# Joint scientific session of the Physical Sciences Division of the Russian Academy of Sciences and the Joint Physical Society of the Russian Federation (26 March 2003) 

A joint scientific session of the Physical Sciences Division of the Russian Academy of Sciences and the Joint Physical Society of the Russian Federation was held on 26 March 2003 at the P N Lebedev Physics Institute, RAS. The following reports were presented at the session:
(1) Ioffe L B, Feĭgel'man M V (L D Landau Institute for Theoretical Physics, RAS, Moscow) "Realization of topologically protected quantum bits in a Josephson junction array";
(2) Saranin A A, Zotov A V, Lifshits V G (Institute of Automation and Control Processes, Far Eastern Branch of RAS, Vladivostok) "Nanostructures on a silicon surface";
(3) Veselago V G (Moscow Institute of Physics and Technology, Dolgoprudnyĭ, Moscow Region; A M Prokhorov Institute of General Physics, Moscow) "Electrodynamics of materials with negative index of refraction".

An abridged version of the reports 1 and 3 is given below.

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## Realization of topologically protected quantum bits in a Josephson junction array

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## 1. Introduction

The present work proposes a new type of topologically nontrivial Josephson lattice, capable of realizing a topologically ordered, spin-liquid-type quantum ground state. At the same time this new state is a superconductor with a charge quantized in units of $4 e$ instead of the usual pair charge $2 e$. The new ground state has a degeneracy of $2^{K}, K$ being the number of topologically nontrivial cycles on the lattice. It is shown that the system we propose possesses the 'built-in' quantum error correction property in the sense that the effect of any type of local noise is exponentially small for a large lattice size. This makes it possible to use the system for creating quantum bits (or arrays thereof) with very large coherence times.

Quantum computing [1, 2], if realized, would be a powerful tool for solving classically intractable problems such as the factorization of large numbers into primes [3] or a search through large databases [4]. The discovery of

[^0]quantum error correction algorithms [5] showed that building a working quantum computer is in principle accomplishable. Doing so, however, remains a formidable challenge because the basic element of the quantum computer - its 'quantum processor' - must be a system with a huge number $2^{K}$ (where $K \sim 10^{6}$ ) of quantum states which are degenerate (or almost degenerate) in energy in spite of external perturbations present in any real system. A quantum algorithm consists in specifying the Hamiltonian of this system as a function of time, and this in such a way that the relative uncertainty of the Hamiltonian parameters is less than $10^{-4}-10^{-6}$ in order that error correction algorithms could be used. If the level of noise and the uncertainty in the Hamiltonian parameters could be reduced even further, by several orders of magnitude, then systems of much smaller size, with $K \sim 10^{4}$, could be used (noise considerably affects how many elements should be added to the processor to ensure that error correction algorithms work) [6]. In other words, the desired system should have the following property: its Hilbert space must have (what we shall call) a protected subspace such that for any local perturbation operator $\hat{O}$ we have $\langle n| \hat{O}|m\rangle=O_{0} \delta_{m n}+o(\exp (-L))$, where $n$ and $m$ are any two of the $2^{K}$ states of the protected state, and $L$ is the linear dimension of the system, which can be made very large. Such a system would realize the nearly complete correction of quantum errors 'physically' rather than 'algorithmically' due to certain properties of the set of its low-lying states.

Although the above requirements appear at first glance to be too unusual for a physical system to meet, recent advances in the theory of strongly correlated quantum systems suggest that things are not so hopeless after all and that examples of systems with somewhat similar properties are in fact already known. Particularly noteworthy work in this connection is that of Wen et al. [7, 8], which shows that the states of the fractional quantum Hall effect on a closed surface with a nontrivial topology have an exponentially weak sensitivity to local perturbations; thus, the state with a filling factor of $1 / 3$ on the surface of a torus should be threefold degenerate. This, of course, should be considered as a purely theoretical result because it is practically impossible to create a two-dimensional electron gas on the surface of a torus and, in addition to that, to apply a strong magnetic field transverse to the surface. Still, this example shows that a quantum system with such properties is in principle conceivable, and the matter simply depends on finding a practically feasible version of this system.

An entirely different theoretical approach to essentially the same problem originates from the long-standing idea of Anderson [9] (see also Ref. [10]) of the unusual spin-liquidtype quantum state which can exist on a triangular lattice of
antiferromagnetically interacting spins $1 / 2$. Even though recent large-scale numerical computations have shown that such a lattice does have a state with antiferromagnetic order (albeit strongly suppressed by quantum fluctuations), similar models were found to exist in which the ground state is indeed spin-liquid in character [11, 12], with toroidal boundary conditions leading to a degenerate ground state in these spin systems also. These ideas were used by Kitaev in his pivotal paper [13], which formulates the general approach to constructing a quantum system with a topologically protected subspace of states, and which proposes a quite simple model Hamiltonian satisfying all necessary requirements. The idea of Kitaev's approach can be formulated as follows: it is necessary to find a quantum system (a) which does not possess a usual (local) order parameter (in other words, a local observer sees it as a 'structureless liquid,' (b) which has a gap in the energy spectrum, and (c) in which a degenerate ground state occurs under topologically nontrivial boundary conditions.

The following picture illustrates clearly what kind of a system we are interested in: imagine an observer traveling over a surface of the closed strip type. There are two possible surfaces of this kind, the cylinder and the Möbius strip. Clearly, if the traveller's range of vision is small compared to the width and length of the strip, he cannot discern one surface from the other. What he needs in order to do this is to 'make a tour round the world,' i.e., to travel around the entire strip along its generatrix. In quantum language, such a trip is analogous to the effect of a nonlocal operator, the only means by which two 'topologically different' states can be distinguished.

The simplest Hamiltonian possessing all these properties has been proposed and solved in Ref. [13]; its important feature is binary quantum variables - spins - associated with the edges of the lattice on which the system is defined. Kitaev showed that in the low-energy limit the system he proposed is described by an effective gauge $Z_{2}$ theory on a lattice with anyons as elementary excitations. He also showed that the anyon character of excitations and the topologydependent ground state degeneracy are interrelated features. The first physical realization of a Hamiltonian with properties similar to those of Kitaev's was proposed in Ref. [14]. This paper showed how an array of Josephson junctions (with a relatively large charging energy $E_{\mathrm{C}}$ ) can be used to realize a model of quantum dimers on a triangular lattice - a model which had already been considered in connection with studies in the spirit of Anderson's approach to the spin liquid problem [12] (to be comprehensive, we note that a number of interesting theoretical studies have most recently been made on spin liquids [15-22]). Another point shown in Ref. [14] is that (in a certain range of parameters) a triangular lattice of quantum dimers does indeed possess all the properties needed to realize a topologically protected qubit - and even a set of such qubits.

However, given the current level of fabrication technology of submicron Josephson junctions, the system proposed in Ref. [14] is rather difficult to realize experimentally. In the present talk we will describe a new approach to the problem, which is more practical and at the same time closer to Kitaev's original model (see Refs [23, 24] for a detailed discussion).

## 2. Josephson lattice and the effective Hamiltonian

The building block of our Josephson array is a rhombus of four superconducting islands connected by identical tunnel-
ing contacts. A transverse magnetic field applied to the system creates within each rhombus a flux $\Phi_{\mathrm{r}}$ equal to $\Phi_{0} / 2$, i.e., half of the superconducting flux quantum. As a result, each of these rhombi taken separately is a bistable quantum system. In the case considered below, of a relatively large Josephson junction energy $E_{\mathrm{J}} \gg E_{\mathrm{C}}$, where $E_{\mathrm{C}}=e^{2} / 2 C$ and $C$ is the capacitance of the tunneling contacts, a rhombus can be described quasi-classically in terms of the superconducting currents it carries. Two quasi-classical states minimizing the Josephson energy differ in the sign of the phase difference on the junctions (which can be equal to $\pm \pi / 4$ ) and hence in the direction in which the current flows around the rhombus. The potential energy barrier for the transition between these two states is of the order of $E_{\mathrm{J}}$. The role of kinetic energy in this case is played by the charging energy $E_{\mathrm{C}}$, and the amplitude of the quantum tunneling process - the transition between states with oppositely directed currents - is of the order of $E_{\mathrm{J}}^{3 / 4} E_{\mathrm{C}}^{1 / 4} \exp \left(-S_{0}\right)$, where the tunneling action $S_{0} \sim$ $\left(E_{\mathrm{J}} / E_{\mathrm{C}}\right)^{1 / 2}$. A quantitative analysis requires using the Lagrangian of a rhombus of the from

$$
\begin{equation*}
\mathcal{L}_{\text {rhomb }}=\sum_{(i j)} \frac{1}{16 E_{\mathrm{C}}}\left(\dot{\phi}_{i}-\dot{\phi}_{j}\right)^{2}+E_{\mathrm{J}} \cos \left(\phi_{i}-\phi_{j}-a_{i j}\right), \tag{1}
\end{equation*}
$$

where $\phi_{i}$ is the phase of the superconducting order parameter in an $i$ th island, and the $a_{i j}$ are chosen in such a way as to ensure the correct value $\Phi_{\mathrm{r}}$ of the magnetic flux through the rhombus; the summation is over $i, j=1,2,3,4$. The potential energy of the rhombus as a function of the phase difference along its major diagonal is obtained by minimizing the Josephson energy with respect to the phases of two 'side' islands, to give

$$
\begin{equation*}
U\left(\phi_{i j}\right)=-2 E_{\mathrm{J}}\left(\left|\cos \frac{\phi_{i j}}{2}\right|+\left|\sin \frac{\phi_{i j}}{2}\right|\right) . \tag{2}
\end{equation*}
$$

This energy has two minima at $\phi_{i j}= \pm \pi / 2$, which can be used to create an elementary 'unprotected' qubit [25]. In either of these states the phase changes by $\pm \pi / 4$ on each junction in the clockwise direction around the rhombus. We will denote these states by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively, and set them to correspond to the projections of the fictitious 'spins' $1 / 2$ described in the basis of Pauli matrix $\sigma^{z}$. The operator of transition between these states will then have the form $v \sigma^{x}$. By numerically solving equations written in the quasi-classical approximation we obtain

$$
v \sim E_{\mathrm{J}}^{3 / 4} E_{\mathrm{C}}^{1 / 4} \exp \left(-S_{0}\right) \ll E_{\mathrm{J}}
$$

$$
\begin{equation*}
S_{0} \approx 1.61\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right)^{1 / 2} \tag{3}
\end{equation*}
$$

for the tunneling amplitude.
The next stage in 'constructing' the array is to assemble superconducting rhombi into a hexagonal lattice with major diagonals of the rhombi as edges (Fig. 1). We assume the linear dimensions of the elements to be chosen in such a way that the same magnetic field ensures the magnetic flux $\Phi_{\text {hex }}=(n+1 / 2) \Phi_{0}$ through each hexagon of the array (here $n$ is any integer; it should also be noted that we assume the condition $\mathcal{L} I_{\mathrm{c}} \ll \Phi_{0}$, with $\mathcal{L}$ the inductance of a rhombus, to be fulfilled everywhere; the same condition should be fulfilled for the hexagon made of rhombi). Let us consider possible


Figure 1. Example of the proposed Josephson lattice. Thick lines represent superconducting wires, with one Josephson junction on each wire as shown in the inset. The array is in a magnetic field, whose flux through each rhombus and each hexagon is equal to a half-integer number of superconducting flux quanta. Thin lines are 'effective' bonds from which the hexagonal lattice is built. The Josephson interaction on these effective bonds has a period of $\pi$ as a function of the phase difference. The lattice has one large hole, which corresponds to the number of 'qubits' $K=1$.
low-energy states of a hexagon constructed from rhombi. The total number of such states $2^{6}=64$ corresponds to the independent enumeration of all the binary states possible for the rhombi. Of these states, however, not all actually have a low energy but only those for which the numbers of 'clockwise' and 'counterclockwise' rhombi in the hexagon are even. Indeed, note that the total accumulation of gaugeinvariant phase difference which arises as we go around the rhombus should be equal to $2 \pi m$ in order that the condition of uniqueness be fulfilled for the superconducting wave function. Noting that in the lowest-energy states of each rhombus the phase differences along the rhombus's major diagonal are $\pm \pi / 2$ and that the magnetic field inside the hexagon adds a phase shift of $\pi$, we conclude that the allowable configurations should contain even numbers (i.e., $0,2,4,6$ ) of rhombi with a phase difference of $\pi / 2$ (and, correspondingly, also with a phase difference of $-\pi / 2$ ). As for the states obtained from the 'allowed' ones by reversing the current in an odd number of rhombi, these unavoidably must contain rhombi with a non-optimal phase difference along the diagonal (to compensate the extra phase difference of $\pi$ which arises as we go around the hexagon). This means that in such states the current around the hexagon is of the order of a fraction of $I_{\mathrm{c}}$, and therefore the energy of these states exceeds that in the 'allowed' states by an amount of the order of $E_{\mathrm{J}}$. We can neglect states with this energy when considering the system's low-energy spectrum. Mathematically, this can be expressed by introducing the projection operator of the hexagon, $\hat{P} \equiv \prod_{a} \sigma_{a}^{z}$, where the product is taken over all six rhombi, and by imposing the condition $\hat{P}|\Psi\rangle=|\Psi\rangle$ on the space of allowable quantum states.

Let us now consider the total effective lattice shown in Fig. 2 and let the Latin letters $a, b, c \ldots$ denote the sites of the triangular lattice dual to the lattice of our hexagons. Then


Figure 2. Arrangement of the effective discrete $Z_{2}$ degrees of freedom of the Josephson lattice depicted in Fig. 1. Spin variables are associated with the bonds of the triangular lattice (shown by thick lines), and constraints on allowable configurations are determined at the sites of this lattice. The contours $\gamma$ and $\gamma^{\prime}$ are used for constructing the operator of the topological order parameter and that of elementary excitations, respectively.
each rhombus is put in correspondence with a pair $(a, b)$ of the neighboring sites of the dual lattice; the rule of correspondence is obvious: the edge $(a, b)$ of the dual lattice intersects the rhombus. The Hilbert space of the low-lying states is determined by the constraint

$$
\begin{equation*}
\hat{P}_{a}|\Psi\rangle=|\Psi\rangle \tag{4}
\end{equation*}
$$

for all sites $a$ on the dual lattice. The Hamiltonian operating in this space has the form

$$
\begin{equation*}
H=-r \sum_{(a b c)} Q_{(a b c)}, \quad Q_{(a b c)}=\sigma_{a b}^{x} \sigma_{b c}^{x} \sigma_{c a}^{x}, \tag{5}
\end{equation*}
$$

and the amplitude $r$ of the simultaneous 'flip' of the three rhombi $a b, b c, c a$ which have an island in common is of the order of

$$
\begin{equation*}
r \approx E_{\mathrm{J}}^{3 / 4} E_{\mathrm{C}}^{1 / 4} \exp \left(-3 S_{0}\right) \tag{6}
\end{equation*}
$$

where $S_{0}$ is defined in Eqn (4). The 'triple' processes described by the Hamiltonian (5) are the lowest order ones (perturbatively) which are compatible with condition (4).

## 3. The ground state, topological degeneracy, and elementary excitations

The Hamiltonian (5), together with the projection condition (4), permits a simple exact solution. This has to do with the fact that all individual projection operators $\hat{P}_{a}$ commute with all the operators $\hat{Q}_{(a b c)}$ entering the Hamiltonian. Therefore the exact ground state wave function can be constructed as follows: take state $|0\rangle$, which is an eigenstate (with an
eigenvalue of +1 ) for all the rhombus operators $\sigma_{a b}^{x}$, and act on it by a projector which separates out only those configurations satisfying Eqn (4),

$$
\begin{equation*}
|G\rangle=\prod_{a} \frac{1+\hat{P}_{a}}{\sqrt{2}}|0\rangle . \tag{7}
\end{equation*}
$$

It is the state $G$ which, as is easily verified, is the exact ground state. In the language of the effective 'spin' Hamiltonian this state is a 'spin liquid' because spin operators in this state do not have nonzero averages (a non-magnetic state). In the case of a topologically trivial lattice, all the remaining states are separated from the ground state $G$ by a gap equal to $2 r$, as will be shown shortly. What we are interested in, however, is lattices with topologically non-trivial cycles because such cycles lead to an exactly degenerate ground state. To see this, consider the operators

$$
\begin{equation*}
\hat{T}_{q}=\prod_{\left(\gamma_{q}\right)} \sigma_{a b}^{z}, \tag{8}
\end{equation*}
$$

where the contour $\gamma_{q}$ starts at one boundary of the lattice and ends at another [for a lattice with $K$ holes, $K$ independent operators of the type (8) can be defined]. Each of these operators commutes with the Hamiltonian as well as with the condition (4). The physical meaning of these operators is simple: they compute the parity of the number of rhombi in the state $|\uparrow\rangle$ which intersect the contour $\gamma_{q}$. Let us now define the quantum states

$$
\begin{equation*}
\left|G_{\mathrm{f}}\right\rangle=\prod_{q} \frac{1+c_{q} \hat{T}_{q}}{\sqrt{2}}|G\rangle, \tag{9}
\end{equation*}
$$

where $c_{q}= \pm 1$ is the eigenvalue of the operator $\hat{T}_{q}$. Each of these states involves only configurations with a definite parity of rhombi in the state $|\uparrow\rangle$ which intersect the contour $\gamma_{q}$, so it can be said that the states $G_{\mathrm{f}}$ possess a definite phase difference (either 0 or $\pi$ ) between the lattice boundaries. It is a set of $2^{K}$ such states on a lattice with $K$ holes, which forms the $2^{K}$-dimensional basis of the topologically protected subspace we are looking for.

Elementary excitations can be constructed in a similar manner. Let us separate out an island $(a b c)$ and define the state $\left|v_{(a b c)}\right\rangle$ by applying an operator defined as $\hat{v}_{(a b c)}=$ $\prod_{\gamma^{\prime}} \sigma_{(d f)}^{z}$, where the product is taken over all rhombi intersecting the path $\gamma^{\prime}$ which starts at the outer boundary of the lattice and terminates at the island $(a b c)$,

$$
\begin{equation*}
\left|v_{(a b c)}\right\rangle=\prod_{q} \frac{1+c_{q} \hat{T}_{q}}{\sqrt{2}} \prod_{a} \frac{1+\hat{P}_{a}}{\sqrt{2}} \hat{v}_{(a b c)}|0\rangle . \tag{10}
\end{equation*}
$$

The operator $\hat{v}_{(a b c)}$ does not commute with only one of the operators $Q_{a b c}$, namely with that at the end of the contour $\gamma^{\prime}$. Therefore this state differs from the ground state only in the respect that this unique $Q$ operator has an eigenvalue -1 instead of +1 , i. e., the excitation energy is $2 r$. It is important to note that this excitation is strictly localized in a definite island. The physical meaning of the excitation is very simply understood by noting that the ground state $G$ of the system describes macroscopically a superconductor with an elementary charge $4 e$ (two Cooper pairs) instead of the usual $2 e$. This occurs because the magnetic flux $\Phi_{0} / 2$ has a frustrating effect on each of the rhombi, thus suppressing the transport of individual Cooper pairs (a similar phenomenon was predicted
[26] for a chain of the same kind of rhombi as ours). As a result, an 'odd' Cooper pair which finds itself on one of the islands is a localized excitation above the ground state. A general excited state is constructed as a set of elementary excitations $\left|v_{(a b c)}\right\rangle$. Note that creating an excitation at one lattice boundary and moving it to another is equivalent to the action of the operator $\hat{T}_{q}$; in other words, this process is equivalent to the operator $\tau_{q}^{z}$ acting in the $2^{K}$-dimensional basis of a protected subspace.

Now consider the matrix elements $O_{\alpha \beta}=\left\langle G_{\alpha}\right| \hat{O}\left|G_{\beta}\right\rangle$ of the local operator $\hat{O}$ between two states of a protected space. The term local will refer to an operator containing a small (compared with the linear dimensions of the system) number of operators $\sigma_{a b}^{x, y, z}$. Let us first see that for $\alpha \neq \beta$ such matrix elements vanish. For this, just note that with the ground states as given by Eqn (9) it is possible to replace $\hat{O}$ by $\hat{O}^{\prime}=\mathcal{P} \hat{O} \mathcal{P}$, where $\mathcal{P}=\prod_{a}\left(1+\hat{P}_{a}\right) / 2$, in such a calculation. It is now easy to see that all the operators involved in $\hat{O}^{\prime}$ commute with the operators $\hat{T}_{q}$ and therefore matrix elements between states differing in the eigenvalues of some of the operators $\hat{T}_{q}$ vanish. Similarly, the difference of the matrix elements of $\hat{O}$ between states of different parity along one of the contours $\gamma_{q}$ is calculated to be zero. Thus, the Hamiltonian we consider describes a system which can represent an ideal quantum memory. The 'reduced' model (4) and (5) we have considered here is actually very similar to the original model of Kitaev [13] and differs from it by the lattice geometry and by replacing one part of the Hamiltonian by the projection condition (4), which is equivalent to neglecting one of the two types of elementary excitations (see below). In a real physical system of Josephson junctions there are of course processes which are not taken into account in the idealized model (4) and (5). Below we show that the effect of such processes (which we assume to be small perturbations) boils down to producing differences between the matrix elements $O_{\alpha \beta}$ and const $\times \delta_{\alpha \beta}$, which are exponentially small in the lattice size.

## 4. Effect of physical perturbations; half-vortices and the phase diagram

Let us now consider corrections to the ideal model (4) and (5). Static corrections are due primarily to two facts: the fluxes through the rhombi, $\Phi_{\mathrm{r}}$, and through the hexagons, $\Phi_{\text {hex }}$, differ from their nominal half-integer values, and the critical currents through different Josephson junctions are different. Assuming these corrections to be small (which is quite realistic for present-day technology of Josephson arrays fabrication) one can obtain the perturbation Hamiltonian in the form

$$
\begin{equation*}
\delta H_{1}=\sum_{(a b)} V_{a b} \sigma_{a b}^{z}+\sum_{(a b),(c d)} V_{(a b),(c d)} \sigma_{a b}^{z} \sigma_{c d}^{z}, \tag{11}
\end{equation*}
$$

where the first sum accounts for the difference between $\Phi_{\mathrm{r}}$ and $\Phi_{0} / 2$, i.e., $V_{a b}=\epsilon=\pi \sqrt{2} E_{\mathrm{J}}\left(\Phi_{\mathrm{r}} / \Phi_{0}-1 / 2\right)$, and the coefficients $V_{(a b),(c d)}$ in the second sum contain contributions proportional to the spatial fluctuations $E_{\mathrm{J}}$ and to uncertainties in specifying the quantities $\Phi_{\text {hex }}$. The summation in the first term runs over all the neighbors $b(a)$ of a given site $(a)$ in the dual (i.e., triangular) lattice, and then also over all its sites (a); thus, this is an independent summation over all rhombi. In the second term, the summation is over all pairs of rhombi which belong to the same hexagon, and then over all the hexagons. The quantities $V_{(a b),(c d)}$ are second order in $\delta E_{\mathrm{J}} / E_{\mathrm{J}}$
and $\delta \Phi_{\text {hex }} / \Phi_{0}$ and hence are generally small in comparison with $V_{(a b)}$. Treating the first term in Eqn (11) perturbatively modifies the ground state wave functions giving

$$
\begin{equation*}
\left|\tilde{G}_{ \pm}\right\rangle=\left|G_{ \pm}\right\rangle+\frac{\epsilon}{4 r} \sum_{(a b)} \sigma_{(a b)}^{z}\left|G_{ \pm}\right\rangle \tag{12}
\end{equation*}
$$

This perturbation theory is applicable as long as $\epsilon<r r$. It is easily seen that the matrix elements of local operators calculated with the modified wave functions (12) are, as before, proportional to $\delta_{\alpha \beta}$. Nontrivial matrix elements arise only in $L$ th order of perturbation theory, when terms containing $L$ operators $\sigma^{z}$ appear in the corrected wave function. Physically, this corresponds to including the tunneling of excitations (i.e., of single Cooper pairs) across the lattice, from its outer to the inner boundary or vice versa. As already mentioned above, such processes correspond to the operator $\tau_{q}^{z}$ in the basis of states with definite eigenvalues of the operators $\hat{T}_{q}$. In particular, they lead to an exponentially small energy splitting, $E_{+}-E_{-} \sim O_{+}-O_{-} \sim(\epsilon / 2 r)^{L}$.

Let us now consider processes leading to transitions of the type $\left|G_{+}\right\rangle \rightarrow\left|G_{-}\right\rangle$between states of definite parity. In our discussion thus far the amplitudes of such processes have been zero, due to the prohibition we imposed from the outset, on 'wrong-parity' configurations violating condition (4). Although this prohibition may in fact be violated, this entails a penalty, in the form of a large energy of the resulting excited state. This state is similar in nature to a usual superconducting vortex, with the difference that the phase accumulation about the center of the vortex is $\pi$ rather than the usual $2 \pi$. The reason is that in our system the energy of an individual 'edge' of the hexagonal lattice has a periodicity of $\pi$ [see Eqn (2)]. The energy of such a vortex on a hexagonal lattice is

$$
\begin{equation*}
E_{\mathrm{v}}(R)=\frac{\pi E_{\mathrm{J}}}{4 \sqrt{6}}(\ln R+c), \quad c \approx 1.2 \tag{13}
\end{equation*}
$$

and contains a logarithmic dependence on the lattice size $R$. Let us consider a single-hole lattice as in Fig. 1. The values of the topological invariant $\hat{T}_{q}$ can change in a process in which two half-vortices of different sign emerge near the inner boundary, one of which tunnels around the entire hole and then annihilates with the other. The amplitude of such a process arises in the order $\Lambda \gg 1$ of perturbation theory in $v / E_{\mathrm{V}}(L)$, where $v$ is flip amplitude for an individual rhombus defined in Eqn (4), and $\Lambda$ the hole perimeter. The processes we have described ultimately lead to terms of the form $h_{x} \tau^{x}$, with $h_{x} \sim\left[v / E_{\mathrm{v}}(L)\right]^{\Lambda}$, in the effective Hamiltonian operating in the protected subspace.

The above picture is valid as long as the condition $E_{\mathrm{C}} \ll E_{\mathrm{J}}$ is fulfilled, i.e., that quantum effects are numerically small. As $E_{\mathrm{C}}$ increases, ultimately (at $E_{\mathrm{C}} \geqslant E_{\mathrm{J}}$ ) a phase transition to an insulating state should occur (see, for example, the review article [27]). On the other hand, when the values of the flux $\Phi_{\mathrm{r}}$ deviate from $\Phi_{0} / 2$, then the unusual state which we found - one with paired Cooper pairs should give rise to the usual superconducting state with charge quantum $2 e$. This transition, as can be shown, is a second-order phase transition equivalent to the orderdisorder transition in the two-dimensional quantum Ising model, and it occurs at $\epsilon \sim r$. The phase diagram of our model is shown schematically in Fig. 3. It is assumed that the quantum phase transition from the 'topologically ordered superconductor' phase we discovered occurs immediately to


Figure 3. Schematic phase diagram for the case of half-integer $\Phi_{\text {hex }}$ at low temperatures. Here $\delta \Phi_{\mathrm{d}}$ is the deviation of the magnetic flux through each rhombus from its ideal value $\Phi_{0} / 2$, and the parameter $\eta \sim 1$. SC denotes the usual superconducting phase, and SCT is a phase with long-range order with respect to the variable $\cos 2 \phi$ in continuous variables and with discrete topological order. The phases SCT and SC are separated by the line of a phase transition equivalent to a quantum transition in the twodimensional Ising model in a transverse field. Ins is the insulating phase.
the usual insulating phase. Generally speaking, various insulating phases, with or without topological order, can exist. This question was considered elsewhere [24] and will not be pursued here.

Let us turn finally to the question of the exchange statistics of excitations in the 'topologically ordered superconductor' phase being studied here. As we have seen, there are two different types of elementary excitations: localized single Cooper pairs (carrying a charge $q_{e}=2 e$ ) and half-vortices (carrying a magnetic flux $\Phi_{\mathrm{v}}=\Phi_{0} / 2=$ $h c / 4 e$ ). Consider a process in which a half-vortex adiabatically moves in a closed contour around a Cooper pair. Then all the coordinates of the system return to their original values, but because of the Aharonov-Bohm effect its full wave function is multiplied by the phase factor $\exp \left(\mathrm{i} q_{\mathrm{e}} \Phi_{\mathrm{v}} / \hbar c\right)=-1$. Thus, the state under study realizes the simplest version of the anyon statistics of elementary excitation [28].

## 5. Quantum manipulations

Let us now consider the simplest quantum operations which can be performed upon the states of a protected subspace. Recall first that the eigenvalues of the operator $\hat{T}_{q}$, equal to $\pm 1$, have a simple physical meaning: they correspond to phase differences of either 0 or $\pi$ between the inner and outer boundaries, respectively, of the system of Fig. 1. This fact enables the rotation operator $U_{q}^{z}=\exp \left(\mathrm{i} \epsilon_{J} \tau_{q}^{z}\right)$ to be realized by closing the outer and inner (index $q$ ) boundaries of the array be means of a weak Josephson junction with energy $\epsilon_{\mathrm{J}}$ for a time $t$. In an entirely similar manner, the two-qubit operator $U_{q, p}^{z}=\exp \left(\mathrm{i}_{\mathrm{J}}^{q p} t \tau_{q}^{z} \tau_{p}^{z}\right)$ can be realized by connecting via a Josephson junction two inner boundaries corresponding to the $q$ th and $p$ th qubits.

The operations described above depend on a continuous parameter and can therefore be performed only with finite accuracy. In our system, a number of discrete quantum operators can also be realized, which can be specified with a much higher accuracy. These include, for example, the operators $\tau_{q}^{z}$ and $\tau_{q}^{x}$ and their square roots, $\sqrt{\tau^{z}}=$ $\left(1+\mathrm{i} \tau^{z}\right) / \sqrt{2 \mathrm{i}}$ and $\sqrt{\tau^{x}} \stackrel{q}{=}\left(1+\mathrm{i} \tau^{x}\right) / \sqrt{2 \mathrm{i}}$; see Ref. [23] for more details on the realization of these operators. Unfortunately, the discrete operations which can be carried out in our
system do not constitute a complete set of operators needed 22 . for universal quantum computing. Nor should they: as shown in Ref. [13], the complete set of exact quantum operators can only be realized in a system with non-Abelian topological order - in a system, in other words, where excitations are non-Abelian anyons. Our lattice, however, produces the 26 simplest Abelian anyons, as already explained above. Thus, ㅌoiz 27. the proposed model can be considered as a model of an ideal quantum memory, but as yet not a model of an ideal quantum computer. A theory of Josephson lattice-based topologicallyordered lattice systems with a non-Abelian gauge group has been given recently in Ref. [29].

## 6. Conclusions

In the present talk, which is the development of the ideas of Ref. [14], we offer a new type of a Josephson lattice, capable of acting as an 'ideal' quantum memory. Compared to the original version outlined in Ref. [14], there are a number of important advantages to the new system, namely: (i) it operates in the parameter range $E_{\mathrm{J}} \gg E_{\mathrm{C}}$, thereby reducing the level of poorly controlled electric noise due to offset charges in the insulating substrate; (ii) it employs only one type of Josephson junctions and is therefore much simpler to fabricate; (iii) in the ideal case it is exactly $2^{K}$-fold degenerate, whereas in the version in Ref. [14] degeneracy was achieved asymptotically with the size of the system, with accuracy $\exp (-c L)$, where $c \sim 1$; for the new system a similar constant is calculated, including small perturbations, to be of the order of $\ln (r / \epsilon) \gg 1$, i.e., an array of smaller size will suffice to achieve the desired accuracy.

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## Electrodynamics of materials with negative index of refraction

## V G Veselago

Over the past few years there has been an avalanche of progress in a new branch of electrodynamics - electrodynamics of materials with negative refraction. Experiments in this area were pioneered by a group of physicists at the University of California at San Diego, USA [1, 2]. They demonstrated the unusual electrodynamic properties of some composite materials, which can be explained purely formally by ascribing a negative index of refraction $n$ to these materials. These composites are the assemblies of small metallic elements arranged into strictly regular crystal-like geometric structures. The structures can be considered continuous for wavelengths considerably longer than the size of and the separation between its constituent elements. The UCSD experiments were performed in the centimeter wavelength range on composites with element size and separation typically of the order of 7 to 10 mm .

The key experimental finding of the San Diego study was a rather unusual manifestation of Snell's law of refraction for such materials. In Fig. 1 is shown the passage of a light ray through the interface between two media with indices of refraction $n_{1}$ and $n_{2}$. If we take $n_{1}=1$ (without loss of generality) then, customarily, a refracted ray takes the path $1-4$. In the San Diego experiments the ray took the path $1-3$.


Figure 1. Refraction of light at the interface between two media. The paths $1-4$ and $1-3$ are taken by the incident and refracted rays in the cases $n_{2}>0$ and $n_{2}<0$, respectively.


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