and time intervals) with a homogeneous diffusion process.

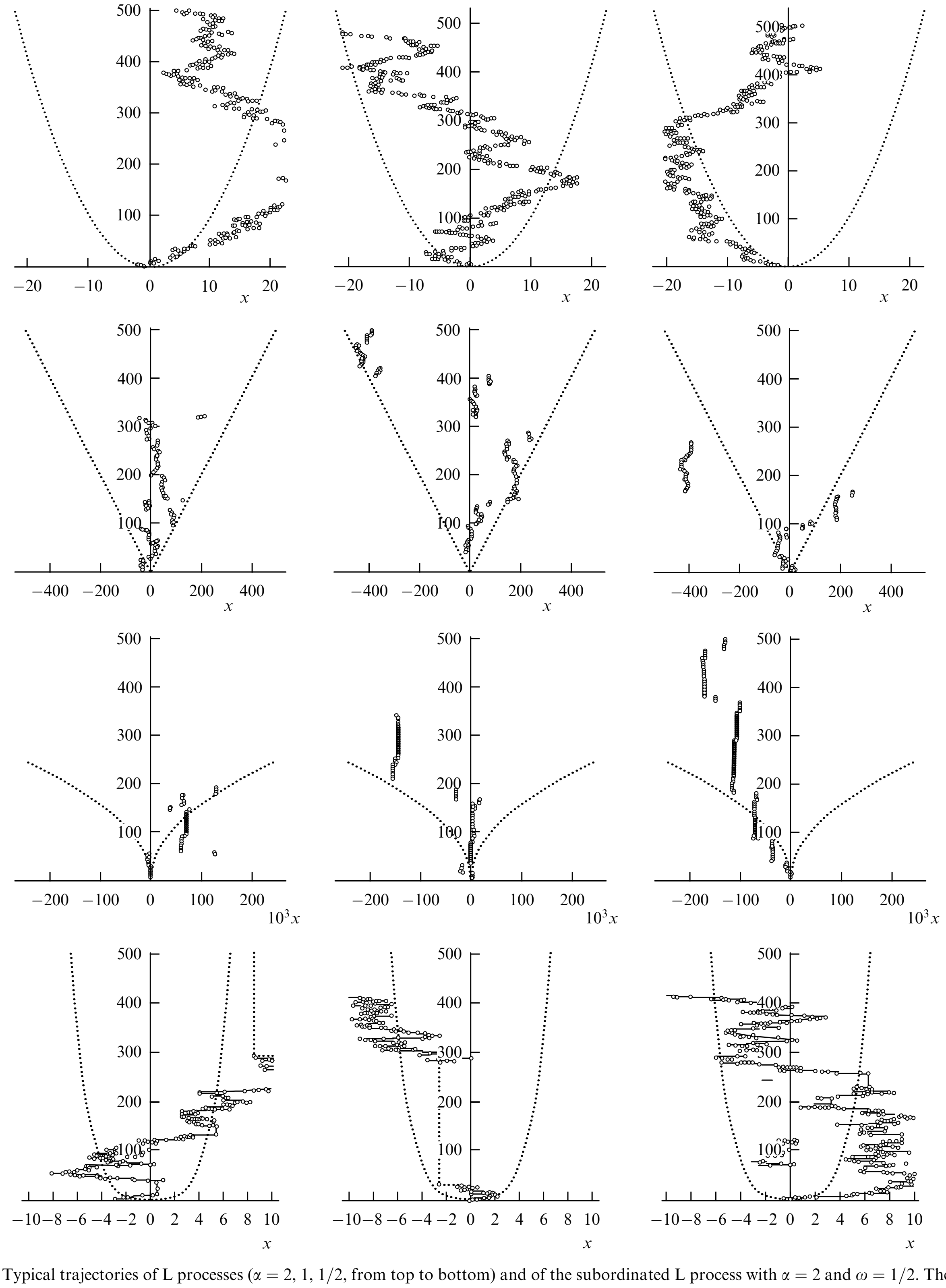

Figure 1. Typical trajectories of L processes ($\alpha = 2$, 1, 1/2, from top to bottom) and of the subordinated L process with $\alpha = 2$ and $\omega = 1/2$. The dotted curves show the variations in the width of the diffusion packet (abscissas) with time (ordinates).

2.4.3 Along with the above-presented FPK equation, another version of the forward equation exists:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x,t)\,p\right] + \frac{1}{2}\frac{\partial}{\partial x}\left[\sigma(x,t)\,\frac{\partial}{\partial x}\big(\sigma(x,t)\,p\big)\right];$$

it follows from Stratonovich's interpretation of the Langevin equation, as the FPK equation itself is Itô's interpretation of the same Langevin equation, which is a nonlinear stochastic differential equation of the form

$$\mathrm{d}X(t) = \mu\big(X(t),t\big)\,\mathrm{d}t + \sigma\big(X(t),t\big)\,\mathrm{d}W(t)\,,$$

where $W(t)$ is a standard ($\mu = 0$, $\sigma = 1$) Wiener process. Various forms of the diffusion equation are discussed by van Kampen [26].

These equations can straightforwardly be generalized on the basis of L processes: it is sufficient to replace $\mathrm{d}W(t)$ with $\mathrm{d}L(t)$, the increment of the L process. Then the FPK equation assumes the form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\big(\mu(x,t)\,p(x,t)\big) + DK\big(x;\,p(\,\cdot\,,t)\big)\,.$$

This equation can be solved by analogy with the solution procedure for the normal FPK equation (see, e.g., Ref. [27]). We will solve it here for the case of $\mu(x,t) = -\nu x$, which admits two different physical interpretations. One of them associates the above equation with a harmonic oscillator acted upon by a series of random pulses, and the other refers to the variation in the velocity of a particle affected by the momenta of 'heavy molecules' in a viscous medium. The characteristic function of this distribution (at $D = 1$ and $\nu = 1$) satisfies the equation (see Section 2.3.2)

$$\frac{\partial \tilde p}{\partial t} + k\,\frac{\partial \tilde p}{\partial k} = -|k|^{\alpha}\left[1 - \mathrm{i}\beta\tan\left(\frac{\alpha\pi}{2}\right)\,\mathrm{sign}\,k\right]\tilde p^{\,(\alpha,\beta)}(k,t)\,.$$

It has a solution in the form

$$\tilde p(k,t) = \exp\left\{-|k|^{\alpha}\big[1-\exp(-\alpha t)\big]\,\frac{M}{\alpha} - \mathrm{i}k\exp(-t)x_0\right\},$$

where x_0 is the initial coordinate of the process, and

$$M = 1 - \mathrm{i}\beta\tan\left(\frac{\alpha\pi}{2}\right)\,\mathrm{sign}\,k\,.$$

The inverse Fourier transform yields

$$p(x,t) = \frac{\alpha^{1/\alpha}}{\big(1-\exp(-\alpha t)\big)^{1/\alpha}}\,g^{(\alpha,\beta)}\left(\frac{\big(x - x_0\exp(-t)\big)\,\alpha^{1/\alpha}}{\big(1-\exp(-\alpha t)\big)^{1/\alpha}}\right).$$

For $t \to \infty$, we obtain a stationary solution

$$p(x,\infty) = \alpha^{1/\alpha} g^{(\alpha,\beta)}(x\alpha^{1/\alpha})\,.$$

Chechkin et al. [28] have obtained this solution directly from the stationary ($\partial\tilde p/\partial t = 0$) equation. The above result is more general and also demonstrates the relaxation of the system toward a uniform distribution, which is now a stable law (with the parameters α and β). At $\alpha = 2$, however, we obtain the well-known result for the Ornstein – Uhlenbeck process.

3. Lévy-stable laws

3.1 Stable distributions as limiting distributions

The discovery of the class of stable distributions by Paul Lévy was likely among the most significant achievements of the 20th century in probability theory. It lifted the limitation imposed on the CLT by the finite-variance requirement and opened up the possibility of summing random variables with infinite variances. It turned out that, if a nondegenerate (not concentrated at one point) limiting distribution of a normalized sum $(\sum_{j=1}^{n} X_j - A_n)/B_n$ exists for $n \to \infty$ and properly chosen sequences A_n and $B_n > 0$, this distribution will necessarily be stable. In this case, $B_n = h(n)n^{1/\alpha}$, where $h(n)$ is a slowly varying function (such as a logarithm or some power of it), and α is the characteristic exponent.

3.1.1 If such A_n and B_n can be found for a random variable (r.v.) X whose distribution function (d.f.) is $F(x)$, the r.v. X [or its d.f. $F(x)$] is said *to belong to the domain of attraction of a stable law*. Otherwise, it does not belong to the attraction domain of any law, since only stable laws possess attraction domains.

To verify whether or not some particular r.v. X belongs to an attraction domain, one first of all has to calculate the variance or simply the second moment $\langle X^2\rangle$. If it is finite, the r.v. X is within the domain of attraction of the normal law. If $\langle X^2\rangle = \infty$, the asymptotics of the distribution tails of X must be checked. If one finds that

$$\mathsf{P}\big(|X| > x\big) \sim h(x)x^{-\alpha}\,, \qquad x \to \infty\,,$$

the answer is positive; if, however, the asymptotics has a different form, the answer is negative. We restrict ourselves here to the case of 'normal' (not to be confused with Gaussian!) attraction, with $h(x) \to \mathrm{const} > 0$ for $x \to \infty$.

Assume now that X is known to belong to the domain of attraction of a stable law. How can the parameters α and $\beta(\theta)$ of this law be determined? The answer is provided by the generalized limit theorem.

3.1.2 Generalized limit theorem (GLT). *Let the random variables X_j be independent, have the same distribution, and obey the conditions*

$$\mathsf{P}(X > x) \sim a_+ x^{-\alpha}\,, \qquad x \to \infty\,,$$
$$\mathsf{P}(X < -x) \sim a_- x^{-\alpha}\,, \qquad x \to \infty\,,$$

$0 < \alpha \leqslant 2$, $a_+ \geqslant 0$, $a_- \geqslant 0$, *and* $a_+ + a_- > 0$. *Then such sequences A_n and $B_n > 0$ can be found that, for $n \to \infty$,*

$$\frac{\sum_{j=1}^{n} X_j - A_n}{B_n} \overset{\mathrm{d}}{\sim} S^{(\alpha,\beta)}\,,$$

where $\beta = (a_+ - a_-)/(a_+ + a_-)$.

3.1.3 Undoubtedly, there exists an infinite set of sequences of normalizing coefficients A_n and B_n that exhibit the same asymptotic behavior for $n \to \infty$. In particular, they can be defined as follows (here, $a = \langle X\rangle$ and $c = a_+ + a_-$):

for $\alpha = 2$, $\quad A_n = na$ and $B_n = \sqrt{cn\ln n}$;

for $\alpha \in (1,2)$, $\quad A_n = na$ and $B_n = \dfrac{(\pi cn)^{1/\alpha}}{\big[2\Gamma(\alpha)\sin(\alpha\pi/2)\big]^{1/\alpha}}$;

for $\alpha = 1$, $\quad A_n = \beta cn\ln n$ and $B_n = \dfrac{\pi cn}{2}$; and

for $\alpha \in (0,1)$, $\quad A_n = 0$ and $B_n = \dfrac{(\pi cn)^{1/\alpha}}{\big[2\Gamma(\alpha)\sin(\alpha\pi/2)\big]^{1/\alpha}}$.

3.2 Properties of Lévy-stable distributions

Let us list the most important properties of stable distributions (in form C).

3.2.1 All stable densities are unimodal.

3.2.2 The variances of all stable distributions other than the Gaussian distribution are infinite.

3.2.3 Mean values of stable distributions with exponents $\alpha \leqslant 1$ do not exist.

3.2.4 The stable densities satisfy the inversion relationship

$$g(x;\alpha,\theta) = g(-x;\alpha,-\theta)\,.$$

3.2.5 The stable densities satisfy the duality relationship: for $\alpha \geqslant 1$,

$$g(x;\alpha,\theta) = x^{-1-\alpha} g(x^{-\alpha};\alpha',\theta')\,,$$

where $\alpha' = 1/\alpha$, $\theta' = \alpha(1+\theta) - 1$.

3.2.6 At the origin, the distribution function

$$G(x;\alpha,\theta) = \int_{-\infty}^{x} g(x';\alpha,\theta)\,\mathrm{d}x',$$

the distribution density, and its derivative have the following values:

$$G(0;\alpha,\theta) = \frac{1-\theta}{2}\,,$$

$$g(0;\alpha,\theta) = \pi^{-1}\Gamma\left(1+\frac{1}{\alpha}\right)\cos\left(\frac{\theta\pi}{2}\right),$$

$$g'(0;\alpha,\theta) = (2\pi)^{-1}\Gamma\left(1+\frac{2}{\alpha}\right)\sin(\theta\pi)\,.$$

3.2.7 If the distribution density $g(x;\alpha,\theta)$ is not an extreme density (i.e., $\theta \neq \pm\theta_\alpha$), both its tails decline according to the law $|x|^{-\alpha-1}$ ('heavy tails'):

$$g(\pm|x|;\alpha,\theta) \sim \frac{\Gamma(1+\alpha)}{\pi}\sin\left[\frac{\alpha(1\pm\theta)\pi}{2}\right]|x|^{-1-\alpha}, \qquad x\to\infty\,.$$

3.2.8 For extreme densities, i.e., for

$$\theta = \pm 1\,, \qquad \alpha < 1\,,$$

$$\theta = \pm\left(\frac{2}{\alpha}-1\right), \qquad \alpha > 1\,,$$

one of the above-presented formulas loses its meaning, since the corresponding tail becomes 'short', with an exponentially declining asymptotics.

3.2.9 As follows from the formula for $G(0;\alpha,\theta)$, most probability is concentrated on the positive semiaxis if the asymmetry coefficient θ is positive and on the negative semiaxis if θ is negative. For $\alpha \leqslant 1$ $G(0;\alpha,1) = 0$ and $G(0;\alpha,-1) = 1$, i.e., the extreme distributions become *one-sided*, concentrated exclusively on the positive or negative semiaxis. For $\alpha \to 1$,

$$G(x;\alpha,1) \to G(x;1,1) = \begin{cases} 0\,, & x<1\,,\\ 1\,, & x>1\,,\end{cases}$$

and the one-sided densities become degenerate:

$$g(x;1,\pm 1) = \delta(x\mp 1)\,.$$

3.2.10 The following stable densities can be represented in terms of elementary functions:
the Gaussian distribution

$$g(x;2,0) = \frac{1}{2\sqrt{\pi}}\exp\left\{-\frac{x^2}{4}\right\},$$

the Cauchy distribution

$$g(x;1,0) = \frac{1}{\pi(1+x^2)}\,,$$

and the Lévy – Smirnov distribution

$$g\left(x;\frac{1}{2},1\right) = \frac{1}{2\sqrt{\pi x^3}}\exp\left\{-(4x)^{-1}\right\}.$$

The distributions representable in terms of special functions are listed in Section 8.1, and some of them are plotted in Fig. 2.

3.2.11 The Mellin transform of the density on the positive semiaxis is

$$\bar g(s;\alpha,\theta) \equiv \int_0^\infty g(x;\alpha,\theta)x^s\,\mathrm{d}x = \frac{\Gamma(s)\Gamma(1-s/\alpha)}{\Gamma(\rho s)\Gamma(1-\rho s)}\,,$$

$$\rho = \frac{1+\theta}{2}\,.$$

3.2.12 The Laplace transform of one-sided distributions is

$$\hat g(\lambda;\alpha,1) \equiv \int_0^\infty g(x;\alpha,1)\exp(-\lambda x)\,\mathrm{d}x = \exp(-\lambda^\alpha)\,, \quad \alpha\leqslant 1\,.$$

3.2.13 The characteristic exponent α is the same in both forms A and C, the parameters of asymmetry β and θ are linked by the formula

$$\beta = \frac{\tan(\theta\alpha\pi/2)}{\tan(\alpha\pi/2)}\,,$$

and the random variables themselves are linked by the relationship

$$S(\alpha,\theta) \stackrel{\mathrm{d}}{=} \left[\cos\left(\frac{\theta\alpha\pi}{2}\right)\right]^{1/\alpha} S^{(\alpha,\beta)}\,.$$

3.2.14 The convenience of form C is that, among other things, for $\theta = 1$ and $\alpha\uparrow 1$,

$$g(x;\alpha,1) \to \delta(x-1)\,,$$

as can be seen from Section 3.2.12. Under these conditions, the probability distribution in form A goes to infinity.

We arrived at stable distributions upon having considered L processes and thereafter sequences of normalized sums of independent random variables; however, direct definitions can also be given to the stable distributions (we consider here only strictly stable ones) and to the random variables determined by them. One such definition reads as follows.

3.2.15 A random variable S is called *stable with an exponent* $\alpha\in(0,2]$ if, for any n,

$$\sum_{i=1}^{n} S_j \stackrel{\mathrm{d}}{=} n^{1/\alpha}S\,,$$

where S_j are random variables, independent of S and of one another, that have the same α-stable distribution.

This definition can easily be extended to random vectors.

3.2.16 The random vector $\mathbf{S} = (S_1,\ldots,S_d)$ is called *stable* if, for any n,

$$\sum_{j=1}^{n} \mathbf{S}_j \stackrel{\mathrm{d}}{=} n^{1/\alpha}S\,.$$

The characteristic function of a d-dimensional stable vector has the form

$$\tilde g_d^{(\alpha,\Gamma)}(\mathbf{k}) = \exp\left\{-\int_{S_d}|\mathbf{ks}|^\alpha\left[1-\mathrm{i}\,\mathrm{sign}(\mathbf{ks})\tan\left(\frac{\alpha\pi}{2}\right)\right]\Gamma(\mathrm{d}\mathbf{s})\right\},$$

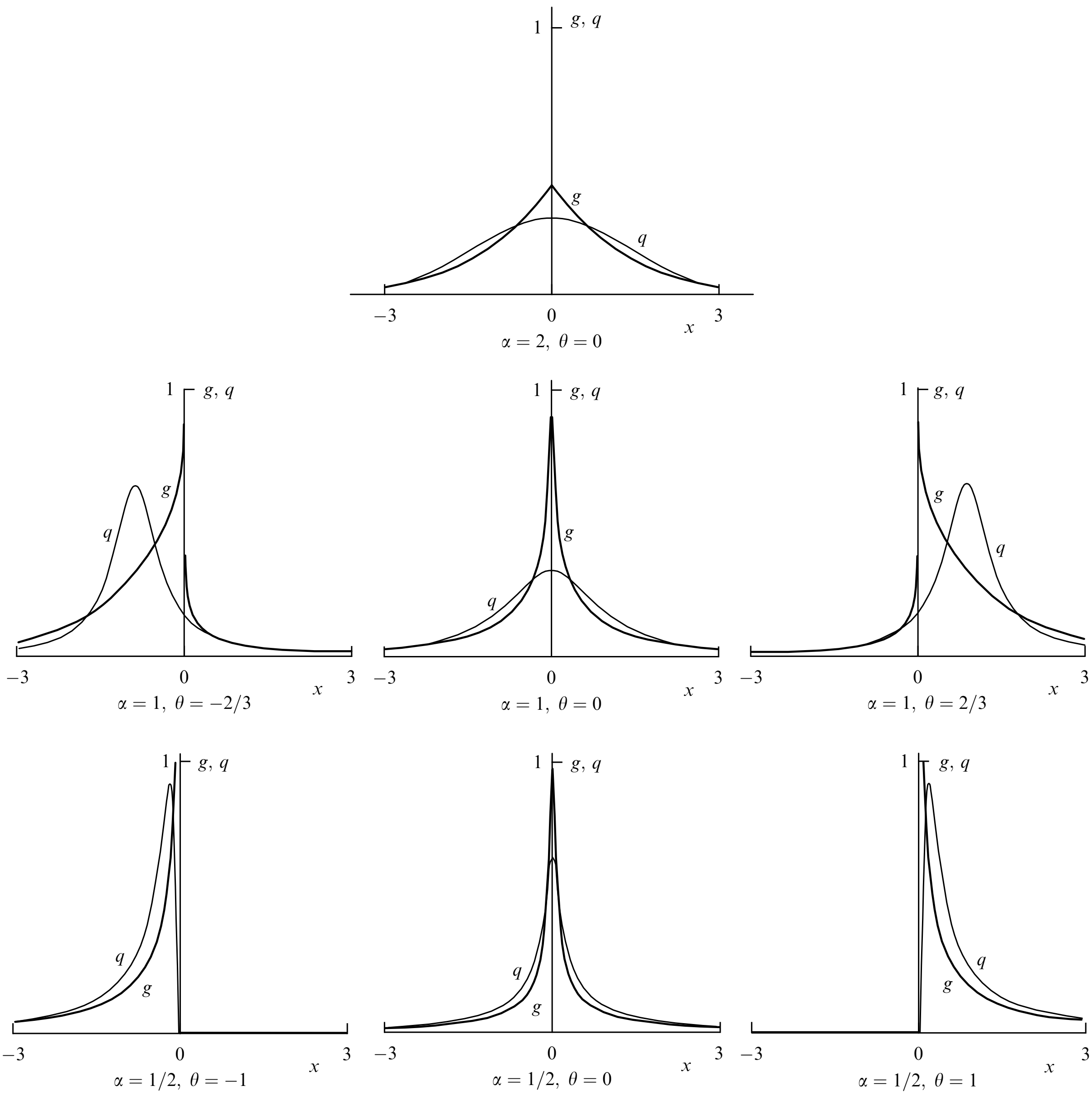

Figure 2. Densities of stable distributions (smoothly peaked light curves) and of fractionally stable distributions with the exponent $\omega = 1/2$ (sharply peaked heavy curves).

where $\Gamma(\mathrm{d}\mathbf{s})$ is a finite measure on the unit-radius sphere S_d in the considered d-dimensional space, which is termed the *spectral measure*. The density

$$g_d^{(\alpha,\Gamma)}(\mathbf{x}) = g_d^{(\alpha,\Gamma)}(x_1,\ldots,x_d) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty}\mathrm{d}k_1\ldots\int_{-\infty}^{\infty}\mathrm{d}k_d \exp\left[-\mathrm{i}(k_1x_1+\ldots+k_dx_d)\right] \times \tilde{g}^{(\alpha,\Gamma)}(k_1,\ldots,k_d)$$

is called the *multidimensional stable density*. If the spectral measure is uniformly distributed over the sphere, this density is isotropic. If the spectral measure is concentrated at the points of intersection of the axes and the sphere, the components $S_1,\ldots,S_d$ of the stable vector $\mathbf{S}$ are mutually independent.

Two important points should be noted here. First, the isotropy of the distribution entails the independence of the components if $\alpha = 2$. At $\alpha < 2$, the components of an isotropically distributed vector are not independent, and a distribution with independent components is not isotropic. Second, in view of the divergence of the variance, the standard correlation technique of analyzing the statistical relationships between the components is not applicable at $\alpha < 2$ [22].

3.3 Fractional stable distributions

We now consider the distributions that will be needed below in the context of solving equations with fractional time derivatives (Section 4.3.).

3.3.1 Let $S(\alpha_1,\theta_1)$ and $S(\alpha_2,\theta_2)$ be mutually independent, strictly stable random variables represented in form C. The random variable

$$Y(\alpha_1,\alpha_2,\theta_1,\theta_2,\mu) \stackrel{\mathrm{d}}{=} \frac{S(\alpha_1,\theta_1)}{\left[S(\alpha_2,\theta_2)\right]^{\langle\mu\rangle}},$$

where

$$S^{\langle\mu\rangle} = |S|^\mu \operatorname{sign} S\,, \qquad -\infty < \mu < \infty\,,$$

and the corresponding density

$$p_Y(x;\alpha_1,\alpha_2,\theta_1,\theta_2,\mu) = \int_{-\infty}^{\infty} g\big(xy^{\langle\mu\rangle};\alpha_1,\theta_1\big)\, g(y;\alpha_2,\theta_2)|\,y|^\mu\, \mathrm{d}y$$

will hereinafter be called *fractional stable*.

Fractional stable random variables are not rare in the literature. In particular, the random variable

$$Y\left(2,\alpha,0,1,-\frac{1}{2}\right) = S(2,0)\sqrt{S(\alpha,1)}\,, \qquad \alpha<1\,,$$

is called *sub-Gaussian* in Ref. [22]; in a more general case,

$$Y\left(\alpha_1,\alpha_2,0,1,-\frac{1}{\alpha_1}\right) = S(\alpha_1,0)\big[S(\alpha_2,1)\big]^{1/\alpha_1},$$

$$0<\alpha_1\leqslant 2\,, \qquad \alpha_2<1\,,$$

is referred to as a *substable* random variable. The following relationships are known:

$$Y(2,2,0,0,1) \stackrel{\mathrm{d}}{=} S(1,0)\,,$$

$$Y\left(\alpha_1,\frac{\alpha}{\alpha_1},0,1,-\frac{1}{\alpha_1}\right) \stackrel{\mathrm{d}}{=} S(\alpha,0)\,,$$

$$Y(1,1,1,0,1) \stackrel{\mathrm{d}}{=} S(1,0)\,.$$

It is understood that

$$Y(\alpha,1,\theta,1,\mu) \stackrel{\mathrm{d}}{=} S(\alpha,0)\,.$$

The distribution of the random variable

$$Z(\alpha,\omega,\theta) = \frac{S(\alpha,\theta)}{\big[S(\omega,1)\big]^{\omega/\alpha}}\,, \qquad \omega<1\,,$$

obtained by Kotulski [29] by asymptotically solving the Montroll – Weiss problem (see Section 4.3)

$$q(x;\alpha,\omega,\theta) \equiv p_Y\left(x;\alpha,\omega,\theta,1,\frac{\omega}{\alpha}\right) = \int_0^\infty g(xy^{\omega/\alpha};\alpha,\theta)\, g(y;\omega,1)\, y^{\omega/\alpha}\,\mathrm{d}y$$

also belongs to the class of fractional stable distributions.

We introduced the term 'fractional stable distributions' in Ref. [30], and their properties were studied in Refs [31 – 33]. Let us note the most important ones.

3.3.2 If $0<\alpha<1$ and $\theta=1$, the density $q(x;\alpha,\omega,1)$ is not zero only on the positive semiaxis. Otherwise, it differs from zero on the entire real axis.

3.3.3 The following inversion property takes place:

$$q(-x;\alpha,\omega,\theta) = q(x;\alpha,\omega,-\theta)\,.$$

If $\theta=0$,

$$q(-x;\alpha,\omega,0) = q(x;\alpha,\omega,0)\,,$$

i.e., a fractional stable distribution with $\theta=0$ is symmetric with respect to the origin.

3.3.4 According to the inversion property, it is sufficient to consider the class of fractional stable densities for all α, ω, and θ only on the positive semiaxis. This allows the introduction of the one-sided Mellin transform

$$\bar{q}(s;\alpha,\omega,\theta) = \int_0^\infty x^s q(x;\alpha,\omega,\theta)\,\mathrm{d}x\,, \qquad -1<\operatorname{Re}s<\alpha\,,$$

which, in view of specific features of fractional stable random variables, is a much more convenient tool for analysis than the traditional characteristic function. We employ this transform and use the expression for the Mellin transform of the stable density,

$$\bar{g}(s;\alpha,\theta) = \rho\,\frac{\Gamma(1+s)\,\Gamma(1-s/\alpha)}{\Gamma(1+\rho s)\,\Gamma(1-\rho s)}\,, \qquad \rho=\frac{1+\theta}{2}\,,$$

to obtain

$$\bar{q}(s;\alpha,\omega,\theta) = \rho\,\frac{\Gamma(1+s)\,\Gamma(1-s/\alpha)\,\Gamma(1+s/\alpha)}{\Gamma(1+\rho s)\,\Gamma(1-\rho s)\,\Gamma(1+\omega s/\alpha)}\,.$$

3.3.5 A fractional stable distribution has moments of all orders only if $\alpha=2$. In this case, the region of θ values shrinks to a single value, $\theta=0$, the distribution becomes symmetric, odd moments vanish, and even moments are

$$m^{(2n)}(2,\omega,0) \equiv \int_{-\infty}^\infty x^{2n} q(x;2,\omega,0)\,\mathrm{d}x = \frac{4^n n!\,\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n\omega+1)}\,.$$

At $1<\alpha<2$, the second and higher moments are infinite; at $\alpha\leqslant 1$, even the mean value does not exist.

3.3.6 At the origin, we have

$$q(0;\alpha,\omega,\theta) = \frac{\Gamma(1+1/\alpha)\,\Gamma(1-1/\alpha)}{\pi\Gamma(1-\omega/\alpha)}\cos\left(\frac{\theta\pi}{2}\right)$$

and

$$Q(0;\alpha,\omega,\theta) \equiv \int_{-\infty}^0 q(x;\alpha,\omega,\theta)\,\mathrm{d}x = \frac{1-\theta}{2}\,.$$

We note that $q(x;\alpha,\omega,\theta)$ has an integrable singularity at the origin if $\alpha\leqslant 1$ and $\omega<1$.

3.3.7 If $\omega\to 1$, the densities $q(x;\alpha,\omega,\theta)$ become stable densities $g(x;\alpha,\theta)$.

3.3.8 If $\omega<1$ and $\theta>0$, then

$$q(x;\alpha,\omega,\theta) \sim \frac{g(x;\alpha,\theta)}{\Gamma(1+\omega)}\,, \qquad x\to\infty\,,$$

i.e., the tails of the fractional stable densities are as heavy (in terms of their power-law behavior) as those of the stable densities.

3.3.9 At $\alpha=2$, a fractional stable density can be represented in terms of a one-sided stable density via the relationship

$$q(x;2,\omega,0) = \frac{1}{\omega|x|^{1+2/\omega}}\, g\left(|x|^{-2/\omega};\frac{\omega}{2},1\right).$$

Fractional stable densities representable in terms of elementary or widely known special functions are listed in Section 8.1. Some of them are plotted in Fig. 2.

3.3.10 In Ref. [33], the characteristic functions of fractional stable distributions were derived,

$$\tilde{q}(k;\alpha,\omega,\theta) = E_\omega\big(-\psi(k;\alpha,\theta)\big)\,,$$

where

$$E_\omega(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+n\omega)}$$

is the Mittag–Leffler function, and

$$\psi(k;\alpha,\theta) = -|k|^\alpha \exp\left\{-\mathrm{i}\alpha\theta\left(\frac{\pi}{2}\right)\operatorname{sign} k\right\}.$$

In the same study, inverse-power series for fractional stable densities

$$q(x;\alpha,\omega,\theta) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{\Gamma(n\rho)\,\Gamma(1-n\rho)\,\Gamma(1+n\omega)}\,x^{-n\alpha-1}$$

were obtained.

3.3.11 One-dimensional fractional stable densities can in a natural way be extended to a multidimensional case [34],

$$q_d(\mathbf{x};\alpha,\omega,\Gamma) = \int_0^\infty g_d\left(\mathbf{x}y^{\omega/\alpha};\alpha,\Gamma\right) g_1(y;\omega,1)\,y^{d\omega/\alpha}\,\mathrm{d}y\,,$$

to describe the distribution of the random vector

$$\mathbf{Z}_d(\alpha,\omega,\Gamma) \equiv \frac{\mathbf{S}_d(\alpha,\Gamma)}{\left[S_1(\omega,1)\right]^{\omega/\alpha}}\,.$$

4. Anomalous diffusion as an asymptotics of jump processes

4.1 Renewal processes

The above scheme of the anomalous-diffusion mechanism was historically developed in a different way, using an asymptotic analysis of jump processes. The groundwork for this approach was laid by Montroll and Weiss [35], and none of the review articles on anomalous diffusion has avoided making a reference to their study (see also an excellent review by Montroll and Schlesinger [36]). We note here the main milestones on this avenue using the terminology of the renewal theory [37].

Let $T_1, T_2, \ldots$ be a sequence of mutually independent positive random variables with a common density of distribution $q(t)$. We will call them the *waiting times*, and the instants of time

$$T(n) = \sum_{j=1}^{n} T_j\,, \qquad T(0) = 0\,,$$

will be referred to as the *times of renewal* (*regeneration*).

4.1.1 The sequence of random variables $\{T(n)\}$ forms a *renewal process*.

Since the random variables T_j are positive, their mean value

$$\tau \equiv \langle T_j\rangle = \int_0^\infty tq(t)\,\mathrm{d}t$$

is also meaningful even if the integral diverges; in this case, $\tau = \infty$.

In physical processes, some zero-duration physical events are usually related to the instants $T(n)$, such as jumpwise transitions from one state to another, collisions between a particle and atoms of the medium, emission or absorption of a photon, etc. We will simply call them *events*. Let us consider the number $N(t)$ of events occurring during the interval of time $(0, t)$.

4.1.2 The random process

$$N(t) = \max\left\{n : T(n) < t\right\}$$

is called a *counting renewal process*, and the function

$$\bar{N}(t) = \left\langle N(t)\right\rangle$$

is called the *renewal function*. The latter can be represented in the form

$$\bar{N}(t) = \sum_{n>0}\mathsf{P}\big(T(n) < t\big) = \sum_{n>0}\int_0^t q^{*n}(s)\,\mathrm{d}s$$

[here, $p^{*n}(x)$ denotes the multiple convolution of the densities $p(x)$, with $p^{*1}(x) \equiv p(x)$ and $p^{*0}(x) = \delta(x)$]. It decreases, being finite, nonnegative, and semi-additive:

$$\bar{N}(t+s) \leqslant \bar{N}(t) + \bar{N}(s)\,, \qquad t, s \geqslant 0\,,$$

and satisfies the *equation of renewal*

$$\bar{N}(t) = \int_0^t \left[1 + \bar{N}(t-s)\right] q(s)\,\mathrm{d}s\,.$$

This equation has an extremely simple interpretation: the mean number of events within the interval $(0, t)$ is equal to the probability of the first event falling in this interval plus the mean number of subsequent events.

4.1.3 The renewal function also satisfies the *elementary renewal theorem*

$$\lim_{t\to\infty}\frac{\bar{N}(t)}{t} = \tau^{-1}\,,$$

and under the assumption that the second moment is finite,

$$m^{(2)} = \int_0^\infty t^2 q(t)\,\mathrm{d}t < \infty\,,$$

the following inequalities are valid:

$$\frac{t}{\tau} \leqslant \bar{N}(t) \leqslant \frac{t}{\tau} + \frac{m^{(2)}}{2\tau^2}\,.$$

Choosing $q(t)$ in the form of an exponential distribution,

$$q(t) = \mu\exp\left(-\mu t\right),$$

yields the simplest renewal process, the *Poisson process*.

4.1.4 *The generalized Poisson process* was introduced by Feller [17, Ch. VI, § 4] using the random sums

$$X(t) = \sum_{j=1}^{N(t)} R_j\,,$$

where R_j are independent random variables with a common distribution $p(x)$, and $N(t)$ is a normal Poisson process,

$$\mathsf{P}\big(N(t)=n\big)=\frac{(\mu t)^n}{n!}\exp(-\mu t)\,.$$

The distribution density $p(x,t)$ of the random sum $X(t)$ has the form

$$p(x,t)=\sum_{n=0}^{\infty}\frac{(\mu t)^n}{n!}\exp(-\mu t)\,p^{*n}(x)\,,$$

and the process itself can be interpreted as the jumpwise random walk of a particle along the x axis. The fact that the summation starts from $n=0$ indicates that the particle is initially at the origin and remains there until the time $t=T_1$ when it makes its first jump. After this jump, the ability of the particle to wait is restored to its initial form, and the process continues.

Feller also modified this process, passing from the process $\{X_1=R_1,\ X_2=X_1+R_2,\ X_3=X_2+R_3,\ \ldots\}$ with independent increments R_j to the Markovian chain $\{X_1,X_2,X_3,\ldots\}$; he termed the resulting process *pseudo-Poissonian*.

4.2 Subordinated processes

The distribution density of a pseudo-Poissonian process is

$$p(x,t)=\sum_{n=0}^{\infty}W_n^0(t)\,\pi(x,n)=\big\langle\pi\big(x,N(t)\big)\big\rangle\,,$$

where

$$W_n^0(t)=\frac{(\mu t)^n}{n!}\exp(-\mu t)$$

and

$$\pi(x,n)=p^{*n}(x)\,.$$

The last equality can be interpreted as a representation of the distribution density of a process with integer time n, called *operational time*. The random variable $N(t)$ is referred to by Feller as *randomized operational time*. In the general treatment, operational time T is not necessarily Poisson distributed and not even necessarily discrete.

4.2.1 *Let $\pi(x,t)$ be the transition density of the Markovian process $\{X(t)\}$, and let $w(\theta,t)\,\mathrm{d}\theta$ be the distribution of randomized time $T(t)$ localized on the positive semiaxis. Then the density $p(x,t)$ for the process $\{X(T(t))\}$ assumes the form*

$$p(x,t)=\int_0^{\infty}\pi(x,\theta)\,w(\theta,t)\,\mathrm{d}\theta\,.$$

We will call this process *subordinated in a broad sense*, to distinguish it from Feller's definition [17, Ch. X, § 7].

4.2.2 *Let $\{X(t)\}$ be a Markovian process with continuous transition probabilities and let $\{T(t)\}$ be a process with nonnegative independent increments. Then $\{X(T(t))\}$ is also a Markovian process.*

We will say that this process *is subordinated to $\{X(t)\}$ with the use of the operational time $\{T(t)\}$*. The process $\{T(t)\}$ is called the *directing (governing) process*.

If the process $\{X(t)\}$ also has independent increments, we arrive again at the above-presented formula for $p(x,t)$. In particular, if $\{X(t)\}$ is Brownian motion with the transition density

$$\pi(x,t)=(2\pi t)^{-1/2}\exp\left(-\frac{x^2}{2t}\right),$$

and the directing process is a Smirnov – Lévy process

$$w(\theta,t)=\frac{t}{\sqrt{2\pi\theta^3}}\exp\left(-\frac{t^2}{2\theta}\right),$$

the subordinated process has the density

$$p(x,t)=\frac{t}{2\pi}\int_{-\infty}^{\infty}\theta^{-2}\exp\left(-\frac{x^2+t^2}{2\theta}\right)\mathrm{d}\theta=\frac{t}{\pi(t^2+x^2)}\,,$$

i.e., it is a Cauchy process. This example can also be extended to a more general case: if $\{X(t)\}$ is a stable process (Lévy motion) with an exponent α_1 and $\{T(t)\}$ is a one-sided stable process with an exponent $\alpha_2<1$ ($\beta=1$), the subordinated process is also a stable process (Lévy motion) with the exponent $\alpha_1\alpha_2$.

All these considerations refer, however, to the case where $\{T(t)\}$ is a process with independent increments, as required by Feller's definition. In contrast, we assume $\{T(t)\}$ to be a counting renewal process with a transition density $q(t)$, which is in no way a process with independent increments (obviously, unless $q(t)$ is an exponential function). It is this point that warrants our departure from Feller's definition.

Thus, if we assume R_j to be independent and $q(t)$ to be arbitrary in our scheme, we obtain

$$p(x,t)=\sum_{n=0}^{\infty}W_n(t)\,p^{*n}(x)\,,$$

where

$$W_n(t)=\mathsf{P}\big(N(t)=n\big)=\mathsf{P}\big(N(t)\geqslant n\big)-\mathsf{P}\big(N(t)\geqslant n+1\big)$$
$$=Q^{*n}(t)-Q^{*(n+1)}(t)\,.$$

Therefore,

$$p(x,t)=\sum_{n=0}^{\infty}\big[Q^{*n}(t)-Q^{*(n+1)}(t)\big]\,p^{*n}(x)\,.$$

This is a very simple model of one-dimensional jumpwise random walk of a particle with independent waiting times continuously distributed according to the same law $q(t)$, and with displacements independent of one another and of the waiting times and distributed with the same density $p(x)$. In the English literature, this process is denoted as CTWR (Continuous Time Random Walks).

4.3 Anomalous-diffusion theorem

The calculation of the distribution $p(x,t)$ under the above-specified assumptions is called the *Montroll – Weiss problem* [35]. It can conveniently be solved using the Fourier transform over the coordinate and the Laplace transform over time:

$$\tilde{p}(k,\lambda)=\int_0^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\,\exp(\mathrm{i}kx-\lambda t)\,p(x,t)\,.$$

Taking into account that

$$\int_{-\infty}^{\infty}\exp(\mathrm{i}kx)\,p^{*n}(x)\,\mathrm{d}x=\big[\tilde{p}(k)\big]^n$$

and

$$\int_0^\infty \exp(-\lambda t)\, Q^{*n}(t)\, \mathrm{d}t = \int_0^\infty \mathrm{d}t \exp(-\lambda t) \int_0^t \mathrm{d}s\, q^{*n}(s)$$

$$= \int_0^\infty \mathrm{d}s \int_s^\infty \mathrm{d}t \exp(-\lambda t)\, q^{*n}(s)$$

$$= \frac{1}{\lambda}\int_0^\infty \exp(-\lambda s) q^{*n}(s)\, \mathrm{d}s = \frac{1}{\lambda}\left[\tilde{q}(\lambda)\right]^n ,$$

we obtain

$$\tilde{p}(k,\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left[\tilde{q}^{\,n}(\lambda) - \tilde{q}^{\,n+1}(\lambda)\right]\tilde{p}^{\,n}(k) = \frac{1-\tilde{q}(\lambda)}{\lambda\left[1-\tilde{q}(\lambda)\,\tilde{p}(k)\right]} .$$

Therefore,

$$p(x,t) = \frac{1}{(2\pi)^2 \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}k \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}\lambda\, \frac{1-\tilde{q}(\lambda)}{\lambda\left[1-\tilde{q}(\lambda)\,\tilde{p}(k)\right]} \times \exp(-\mathrm{i}kx + \lambda t) .$$

Direct calculations according to this formula require specifying particular distributions $q(t)$ and $p(x)$; if, however, we restrict ourselves to an asymptotic analysis for $t \to \infty$, then knowing only the asymptotics of $q(t)$ and $p(x)$ for large arguments will be sufficient. In a one-dimensional formulation, this problem was solved by Kotulski [29]. We present here his result in the form of a theorem formulated in Ref. [30].

Let $T_1, T_2, \ldots$ be a sequence of independent random variables with a common distribution function, which belongs to the domain of attraction of a stable law with the parameters $\omega, \theta = 1$, and let $R_1, R_2, \ldots$ be another, independent sequence of independent random variables with a common distribution function, which belongs to the domain of attraction of a stable law with the parameters α, θ. Then a finite positive constant $c = c(\alpha, \omega, \theta)$ exists, such that

$$\mathsf{P}\left(\frac{X(t)}{c(\alpha,\omega,\theta)\, t^{\omega/\alpha}} < x\right) \Rightarrow Q(x;\alpha,\omega,\theta) , \qquad t \to \infty ,$$

where

$$Q(x;\alpha,\omega,\theta) = \int_0^\infty G(xy^{\omega/\alpha};\alpha,\theta)\, \mathrm{d}G(y;\omega,1) .$$

We differentiate this relationship with respect to x, thus passing to the density

$$q(x;\alpha,\omega,\theta) = \int_0^\infty g(xy^{\omega/\alpha};\alpha,\theta)\, g(y;\omega,1)\, y^{\omega/\alpha}\, \mathrm{d}y .$$

Therefore, the distribution density of the one-dimensional process $\{X(t)\}$ has the following form in the long-time asymptotics:

$$p^{as}(x,t) = q\left(\frac{xt^{-\omega/\alpha}}{c} ;\, \alpha,\omega,\theta\right) \frac{t^{-\omega/\alpha}}{c} .$$

This result can be obtained in two ways: first, by treating the process $X(t)$ as the sum of a random number of random summands and by employing a generalized version of the central limit theorem, which leads to stable laws [29, 30], or, second, by performing an asymptotic inversion with the use of the Tauberian theorems, as done by Saichev and Zaslavsky [38] for the one-dimensional case and by Uchaĭkin and Zolotarev [39] for the three-dimensional symmetric case. For d-dimensional spherically symmetric walk,

$$p_d^{as}(\mathbf{x},t) = q_d\left(\frac{\mathbf{x}t^{-\omega/\alpha}}{c} ;\, \alpha,\omega\right) \frac{t^{-d\omega/\alpha}}{c^d} ,$$

where

$$q_d(\mathbf{x};\alpha,\omega) = \int_0^\infty g_d(\mathbf{x}y^{\omega/\alpha};\alpha)\, g_1(y;\omega,1)\, y^{d\omega/\alpha}\, \mathrm{d}y ,$$

$\mathbf{x} \in \mathbf{R}^d$, and $g_d(\mathbf{x},\alpha)$ is the density of the spherically symmetric, d-dimensional, stable distribution whose characteristic function has the form

$$\tilde{g}_d(\mathbf{k};\alpha) = \exp\left(-|\mathbf{k}|^\alpha\right) , \qquad \mathbf{k} \in \mathbf{R}^d .$$

4.4 Equations of anomalous diffusion

The simplest way to obtain the asymptotics of the considered process going from the transform

$$\tilde{p}(k,\lambda) = \frac{1-\tilde{q}(\lambda)}{\lambda\left[1-\tilde{q}(\lambda)\,\tilde{p}(k)\right]}$$

seems to be based on using the stable densities $g(x;\alpha,\theta)$ and $g(x;\omega,1)$ $(0 < \alpha \leqslant 2, \omega \leqslant 1)$ themselves as $p(x)$ and $q(t)$. This is admissible, since any stable law belongs to its domain of attraction, and the condition of the theorem in Section 4.3.1 is thus satisfied. At the same time, the asymptotic representation of $p(x,t)$ is determined only by the asymptotics of the functions $p(x)$ and $q(t)$, while all distributions in a fixed domain of attraction, including the stable distribution, have, within a constant factor, the same asymptotics. Therefore, we can choose stable densities as $p(x)$ and $q(t)$ without any loss in generality.

According to the Tauberian theorems, the major role in recovering the distribution $p(x,t)$ for large x and t is played by the behavior of the Fourier–Laplace transform $\tilde{p}(k,\lambda)$ in the region of small arguments, where

$$\tilde{p}(k) = \tilde{g}(k;\alpha,\theta) = \exp\left\{-|k|^\alpha \exp\left[-\frac{\mathrm{i}\alpha\theta\pi}{2}\,\mathrm{sign}\,k\right]\right\}$$

$$\sim 1 - |k|^\alpha \exp\left\{-\frac{\mathrm{i}\alpha\theta\pi}{2}\,\mathrm{sign}\,k\right\} , \qquad k \to 0 ,$$

and

$$\tilde{q}(\lambda) = \tilde{g}(\mathrm{i}\lambda;\omega,1) = \exp(-\lambda^\omega) \sim 1 - \lambda^\omega , \qquad \lambda \to 0 .$$

Simple rearrangement yields the following relationship for the principal asymptotics:

$$\lambda^\omega\, \tilde{p}^{\,\mathrm{as}}(k,\lambda) = -|k|^\alpha \exp\left\{-\frac{\mathrm{i}\alpha\theta\pi}{2}\,\mathrm{sign}\,k\right\} \tilde{p}^{\,\mathrm{as}}(k,\lambda) + \lambda^{\omega-1} .$$

Let the Fourier transforms of the functions $f(x)$ and $p(x)$ be linked by the relationship

$$\tilde{f}(k) = -|k|^\alpha \exp\left\{-\frac{\mathrm{i}\alpha\theta\pi}{2}\,\mathrm{sign}\,k\right\} \tilde{p}(k) .$$

After some elementary manipulation, we recast it into the form

$$\tilde{p}(k) = \left[(u+v)\cos\left(\frac{\alpha\pi}{2}\right) + \mathrm{i}(u-v)\sin\left(\frac{\alpha\pi}{2}\right)\operatorname{sign}k\right] \times |k|^{-\alpha}\tilde{f}(k)\,,$$

$$u = \frac{\sin\big((1+\theta)\alpha\pi/2\big)}{\sin(\alpha\pi)}\,,$$

$$v = \frac{\sin\big((1-\theta)\alpha\pi/2\big)}{\sin(\alpha\pi)}\,,$$

implying, according to Eqns (12.11)–(12.14) and (12.25) in Ref. [25], that the originals of these transforms are related by the *Feller operator* I_δ^α, $\delta = \theta\pi/2$,

$$p(x) = I_\delta^\alpha f(x) = \frac{1}{2\Gamma(\alpha)}\int_{-\infty}^{\infty}\frac{u+v+(u-v)\operatorname{sign}(x-\xi)}{|x-\xi|^{1-\alpha}}\,f(\xi)\,\mathrm{d}\xi\,,$$

which is a linear combination of fractional-order integral operators.

The inverse operator is written as

$$(I_\delta^\alpha)^{-1} = \frac{\alpha}{2A\Gamma(1-\alpha)}\int_{-\infty}^{\infty}\frac{u+v+(u-v)\operatorname{sign}(x-\xi)}{|x-\xi|^{1-\alpha}} \times \big[p(x)-p(\xi)\big]\,\mathrm{d}\xi = \frac{\alpha}{2A\Gamma(1-\alpha)}\int_0^{\infty}\big[(u+v)\,p(x) - up(x-\xi) - vp(x+\xi)\big]\,\xi^{-1-\alpha}\,\mathrm{d}\xi\,,$$

$$A = \left[(u+v)\cos\left(\frac{\alpha\pi}{2}\right)\right]^2 + \left[(u-v)\sin\left(\frac{\alpha\pi}{2}\right)\right]^2.$$

Let us return to the equation for $\tilde{p}^{as}$, which yields the following inverse Fourier transform in k:

$$\lambda^\omega\tilde{p}^{\,\mathrm{as}}(x,\lambda) = -(I_\delta^\alpha)^{-1}\tilde{p}^{\,\mathrm{as}}(x,\lambda) + \lambda^{\omega-1}\,.$$

The product $\lambda^\omega\,\tilde{p}(x,\lambda)$ and the function $\lambda^{\omega-1}$ are the Laplace transforms of the *fractional Riemann – Liouville derivative*

$$\frac{\partial^\omega p(x,t)}{\partial t^\omega} = \frac{1}{\Gamma(1-\omega)}\,\frac{\partial}{\partial t}\int_0^t\frac{p(x,\tau)\,\mathrm{d}\tau}{(t-\tau)^\omega}$$

and the *generalized function*

$$\frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(x)\,.$$

Thus, the principal asymptotic (for $t\to\infty$) part of the distribution $p(x,t)$ satisfies the fractional-order partial differential equation

$$\frac{\partial^\omega p^{as}(x,t)}{\partial t^\omega} = -(I_\delta^\alpha)^{-1}p^{as}(x,t) + \frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(x)\,,$$

which we will call the *equation of anomalous diffusion* (*AD equation*). Here, the variables x and t are scaled in a special way. Generally, the AD equation has the form

$$\frac{\partial^\omega p^{\mathrm{as}}(x,t)}{\partial t^\omega} = DK\big(x;p^{\mathrm{as}}(\,\cdot\,,t)\big) + \frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(x)\,,$$

where D is a positive constant (the diffusion coefficient), and the operator K is specified as

$$K\big(x;\,p(\,\cdot\,,t)\big) \equiv -(I_\delta^\alpha)^{-1}\,p(x,t)\,.$$

The multidimensional analogue of the AD equation in the isotropic case includes a fractional-order Laplacian:

$$\frac{\partial^\omega p^{\mathrm{as}}(\mathbf{x},t)}{\partial t^\omega} = -D(-\Delta)^{\alpha/2}p^{\mathrm{as}}(\mathbf{x},t) + \frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(\mathbf{x})\,.$$

Various aspects of employing fractional-order equations are discussed in Refs [40–42] etc.

4.5 Generalized Fick's law

Upon being recast [by taking the $(1-\omega)$th fractional derivatives of both sides] into the form

$$\frac{\partial p^{\mathrm{as}}}{\partial t} = -D(-\Delta)^{\alpha/2}\,\frac{\partial^{1-\omega}p^{\mathrm{as}}(\mathbf{x},t)}{\partial t^{1-\omega}} + \delta(\mathbf{x})\delta(t)\,,$$

the AD equation can be represented as the system of two equations,

$$\frac{\partial p^{\mathrm{as}}}{\partial t} = -\operatorname{div}\mathbf{j}\,,$$

$$\operatorname{div}\mathbf{j} = D(-\Delta)^{\alpha/2}\,\frac{\partial^{1-\omega}p^{\mathrm{as}}(\mathbf{x},t)}{\partial t^{1-\omega}}\,.$$

The first one is the continuity equation, which determines the vector of the probability-current density $\mathbf{j}$, and the second one is an implicit form of the *generalized Fick's law*. At $\alpha=2$, it can be written explicitly,

$$\mathbf{j} = -D\operatorname{grad}\frac{\partial^{1-\omega}p^{\mathrm{as}}(\mathbf{x},t)}{\partial t^{1-\omega}}\,,$$

and assumes the normal form of Fick's law at $\omega=1$. The presence of the fractional-order time derivative

$$\frac{\partial^{1-\omega}p^{\mathrm{as}}(\mathbf{x},t)}{\partial t^{1-\omega}} = \frac{1}{\Gamma(\omega)}\,\frac{\partial}{\partial t}\int_0^t\frac{p^{\mathrm{as}}(\mathbf{x},\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\omega}}$$

on the right-hand side reflects the effect of the memory of the process on the propagation of probability: the current density is determined at a given time not by the local density gradient at this time but rather by the variation of the density during the entire evolutionary period elapsed. At $\alpha=2$, it suffices to know only the variation of the density gradient at the observation point $\mathbf{x}$ (i.e., the spatial locality is still preserved), but if $\alpha<2$, the current density at point $\mathbf{x}$ at time t is determined by the evolution of $p(\mathbf{x},t)$ in the entire space (the effect of long paths called *Lévy flights*) and during the entire period of diffusion.

We note that, as $\omega\to1$, the factor $t^{-\omega}/\Gamma(1-\omega)$ becomes the Dirac delta function $\delta(t)$ [43, p. 487], and the AD equation changes into the superdiffusion equation considered in Ref. [23]. As $\alpha\to2$, the Riesz operator becomes the normal Laplacian, and the AD equation changes into the subdiffusion equation studied in Ref. [44]. If both these conditions are satisfied, we have the regular diffusion equation, with considering the one-dimensional version of which we started our review.

5. Memory, noise, and fractals

5.1 Fractional derivatives

The fate of fractional calculus resembles the fate of stable laws: this calculus still remains somewhat exotic for many physicists. It was my good fortune to meet Professor V M Zolotarev of Moscow University and Professor R Nigmatullin of Kazan University, who molded this exotic into a real tool for me. I cannot but remember here my meeting with Professor P S Landa of Moscow University, the author of an excellent monograph [45], at the International Conference on Nonlinear Dynamics and Chaos in Saratov, 1996. With her youthful curiosity (which is so rare among famous scholars), Polina Solomonovna quizzed me: the first derivative characterizes the slope of the curve at a given point; the second derivative, the convexity or concavity of the curve; but what is characterized by the derivative of order, e.g., 3/2? My explanations (probably not quite intelligible) seemed to disappoint her. Perhaps today I could give a more correct answer, but, by and large, this issue still remains debatable. A comprehensive exposition of the theory of fractional integration and differentiation can be found in a book by Samko et al. [25], which has become an encyclopedia on this subject and is widely cited by both Russian and foreign authors. I will restrict myself to only one remark.

A fractional-order derivative is a nonlocal characteristic of the function; it depends not only on the behavior of the function in the neighborhood of the considered point x but also on the values of the function over the entire interval (a, x) [or (x, b)]. Similar to a normal derivative, a fractional-order derivative can be represented as the limiting value of the ratio of increments; to this end, however, one has to extend the notion of integer-order increment to the case of a fractional order. All told, many curious details present themselves here; moreover, many variously constructed fractional derivatives are considered, so that the practical use of these techniques requires some training, without which one can get into trouble.

A fragment from Ref. [46] can be given as an example: "The definition of the fractional Hölder derivative of order v

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(\Delta x)^v} = \frac{\partial^v f}{\partial x^v} ,$$

can be used to obtain an expression for the current,

$$j(x,t) = -\lim_{a \to 0} \tilde{k}\, \frac{T(x,t) - T(x-a,t)}{a^v} = -k\, \frac{\partial^v T(x,t)}{\partial x^v} .$$

Thus, the equation of heat conduction on fractals is

$$\frac{\partial T(x,t)}{\partial t} = \tilde{k}\, \frac{\partial^{1+v} T(x,t)}{\partial x^{1+v}} ."$$

On the next page, 357, the authors write: "Secondly, the asymptotic behavior of heat transfer on fractals for long times can easily be determined:

$$T(x,t) = T_0 \exp\left[-\left(\frac{x^{1+v}}{\tilde{k}t} \right)^{1/v} \right] ."$$

It is obvious, however, that the function $T(x,t)$ is differentiable for any $x > 0$, so that

$$T(x+\Delta x, t) - T(x,t) \propto \Delta x , \qquad \Delta x \to 0 ,$$

and the ratio

$$\frac{\Delta x}{(\Delta x)^v}$$

for $\Delta x \to 0$ can only approach zero ($v > 1$) or infinity ($v < 1$), whereas the time derivative on the left-hand side of the heat-conduction equation is finite at positive times for the above-specified function.

By way of illustration, we give here some examples of fractional Riemann–Liouville derivatives for simple functions:

$$\frac{\mathrm{d}^v x^\lambda}{\mathrm{d}x^v} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}\, x^{\lambda-\alpha} , \qquad x > 0 , \qquad \lambda > -1 ;$$

$$\frac{\mathrm{d}^v 1}{\mathrm{d}x^v} = \left[\Gamma(1-\alpha)\right]^{-1} x^{-\alpha} ;$$

$$\frac{\mathrm{d}^v \exp(ax)}{\mathrm{d}x^v} = \left[\Gamma(\alpha)\right]^{-1} \exp(ax)\, \gamma(\alpha, ax) .$$

Note that the derivative of a constant vanishes only if its order is integer, while it is not zero for fractional α. The situation changes, however, if the derivative is taken not from zero to x but rather from $a \neq 0$ to x: a fractional derivative, like an integral, depends on the length of the interval.

5.2 Memory and intermittency

To better comprehend the situation that leads to fractional derivatives in an analysis of the asymptotics of jump processes, we consider the integral equation for the 'counting rate' $f(t) = \mathrm{d}N(t)/\mathrm{d}t$:

$$f(t) - \int_0^t q(t-\tau)\, f(\tau)\, \mathrm{d}\tau = q(t) .$$

If the intervals between pulses are distributed exponentially,

$$q_T(t) = \tau^{-1} \exp\left(-\frac{t}{\tau} \right) ,$$

we have a Poisson process. It is Markovian and has no memory. What does this statement mean? Let T be the waiting time for the first pulse and $q_{T-\theta}(t|\theta)$ be the conditional density of the waiting time $T - \theta$ remaining by time T on condition that the interval $(0, \theta)$ is empty. Obviously,

$$q_{T-\theta}(t|\theta) = \frac{q_T(\theta + t)}{\int_\theta^\infty q_T(t)\, \mathrm{d}t} = \tau^{-1} \exp\left(-\frac{t}{\tau} \right) .$$

Thus, we obtain the same distribution density for the first pulse — an exponent with the parameter τ — regardless of whether we measure time from $t = 0$ (the beginning of the process) or from some $\theta > 0$ [provided that the interval $(0, \theta)$ is empty]. If, however, we assume that, e.g.,

$$q_T(t) \sim B t^{-v-1} ,$$

then

$$q_{T-\theta}(t|\theta) \sim v\theta^v (\theta + t)^{-v-1} , \qquad t \to \infty .$$

The conditional density of the remaining waiting time depends on θ; precisely this fact implies that the process is not Markovian, or, in the physical language, it possesses memory.

To examine the asymptotics of the process on the whole, we use the Laplace transform of the equation for $f(t)$,

$$\left[1-\tilde{q}(\lambda)\right]\tilde{f}(\lambda)=\tilde{q}(\lambda)\,.$$

According to the Tauberian theorems (see, e.g., Ref. [17]), the behavior of the function $f(t)$ at large t is determined by the behavior of its transform

$$\tilde{f}(\lambda)=\int_0^\infty \exp\left(-\lambda t\right) f(t)\,\mathrm{d}t$$

in the region of small λ; specifically, the transform of any monotonic (starting from some t) function has the asymptotic representation

$$\tilde{f}(\lambda)\sim C\lambda^{-\rho}\,,\qquad \lambda\to 0\,,$$

if and only if

$$f(t)\sim\frac{C}{\Gamma(\rho)}\,t^{\rho-1}\,,\qquad \rho>0\,,\qquad t\to\infty\,.$$

It can easily be shown that

$$1-\tilde{q}(\lambda)\sim\begin{cases}\dfrac{B}{\nu}\,\Gamma(1-\nu)\lambda^{\nu}\,, & \nu<1\,,\\ \langle T\rangle\lambda\,, & \nu>1\,.\end{cases}$$

Thus,

$$\tilde{f}(\lambda)\sim\begin{cases}\dfrac{\nu}{B\Gamma(1-\nu)}\,\lambda^{-\nu}\,, & \nu<1\,,\\ \langle T\rangle^{-1}\lambda^{-1}\,, & \nu>1\,,\end{cases}$$

and the inverse transform yields

$$\tilde{f}(t)\sim\begin{cases}Ct^{\nu-1}\,, & \nu<1\,,\\ \langle T\rangle^{-1}\,, & \nu>1\,.\end{cases}$$

If, however, we calculate the inverse Laplace transform of the expressions

$$\lambda^{\nu}\tilde{f}(\lambda)=\frac{\nu}{B\Gamma(1-\nu)}$$

and

$$\lambda\tilde{f}(\lambda)=\langle T\rangle^{-1}\,,$$

then, according to the properties of the fractional Riemann–Liouville derivative, we obtain

$$\frac{\mathrm{d}^{\nu}f}{\mathrm{d}t^{\nu}}=0\,,\qquad \nu<1\,,$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}t}=0\,,\qquad \nu>1\,.$$

Thus, at $\nu>1$, when the mean $\langle T\rangle$ is finite, $f(t)$ levels off as $t\to\infty$, i.e., it behaves precisely as a Poisson process in the large-t asymptotics. In other words, it *loses its memory* with time (*asymptotic sclerosis* is observed). If, however, $\nu<1$, and the mean is accordingly infinite, the density decreases as $t^{\nu-1}$ for as large a t as desired, i.e., the *memory is preserved*.

A direct simulation of this process for $\nu<1$ also reveals another property, *intermittency*. Whatever the scale on which the distribution of points along the axis is observed, it looks interruptive. Regions of condensations alternate with voids (*are intermittent*). The mean number of points on the interval $(0,t)$ grows proportionally to t^{γ}, i.e., we deal here with a stochastic fractal whose fractal dimension is γ. It is remarkable that the fractal dimension coincides with the order of the fractional derivative γ. If $\gamma>1$, the intermittency disappears, and the distribution of points on large scales looks uniform.

Obviously, this interpretation of fractional derivatives is not the only imaginable one (see Refs [47–50]).

5.3 Hereditary processes

The above-noted nonlocal nature of fractional derivatives implies that the variation of density depends not only on its values in the neighborhood of the point considered (as in the case of normal diffusion) but also on its values at remote points in space. This fact can be expressed as follows:

$$\frac{\partial p(x,t)}{\partial t}=K\big(x;p(\,\cdot\,,t)\big)$$

(for brevity, we omit the characteristic exponents α and β; the diffusion coefficient is $D=1$). Here, $K(x;p(\,\cdot\,,t))$ is not simply an abbreviated notation for the right-hand side but it is a functional on the set of densities $\{p(x',t)\}$ (at fixed parameters x and t), which establishes a correspondence between any density $p(x',t)$ and some number $K(x;p(\,\cdot\,,t))$. We note, however, that $\partial p(x,t)/\partial t$ depends solely on the density at the same instant of time, and information on previous states of the process is not used.

It seems quite natural to extend the above functional equation to the case where not only the coordinates but also time is an active variable:

$$\frac{\partial p(x,t)}{\partial t}=Q\big(x,t;\,p(\,\cdot\,,\cdot)\big)\,.$$

A random process in which the rate of density variation depends on the density values at preceding times is called a *hereditary* process.

Such notions as *memory, aftereffect, and time lag* are closely related to hereditarity. Vito Volterra, who devoted a number of his studies and several chapters in his books [51, 52] to the development of the notion of hereditarity and to its applications to physical and ecological problems, remarked that Picard was the first to introduce (in 1907) the notion of aftereffect in physics, although such phenomena as the fatigue of metals, magnetic hysteresis, and some other hereditary processes had certainly been known very long before.

By far the majority of applications are restricted to the *linear hereditarity*, which, in the model at hand, yields the equation

$$\frac{\partial p(x,t)}{\partial t}=\int_{-\infty}^{\infty}q(t,t')\,K\big(x;\,p(\,\cdot\,,t')\big)\,\mathrm{d}t'\,.$$

The linear hereditarity is called *invariant* if the kernel of the integral operator on the right-hand side depends solely on the difference of times $t-t'$. We consider only positive times, assuming $p(x,t)=0$ for $t<0$; the equation of the corresponding invariant hereditary process is

$$\frac{\partial p(x,t)}{\partial t}=\int_0^t q(t-t')\,K\big(x;\,p(\,\cdot\,,t')\big)\,\mathrm{d}t'$$

with the initial condition

$$p(x,0) = \delta(x)\,.$$

We recall that the functionals $K^{(\alpha,\beta)}$ introduced in Section 2.3.2 are homogeneous functions of the variable x:

$$K^{(\alpha,\beta)}\big(ax;\, p^{(\alpha,\beta)}(\,\cdot\,,t)\big) = a^{-\alpha}K^{(\alpha,\beta)}\big(x;\, p^{(\alpha,\beta)}(\,\cdot\,,t)\big)\,,$$

which is precisely the reason for the self-similarity of the fundamental solution. To preserve this property for the considered type of hereditary process as well, it is sufficient to demand that the kernel $q(t)$ satisfy the condition

$$q(t) = \frac{t^{-\omega-1}}{\Gamma(-\omega)}\,.$$

In this case,

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial^{1-\omega}}{\partial t^{1-\omega}}\,K\big(x;\,p(\,\cdot\,,t)\big) + \delta(x)\delta(t)$$

or

$$\frac{\partial^{\omega} p(x,t)}{\partial t^{\omega}} = K\big(x;\,p(\,\cdot\,,t)\big) + \frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(x)\,.$$

Then, the self-similar solution has the form

$$p(x,t) = t^{-\beta/\alpha}p(xt^{-\beta/\alpha},1)\,.$$

In particular, at $\alpha = 2$ we arrive at the equation

$$\frac{\partial^{\omega} p(x,t)}{\partial t^{\omega}} = \frac{\partial^{2} p(x,t)}{\partial x^{2}} + \frac{t^{-\omega}}{\Gamma(1-\omega)}\,\delta(x)\,,$$

which describes subdiffusion [44]; for ω approaching unity, in which case $t^{-\omega}/\Gamma(1-\omega) \to \delta(t)$, we obtain the normal equation of diffusion.

5.4 Fractional diffusion processes and noise

Another way of introducing hereditarity into self-similar processes is based on using stochastic integrals of the random measure $\mathrm{d}L(t)$ that describes the random increment of an L process over the time interval $(t, t+\mathrm{d}t)$. In this terminology (see Section 2.4.3),

$$X(t) = \int_0^t \mathrm{d}L(\tau)\,, \qquad t > 0\,,$$

$$X(t+\tau) - X(t) = \int_t^{t+\tau} \mathrm{d}L(\tau') \overset{\mathrm{d}}{=} \tau^{1/\alpha}S^{(\alpha,\beta)}\,.$$

Here, the hereditarity is introduced using the function $h(t,\tau)$, which determines the contribution of the unit measure at time τ to the state of the process at time t:

$$X(t) = \int_{-\infty}^{\infty} h(t,\tau)\,\mathrm{d}L(\tau)\,.$$

If the function $h(t,\tau)$ is invariant with respect to translations in time,

$$h(t,\tau) = h(t-\tau)\,,$$

such a process is referred to as a *moving-average process* (MA process).

An example of an MA process is the Ornstein – Uhlenbeck – Lévy process, which can be written as

$$X(t) = \int_{-\infty}^{t} \exp\big[-\lambda(t-\tau)\big]\,\mathrm{d}L(\tau)\,.$$

The process

$$X_\alpha^H(t) = \int_{-\infty}^{\infty} \big(|t-\tau|^{H-1/\alpha} - |\tau|^{H-1/\alpha}\big)\,\mathrm{d}L(\tau)\,,$$

with $0 < H < 1$ and $H \neq 1/\alpha$, which is constructed according to the same principle, is called *linear fractional stable motion*, since it can be obtained from a stable process by fractional-order integration. Note two important properties of the process $\{X_\alpha^H(t)\}$. First, it is self-similar with the parameter H, i.e., for any $a > 0$ and $t_1, \ldots, t_n$,

$$\big(X_\alpha^H(at_1), \ldots, X_\alpha^H(at_n)\big) \overset{\mathrm{d}}{=} \big(a^H X_\alpha^H(t_1), \ldots, a^H X_\alpha^{(H)}(t_n)\big)\,.$$

Second, its increments are stationary,

$$X_\alpha^H(t) - X_\alpha^H(0) \overset{\mathrm{d}}{=} X_\alpha^H(t+\tau) - X_\alpha^H(\tau)\,.$$

In the particular case of $\alpha = 2$, we deal with *fractional Brownian motion*. If its mean value is zero (at $H \neq 1$), the variance is

$$\big\langle \big[X_2^H(t)\big]^2 \big\rangle = t^{2H}\sigma^2\,, \qquad \sigma^2 = \big\langle \big[X_2^H(1)\big]^2 \big\rangle\,,$$

and the autocovariance function is

$$C_2^H(t_1,t_2) = \big\langle X_2^H(t_1) X_2^H(t_2) \big\rangle = \big(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\big)\,\frac{\sigma^2}{2}\,.$$

The case of $H = 1/2$ and

$$C_2^{1/2}(t_1,t_2) = \begin{cases} \sigma^2 \min(t_1,t_2) & \text{if } t_1 \text{ and } t_2 \text{ are of like sign}\,, \\ 0 & \text{if } t_1 \text{ and } t_2 \text{ are of opposite sign} \end{cases}$$

corresponds to regular Brownian motion. Subdiffusion takes place at $H < 1/2$, superdiffusion at $1/2 < H < 1$, and a ballistic regime at $H = 1$. We note that the distribution of probability remains Gaussian in all regimes:

$$p(x,t) = \frac{1}{2\sqrt{\pi}\,\sigma t^H}\exp\left(-\frac{x^2}{4\sigma^2 t^{2H}}\right)\,.$$

Since Brownian motion has stationary increments, the sequence

$$\big\{Z_j = X_2^H(j+1) - X_2^H(j)\,, \quad j = \ldots,-1,0,1,\ldots\big\}$$

is stationary and is called *fractional Gaussian noise*. Its autocovariance function is

$$R_j = \big(|j+1|^{2H} - 2|j|^{2H} + |j-1|^{2H}\big)\,\frac{\sigma_0^2}{2} \sim \sigma_0^2 H(2H-1)\,j^{2H-2}\,, \quad j \to \infty\,, \quad H \neq \frac{1}{2}\,,$$

and the spectral density $P(\nu)$ obeys the asymptotic power law

$$P(\nu) \propto \nu^{-\gamma}, \qquad \gamma = 2H - 1.$$

The case of $\gamma = 0$ describes white noise, which is a model for electron and photon shot noise and for thermal noise. The cases of $\gamma = 1$, 2, and $\gamma > 2$ correspond to pink, brown, and black noise, respectively, which also have applications to natural and social processes. In particular, black spectra describe various catastrophes, such as the floods of rivers, droughts, failures of electricity, and financial crises [53].

5.5 Stochastic fractals

The spatial analogue of the integral equation presented in Section 5.1 has the form

$$f(\mathbf{x}) - \int p(\mathbf{x}-\mathbf{y})\, f(\mathbf{y})\, \mathrm{d}\mathbf{y} = p(\mathbf{x}).$$

Physically, it describes the mean density of the number of collisions of a particle that starts moving at the origin with the transition density $p(\mathbf{x})$ and does not disappear. This, however, is not the sole possible interpretation: if we assume that, during each collision, the particle disappears with a probability of $1 - 1/n$ and disintegrates into n particles at a probability of $1/n$ and the resulting particles continue moving with the same transition probabilities independently of one another, then we will arrive at the same equation for the mean collision density.

In this case, we are not interested in the motion itself but we will consider only the trace produced by the particle in the form of a finite or infinite set of points in space (we will call it *dust*) — the points where the particle underwent collisions. Such a consideration in terms of the random walk of a particle allows one to better imagine the overall pattern of correlations in this set of random points and to use the previously developed mathematical techniques of the theory of transfer [54].

The solution of this problem for isotropic spectral density

$$p(\mathbf{x}) = \frac{1}{4\pi r^2} \exp(-r), \qquad r = |\mathbf{x}|,$$

is well known [55]. Asymptotically, $f(\mathbf{x})$ behaves as

$$f^{\,\mathrm{as}}(\mathbf{x}) \propto r^{-1}, \qquad r \to \infty.$$

We also obtain the same result for any other choice of the transition density with the condition

$$\sigma^2 \equiv \int p(\mathbf{x}) |\mathbf{x}|^2\, \mathrm{d}\mathbf{x} < \infty$$

satisfied. The situation will change if the variance is infinite; this category also includes random walk with the transition density

$$p(\mathbf{x}) \propto r^{-3-\alpha}, \qquad r \to \infty, \qquad \alpha < 2,$$

called *Lévy flights*.

The direct Fourier transform of the integral equation

$$\left[1 - \tilde{p}(\mathbf{k})\right] \tilde{f}^{\,as}(\mathbf{k}) = 1$$

in the region of small $|\mathbf{k}|$, with

$$1 - \tilde{p}(\mathbf{k}) \propto |k|^{\alpha},$$

yields the relationship

$$|\mathbf{k}|^{\alpha}\, \tilde{f}^{\,as}(\mathbf{k}) = 1,$$

equivalent to the equation with a fractional Laplacian,

$$-(-\Delta)^{\alpha/2} f^{\mathrm{as}}(\mathbf{x}) = \delta(\mathbf{x}).$$

This is the stationary version of the equation of anomalous isotropic diffusion. Its solution is

$$f^{\,\mathrm{as}}(\mathbf{x}) = C(\alpha) r^{-3+\alpha},$$

where $C(\alpha)$ is a normalizing constant (see Section 25 in Ref. [25]). Note the change in the sign at α: as in the one-dimensional case, α appears in the expression for the transition density with a minus sign and in the formula for the density of all points with a plus sign. Such a behavior results from the infinite length of the trajectory. If the trajectory has a finite number of nodes (either determinate random with a finite mean), the asymptotics $f^{\mathrm{as}}(\mathbf{x})$ will be similar to $p(\mathbf{x})$, i.e., $\propto r^{-3-\alpha}$, although if this number is (on average) large, the function $f^{\mathrm{as}}(\mathbf{x})$ will have a clearly pronounced interval where it behaves as $r^{-3+\alpha}$, followed by an interval where $f^{\mathrm{as}}(\mathbf{x}) \propto r^{-3-\alpha}$ [56].

For an infinite trajectory, the mean number of its nodes within a sphere of radius R increases according to the asymptotic law

$$\left\langle N(R) \right\rangle \propto R^{\alpha}, \qquad \alpha \in (0, 2].$$

This gave ground to B Mandelbrot to call this set of points a *stochastic fractal*. He used Levy flights to model the distribution of galaxies (nodes) and achieved qualitative agreement with observations, which gave rise to another term popular in recent years, *fractal cosmology* [57].

To conclude, we emphasize that the averaged conditional density (i.e., the density in a coordinate system that has its origin at one of the random points) satisfies the equation with the fractional operator $(-\Delta)^{\alpha/2}$, and its exponent α coincides (in the case of infinite trajectories) with the fractal dimension.

5.6 One-dimensional fractal gas

Barkai et al. [58] use the term *one-dimensional Lorentz gas* to denote the random distribution $\{X_j\} = \ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ of point atoms over a straight line with the following properties:

(1) $X_0 = 0$;

(2) $X_i < X_j$ if $i < j$;

(3) $X_j - X_{j-1} = R_j$ are mutually independent random variables with a common distribution function $F(x)$.

It can easily be seen that the probability distribution for the number of atoms $N_+(x)$ over the interval $(0, x]$ can be represented in terms of multiple convolutions of the distribution $F(x)$ via the relationship

$$W(n, x) \equiv \mathsf{P}\big(N_+(x) = n\big) = F_n(x) - F_{n+1}(x).$$

A similar relationship holds for the distribution of the number of particles $N_-(x)$ over the interval $[-x, 0)$. The total number of atoms within the segment $[-x, x]$ is

$$N(x) = N_+(x) + N_-(x) + 1.$$

Choosing various distribution functions $F(x)$ yields various models of random media. In particular, using the

Heaviside step function

$$F(x) = \mathsf{H}(x-a) \equiv \begin{cases} 0, & x < a, \\ 1, & x \geqslant a, \end{cases}$$

results in a one-dimensional determinate lattice, while the exponential distribution

$$F(x) = 1 - \exp(-\mu x)$$

leads to the Poisson model of independently distributed atoms.

In any case, if the mean R is finite, in the asymptotics of large x we obtain $\langle N(x)\rangle \propto x$ and find for the relative fluctuation that $\Delta N(x)/\langle N(x)\rangle \to 0$. This means that if $f(N(x), x)$ is a certain smooth function of the random variable N, then for $x \to \infty$ we have $f(N(x), x) \to f(\langle N(x)\rangle, x)$, i.e., as the depth of the layer x increases, *self-averaging* takes place:

$$\big\langle f\big(N(x), x\big)\big\rangle \to f\big(\langle N(x)\rangle, x\big).$$

Now, assume that

$$1 - F(x) \sim \frac{A}{\Gamma(1-\alpha)}\, x^{-\alpha}, \qquad x \to \infty, \qquad \alpha < 1.$$

In this case, the mean distance between atoms is infinite, while the actual distances are finite in any realization of the model of a random medium. Since the mean R value is infinite, condensations interspaced with voids (i.e., intermittency) will be observed on any scale (Fig. 3). Applying the generalized limit theorem gives the following result:

$$\sum_{i=1}^{n} W(i, x) \sim \int_0^z w_\alpha(z)\, \mathrm{d}z, \qquad x \to \infty,$$

where $z = n/\langle N(x)\rangle$, and

$$w_\alpha(z) = \frac{z^{-1-1/\alpha}}{\alpha\Gamma(1+\alpha)}\, g\left(\frac{z^{-1/\alpha}}{\Gamma(1+\alpha)}\,;\, \alpha, 1\right).$$

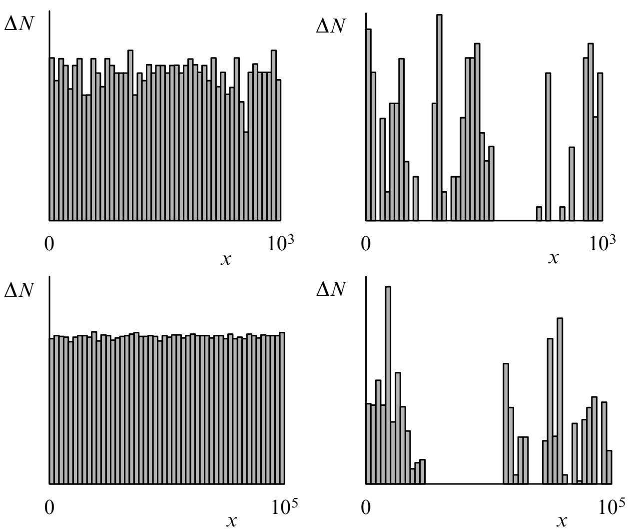

Figure 3. Regular (left) and fractal (right; $\alpha = 0.75$) distributions of atoms over a straight line on various scales.

It can easily be seen that the Lorentz gas has the following properties:

(1) all atoms are indistinguishable, and all processes $N(x)$ with various initial atoms are statistically equivalent, $N \overset{\mathrm{d}}{=} N(x)$;

(2) the number of atoms averaged over the ensemble increases with the layer depth measured from one of the atoms and obeys the power law

$$\big\langle N(x)\big\rangle \sim N_1 x^\alpha, \qquad 0 < \alpha < 1;$$

(3) the relative fluctuations in the number of atoms in this layer do not decrease with the layer depth but approach a constant value.

These properties warrant that the obtained structure be termed a *stochastic fractal* (or a fractal gas), which, in a probabilistic sense, is a self-similar set with a fractal dimension α. For a fractal gas, we have

$$\big\langle f\big(N(x), x\big)\big\rangle \sim \int_0^\infty f(N_1 x^\alpha z, x)\, w_\alpha(z)\, \mathrm{d}z$$

as $x \to \infty$. This relationship indicates that self-averaging does not take place on fractal structures, which is the principal difference of the random walk on fractals from the random walk in a regular medium.

Let us similarly construct a random set of points $\{T_j\}$ on the positive semiaxis of time, to characterize the random times of the jumps of the walking particle from one atom to another. We will assume that the random time intervals between these events are independent and distributed with a common distribution function $Q(t)$. If $Q(t) = 1 - \exp(-\mu t)$ and $\mu > 0$, the random set $\{T_i\}$ forms a uniform *Poisson stream*. This means that the probability of a jump of the particle in the interval $(t, t + \mathrm{d}t)$ does not depend on the time of the preceding jump; in other words, the particle does not possess memory. In all other cases, the particle is considered to have memory and, if

$$1 - Q(t) \sim \frac{B}{\Gamma(1-\omega)}\, t^{-\omega}, \qquad t \to \infty, \qquad \beta < 1,$$

to have fractal memory. All the above considerations concerning the ensemble $\{X_i\}$ are also valid for the ensemble $\{T_i\}$, including the averaging rule. If $K(t)$ is the random number of jumps within a fixed interval $(0, t]$, the function $h(K(t), t)$ averaged over the statistical ensemble $\{T_i\}$ satisfies the asymptotic relationship

$$\big\langle h\big(K(t), t\big)\big\rangle \sim \int_0^\infty h(K_1 t^\omega z, t)\, w_\omega(z)\, \mathrm{d}z, \qquad t \to \infty.$$

5.7 Random walk on a one-dimensional fractal

We now discuss the random walk of a particle on the above-considered fractal. Initially, the particle is located at the origin. It jumps equiprobably to one of the two neighboring atoms after a random time T_1 has elapsed and remains there for some random time, after which it jumps again to a neighboring atom (this may prove to be the atom situated at the origin, from which the particle started its motion).

If we choose the order number of atom i instead of x as the coordinate and the order number of the jump time j instead of t as the time, then, according to the central limit theorem, we obtain

$$\mathsf{P}(I < i | J = j) \sim \frac{1}{\sqrt{2\pi j}} \int_{-\infty}^{i} \exp\left(-\frac{x^2}{2j}\right) \mathrm{d}x, \qquad j \to \infty.$$

This is the result of averaging over the ensemble of random trajectories of particles with fixed nodes and jump times. To find the required distribution function, we have to perform averaging over two independent statistical ensembles $\{X_i\}$ and $\{T_j\}$, i.e., over the random values I and J of the subscripts i and j, respectively:

$$F(x,t) = \left\langle\left\langle \mathsf{P}(I < i | J = i)\right\rangle\right\rangle .$$

After this averaging and some manipulation, we obtain [59]

$$F(x,t) \sim \Xi^{(\alpha,\omega)}\left((Ct)^{-\omega/(2\alpha)}x\right), \qquad t \to \infty, \qquad x \to \infty,$$

where $C = \text{const}$,

$$\Xi^{(\alpha,\omega)}(x) = \int_0^\infty Q(xy^{-\alpha}; 2, \omega, 0)\, g(y; \alpha, 1)\, \mathrm{d}y .$$

The corresponding relationship for density $p(x,t) = \partial F(x,t)/\partial x$ is

$$p(x,t) = (Ct)^{-\omega/(2\alpha)} \xi^{(\alpha,\omega)}\left((Ct)^{-\omega/(2\alpha)}x\right),$$

where

$$\xi^{(\alpha,\omega)}(x) = \int_0^\infty q(xy^{-\alpha}; 2, \omega, 0)\, g(y; \alpha, 1) y^{-\alpha}\, \mathrm{d}y .$$

As $\omega \to 1$, the distribution changes into the normal one. If both these conditions are satisfied, we obtain a Gaussian form for the density $p(x,t)$ itself, which corresponds to the normal diffusion in a regular medium.

A comparison of the spatial distribution of a particle walking on a fractal with the solution of the AD equation for the fractal walk (Fig. 4) indicates that this equation cannot generally be interpreted as an equation describing the walk on fractals: in the first case, the diffusion packet spreads according to the law $t^{\omega/(2\alpha)}$, while in the latter, the spreading is much faster and follows the law $t^{\omega/\alpha}$. For the random walk on fractals, the exponent $H = \omega/(2\alpha)$ is within the interval $(0, 1/2)$, and a superdiffusion regime ($\gamma > 1/2$) does not set in at all. The reason for this difference can be seen from Fig. 5: a fractally walking particle, after having left an atom, can in any case move far away, whereas a particle walking on a fractal may be captured between neighboring clusters, undergoing numerous transitions between them.

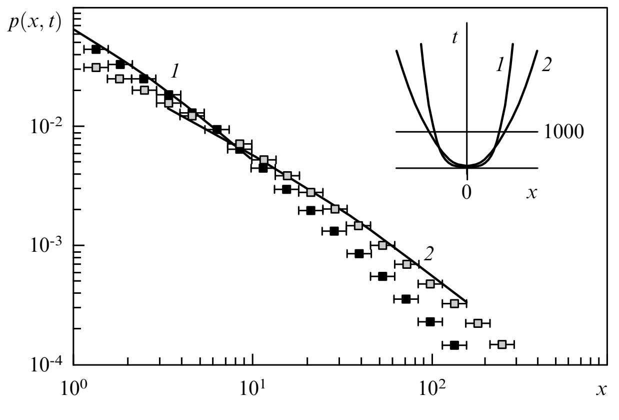

Figure 4. Densities of symmetric ($\theta = 0$) distributions $p(x,t;\alpha,\omega)$ with $\alpha = 1/2$, $\omega = 1/4$, $t = 10^3$ for (*1*) random walk on a fractal and (*2*) fractal random walk. Curves: numerical simulations; squares: Monte Carlo computations. In the inset, the thickness of diffusion packets is shown as a function of time.

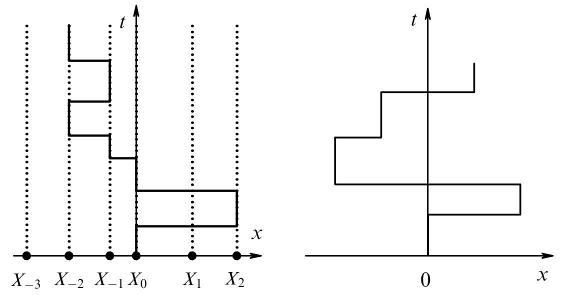

Figure 5. Random walk on a fractal (left) and fractal random walk (right).

The distributions $\xi(x;\alpha,\omega)$ and $q(x;\alpha,\omega)$ themselves have different shapes in the cases considered here. We considered the angular distribution of particles multiply scattered on a fractal in Ref. [60].

To conclude, we emphasize that our inferences are valid for a statistical ensemble of one-dimensional 'frozen' distributions. For a multidimensional random walk, the correlation between consecutive free paths may be less pronounced and may affect the differences less. Moreover, the situation can alter dramatically if the disposition of atoms undergoes considerable changes during the characteristic time of the particle's stay at one of the atoms (as is the case for diffusion in a turbulent medium, which gives rise to superdiffusion).

6. Kinetics at finite free-motion speeds

6.1 Mesodiffusion

A specific type of hereditarity arises if a particle makes jumps at a finite velocity of free motion. In this case, we obtain a nondegenerate distribution even in the absence of traps, i.e., for continuous motion of the particle at some velocity constant in magnitude and variable in direction. Interest in such processes has been somewhat cooled since the time of active investigations of nonstationary regimes in nuclear reactors, but it was rekindled after the new field of research — the physics of mesoscopic systems [61–63] — was opened up. Mesoscopic nanostructures offer a unique opportunity for experimental investigations of transport processes in a medium with a well-defined potential field not distorted by casual admixtures or other defects. It proves to be very important that three time regions with different transport regimes exist in this case: the interval $(0, t_1)$ where ballistic transport dominates, the region (t_2, ∞), $t_2 > t_1$ in which the regime of normal (Gaussian) diffusion settles, and the intermediate region (t_1, t_2) where the transport regime called mesoscopic diffusion [64] (see also Ref. [65]) takes place. We will also use the term *mesodiffusion*.

The quantum-mechanical analysis of the one-dimensional problem carried out by Godoy and Garcia-Colin [64] demonstrates that a distinctive feature of mesodiffusion is a departure from Fick's law

$$j = -D\,\frac{\partial p}{\partial x},$$

which is replaced in this region by the Maxwell–Cattaneo relationship

$$j = -D\,\frac{\partial p}{\partial x} - \theta\,\frac{\partial j}{\partial t}\,, \qquad \theta > 0\,.$$

Together with the continuity equation

$$\frac{\partial p}{\partial t} = -\frac{\partial j}{\partial x}\,,$$

it yields the following equation for the density of particle distribution $p(x,t)$:

$$\frac{\partial p}{\partial t} + \theta\,\frac{\partial^2 p}{\partial t^2} = D\,\frac{\partial^2 p}{\partial x^2}\,,$$

called the *telegraph* equation. A characteristic feature of the solution to this equation for a source localized in time (e.g., an instantaneous source) is the presence of a diffusion front, outside which no diffusing particles are present and in the neighborhood of which they move in a ballistic regime. At larger times, the ballistic component decays, while the remaining part of the solution in large-size samples transforms into a Gaussian packet, which satisfies the normal equation of diffusion. This process is studied in detail in Refs [64–68].

6.2 General solution

The probability distribution for a particle that walks along the OX axis at a velocity v, whose free-path distribution is

$$\mathsf{P}(R > x) = P(x) = \int_x^\infty p(x)\,\mathrm{d}x$$

and whose postcollisional directions are equiprobable, is characterized by the density

$$p(x,t) = \frac{1}{2}\int_0^t \big[f(x - v\tau, t-\tau) + f(x + v\tau, t - \tau)\big]P(v\tau)\,\mathrm{d}\tau\,,$$

where $f(x,t)$ is the density of collisions (the mean number of collisions in a unit interval per unit time); it satisfies the equation

$$f(x,t) = \frac{1}{2}\int_0^t \big[f(x - v\tau, t - \tau) + f(x + v\tau, t-\tau)\big]\,p(v\tau)\,\mathrm{d}\tau + \delta(x)\delta(t)\,.$$

The Fourier and Laplace transforms (in x and t, respectively) yield the equations

$$\tilde{p}(k,\lambda) = \frac{1}{v}\,W(k,\lambda)\,\tilde{f}(k,\lambda)\,,$$
$$\tilde{f}(k,\lambda) = 1 + w(k,\lambda)\,\tilde{f}(k,\lambda)\,,$$

where

$$W(k,\lambda) = \int_0^\infty P(x)\cos(kx)\exp\left(-\frac{\lambda}{v}\,x\right)\mathrm{d}x\,,$$

$$w(k,\lambda) = \int_0^\infty p(x)\cos(kx)\exp\left(-\frac{\lambda}{v}\,x\right)\mathrm{d}x\,.$$

The solution of these equations is

$$\tilde{p}(k,\lambda) = \frac{W(k,\lambda)}{v\big[1 - w(k,\lambda)\big]}\,,$$

so that the density itself is represented by the relationship

$$p(x,t) = \frac{1}{(2\pi)^2 i}\int_{-\infty}^\infty \mathrm{d}k \int_C \mathrm{d}\lambda\;\frac{W(k,\lambda)}{v\big[1 - w(k,\lambda)\big]}\,\exp(-\mathrm{i}kx + \lambda t)\,.$$

This representation is more convenient for asymptotic analysis than for numerical calculations.

To obtain another form of the general solution [i.e., of the solution for an arbitrary distribution of free paths between collisions, $p(x)$], we introduce the distribution of paths *between the collisions changing the direction of motion*. The density of this distribution $g(x)$ is related to $p(x)$ as follows:

$$g(x) = \sum_{n=1}^\infty \left(\frac{1}{2}\right)^n p^{*n}(x)\,,$$

where

$$p^{*n}(x) = \int_0^x p^{*(n-1)}(x - x')\,p(x')\,\mathrm{d}x'$$

is the $(n-1)$-fold convolution of the densities $p(x)$. We also introduce indices 1 and 2 to denote the direction of motion of the particle along the OX axis (leftward and rightward, respectively); let also $p_{ij}(x,t)$ be the distribution density for the particle in state i provided that it started its motion from the origin at time $t = 0$ being in state j. Obviously,

$$\sum_{i=1}^2 \int_{-\infty}^\infty p_{ij}(x,t)\,\mathrm{d}x = 1\,,$$

$$p(x,t) = \frac{1}{2}\sum_{i,j} p_{ij}(x,t)\,.$$

Yarovikova [71] has shown that

$$p_{11}(x,t) = \frac{1}{2}\sum_{n=0}^\infty \Delta G^{*n}(\xi_1)\,g^{*n}(\xi_2)\,,$$

$$p_{21}(x,t) = \frac{1}{2}\sum_{n=0}^\infty g^{*(n+1)}(\xi_1)\,\Delta G^{*n)}(\xi_2)\,,$$

where

$$g^{*(n+1)}(\xi) = \int_0^\xi g^{*n}(\xi - x)\,g(x)\,\mathrm{d}x$$

is the n-fold convolution of the density $g(x)$, and

$$G^{*n}(\xi) = \int_0^\xi g^{*n}(x)\,\mathrm{d}x\,,$$

$$\Delta G^{*n}(\xi) = G^{*n}(\xi) - G^{*(n+1)}(\xi)\,,$$

$$\xi_1 = \frac{vt - x}{2}\,, \qquad \xi_2 = \frac{vt + x}{2}\,.$$

Note that the distributions presented here include also δ-type singularities resulting from the terms with $n = 0$, so that the following normalizing condition is valid:

$$\int_{-vt}^{vt}\left[p_{1j}(x,t)+p_{2j}(x,t)\right]\mathrm{d}x=1\,.$$

6.3 Exact solutions

The second representation of $p(x,t)$ reduces the problem to the calculation of multiple convolutions of the distributions of free paths between the points where the direction of motion changes. There exist distributions whose convolutions can be represented in terms of elementary or special functions. We will denote the corresponding solutions as *exact* ones.

Consider the classic, exponential distribution of free paths

$$p(x)=\mu\exp(-\mu x)\,.$$

In this case,

$$p^{*n}(x)=\mu\,\frac{(\mu x)^{n-1}}{(n-1)!}\exp(-\mu x)\,,$$

$$g(x)=\frac{\mu}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mu x}{2}\right)^n\exp(-\mu x)=\sigma\exp(-\sigma x)\,,\qquad\sigma=\frac{\mu}{2}\,.$$

Furthermore,

$$g^{*n}(\xi)=\sigma\,\frac{(\sigma\xi)^{n-1}}{(n-1)!}\exp(-\sigma\xi)\,,$$

$$\Delta G^{(n)}(\xi)=\frac{(\sigma\xi)^n}{n!}\exp(-\sigma\xi)\,.$$

We insert these expressions into the general formula and represent the power series in terms of modified Bessel functions, to obtain

$$p(x,t)=\exp(-\sigma vt)\,\frac{1}{2}\left\{\delta(x+vt)+\delta(x-vt)+\sigma I_0\left(\sigma\sqrt{(vt)^2-x^2}\right)+\frac{v\,tI_1\left(\sigma\sqrt{(vt)^2-x^2}\right)}{\sqrt{(vt)^2-x^2}}\right\}.$$

The delta functions describe here the distribution of nonscattered particles that was produced by an instantaneous, point source, while the continuous part of the solution refers to the scattered component.

Naturally, the n-fold convolution of the exponential density changes its form. However, if we consider the class of gamma distributions on the whole,

$$g_{\sigma,\nu_1}*g_{\sigma,\nu_2}(x)=g_{\sigma,\nu_1+\nu_2}(x)\,,$$

the convolution operation will change only the index ν of the gamma distribution. This class is said to be closed with respect to the convolution. This property is also called *reproducibility* [69].

Gamma distributions are not the only reproducible distributions. The distributions

$$g_\nu(x)=\nu x^{-1}\exp(-x)\,I_\nu(x)\,,$$

produced by the modified Bessel functions $I_\nu(x)$, also form a class of reproducible distributions

$$g_\nu^{*n}(x)=\nu n x^{-1}\exp(-x)\,I_{\nu n}(x)=g_{n\nu}(x)\,.$$

Obviously, any strictly stable distribution $g(x;\alpha,\theta)$ generates a class of reproducible distributions by virtue of the property

$$g^{*n}(x;\alpha,\theta)=n^{-1/\alpha}g(xn^{-1/\alpha};\alpha,\theta)\,.$$

A particular case of the sort is the δ density

$$g(x;1,1)=\delta(x-1)\,,$$

which describes random walk on a one-dimensional lattice. Note that reproducibility is not equivalent to stability: in the latter case, convolving the distributions does not change the determinant parameters α and θ but only affects the scaling factor.

Thus, the fact that $f(x)$ belongs to the class of reproducible distributions is a sufficient condition for the existence of an exact solution — self-evidently, if the densities can be expressed in terms of elementary or special functions. There exist, however, other densities that also lead to exact solutions. An example is the following distribution [17] uniform on $[0,a]$:

$$g(x)=\frac{1}{a}\,,\qquad 0\leqslant x\leqslant a\,,$$

$$g^{*n}(x)=\frac{1}{a^n(n-1)!}\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}(x-\nu a)_+^{n-1}\,,\qquad x\leqslant na\,,$$

where $(x-a)_+$ is zero for $x\leqslant na$ and $x-a$ for $x\geqslant a$.

6.4 Telegraph equation

The telegraph equation was derived (as Garcia-Pelayo [68] notes) by Lord Kelvin in relation to laying the first trans-Atlantic cable. In nondimensional units of time t, it has the form

$$\frac{\partial^2 f_v}{\partial t^2}+\frac{\partial f_v}{\partial t}=v^2\,\frac{\partial^2 f_v}{\partial x^2}\,.$$

A solution $f_v(x,t)$ to this equation has the meaning of the current at point x in the conductor at time t, and the parameter v is related to the inductance and resistance per unit length of the conductor.

For the initial conditions

$$f_v(x,0)=\delta(x)\,,\qquad\left[\frac{\partial f_v(x,t)}{\partial t}\right]_{t=0}=0$$

the solution consists of two terms,

$$f_v(x,t)=f_v^{(0)}(x,t)+f_v^{(s)}(x,t)\,.$$

The first one describes two instant pulses that travel from the coordinate origin in opposite directions at speed v:

$$f_v^{(0)}(x,t)=\frac{1}{2}\left[\delta(x-vt)+\delta(x+vt)\right]\exp\left(-\frac{t}{2}\right).$$

The second one is the continuous part of the solution, which fills the gap between these pulses $(-vt<x<vt)$:

$$f_v^{(s)}(x,t)=\frac{1}{4v}\left[I_0\left(\sqrt{\frac{t^2-x^2/v^2}{4}}\right)+\frac{tI_1\left(\sqrt{(t^2-x^2/v^2)/4}\right)}{\sqrt{t^2-x^2/v^2}}\right]\exp\left(-\frac{t}{2}\right).$$

Note the following properties of the above-presented solution:

$$f_v(x,t) > 0\,, \qquad |x| \leqslant vt\,,$$
$$f_v(x,t) = 0\,, \qquad |x| > vt\,,$$
$$\int_{-vt}^{vt} f_v(x,t)\,\mathrm{d}x = 1\,,$$
$$\int_{-vt}^{vt} x^2 f_v(x,t)\,\mathrm{d}x = 2v^2\left[t + \exp(-t) - 1\right],$$
$$f_v(x,t) \cong \frac{1}{\sqrt{4\pi v^2 t}}\exp\left(-\frac{x^2}{4v^2 t}\right), \qquad t \to \infty\,.$$

The reader has already noted that an expression of the type of $f_v(x,t)$ appeared in the preceding section as an exact solution to the problem of the one-dimensional random walk of a particle at a constant speed and an exponential distribution of free paths. At $\sigma v = 1/2$, the parameter v has the meaning of the free-motion speed of the particle, $f_v(x,t)$ represents the density of the probability distribution at time t, and $f_v^{(0)}(x,t)$ describes the distribution of the particles that have not changed the direction of their motion until time t. The particles are at the points $x = vt$ and $x = -vt$, forming the front of a diffusion packet, which occupies the segment $[-vt, vt]$; the probability of detecting the particle outside this segment is zero. As $t \to \infty$, the solution $f_v(x,t)$ approaches the normal distribution $g_v(x,t)$ with the variance $2v^2 t$ [70], which satisfies the normal equation of diffusion

$$\frac{\partial g_v}{\partial t} = v^2\,\frac{\partial^2 g_v}{\partial x^2}$$

with the initial condition

$$g_v(x,0) = \delta(x)\,.$$

In view of its approximate (asymptotic) character, this solution no longer contains information on the diffusion front but describes only the central (fairly extensive, however) part of the diffusion packet.

To obtain the telegraph equation from the random-walk problem itself, we go back to the Fourier–Laplace transform $\tilde{p}(k,\lambda)$ and calculate the functions w and W, which appear in this transform, under the assumption of the exponential distribution of free paths, $p(x) = \mu\exp(-\mu x)$:

$$w\left(k, \frac{\lambda}{v}\right) = \mu W\left(k, \frac{\lambda}{v}\right) = \frac{\mu(\mu + \lambda/v)}{(\mu+\lambda/v)^2 + k^2}\,.$$

As a result, we obtain the expression

$$\left[\lambda^2 + \mu v\lambda + v^2 k^2\right]\tilde{p}(k,\lambda) = \mu v + \lambda\,,$$

which is nothing but the Fourier–Laplace transform of the telegraph equation for the density $p(x,t)$ that satisfies the initial conditions

$$p(x,0) = \delta(x)\,, \qquad \left(\frac{\partial p}{\partial t}\right)_{t=0} = 0\,.$$

We note that, in contrast to the FPK equation (see Section 2.4.2), which contains the first and second coordinate derivatives as well as the first time derivative, the telegraph equation contains the first and second time derivatives and the second coordinate derivative.

6.5 Method of moments

To numerically determine the transform of the distribution density

$$p(x,\lambda) = \int_0^\infty \exp(-\lambda t)\,p(x,t)\,\mathrm{d}t$$

for an arbitrary distribution of free paths, it is convenient to use the method of moments.

To this end, we introduce the notation

$$m_{2n}(t) = \int_{-\infty}^{\infty} x^{2n} p(x,t)\,\mathrm{d}x\,,$$
$$\tilde{m}_{2n}(\lambda) = \int_{-\infty}^{\infty} x^{2n}\tilde{p}(x,\lambda)\,\mathrm{d}x = \int_0^\infty \exp(-\lambda t)\,m_{2n}(t)\,\mathrm{d}t\,,$$
$$\tilde{p}_n(\mu) = \int_0^\infty \exp(-\mu x)\,x^n p(x)\,\mathrm{d}x\,,$$

insert the expansions

$$\tilde{p}(k,\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,m_{2n}(\lambda)k^{2n}\,,$$
$$w(k,\lambda) = \sum_{n=0}^{\infty} a_n(\lambda)k^{2n}\,,$$
$$W(k,\lambda) = \sum_{n=0}^{\infty} A_n(\lambda)\,k^{2n}$$

into the equation

$$\left[1 - w(k,\lambda)\right]p(k,\lambda) = \frac{1}{v}\,W(k,\lambda)\,,$$

and equate the coefficients at equal powers of k. Simple manipulation results in the recurrent relationship

$$\tilde{m}_{2n}(\lambda) = \frac{(-1)^n(2n)!}{1 - a_0(\lambda)}\left[\frac{1}{v}\,A_n(\lambda) + \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{(2\nu)!}\,a_{n-\nu}(\lambda)\tilde{m}_{2\nu}(\lambda)\right],$$
$$\tilde{m}_0(\lambda) = \frac{1}{\lambda}\,.$$

Here,

$$a_n(\lambda) = \frac{(-1)^n}{(2n)!}\,\tilde{p}_{2n}\left(\frac{\lambda}{v}\right), \qquad A_0(\lambda) = \frac{v}{\lambda}\left[1 - p_0\left(\frac{\lambda}{v}\right)\right],$$
$$A_1(\lambda) = \left(\frac{v}{2\lambda}\right)\tilde{p}_2\left(\frac{\lambda}{v}\right) + \left(\frac{v}{\lambda}\right)^2\tilde{p}_1\left(\frac{\lambda}{v}\right) + \left(\frac{v}{\lambda}\right)^3\tilde{p}_1\left(\frac{\lambda}{v}\right) - \left(\frac{v}{\lambda}\right)^3,$$

etc.

Let us dwell on the second moment

$$\tilde{m}_2(\lambda) = -\frac{2\left[A_2(\lambda)/v + a_2(\lambda)/\lambda\right]}{1 - a_0(\lambda)}\,.$$

For the exponential distribution of free paths $(p(x) = \mu\exp(-\mu x))$, we have

$$\tilde{m}_2(\lambda) = \frac{2v^2}{\lambda^2(\lambda + \mu v)}\,.$$

The inverse Laplace transform obtained using the residue theorem yields the expression

$$m_2(t) = \frac{2}{\mu^2}\left[\exp(-\mu v t) + \mu v t - 1\right],$$

which corresponds exactly to the solution of the telegraph equation.

In the general case, the inversion $\tilde{m}_2(\lambda) \to m_2(t)$ cannot be elementarily fulfilled, but in the asymptotic region ($t \to \infty$, $\lambda \to 0$), the use of the Tauberian theorems is sufficient [17]. In particular, for the one-sided stable distribution of free paths

$$p(x) = c^{-1/\alpha} g(c^{-1/\alpha}x; \alpha, 1), \qquad c > 0, \qquad \alpha \leqslant 1,$$

we obtain

$$\tilde{m}_2(\lambda) = \frac{2v^2}{\lambda^2}\left[\frac{1}{\lambda}\,\frac{c\alpha(\lambda/v)^{\alpha-1}}{\exp\{c(\lambda/v)^{\alpha}\} - 1}\right] \sim \frac{2(1-\alpha)v^2}{\lambda^3}, \quad \lambda \to 0,$$

and, accordingly,

$$m_2(t) \sim (1-\alpha)(vt)^2, \qquad t \to \infty.$$

Higher moments are calculated and used to reconstruct the distributions in Refs [66, 67, 71 – 74]. The influence of traps on these distributions is also analyzed there. It is, in particular, shown that, in traps of the power-law type with an exponent $\omega < 1$, the distribution becomes subdiffusive with a diffusion coefficient containing the speed v.

6.6 Fractional extension of the telegraph equation

As noted above, the normal telegraph equation with the Fourier – Laplace image

$$\left[\lambda^2 + \mu v \lambda + v^2 k^2\right]\tilde{p}(k, \lambda) = \mu v + \lambda$$

exactly describes the one-dimensional random walk of a particle at a finite velocity v, with an exponential distribution $p(x) = \mu \exp(-\mu x)$ of free paths. Thus, the equation

$$\left[1 - w(k, \lambda)\right]\tilde{p}(k, \lambda) = \frac{1}{v}\, W(k, \lambda)$$

derived in Section 6.2 can be considered a generalization of the telegraph equation for the case of an arbitrary distribution of free paths. Its solution was considered above; however, the direct inversion of this equation, i.e., the transition from the variables k and λ to x and t, yields an integral equation that does not reduce to a differential equation, as is the case for exponential distributions of free paths

$$p(x, t) = P(x)\delta(x - vt) + \int_{-\infty}^{\infty} p(x')\, p\left(x - x', t - \frac{x'}{v}\right) \mathrm{d}x'.$$

A fractional extension of the telegraph equation arises in the case where the free paths are asymptotically distributed according to a power law with a heavy tail, $p(x) \propto x^{-\alpha-1}$, $0 < \alpha < 1$. For the convenience of calculations, we use, as such a density, the one-sided stable density with the exponent α and scaling factor c, $\tilde{p}(\lambda) = \exp\{-c\lambda^{\alpha}\}$ (already employed in the preceding section).

The use of the representation

$$w(k, \lambda) = \frac{1}{2}\left[\tilde{p}\left(\frac{\lambda}{v} - \mathrm{i}k\right) + \tilde{p}\left(\frac{\lambda}{v} + \mathrm{i}k\right)\right]$$

and the asymptotic expansion

$$\exp(-c\lambda^{\alpha}) \sim \frac{1}{1 + c\lambda^{\alpha} + \ldots}$$

leads to the equation

$$T^{(\alpha,\alpha)}(k, \lambda)\,\tilde{p}(k, \lambda) = \frac{1}{v}\, T^{(\alpha-1,\alpha)}(k, \lambda),$$

where

$$T^{(\alpha,\beta)}(k, \lambda) = \frac{v^2}{c}\ \mathrm{Re}\left\{\left(\frac{\lambda}{v} - \mathrm{i}k\right)^{\alpha}\left[1 + c\left(\frac{\lambda}{v} + \mathrm{i}k\right)^{\beta}\right]\right\}.$$

The operator specified by this image at integer α and β can be regarded as a fractional integrodifferential operator, and the equation

$$T^{(\alpha,\beta)}\, p(x, t) = \frac{1}{v}\, T^{(\alpha-1,\alpha)}1$$

as a fractional extension of the telegraph equation. Indeed, as $\alpha \to 1$,

$$T^{(\alpha,\alpha)}(k, \lambda) \to \lambda^2 + \frac{v}{c}\,\lambda + v^2 k^2,$$

$$T^{(\alpha-1,\alpha)}(k, \lambda) \to \left(\lambda + \frac{v}{c}\right) v,$$

and we arrive at the normal telegraph equation, whose solution given in Section 6.4 is characterized by a Gaussian asymptotics for $t \to \infty$.

The corresponding asymptotics for a fractional equation with a characteristic exponent of $\alpha = 1/2$ can easily be obtained if the term $c(\lambda/v + \mathrm{i}k)^{1/2}$ in the second factor of the operator $T^{(\alpha,\beta)}$ is neglected. Then,

$$\tilde{p}(k, \lambda) = \frac{1}{\sqrt{\lambda^2 + (vk)^2}}$$

and, therefore,

$$p(x, t) = \frac{1}{\pi\sqrt{(vt)^2 - x^2}}.$$

This result was confirmed by Monte Carlo computations. The second moment of this distribution, $m_2(t) = (vt)^2/2$, is in agreement with the expression given at the end of Section 6.5.

It should be noted that the above-presented extension of the telegraph equation is not unique, if only since the original equation describes completely different physical processes, such as electromagnetic waves in a conducting medium and one-dimensional random walk of a particle. Fractional extensions of these processes can be different. On the other hand, other extensions of the telegraph equation appear even in the problem of random walk (see, e.g., Ref. [12]), although their interpretation is not so explicit.

6.7 Finite-speed effects

Let us briefly describe the effects of finite speed of free motion on anomalous kinetics, assuming that both the free paths and the times of stay in traps are distributed according to inverse-power laws with indices α and ω.

As a preliminary, we briefly consider the situation with an infinite speed. As noted above, the width of the diffusion packet increases in this case as t^H, where $H = \omega/\alpha$. The region

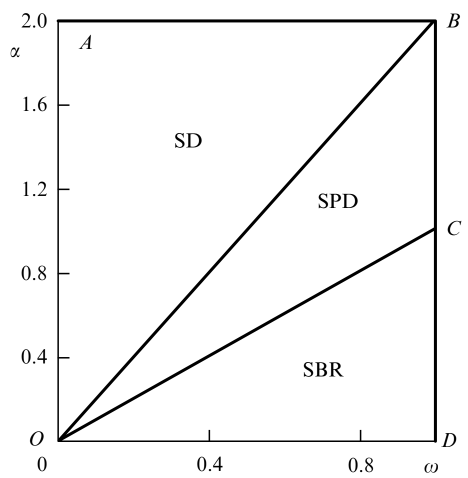

Figure 6. Structure of the anomalous-diffusion region in a model with infinite free-motion speed of the particle.

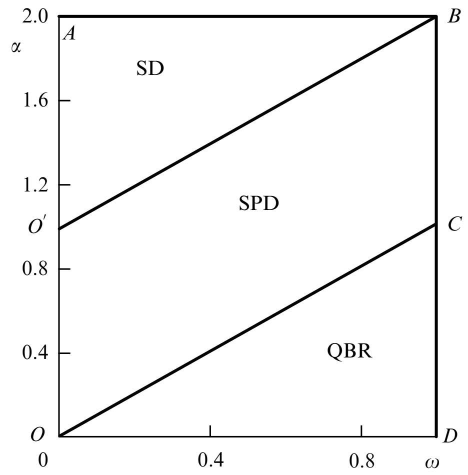

Figure 7. Structure of the anomalous-diffusion region in a model with a finite free-motion speed.

where $H < 1/2$ corresponds to subdiffusion (SD); $1/2 < H < 1$, to superdiffusion (SPD); and $H > 1$, to the regime that can be called *superballistic* (SBR): the packet spreads in space more rapidly than in the case of the free motion of particles. Godoy and Garcia-Colin [64] denote this regime as turbulent (see also Ref. [75]). The disposition of the above-mentioned regions is illustrated in Fig. 6. At the boundary between the zones of superdiffusion and subdiffusion, represented by a segment of the line $\alpha = 2\omega$, the exponent H corresponds to a normal diffusion process; however, only the point B represents normal diffusion (ND), while all other points of segment OB represent *quasi-normal* diffusion with the packet-spreading law $t^{1/2}$ and the shape of the packet differing from the normal one. For these same reasons, we use the term *quasi-ballistic* regime (QBR) for the regime represented by the segment OC.

The most important distinction of a finite-speed process from the above-considered one is the finiteness of all moments of the distribution $p(x, t)$, since outside the segment $[-vt, vt]$ the particle can in no way be found at time t. In this situation, the method of moments becomes efficient again. An asymptotic analysis indicates that

$$m_{2n}(t) \equiv \int_{-vt}^{vt} x^{2n} p(x,t)\,\mathrm{d}x \propto (vt)^{\gamma^{(2n)}},$$

$$\gamma^{(2n)} = \begin{cases} 2n, & \alpha \leqslant \omega, \\ 2n - \alpha + \omega, & \alpha > \omega. \end{cases}$$

The diffusion type $H \equiv \gamma^{(2)}/2$ is determined by the lowest index. It can be seen that the process has the form of superdiffusion at $\omega < \alpha < \omega + 1$ and any ω, of a quasi-normal regime at $\alpha = \omega + 1$, and of subdiffusion at $\alpha > \omega + 1$. The region of the superballistic regime has disappeared, and the quasi-ballistic regime is now the limiting regime of the maximum rate of spread of the diffusion packet (the triangle OCD, Fig. 7).

Quasi-normal diffusion ($H = 1/2$) is now represented by a segment of the line $\alpha = \omega + 1$ instead of $\alpha = 2\omega$, which was the case for infinite speed.

A comparison of Figs 6 and 7 indicates that quasi-normal diffusion at infinite speed takes place at those values of α and ω that result in superdiffusion at finite speeds. For $0 < \omega < 1/2$, the shape of the finite-speed particle distribution differs substantially from that of the infinite-speed particle distribution. In contrast, for $1/2 < \omega < 1$, the distributions have the same shape in both cases.

Thus, our analysis of the results obtained has led to the following inferences:

(1) unlike the second moment, higher-order moments prove to depend on the dimension of space. For any t, however, finite limiting moments of the projection of the position vector onto each coordinate axis exist;

(2) the finiteness of the free-motion speed v reduces the number of possible (for $v = \infty$) diffusion regimes from five to four and changes their regions in the diagram on the (α, ω) plane;

(3) in the case of subdiffusion, the finiteness of the speed has no effect on the shape of the asymptotic distribution of particles: for large times, it makes no difference whether the particles move at finite or infinite speed between the periods of their stay in traps. The distribution itself is described by a fractional stable distribution;

(4) in the case of superdiffusion ($\alpha > 1$), the finite speed slows down the expansion of the diffusion packet of particles; however, this affects only the diffusion coefficient $D_v = (1 + \mu a/v)^{-1} D_\infty$ rather than the shape of the distribution.

The opposite situation occurs if $\alpha < 1$: the kinematic restriction becomes a predominant factor in the formation of the asymptotic behavior of the process. The diffusive distribution, attempting to expand more rapidly than the ballistic region does, thus proves to be confined to the corridor $[-vt, vt]$ and, as $t \to \infty$, becomes almost completely concentrated near its walls (Fig. 8).

7. Conclusions

When completing this review, I prepared myself to hear the following question most likely asked by a reader who is not quite familiar with Lévy-stable laws and processes: Why are these rich and efficient techniques so seldom employed in applications, even remaining little known? In fact, they are not too seldom employed. A multitude of various applications can be found in review articles, books, and collections of papers [6–12, 35, 39]. Nevertheless, some reasons for the 'retarded diffusion' of stable laws into applications really do exist. In my opinion, there are three such reasons.

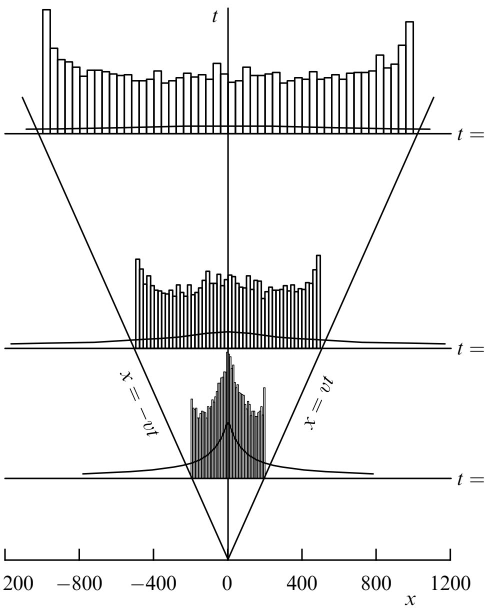

Figure 8. Distributions $p(x,t)$ for one-dimensional diffusion ($\alpha = 1/2$) at $v = 10$ (histograms representing Monte Carlo computations) and $v = \infty$ (solid curves). In both cases, traps with a mean confinement time of unity are present.

The first one is of a fundamental nature: infinite variance (not to mention infinite mean) seems to defy common sense to many people, since a definite physical meaning is imparted in many cases to the variance and in all cases to the mean. Actually, this is more a tribute to traditions than a requirement of logic: we constantly deal with particular numerical values in measuring physical quantities and do not encounter divergences. Divergences originate only when integrals appear, and this is a property of the very physical theories based either on the notion of a Gibbs ensemble with a infinite number of copies or on the possibility of an infinitely large number of independent changes. The normal remedy is to find a physical reason for the existence of a finite minimum value of the quantity considered. The distribution, cut off by this value, possesses all moments and lies in the attraction domain of the normal law. However, such an approach does not rule out employing other stable laws. Experience suggests that, if a distribution is cut off at a large distance, and this point is preceded by a large interval where a power law is valid, then the sum of independent random variables selected from this distribution will at a certain stage (as the number of summands increases) behave as if there were no cutoffs, i.e., it will be distributed according to the corresponding stable law, and will reach a normal regime only with further increases in the number of summands. We term this phenomenon *intermediate asymptotics* [76].

The other two reasons for the cautious dealings of applied physicists with Lévy-stable laws are quite trivial: the lack of explicit analytical expressions for densities (which, however, is no longer a valid reason in our computerized age) and the deficiency of information on these laws and processes. I hope that this review article will, to a certain extent, fill the gap in the available information and will be beneficial for more widely using the modern tools of probability theory in applied problems.

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8. Appendices

8.1 Fractional stable densities representable in terms of elementary and special functions

a) $q(x;2,1,0) = \dfrac{1}{2\sqrt{\pi}}\exp\left(-\dfrac{x^2}{4}\right), \qquad -\infty < x < \infty\,;$

b) $q\left(x;2,\dfrac{2}{3},0\right) = \dfrac{1}{2\pi}\sqrt{|x|}\,K_{1/3}\left(\dfrac{2}{3\sqrt{3}}\,|x|^{3/2}\right),$

$-\infty < x < \infty\,;$

c) $q\left(x;\dfrac{3}{2},1,\dfrac{1}{3}\right)$

$$= \begin{cases} \sqrt{\dfrac{3}{\pi}}\,x^{-1}\exp\left(-\dfrac{2}{27}\,x^3\right)W_{1/2,\,1/6}\left(\dfrac{4}{27}\,x^3\right), & x > 0\,,\\[2mm] \dfrac{1}{2\sqrt{3\pi}}|x|^{-1}\exp\left(\dfrac{2}{27}|x|^3\right)W_{-1/2,\,1/6}\left(\dfrac{4}{27}|x|^3\right), & x < 0\,; \end{cases}$$

d) $q\left(x;\dfrac{3}{2},1,-\dfrac{1}{3}\right)$

$$= \begin{cases} \dfrac{1}{2\sqrt{3\pi}}\,x^{-1}\exp\left(\dfrac{2}{27}\,x^3\right)W_{-1/2,\,1/6}\left(\dfrac{4}{27}\,x^3\right), & x > 0\,,\\[2mm] \sqrt{\dfrac{3}{\pi}}|x|^{-1}\exp\left(-\dfrac{2}{27}|x|^3\right)W_{1/2,\,1/6}\left(\dfrac{4}{27}|x|^3\right), & x < 0\,; \end{cases}$$

e) $q(x;1,1,0) = \left[\pi(1+x^2)\right]^{-1}, \qquad -\infty < x < \infty\,;$

f) $q\left(x;1,\dfrac{1}{2},0\right) = \dfrac{1}{2\sqrt{\pi^3}}\exp\left(\dfrac{x^2}{4}\right)E_1\left(\dfrac{x^2}{4}\right),$

$-\infty < x < \infty\,;$

g) $q\left(x;\dfrac{2}{3},1,1\right)$

$$= \begin{cases} \sqrt{\dfrac{3}{\pi}}\,x^{-1}\exp\left(-\dfrac{2}{27}\,x^{-2}\right)W_{1/2,\,1/6}\left(\dfrac{4}{27}\,x^{-2}\right), & x > 0\,,\\[2mm] 0\,, & x < 0\,; \end{cases}$$

h) $q\left(x;\dfrac{2}{3},1,0\right) = \dfrac{1}{2\sqrt{3\pi}}|x|^{-1}\exp\left(\dfrac{2}{27}\,x^{-2}\right)$

$\times\, W_{-1/2,\,1/6}\left(\dfrac{4}{27}\,x^{-2}\right), \qquad -\infty < x < \infty\,;$

i) $q\left(x;\dfrac{2}{3},1,-1\right)$

$$= \begin{cases} 0\,, & x > 0\,,\\[2mm] \sqrt{\dfrac{3}{\pi}}\,|x|^{-1}\exp\left(-\dfrac{2}{27}\,x^{-2}\right)W_{1/2,\,1/6}\left(\dfrac{4}{27}\,x^{-2}\right), & x < 0\,; \end{cases}$$

j) $$q\left(x;\dfrac{1}{2},1,1\right) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\,x^{-3/2}\exp\left(-\dfrac{1}{4x}\right), & x > 0\,,\\[2mm] 0\,, & x < 0\,; \end{cases}$$

k) $$q\left(x;\dfrac{1}{2},1,-1\right) = \begin{cases} 0\,, & x > 0\,,\\[2mm] \dfrac{1}{2\sqrt{\pi}}\,|x|^{-3/2}\exp\left(-\dfrac{1}{4|x|}\right), & x < 0\,; \end{cases}$$

l) $q\left(x; \frac{1}{2}, \frac{1}{2}, 1\right) = \begin{cases} \left[\pi\sqrt{x}\,(1+x)\right]^{-1}, & x > 0, \\ 0, & x < 0; \end{cases}$

m) $q\left(x; \frac{1}{2}, \frac{1}{2}, -1\right) = \begin{cases} 0, & x > 0, \\ \left[\pi\sqrt{|x|}\,(1+|x|)\right]^{-1}, & x < 0; \end{cases}$

n) $q\left(x; \frac{1}{3}, 1, 1\right) = \begin{cases} \frac{1}{3\pi}\, x^{-3/2} K_{1/3}\left(\frac{2}{3\sqrt{3}}\, x^{-1/2}\right), & x > 0, \\ 0, & x < 0; \end{cases}$

o) $q\left(x; \frac{1}{3}, 1, -1\right) = \begin{cases} 0, & x > 0, \\ \frac{1}{3\pi}\, |x|^{-3/2} K_{1/3}\left(\frac{2}{3\sqrt{3}}\, |x|^{-1/2}\right), & x < 0. \end{cases}$

In the above formulas,

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2)} \left(\frac{z}{2}\right)^\nu \int_1^\infty \exp(-zt)(t^2 - 1)^{\nu - 1/2}\, \mathrm{d}t,$$

$$W_{\lambda,\mu}(z) = \frac{z^{\mu+1/2} \exp(-z/2)}{\Gamma(\mu - \lambda + 1/2)} \times \int_0^\infty \exp(-zt)\, t^{\mu-\lambda-1/2}(1+t)^{\mu+\lambda-1/2}\, \mathrm{d}t,$$

$$\dot{E}_1(z) = \int_z^\infty \frac{\exp(-t)}{t}\, \mathrm{d}t.$$

8.2 Fractional integrodifferential operators

Fractional Riemann–Liouville integrals ($\alpha > 0$):

$$(I_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{f(\xi)\, \mathrm{d}\xi}{(x-\xi)^{1-\alpha}}, \tag{A.1}$$

$$(I_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\xi)\, \mathrm{d}\xi}{(x-\xi)^{1-\alpha}}, \tag{A.2}$$

$$(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(\xi)\, \mathrm{d}\xi}{(\xi - x)^{1-\alpha}}. \tag{A.3}$$

Fractional Riemann–Liouville derivatives ($0 < \alpha < 1$):

$$\left(D_+^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^x \frac{f(\xi)\, \mathrm{d}\xi}{(x-\xi)^\alpha}, \tag{A.4}$$

$$\left(D_{0+}^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{f(\xi)\, \mathrm{d}\xi}{(x-\xi)^\alpha} \equiv \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}, \tag{A.5}$$

$$\left(D_-^\alpha f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^\infty \frac{f(\xi)\, \mathrm{d}\xi}{(\xi - x)^\alpha}.$$

Fractional Marchaud derivatives ($0 < \alpha < 1$):

$$\left(D_+^\alpha f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-\xi)}{\xi^{1+\alpha}}\, \mathrm{d}\xi = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}}\, \mathrm{d}\xi, \tag{A.6}$$

$$\left(D_-^\alpha f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x+\xi)}{\xi^{1+\alpha}}\, \mathrm{d}\xi. \tag{A.7}$$

Riesz potential ($\alpha > 0$, $\alpha \neq 1, 3, 5, \ldots$):

$$\left(I^\alpha f\right)(x) = \frac{1}{2\cos(\alpha\pi/2)} \Big(\left(I_+^\alpha f\right)(x) + \left(I_-^\alpha f\right)(x)\Big) = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^\infty \frac{f(\xi)\, \mathrm{d}\xi}{|x-\xi|^{1-\alpha}}, \tag{A.8}$$

where I_+^α and I_-^α are given by expressions (A.1) and (A.3), respectively.

Riesz derivative ($0 < \alpha < 1$):

$$D^\alpha f \equiv (I^\alpha)^{-1} f = \frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)} \times \int_{-\infty}^\infty \frac{f(x) - f(x-\xi)}{|\xi|^{1+\alpha}}\, \mathrm{d}\xi = \frac{\alpha}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} \times \int_0^\infty \frac{2f(x) - f(x-\xi) - f(x+\xi)}{\xi^{1+\alpha}}\, \mathrm{d}\xi = \left[2\cos\left(\frac{\alpha\pi}{2}\right)\right]^{-1} \left(D_+^\alpha f + D_-^\alpha f\right), \tag{A.9}$$

where D_+^α and D_-^α are given by expressions (A.6) and (A.7), respectively.

Feller potential ($0 < \alpha < 1$):

$$\left(M_{u,v}^\alpha f\right)(x) = u\left(I_+^\alpha f\right)(x) + v\left(I_-^\alpha f\right)(x) = \int_{-\infty}^\infty \frac{u + v + (u-v)\,\mathrm{sign}\,(x-\xi)}{|x-\xi|^{1-\alpha}} f(\xi)\, \mathrm{d}\xi, \tag{A.10}$$

where $u^2 + v^2 \neq 0$. In particular,

$$M_{u,u}^\alpha = 2u\cos\left(\frac{\alpha\pi}{2}\right) I^\alpha,$$

where I^α is given by expression (A.8).

Inverse Feller potential ($0 < \alpha < 1$):

$$\left(M_{u,v}^\alpha\right)^{-1} f = \frac{\alpha}{2A\Gamma(1-\alpha)} \times \int_{-\infty}^\infty \frac{u + v + (u-v)\,\mathrm{sign}\,(x-\xi)}{|x-\xi|^{1+\alpha}} \left[f(x) - f(\xi)\right] \mathrm{d}\xi = \frac{\alpha}{2A\Gamma(1-\alpha)} \int_0^\infty \big[(u+v) f(x) - u f(x-\xi) - v f(x+\xi)\big]\, \xi^{-1-\alpha}\, \mathrm{d}\xi, \tag{A.11}$$

where

$$A = \left[(u+v)\cos\left(\frac{\alpha\pi}{2}\right)\right]^2 + \left[(u-v)\sin\left(\frac{\alpha\pi}{2}\right)\right]^2.$$

In particular,

$$\left(M_{1,0}^\alpha\right)^{-1} = D_+^\alpha,$$

$$\left(M_{0,1}^\alpha\right)^{-1} = D_-^\alpha,$$

$$\left(M_{u,u}^\alpha\right)^{-1} f = \left[2u\cos\left(\frac{\alpha\pi}{2}\right)\right]^{-1} D^\alpha f,$$

where D^α is given by expression (A.9).

Multidimensional Riesz integrodifferentiation is done by the operators

$$(-\Delta_n)^{-\alpha/2} f = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{f(\xi)\,\mathrm{d}\xi}{|x-\xi|^{n-\alpha}}\,, \tag{A.12}$$

where

$$\alpha > 0\,, \quad \alpha \neq n,\, n+2,\, n+4, \ldots,$$

$$\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma\big((n-\alpha)/2\big)}\,,$$

and

$$(-\Delta_n)^{\alpha/2} f = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x-k\xi)|\xi|^{-n-\alpha}\,\mathrm{d}\xi\,, \tag{A.13}$$

where

$$\alpha > 0\,, \quad l = [\alpha]+1\,,$$

$$d_{n,l}(\alpha) = \frac{\pi^{1+n/2}}{2^\alpha \Gamma(1+\alpha/2)\,\Gamma\big((n+\alpha)/2\big)\sin(\alpha\pi/2)} \times \sum_{k=0}^{l}(-1)^k \binom{l}{k} k^\alpha\,.$$

In particular, if $n = 1$, then

$$\gamma_1(\alpha) = 2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right),$$

$$d_{1,1}(\alpha) = -2\Gamma(-\alpha)\cos\left(\frac{\alpha\pi}{2}\right), \qquad \alpha < 1\,,$$

and the operators (A.12) and (A.13) coincide with (A.8) and (A.9), respectively.

The Fourier transform

$$\hat{F}_n f \equiv \int_{R^n} \exp(\mathrm{i}kx)\, f(x)\,\mathrm{d}x\,, \qquad \hat{F}_1 \equiv \hat{F}\,, \tag{A.14}$$

of fractional integrals and derivatives has the form

$$\hat{F}(I^\alpha_\pm f) = |k|^{-\alpha} \exp\left\{\pm \mathrm{i}\alpha\,\frac{\pi}{2}\,\mathrm{sign}\,k\right\} \hat{F} f\,, \qquad 0 < \alpha < 1\,; \tag{A.15}$$

$$\hat{F}(D^\alpha_\pm f) = |k|^{\alpha} \exp\left\{\mp \mathrm{i}\alpha\,\frac{\pi}{2}\,\mathrm{sign}\,k\right\} \hat{F} f\,, \qquad \alpha \geqslant 0\,; \tag{A.16}$$

$$\hat{F}(M^\alpha_{u,v} f) = \left[(u+v)\cos\left(\frac{\alpha\pi}{2}\right) + \mathrm{i}(u-v)\sin\left(\frac{\alpha\pi}{2}\right)\mathrm{sign}\,k\right]|k|^{-\alpha}\hat{F} f\,, \qquad 0<\alpha<1\,; \tag{A.17}$$

$$\hat{F}_n\big((-\Delta_n)^{\alpha/2} f\big) = |k|^\alpha \hat{F}_n f\,. \tag{A.18}$$

In particular,

$$\hat{F}_1\big((-\Delta_1)^{-\alpha/2} f\big) \equiv \hat{F}_1(I^\alpha f) = |k|^{-\alpha}\hat{F}_1 f\,. \tag{A.19}$$

The Laplace transform

$$\hat{L} f \equiv \int_0^\infty \exp(-\lambda x)\, f(x)\,\mathrm{d}x$$

of fractional integrals and derivatives is given by the formula

$$\hat{L}\big(I^\alpha_{0+} f\big) = \lambda^{-\alpha}\big(\hat{L} f\big)\,, \tag{A.20}$$

$$\hat{L}\big(D^\alpha_{0+} f\big) = \lambda^{\alpha}\big(\hat{L} f\big)\,. \tag{A.21}$$

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