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Correlation and percolation properties of turbulent diffusion

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Contents

Abstract. Ideas on characteristic behavior of correlation functions underlie all models of turbulent diffusion. This paper sets forth a consistent analysis of these correlation ideas, beginning with Taylor's work of 1921, which pioneered the use of the autocorrelation function, and ending with works on the percolation theory of turbulent diffusion. Despite the fact that specific physical problems are significantly different, the commonality of the theoretical notions involved is emphasized. It is shown how the ideas of 'long-range' correlations and fractality enter into the percolation method. The 'universality' of the percolation approach to the description of turbulent diffusion is discussed at some length.

1. Introduction

The aim of this paper is to discuss some theoretical ideas, which have played an important part in the understanding of turbulent diffusion. This scientific problem has an eighty-year history. An abundance of papers is devoted to the problem, but its complete solution is still a long way off. Turbulence is one of the fundamental phenomena occurring widely in nature. It manifests itself in quite different forms, depending on whether a study is made of the turbulence of a liquid, an atmosphere, or a plasma. A wide variety of plasma instabilities are responsible for the development of different types of plasma turbulence: Langmuir, ion-sound, drift, etc. The diversity of forms necessitates not only new special description methods, but also an analysis of behavior mechanisms common to different types of turbulence. Exactly 40 years have passed since Vedenov, Velikhov, Sagdeev and Drummond, and Pines constructed the quasi-linear theory for weakly turbulent plasmas [1, 2]. We note that the decades devoted to the development of quasi-linear ideas fall in the middle of the historical space of time counted from Taylor's pioneering work [3]. One can see that the analysis of correlation effects and the interrelation between the diffusion coefficient and the autocorrelation function have been of major importance. It would therefore be instructive to trace the relation between Taylor's work that introduced the autocorrelation function with the works on percolation diffusion [4 – 6], whose foundation is the ideas of 'long-range correlations' borrowed from the theory of phase transitions. The works considered in my paper were selected with precisely the aim of demonstrating that this relation is of prime importance for the understanding of the problem of turbulent diffusion as a whole. Owing to the brevity of this presentation, many issues will be omitted. However, both the monographic literature [7 – 16, 73, 74] and *Physics – Uspekhi* contain excellent reviews [17 – 21, 72] which shed light on different aspects of the problem under consideration.

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2. Taylor's and Richardson's results

In 1921 Taylor published a paper [3] in which he put forward a formula establishing a direct relationship between the diffusion coefficient and the velocity autocorrelation function. A radically new 'instrument' was in fact proposed for the analysis of diffusion. Continuing in the spirit of Langevin's and Einstein's works, Taylor wrote a stochastic equation of motion of a probe Lagrangian particle in a random field:

$$x(t) = \int_0^t v(a, \tau)\, d\tau\,, \tag{1}$$

where x is the coordinate of the point, $v(a, t)$ is the random function of Lagrangian velocity, and a is the initial coordinate of the Lagrangian particle. The object of his calculations was the average square of a random particle displacement:

$$\langle x^2 \rangle = \left\langle \int_0^t v(a, t_1)\, dt_1 \int_0^t v(a, t_2)\, dt_2 \right\rangle. \tag{2}$$

Here, the brackets $\langle\ \rangle$ indicate an average over the ensemble of Lagrangian trajectories. We omit the calculations, which are given in detail in scientific and educational literature [7, 10, 11]. The final result of the calculations is represented in the form

$$\langle x^2 \rangle = 2 \int_0^t dt_1 \int_0^{t_1} C(\tau)\, d\tau\,, \tag{3}$$

where $C(\tau)$ is the Lagrange correlation function:

$$C(\tau) = \big\langle v(a, t)\, v(a, t+\tau) \big\rangle\,. \tag{4}$$

More recently, Kampe de Feriet [7] proposed a somewhat different form of this formula:

$$\langle x^2 \rangle = 2 \int_0^t (t - \tau)\, C(\tau)\, d\tau\,. \tag{5}$$

Estimates of the coefficient of turbulent diffusion in Taylor's approach lead to the expression

$$D_{\mathrm{T}}(t) = \frac{1}{2} \frac{d}{dt} \langle x^2 \rangle = \int_0^t C(\tau)\, d\tau\,. \tag{6}$$

From the standpoint of the modern theory of non-equilibrium systems, in this formula it is already possible to 'envisage' the canonical Kubo – Green result [22] for the coefficient of turbulent diffusion:

$$D \propto \int_0^\infty C(\tau)\, d\tau\,. \tag{7}$$

The specific form of the expression for the coefficient of turbulent diffusion $D_{\mathrm{T}}(t)$ depends on the form of the correlation function $C(t)$. Most often recourse is made to the exponential correlation function

$$C(t) = V_0^2 \exp\left(-\frac{|t|}{\tau}\right),$$

where V_0 is the characteristic velocity and τ is the characteristic correlation time. In addition, there are two asymptotic cases of significance.

In the first case, when $t \gg \tau$, upon simple rearrangement of formula (5) we can obtain

$$\langle x^2 \rangle = 2V_0^2 t\tau - 2\int_0^\infty \tau C(\tau)\, d\tau \approx (2V_0^2 \tau)\, t\,.$$

This representation coincides with the well-known Einstein law for the root-mean-square displacement $R^2 \propto t$.

In the case where $t \to 0$, one can write the simplest approximation of the correlation formula in the form

$$C(t) \approx V_0^2 \left(1 - \frac{t^2}{\tau^2}\right).$$

Upon substitution of this expression into formula (5) we obtain the law of ballistic motion $R \propto t$ in the form $\langle x^2 \rangle = 2V_0^2 t^2$.

The calculation theory of autocorrelation functions was actively developed during the last eighty years [6, 7, 10, 11] in connection with an ever-increasing demand for the temporal series analysis. We therefore restrict ourselves to only several estimates important to the subsequent discussion. Note that from formulas (6) and (7) there follows a dimensional estimate of the diffusion coefficient D_{T} different from the 'Brownian' one D_0:

$$D_{\mathrm{T}} \approx V_0^2 \tau\,, \quad D_0 \approx \frac{\Delta^2}{\tau}\,. \tag{8}$$

Another important relationship, which will be used in the subsequent discussion, is the expression

$$\frac{d^2}{dt^2} \langle x^2 \rangle \bigg|_{t=\tau} = 2C(\tau)\,. \tag{9}$$

Even from general considerations it is clear that the correlation function is a more 'flexible' instrument of investigation than the constant diffusion coefficient. The problem formulated by Taylor [3] turned out to be particularly topical in connection with the investigations of turbulent diffusion performed by Richardson in 1926 [23]. He revealed a significant dissimilarity of the law of atmospheric diffusion (the 'relative' diffusion of two initially close particles) from the classical one:

$$R^2 \propto t^3 \gg t\,, \quad \text{or} \quad D \approx \frac{R^2}{t} \propto R^{4/3}\,. \tag{10}$$

The works of Taylor and Richardson undoubtedly opened up a fundamentally new avenue of investigations and had a profound effect on the subsequent development of the theory of transfer processes.

3. The Monin equation

The notions of not only the diffusion coefficient, but also of the form of diffusion equations underwent significant changes in the theory of turbulent diffusion. Monin's paper [24] came to be a significant work in this field. He took advantage of the Einstein – Smoluchowski functional [7, 14] for the diffusing-particle density

$$\frac{\partial n(x, t)}{\partial t} = \int_{-\infty}^{+\infty} \big[K(x', x)\, n(x', t) - K(x, x')\, n(x, t)\big]\, dx'\,. \tag{11}$$

Here, $K(x, x')\, dx'$ is the probability that a particle residing at a point in time t at a point x executes a transition to an interval $x' + dx'$ in a time dt. Let us assume that

$$G(x', x) = K(x', x) - \delta(x - x') \int_{-\infty}^{+\infty} K(x, x')\, dx'\,. \tag{12}$$

Here, δ is the symbol of the Dirac function. Then, for a uniform and isotropic medium we obtain $G(x'-x) = G(|x-x'|)$. In the simplest case under consideration, the functional (11) is in the form

$$\frac{\partial n(x,t)}{\partial t} = \int_{-\infty}^{+\infty} G(x-x')\, n(x',t)\, \mathrm{d}x' . \qquad (13)$$

Here, the Fourier representation in the x variable is conveniently employed for $n(x,t)$. Then, upon performing formal calculations we arrive at the expression

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = \widetilde{G}(k)\, \tilde{n}_k(k,t) . \qquad (14)$$

Here, $\sim$ symbolizes the Fourier transform. This expression is indicative of the absence of memory effects for the Fourier harmonics. Here, to the classical diffusion equation there corresponds the expression

$$\widetilde{G}(k)\, \tilde{n}_k(t) = -Dk^2 \tilde{n}_k(t) . \qquad (15)$$

It is pertinent to note that this approach was pursued by Levy and Khintchine [25]. In Eqns (14) and (15) they employed an approximation for $\widetilde{G}(k)$ in the form

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^{\alpha} \tilde{n}(k,t) , \quad 0 < \alpha \leqslant 2 . \qquad (16)$$

It is easily seen that we obtain the Gaussian case (a conventional diffusion equation) for $\alpha = 2$. For $\alpha = 1$ we obtain the Cauchy distribution and for $\alpha = 3/2$ the Holtzmark distribution [26]. It is significant that all probability densities with $\alpha < 2$ possess power-law 'tails'.

Monin's work [24] along these lines anticipated the present-day development of ideas regarding invoking additional fractional partial derivatives in diffusion equations. Monin leaned upon Kolmogorov's ideas concerning the universal properties of fully developed isotropic turbulence [7]. In this formulation, all statistical characteristics are defined only by the special scale length $l_k \approx 1/k$ and the average rate of energy dissipation in a turbulent flow $\varepsilon = [L^2/T^3]$. From considerations of dimension, Monin obtained an expression for the kernel of the nonlocal functional describing the turbulent diffusion (13) and (14):

$$\widetilde{G}(k) \propto \widetilde{G}(\varepsilon, k) = \varepsilon^{1/3} k^{2/3} . \qquad (17)$$

This representation actually satisfies Richardson's results of 1926 [23]: if it is assumed that

$$\widetilde{G}(k) = -D(k)\, k^2 ,$$

then

$$D(k) \approx \frac{R^2}{t} \propto R^{4/3} \propto k^{-4/3} . \qquad (18)$$

Furthermore, according to present-day terminology [28], equation

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^{2/3}\, \tilde{n}_k\,(t) \qquad (19)$$

is an equation with a fractional derivative with respect to x:

$$\frac{\partial^{\alpha} n}{\partial x^{\alpha}} \propto \frac{n}{(\Delta x)^{\alpha}} ,$$

where $\alpha = 2/3$ [see formula (16)]. From physical considerations, such probability density was derived for the first time. However, Monin did not content himself with the 'symbolic' form of the equation. There remained the possibility to rearrange the equation to the conventional form. The point is that Davydov [27] had earlier proposed the use of the equation of telegraphy (which includes 'memory effects' [7]) to describe turbulent diffusion:

$$\frac{\partial n}{\partial t} + \tau \frac{\partial^2 n}{\partial t^2} = D \frac{\partial^2 n}{\partial x^2} . \qquad (20)$$

Here, τ is the characteristic correlation time. This is an equation of the hyperbolic type, which opens up additional possibilities of employing characteristics for the description of nonlocal effects. Davydov proposed its use for taking into account the final velocity of particles v in molecular diffusion. The classical diffusion equation of the parabolic type

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}$$

results from the equation of telegraphy in the limit $\tau \to 0$; $D \approx v^2 \tau \to \mathrm{const}$. As would be expected, in the ordinary case

$$v \propto \sqrt{\frac{D}{\tau}} , \quad \text{or} \quad R^2 \propto t ; \quad v \propto \frac{1}{\sqrt{\tau}} \to \infty .$$

Endeavoring to derive an equation as lucid as the equation of telegraphy, Monin resorted to double differentiation with respect to time to bring Eqn (19) to the form

$$\frac{\partial^3 n}{\partial t^3} = \varepsilon \frac{\partial^2 n}{\partial x^2} . \qquad (21)$$

It is easy to generalize the nonlocal equation (14) by representing the effects of memory and nonlocality in one convolution-containing equation:

$$\frac{\partial \tilde{n}_k(t)}{\partial t} = -k^2 \int_0^t \tilde{n}_k(t')\, \widetilde{D}_k(k, t-t')\, \frac{\mathrm{d}t'}{\tau} = -k^2 \widetilde{D}_k(k,t) * \tilde{n}_k(t). \qquad (22)$$

On applying the Laplace transformation with respect to time we establish the fact that expression (15) has acquired a more general form:

$$-Dk^2 n_{k,\omega} \to -k^2 \widetilde{D}_{k,\omega}(k,\omega)\, n_{k,\omega} . \qquad (23)$$

We now see that many years after the theoretical works of Davydov and Monin the diffusion equations have been repeatedly 'complemented' with various partial derivatives:

$$\frac{\partial^2 n}{\partial t^2} , \quad \frac{\partial^3 n}{\partial t^3} , \quad \frac{\partial^{\alpha} n}{\partial t^{\alpha}} , \quad \frac{\partial^{\beta} n}{\partial x^{\beta}} , \qquad (24)$$

with the aim of describing the effects of nonlocality and memory [28].

4. The Howells formula and Peclet number

In the dimensional analysis of the equation of turbulent diffusion Monin employed only one quantity ε — the rate of energy dissipation. However, a significant part in the description of turbulence [7] is played by the spectral energy function $E(k)$:

$$\frac{\langle v^2 \rangle}{2} = \int_0^\infty E(k)\, \mathrm{d}k\,. \tag{25}$$

Howells [29] managed to derive a very important formula which established a relationship between the coefficient of turbulent diffusion D and the spectrum $E(k)$. However, his result presumably became widely known only after the publication of Moffatt's review [30]. We will follow the presentation of Refs [30, 31]. Let us consider the 'local' diffusion coefficient $\delta D(k)$ related to a specific scale length $l_k \approx 1/k$ of the vortices with velocity V_k:

$$\delta D(k) \approx V_k^2 \tau_0\,, \quad V_k^2 \approx E(k)\, \delta k\,. \tag{26}$$

Here, δk is a small interval of wave numbers and τ_0 is the characteristic correlation time. τ_0 will be considered to arise from molecular diffusion:

$$\tau_0 \approx \frac{1}{k^2 D_0}\,.$$

We arrive at the expression

$$\frac{\delta D(k)}{\delta k} = \frac{E(k)}{k^2 D_0}\,, \tag{27}$$

which is differential in form. We take into consideration that the value of $D(k)$ should be taken into account along with D_0. We then obtain

$$\frac{\mathrm{d}D(k)}{\mathrm{d}k} = \frac{E(k)}{k^2\big(D_0 + D(k)\big)}\,. \tag{28}$$

Upon solving this equation we obtain the expression for the coefficient of turbulent diffusion, which takes into account the effect of different scales:

$$\big(D(k) + D_0\big)^2 = \int_k^\infty \frac{E(k)}{k^2}\, \mathrm{d}k + D_0^2\,. \tag{29}$$

We have assumed that $D(\infty) = 0$. With neglect of molecular diffusion effects we obtain an expression which will be encountered several times:

$$D^2 = \int_k^\infty \frac{E(k)}{k^2}\, \mathrm{d}k\,. \tag{30}$$

From the viewpoint of dimensional estimates, the resultant expression is notably different from that introduced by Taylor. The ordinary dimensional estimate of the diffusion coefficient is a formula closely associated with the model of random walk $D_0 \approx \Delta^2/\tau$. Here, λ is the characteristic spatial correlation scale length and τ_0 is the characteristic correlation time. An analogous estimate in Taylor's formula is $D_\mathrm{T} \approx V^2 \tau$. For the formula proposed by Howells we obtain a different type of estimate for the diffusion coefficient:

$$D_\mathrm{H} \approx V\lambda\,.$$

It is possible to determine a relationship between these expressions. We consider the Peclet number [30, 31]:

$$\mathrm{Pe} = \frac{\lambda V}{D_0}\,. \tag{31}$$

This dimensionless quantity is analogous to the well-known Reynolds number $\mathrm{Re} = V\lambda/\eta$ and has the same significance. Here, η is the coefficient of viscosity. The Peclet number enables us to estimate the fraction of convective transfer in comparison with the diffusion one. In terms of the Peclet number we obtain

$$D_0 = D_0\, \mathrm{Pe}^0 \equiv D_0\,, \quad D_\mathrm{T} = D_0\, \mathrm{Pe}^2\,, \quad D_\mathrm{H} = D_0\, \mathrm{Pe}\,. \tag{32}$$

This form of presentation of results is now in wide use [4–6, 30, 31]:

$$D_\mathrm{eff} = D_0\, \mathrm{Pe}^\alpha\,. \tag{33}$$

5. Corrsin's assumption

The definition of the correlation function proposed by Taylor [3] is based on the use of Lagrangian velocities:

$$C(\tau) = \big\langle v(a,t)\, v(a, t+\tau) \big\rangle\,. \tag{34}$$

However, the experimental determination of the Lagrangian velocities that enter into this formula is a serious problem. That is why use is made of the Eulerian representation for the correlation function involving consideration of the velocity correlation at points separated by a distance Δ:

$$C_\mathrm{E}(\Delta, \tau) = \big\langle u(a,t)\, u(a+\Delta, t+\tau) \big\rangle\,. \tag{35}$$

In this formulation, the formula for the correlation function proves to be more convenient for experimenters. We can also write down the Lagrangian correlation function in terms of the Eulerian velocity:

$$C(\tau) = \Big\langle u(a,t)\, u\big(x(a, t+\tau), t+\tau\big) \Big\rangle\,. \tag{36}$$

However, there is no one-to-one correspondence between the Lagrangian and Eulerian correlation functions. This circumstance was repeatedly emphasized in the works of Lamley and Corrsin [32]. Indeed, in formula (35) the Lagrangian constraint on the points a and $a + \Delta$ is absent. Here, Δ is merely 'some' arbitrary displacement.

In 1959, Corrsin proposed an approximation formula [32], expressing the Lagrangian correlation function in terms of the Eulerian correlation function:

$$C(\tau) = \int \rho(\Delta, \tau)\, C_\mathrm{E}(\Delta, \tau)\, \mathrm{d}\Delta\,. \tag{37}$$

In fact, use was made of the randomization procedure here. However, a more important point is the idea of the diffusion nature of the displacement Δ, because for $\rho(\Delta, \tau)$ Corrsin employed the classical solution of the diffusion equation in d-dimensional space in the form

$$\rho = \frac{1}{(2\pi D t)^{d/2}} \exp\left(-\frac{\Delta^2}{4Dt}\right). \tag{38}$$

One can see that the idea of the diffusion spreading of Lagrangian trajectories was put forward by Corrsin eight years prior to the publication of Dupree's papers [34–36]. However, it was precisely after the publication of Refs [34–36] that this idea received wide recognition.

6. The quasi-linear approximation

Mention was made of the great significance of the quasi-linear approximation in the Introduction of the present paper. Quasi-linear equations were first considered [1, 2] as applied to the problem of describing diffusion in the phase space arising from wave – particle interaction. For our purposes it would suffice to invoke only some of the ideas of the work of Vedenov, Velikhov, and Sagdeev related to equations averaging [8, 9, 15, 33].

Let us consider the continuity equation for the density of a passive scalar

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} = 0\,. \tag{39}$$

Here, $n(x,t)$ is the spatial density of a passive scalar and $v(x,t)$ is the random velocity field. We apply averaging to Eqn (39). We assume that the density field can be represented as a sum of the average value $n_0 = \langle n \rangle$ and the fluctuation part $n_1 = n - \langle n \rangle$:

$$n = n_0 + n_1\,;$$

it was assumed here that $\langle n_1 \rangle = 0$ and $\langle v \rangle = 0$. After simple calculations, which have repeatedly been given in the literature [8, 9, 15, 33], we arrive at two equations:

$$\frac{\partial n_0}{\partial t} + \left\langle v\frac{\partial n_1}{\partial x} \right\rangle = 0\,, \tag{40}$$

$$\frac{\partial n_1}{\partial t} + v\,\frac{\partial n_0}{\partial x} + v\,\frac{\partial n_1}{\partial x} - \left\langle v\frac{\partial n_1}{\partial x} \right\rangle = 0\,. \tag{41}$$

The fluctuations n_1 and v are assumed to be of the order of smallness δ in comparison with the average field n_0. In the equations (40) and (41) under consideration, terms of the order of δ^2 are still retained. The quasi-linearity of the approximation lies in the fact that we keep the nonlinear term of the order of δ^2 in the equation for n_0 (40), but keep only the terms of the order of δ in the equation for n_1 (41). Then, from the equation for n_1 we obtain

$$\frac{\partial n_1}{\partial t} + v\frac{\partial n_0}{\partial x} = 0\,, \quad n_1 = -\int_{-\infty}^{t} v\,\frac{\partial n_0}{\partial x}\,\mathrm{d}t'\,. \tag{42}$$

We substitute this expression for n_1 into Eqn (40). On simple rearrangement [8, 9] we obtain

$$\frac{\partial n_0}{\partial t} = \int_0^{\infty} \big\langle v(0)\,v(s) \big\rangle\,\mathrm{d}s\,\frac{\partial^2 n_0}{\partial x^2}\,. \tag{43}$$

Therefore, in the quasi-linear approximation for the diffusion of a passive admixture we arrive at the well-known Kubo – Green formula (7)

$$D = \int_0^{\infty} C(\tau)\,\mathrm{d}\tau = \int_0^{\infty} \big\langle v(0)\,v(t) \big\rangle\,\mathrm{d}t\,. \tag{44}$$

The 'weak link' of the quasi-linear theory is the inconsistency of retaining the quasi-linear term in the equation for n_0 and discarding the nonlinear terms in the equation for n_1. The authors of numerous papers have endeavored to 'touch up' the quasi-linear approximation. Their comprehensive analysis can be found elsewhere [15, 33]. The greatest interest in this field was aroused by Dupree's papers [34 – 36]. He invoked the idea of diffusion trajectory spreading close in meaning to Corrsin's assumption (37), (38). Indeed, the equation for n_1 is linear and hyperbolic and retains the Lagrangian character of correlations. This opens up the possibility to describe the neglected correlation effects by employing the diffusion approximation [34 – 36]. The papers that exploit this idea will be discussed in the following sections.

7. The Taylor – McNamara correlation model

Taylor and McNamara considered the problem of the calculation of the Lagrangian correlation function for the description of a strongly magnetized plasma [37]. However, we set forth their heuristic method regardless of plasma models. The basis for their calculations is the Fourier representation of Lagrangian velocities appearing in the correlation function

$$C(t) = \big\langle v(x(t),t)\,v(x(0),0) \big\rangle = \sum_{k,k'} \Big\langle \tilde{v}_k(t)\exp\big[\mathrm{i}kx(t)\big]\,\tilde{v}_{k'}(0)\exp\big[\mathrm{i}k'x(0)\big] \Big\rangle\,. \tag{45}$$

Here, recourse was made to the Fourier transformation with respect to the spatial x-coordinate. The next step in correlation splitting in expression (45) has come to be known as the 'independence hypothesis':

$$C(t) = \sum_k \big\langle \tilde{v}_k(t)\,\tilde{v}_{k'}(0) \big\rangle \big\langle \exp\big\{\mathrm{i}k\big[x(t) - x(0)\big]\big\} \big\rangle\,. \tag{46}$$

The next step made by Taylor and McNamara relied on Dupree's ideas of 'diffusion spreading of trajectories' $[x(t) - x(0)]^2 \approx 2Dt$. This is in fact a 'recipe for calculating the average' of the quantity $\exp[\mathrm{i}k\Delta x(t)]$ in accordance with the formula

$$\langle \exp A \rangle = \exp\left[\frac{\langle A^2 \rangle}{2}\right]. \tag{47}$$

Performing calculations gives

$$\big\langle \exp\big[\mathrm{i}k\Delta x(t)\big] \big\rangle = \exp\big[-k^2 Dt\big]\,. \tag{48}$$

These steps are hard to substantiate rigorously, but the result of these calculations convinces us of the need to regard this assumption as a serious one. Taylor and McNamara [37] obtained an expression for the correlation function in the form

$$C(t) = \sum_k \langle \tilde{v}_k^2 \rangle \exp(-k^2 Dt)\,. \tag{49}$$

Prior to proceeding with the presentation of the results of Ref. [37], we consider the resultant expression (49) from the 'correlation' standpoint. This expression may be interpreted as the sum of Gaussian exponential correlation functions with 'weight factors' proportional to the turbulence spectrum $E(k)$:

$$C(t) = \sum_k E(k)\,\exp\left(-\frac{t}{\tau_k}\right). \tag{50}$$

In this case it is good to bear in mind that this sum of a large number of exponents may turn out to be a function that is by no means exponential in form. Also evident is the propinquity of this expression to the Laplace transformation. Such expressions are most extensively employed to obtain power correlation functions [17, 18]. Another important property of formula (50) is the following expression for the diffusion coefficient

$$D = \int_0^{\infty} C(t)\,\mathrm{d}t = \frac{1}{D}\sum_k \frac{E(k)}{k^2}\int_0^{\infty} \exp(-k^2 Dt)\ \mathrm{d}(k^2 Dt)$$
$$\approx \frac{1}{D}\int \frac{E(k)}{k^2}\,\mathrm{d}k\,, \qquad (51)$$

which results from this formula.

We take notice of the similarity of the resultant expression to Howells's result

$$D^2 = \int_k^{\infty} \frac{E(k)}{k^2}\,\mathrm{d}k\,. \qquad (52)$$

It is also significant that the exponential form in formula (50) is not a necessary condition. Another functional dependence on the parameter $z = k^2 Dt$ could also suit us to ensure the derivation of expression (52).

One can see from the above analysis that the ideas of the interplay of scales [29, 30] and the correlation ideas of 'diffusion trajectory spreading' [34–36] are closely interrelated. Taylor and McNamara performed elegant calculations in their work to obtain a formula close to Howells's result. They introduced a quantity $R(t) = [\Delta x(t)]^2$. Then,

$$C(t) \approx \frac{\mathrm{d}}{\mathrm{d}t}\,D \approx \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\Delta x^2}{t} \approx \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,R(t)\,. \qquad (53)$$

On the other hand, from formula (49) we obtain

$$C(t) = \int E(k)\exp\left[-k^2 R(t)\right]\mathrm{d}k\,. \qquad (54)$$

In fact, we have a 'Newtonian'-type differential equation

$$\frac{\partial^2}{\partial t^2}\,R(t) = C\{R(t)\}\,. \qquad (55)$$

We apply a formal procedure to obtain the final solution [37]

$$D^2 \approx \frac{\mathrm{d}}{\mathrm{d}t}\,R(t) = \int E(k)\left\{1 - \exp\left[-k^2 R(t)\right]\right\}\frac{\mathrm{d}k}{k^2}\,. \qquad (56)$$

Consideration of the case $k^2 R(t) \gg 1$ is sufficient for obtaining expression (52).

However, it is pertinent to note that Ref. [37] does not contain a reference to Howells's paper [29]. It is likely that Howells's result became widely known more recently [30, 31].

8. The Dreizin – Dykhne superdiffusion model

The problem of calculating the coefficient of turbulent diffusion is intimately related to the problem of the behavior of the correlation function. In everyday language, 'correlation' means some relation of events. The probability theory employs the rigorous mathematical notion of 'return of a roaming particle' to the initial point [11, 17] as a possibility to describe correlation effects. In 1971, Corrsin made a very interesting report [38] devoted to the probabilistic problems of turbulence, wherein he formulated several problems calling for solution. One was the inclusion of 'returns' [11, 17], which diffuse in the turbulent particle flow. Virtually simultaneously published was a paper by Dreizin and Dykhne [39] concerned with conduction in anisotropic media. In this paper Dreizin and Dykhne proposed and considered a physically clear model of the behavior of a particle under the action of a sharply anisotropic diffusion. We select the longitudinal direction (related to the direction of the magnetic field) and assume that a 'seed' diffusion with the coefficient D_0 acts in this direction. In the transverse direction, the diffusing particle experiences random pulsations, which produce narrow convective flows with a velocity V_0 and a width a (Fig. 1).

Dreizin and Dykhne came up with a simple model for calculating the coefficient of diffusion in the transverse direction $D_\perp$:

$$D_\perp \approx \frac{\lambda_\perp^2}{t}\,, \quad \lambda_\perp \approx V_0 t P\,. \qquad (57)$$

Here, $P = \delta N/N$ is the fraction of 'uncompensated' pulsations and the quantity $N \approx \sqrt{2D_0 t}/a$ is the number of flows of width a intersected by the particle. The authors of Ref. [39] made an estimate with the aid of 'Gaussian statistics' $\delta N \approx \sqrt{N}$ to obtain

$$D_\perp \approx V_0^2 a \sqrt{\frac{t}{D_0}}\,. \qquad (58)$$

This is in fact a superdiffusion mode:

$$\lambda_\perp \approx \frac{V_0^2 a}{\sqrt{D_0}}\,t^{3/4} \gg t^{1/2}\,. \qquad (59)$$

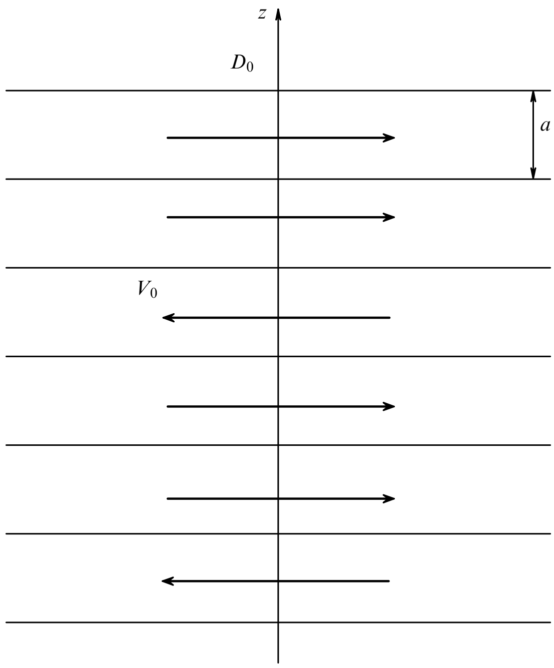

Figure 1. Model of Dreizin – Dykhne anisotropic superdiffusion: D_0 — 'seed' diffusion coefficient; V_0 — velocity of transverse pulsations; a — transverse dimension of pulsations.

To explain this result, Dreizin and Dykhne considered the correlation function of the Eulerian form

$$C(t_1, t_2) = \int_{-\infty}^{\infty} \langle V(0)\, V(z) \rangle\, \Phi(z, t_2 - t_1)\, \mathrm{d}z\,, \tag{60}$$

where

$$\Phi = \frac{1}{\left(4\pi D_0 (t_2 - t_1)\right)^{1/2}} \exp\left(-\frac{z^2}{4D_0 t}\right). \tag{61}$$

This representation is in complete agreement with Corrsin's idea of the diffusion nature of 'decorrelations' (37), (38).

However, invoking the hypothesis of the significant role of 'returns' [17] in the diffusion model under consideration became the main link in the description of the anomalous diffusion nature. Under these assumptions we study the situation with $z \to 0$. Then,

$$C(t_1, t_2) = C(\tau) \approx \frac{V_0^2 a}{\sqrt{2\pi D_0 \tau}}\,, \quad \tau = t_2 - t_1\,. \tag{62}$$

The result (59) is now easy to obtain by performing calculations [see expression (3)]:

$$\langle \lambda_\perp^2 \rangle \approx \frac{V_0^2 a}{\sqrt{2\pi D_0}} \int_0^t \int_0^t \frac{\mathrm{d}t_1\, \mathrm{d}t_2}{\sqrt{t_1 - t_2}} \approx \frac{V_0^2 a}{\sqrt{2\pi D_0}}\, t^{3/2}\,. \tag{63}$$

As a result of analysis and calculations we see that the nature of superdiffusion in the Dreizin – Dykhne model [39] is related to the significant role of 'returns', which are responsible for the correlated nature of movements of the diffusing particle. An interesting analysis of the part played by 'returns' in 'strip-like' flows was performed by Chukbar in Ref. [40]. Also given in Ref. [40] was the diffusion equation corresponding to the model of Ref. [39] and containing fractional derivatives:

$$\frac{\partial^2}{\partial t^2} \int_0^t \frac{n(t', x)\, \mathrm{d}t'}{\sqrt{\pi(t - t')}} - \frac{V_0^2 a}{\sqrt{2D}} \frac{\partial^2 n(t, x)}{\partial x^2} = -\frac{n_0(0, x)}{2\sqrt{\pi}\, t^{3/2}}\,. \tag{64}$$

One more remark concerning the findings of Ref. [39] is appropriate here. A close look at formula (62) will show that the number of flows intersected by the particle enters explicitly into the formula. In the one-dimensional case involved this number is

$$N_{\mathrm{B}}(\tau) \propto \sqrt{\frac{D_0 \tau}{a}}\,. \tag{65}$$

We now can endeavor to generalize the result of Dreizin and Dykhne to the case of a more general 'topology' of the flows with $d > 1$:

$$C(\tau) \approx \frac{V_0^2}{N_{\mathrm{B}}(\tau)}\,. \tag{66}$$

By way of simple calculations using this formula it is possible to obtain other scalings describing superdiffusion. In particular, an estimate $N_{\mathrm{B}} \propto t^{2/3}$ is well known [17] for $d \geqslant 2$, and hence we arrive at a scaling $x \propto t^{2/3}$.

A similar formula for the correlation function is encountered in statistical physics [22] and has a clear physical meaning:

$$C(\tau) \approx \langle V_0 V_\tau \rangle \approx V_0 \frac{V_0}{N_I(\tau)}\,, \tag{67}$$

where N_I is the number of particles that the 'probe' particle had interacted with,

$$N_I(t) \approx n R_I(t) \propto n (Dt)^{d/2}\,. \tag{68}$$

Here, R_I is the interaction radius calculated on the basis of the diffusion model in the d-dimensional space.

9. The Kadomtsev – Pogutse hydrodynamic approximation

In this section we briefly outline only some of the results of the well-known paper by Kadomtsev and Pogutse [41]. This paper is concerned with the problem of anomalous electron transport in a stochastic magnetic field [42]. In connection with the question of the effect of correlations on turbulent transport, a particular role is played by the model of magnetic field diffusion under conditions whereby the quasi-linear approximation is invalid, which was considered by Kadomtsev and Pogutse. They considered a three-dimensional problem in which a weak random field $\mathbf{B}'(B_x, B_y, 0)$ is superimposed on a strong constant field $\mathbf{B}(0, 0, B_0)$ aligned with the z-axis. As regards the way of describing anisotropy, this model is very close to that of Dreizin – Dykhne. However, in the case considered by Kadomtsev and Pogutse, the motion along the z-axis is not diffusive in character. That is why the correlation effects in this formulation will not depend directly on 'returns'.

The analysis of diffusion in the quasi-linear limit was based on the stochastic equations for field lines

$$\frac{\mathrm{d}\mathbf{r}_\perp}{\mathrm{d}t} = \mathbf{b}(z, \mathbf{r}_\perp)\,, \quad \mathbf{b} = \frac{\mathbf{B}'}{B_0} \approx b_0\,.$$

Here, b_0 is the characteristic relative perturbation scale. For the transverse diffusion coefficient of magnetic field lines, then, we obtain on averaging

$$D_F = \frac{1}{4} \int_{-\infty}^{\infty} \langle \mathbf{b}(z, 0)\, \mathbf{b}(0, 0) \rangle\, \mathrm{d}z \propto b_0^2 \Delta_z\,. \tag{69}$$

Here, Δ_z is the longitudinal correlation scale length. It is evident that this representation will be valid only when the diffusion displacement in the transverse direction is far less than the transverse correlation scale length: $b_0 \Delta_z \ll \Delta_\perp$.

Kadomtsev and Pogutse considered the opposite case as well, when $b_0 \Delta_z \geqslant \Delta_\perp$. The authors of Ref. [41] introduced the continuity equation for the density of magnetic field lines

$$\frac{\partial n_b}{\partial z} + \mathbf{b} \nabla n_b(r_\perp, z) = 0\,. \tag{70}$$

They represented n_b as the sum of the average value $n_0 = \langle n_b \rangle$ and the fluctuation part n_1:

$$n_b = n_0 + n_1\,.$$

In this formulation, the problem is close to the problem of quasi-linear diffusion of a passive scalar (40) and (41). Indeed, the authors of Ref. [41] took advantage of the traditional form of the equation for n_0:

$$\frac{\partial n_0}{\partial z} + \nabla \langle \mathbf{b} n_1 \rangle = 0\,. \tag{71}$$

However, in the equation for n_1 they replaced the previously discarded small second-order terms with a term diffusive in form. In essence, following the ideas by Dupree, they related the previously neglected correlation effects to the diffusion spreading of trajectories:

$$\frac{\partial n_1}{\partial z} - D_F \nabla_\perp^2 n_1 = -\mathbf{b}\nabla n_0 . \tag{72}$$

The equation has retained linearity, but instead of the hyperbolic form (42) it assumed a parabolic form. Applying the technique of Green's function to Eqn (72)

$$\frac{\partial G}{\partial z} - D_F \nabla_\perp^2 G = \delta(\mathbf{r} - \mathbf{r}') , \tag{73}$$

Kadomtsev and Pogutse derived the final expression for n_0:

$$\frac{\partial n_0}{\partial z} = D_F \nabla_\perp^2 n_0 , \tag{74}$$

$$D_F = \frac{1}{2} \int \frac{b^2(\mathbf{k})}{\mathrm{i}k_z + k_\perp^2 D_F} \, \mathrm{d}\mathbf{k} ,$$

$$b^2(\mathbf{k}) = \frac{1}{(2\pi)^2} \int \langle b(0)\, b(r) \rangle \exp(-\mathrm{i}\mathbf{kr}) \, \mathrm{d}\mathbf{r} . \tag{75}$$

For $\Delta k_z > k_\perp^2 D_F$, it was possible to derive a quasi-linear expression [8, 9]

$$D_F = \frac{\pi}{2} \int b^2(\mathbf{k}) \, \delta(k_z) \, \mathrm{d}\mathbf{k} \propto b_0^2 \Delta_z . \tag{76}$$

In the case of strong transverse correlations $\Delta k_z < k_\perp^2 D_F$, we arrive at the expression resembling Howells's result [29] [see expression (30)]:

$$D_F^2 = \frac{1}{2} \int \frac{b^2(\mathbf{k})}{k_\perp^2} \, \mathrm{d}\mathbf{k} . \tag{77}$$

This result of Kadomtsev and Pogutse demonstrated once again the significance of including correlation effects, which are neglected in the quasi-linear approach, and their intimate relationship with the problem of including the effects of different scales in the theory of turbulent transport. The ideas concerning detailed consideration of scale hierarchy were subsequently developed in the works on continuous percolation [4 – 6].

10. Continuous percolation and diffusion

Kadomtsev and Pogutse were the first to propose the use of a percolation approach for the description of anomalous diffusion in plasmas [41]. Late in the 1970s, the ideas of scaling, fractality, and percolation received wide acceptance [43 – 45]. A physically clear presentation of these ideas can be found in review papers [17, 18] and in books [16, 46]. In the subsequent discussion we assume that the reader is familiar with the basic definitions given therein. In the context of this approach, the lines of flow $\Psi = \Psi(x, y)$ are treated as coastlines arising from flooding a hilly landscape with water. It is anticipated that there is a sharp transition from isolated lakes on boundless dry land to isolated isles in the infinite ocean. The percolation theory necessitates the existence of at least one coastline of infinite length.

Kadomtsev and Pogutse [41] related the anomalous character of diffusion to the fractal character of behavior of the lines of two-dimensional flow near the percolation threshold (Fig. 2). They proposed the use of scaling for the length of a percolation line of flow:

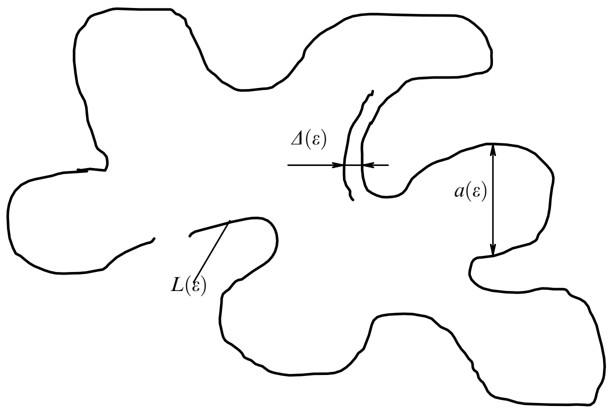

Figure 2. Fractal line of flow: $L(\varepsilon)$ — length of a percolation line of flow; $\Delta(\varepsilon)$ — width of the percolation layer; $a(\varepsilon)$ — correlation dimension; ε — small parameter of the percolation theory.

$$L(\varepsilon) \propto \frac{1}{\varepsilon^{2.4}} . \tag{78}$$

Here, ε is a small quantity which characterizes the degree of departure of the system from the critical state (the percolation threshold);

$$\varepsilon \approx \frac{h}{\lambda V_0} ,$$

where h has the dimension of the stream function Ψ, λ is the characteristic dimension, and V_0 is the characteristic velocity.

The subsequent progress of research on diffusion processes in systems with complex structures, like convective cells (Fig. 3) [31], led to the understanding of the significance of the percolation layer width Δ (the stochastic layer 'width') [47]. The percolation theory contains a scaling for the effective diffusion coefficient [17]

$$D_{\mathrm{eff}}(\varepsilon) = D_0 P_\infty(\varepsilon) . \tag{79}$$

Here, D_0 is the 'seed' diffusion coefficient and P_∞ is the fraction of space occupied with the percolation cluster. In the

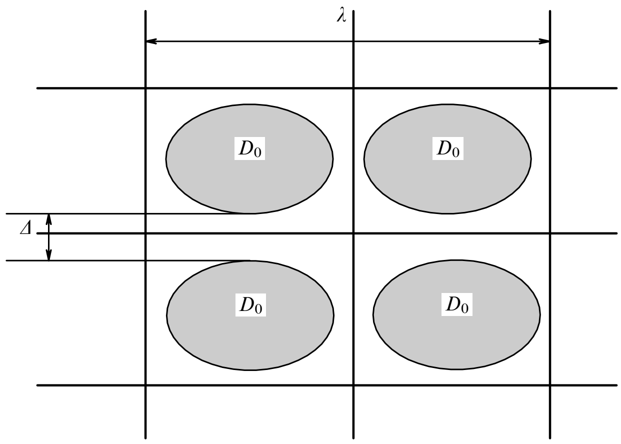

Figure 3. Model of turbulent diffusion on convective cells: D_0 — 'seed' diffusion coefficient; λ — typical cell dimension; Δ — boundary layer thickness.

case of convective cells, the magnitude of P_∞ is easy to estimate when a parameter λ — the cell dimension — is introduced:

$$P_\infty = \frac{L\Delta}{\lambda^2}\,. \tag{80}$$

This formula reflects the simple 'topology' of percolation diffusion in a plane.

Osipenko, Pogutse, and Chudin [47] proposed a diffusion estimate of the stochastic layer width Δ

$$\Delta = \sqrt{\frac{D_0\lambda}{V_0}} \approx \sqrt{D_0\tau}\,. \tag{81}$$

Here, τ is the correlation time scale. This formula has the lucid physical significance of particle number balance. From a convective cell there escape $D_0(n/\Delta)\lambda$ particles per unit time. Convection along the boundary layer carries away $nV_0\Delta$ particles. Taking into account that the convective flow exists only in a Δ/λ fraction of space, we obtain

$$D_{\text{eff}} \propto \lambda V_0 \frac{\Delta}{\lambda}\,. \tag{82}$$

The authors of Ref. [47] eventually arrived at the following estimate for the coefficient of turbulent diffusion:

$$D_{\text{eff}} = \text{const}\sqrt{D_0 V_0 \lambda}\,. \tag{83}$$

Somewhat later the same estimate was obtained in Ref. [48]. In terms of the Peclet number this formula takes on the form

$$D_{\text{eff}} = \text{const}\,\frac{\lambda^2}{\tau}\sqrt{\frac{V_0\lambda}{D_0}} \approx D_0\,\text{Pe}^{1/2}\,. \tag{84}$$

Close examination of formula (83) shows that, despite the substantial advance made in Ref. [47], the anticipated percolation character inherent in the $D_{\text{eff}} = D_{\text{eff}}(\varepsilon)$ dependence was lost. This in fact signifies the loss of the direct relation to 'long-range correlation' effects, which underlie the percolation approach based on the power-law behavior of the correlation scale: $a(\varepsilon) = \lambda|\varepsilon|^{-v}$.

11. Percolation in the stationary case

The team of authors of Ref. [4] was able to realize the potentiality of the Kadomtsev–Pogutse percolation approach. They considered a two-dimensional stationary flow with a zero average velocity, the flow being defined by the bounded stream function $\Psi(x,y)$ of 'general position'. They implied an isotropic-on-average oscillating function with a quasi-random location of saddle points in height. The following scales were selected:

$$\Psi_0 \approx \lambda V_0\,, \quad \lambda \approx \left|\frac{\Psi}{\nabla\Psi}\right|.$$

The authors of Ref. [4] based themselves on formula (79) for effective diffusion and invoked the notion of the convective nature of the flow along the percolation streamline:

$$D_{\text{eff}}(\varepsilon) \approx \frac{a^2}{\tau}\,\frac{L(\varepsilon)\,\Delta(\varepsilon)}{a^2}\,. \tag{85}$$

Here, τ is the correlation time scale and a is the only parameter which characterizes the spatial scale in the percolation theory [49, 50]. It is precisely through a that the 'long correlation' effects enter into the expression for the diffusion coefficient. This is in fact a formula from Ref. [47]. However, continuing in the spirit of the works on percolation theory, the authors of Ref. [4] suggested a 'renormalization', i.e., a way of calculating the universal value of the small parameter ε in their percolation diffusion theory. They identified the small 'width' of a percolation streamline with the small parameter of the percolation theory (see Fig. 2):

$$\Delta(\varepsilon) = \lambda\varepsilon\,. \tag{86}$$

We employ the well-known expressions (81) to obtain the equation for the determination of the 'universal' value of $\varepsilon_* = h_*/(\lambda V_0)$ as a function of the flow parameters D_0, V_0, and λ:

$$\sqrt{\frac{D_0 L(h)}{V_0}} = \frac{h}{V_0}\,. \tag{87}$$

It is possible to bring the calculations to completion if advantage is taken of the rigorous scaling results of the percolation theory [16, 17, 51–53] obtained for the correlation scale a and the length of a fractal streamline L as functions of ε:

$$a(\varepsilon) = \lambda\varepsilon^{-v}\,, \quad L(\varepsilon) = \lambda\left(\frac{a}{\lambda}\right)^{D_h}, \quad v = \frac{4}{3}\,, \quad D_h = 1 + \frac{1}{v}\,. \tag{88}$$

The functional form of these dependences reflects the fractal behavior of streamlines introduced by Kadomtsev and Pogutse. Calculations in terms of the Peclet number lead to the expressions

$$h_* = \lambda V_0\,\text{Pe}^{-3/13}, \tag{89}$$

$$D_{\text{eff}} = D_0\,\text{Pe}^{10/13}. \tag{90}$$

It is pertinent to make several general remarks here. Formula (90) derived in Ref. [4] possesses broad generality, which may be comparable to that of Bohm's scaling for plasma diffusion. Like Bohm's scaling, it rests upon the 'elimination of the characteristic dimension' (86) (see also Krommes's comment on Bohm's scaling [15]). Some arbitrariness in expression (86) in the selection of the value $\lambda\varepsilon$, and not $\lambda\varepsilon^2$ or $\lambda\varepsilon^3$, may be interpreted as a will to have a universal small parameter, just as there exists a single characteristic dimension — the correlation length in the theory of phase transitions. Here it should be particularly emphasized that the fractal percolation streamline is not infinitely long in the framework of Ref. [4]: the small parameter ε_* does not tend to zero, but has a final value $\varepsilon_* = h_*/(\lambda V_0)$ for all types of flow with the characteristic D_0, V_0, and λ values. Therein lies the universality of formula (90). Apart from the scalings (89) and (90), we can also obtain some additional information useful for the subsequent analysis. We note that the percolation mode is intermediate in terms of the Peclet number between the mode of convective cells from Ref. [47] and the purely convective mode $D_{\text{eff}} \approx \lambda V_0$. The volume fraction occupied with percolation streamlines is estimated as

$$P_\infty = \frac{L(\varepsilon)\,\Delta(\varepsilon)}{a(\varepsilon)^2} \approx \varepsilon^{4/3} \propto \frac{1}{a}\,. \tag{91}$$

This result was later used as a 'universal' one in another work on transient percolation [5].

The correlation time is described by the following scaling:

$$\tau \approx \frac{L(\varepsilon)}{V_0} \approx \frac{\lambda}{V_0} \frac{1}{\varepsilon^{7/3}} . \tag{92}$$

It is instructive to establish a closer relation of the results of Ref. [4] to the formulas of the percolation theory [16, 17]. We prescribe the scaling relationships for

$$P_\infty \propto \varepsilon^{\beta} , \quad D_{\text{eff}} \propto \varepsilon^{\mu} . \tag{93}$$

Comparing formulas (91) and (93) gives $\beta = v = 4/3$ for P_∞. We employ formula (85) and the scaling for D_{eff} from formula (93) to obtain $\mu = 1$. The complete formula for D_{eff} with $D_0 \propto 1/r^{\theta}$ gives the expression

$$D_{\text{eff}} \approx r^{-\theta} \varepsilon^{\mu} \approx \varepsilon^{\theta v + \beta} . \tag{94}$$

Therefore, we can calculate the internal dimensionality of random walking [16, 17] in the case of fractal streamlines:

$$d_w = 2 + \theta = -\frac{7}{4} , \quad \text{or} \quad D_0 \approx \frac{a^2}{\tau} \approx a^{1/4} .$$

Some other estimates are also possible (see Refs [16, 17]).

It is interesting to note that abandoning the relation $\lambda\varepsilon$ in favor of relation $\lambda\varepsilon^{\chi}$ (with some arbitrary index χ) would result in the transition from the $D_{\text{eff}} = D_0 \,\text{Pe}^{10/13}$ mode to the $D_{\text{eff}} = D_0 \,\text{Pe}^{1/2}$ mode only for $\chi > 7$. A transition to the $D_{\text{eff}} = D_0 \,\text{Pe}$ mode would necessitate values $\chi \to 0$.

Concluding the analysis of the results of Ref. [4] we remark that we sought to make the presentation simple, but the simplicity of the results is deceptive. It will suffice to recall in this connection the entire 'hierarchy' of scales used by the authors of Ref. [4] for their analysis:

$$\frac{a}{\varepsilon} \approx L \gg a \gg \lambda \gg \Delta \approx \lambda\varepsilon \approx \frac{h}{V_0} . \tag{95}$$

Here,

$$L(\varepsilon_*) \approx \lambda \frac{1}{\varepsilon_*^{7/3}} , \quad a(\varepsilon_*) = \lambda \frac{1}{\varepsilon_*^{4/3}}$$

are not infinitely large.

12. Percolation in the transient case

Continuing in the spirit of Ref. [4], Gruzinov, Isichenko, and Kalda [5] considered the percolation limit of the turbulent diffusion of a scalar admixture in a transient incompressible two-dimensional flow. On estimating the time it takes the flow pattern to change completely as $T_0 \approx 1/\omega$, the authors of Ref. [5] focused their attention on the low-frequency case $\omega \ll V_0/\lambda$, or $\lambda \ll V_0 T_0$.

In this formulation of the problem, the main parameter is the lifetime of an individual percolation streamline τ. For the diffusion coefficient, advantage can be taken of the conventional expression

$$D_*(\varepsilon) \approx \frac{a^2}{\tau} . \tag{96}$$

In the context of this problem,

$$\tau \approx \varepsilon \frac{1}{\omega} \approx \varepsilon T_0 ,$$

where ε is the small parameter of the problem similar to that of Ref. [4]. In the transient case under consideration, one would expect a universal result provided that a specific 'universal' value of ε_* were possible to calculate. For this purpose, the authors of Ref. [5] proposed a simple expression which takes into account the convective nature of motion along the percolation streamline during the lifetime of this streamline:

$$\tau \approx \varepsilon \frac{1}{\omega} = \frac{L(\varepsilon)}{V_0} . \tag{97}$$

Here, it is easy to see the analogy with formula (87). Employing the scaling from the percolation theory for $L(\varepsilon)$:

$$a(\varepsilon) = \lambda\varepsilon^{-v} , \quad L(\varepsilon) = \lambda \left(\frac{a}{\lambda}\right)^{D_h} , \quad v = \frac{4}{3} , \quad D_h = 1 + \frac{1}{v} , \tag{98}$$

we easily obtain $\varepsilon_* = h_*/(\lambda V_0)$ as a function of the flow parameters ω, V_0, and λ:

$$\varepsilon_* = \left(\frac{\lambda\omega}{V_0}\right)^{1/(2+v)} = \text{Ku}^{-3/10} \propto \omega^{3/10} . \tag{99}$$

Here, we have conveniently introduced the Kubo number $\text{Ku} = V_0/(\lambda\omega)$. The expression for D_* is obtained by direct substitution in expression (96):

$$D_*(\varepsilon_*) \approx \frac{a(\varepsilon_*)^2}{\tau(\varepsilon_*)} . \tag{100}$$

We note, however, that the dependence on $T_0 \approx 1/\omega$ appears quite odd:

$$D_* \propto T_0^{1/10} \approx \frac{1}{\omega^{1/10}} . \tag{101}$$

The slow 'restructuring' of the flow is unlikely to result in a significant growth of turbulent diffusion. The reason lies in the fact that we have not taken into account the fraction P_∞ of percolation streamlines in the total flow:

$$P_\infty = \frac{L(\varepsilon)\,\Delta(\varepsilon)}{a(\varepsilon)^2} . \tag{102}$$

It is now evident that we need additional information on the magnitude of $\Delta(\varepsilon)$, despite the fact that we have calculated $\varepsilon_* = h_*/(\lambda V_0)$. Unlike Ref. [4], in addition to expression (97) we are forced to make additional assumptions here. The authors of Ref. [5] adopt $\Delta(\varepsilon) = \lambda\varepsilon$, similarly to Ref. [4]. In fact, use is made of formulas (86) and (91):

$$\Delta \approx \varepsilon_* \lambda , \quad P_\infty = \varepsilon^{4/3} \propto \frac{1}{a(\varepsilon)} . \tag{103}$$

Calculations now lead to the final expression for D_{eff}:

$$D_{\text{eff}} = D_0 \,\text{Ku}^{7/10} \propto \omega^{3/10} . \tag{104}$$

The resultant formula accounts for the universal character of percolation diffusion in transient flows, since the value

$\varepsilon_* = h_*/(\lambda V_0)$ depends only on the flow parameters ω, V_0, and λ.

Additional estimates should be made of the effect of diffusion escape of particles from the lines of flow. The formulas of Ref. [5] were obtained assuming that $\tau < \tau_D$. To make estimates, we make use of the streamline diffusion coefficient D_ψ and relate it to the 'seed' diffusion coefficient D_0:

$$\tau_D \approx \frac{h^2}{D_\psi}, \quad D_\psi = V_0^2 D_0 . \qquad (105)$$

The applicability condition for the results of Ref. [5] takes on the form

$$\frac{1}{\omega}\,\frac{h}{\lambda V_0} < \frac{h}{V_0^2 D_0} . \qquad (106)$$

This is in fact a limitation on the magnitude of the 'seed' diffusion coefficient D_0:

$$D_0 < \frac{\lambda}{V_0}\, h\omega(h) \propto h^{v+3} . \qquad (107)$$

To conclude the discussion of this issue, we give the set of characteristic times in the problem on percolation in a transient flow:

$$\frac{1}{\omega}\,\frac{h}{\lambda V_0} \ll \frac{h^2}{V_0^2 D_0} \ll \frac{1}{\omega} \approx T_0 . \qquad (108)$$

In a similar manner to the above-considered scale hierarchy in the stationary percolation (95), this set of characteristic times makes it possible to distinguish the flow modes wherein the effects of 'long-range' correlations become the principal ones.

13. Conclusions

Even a brief consideration of the above papers shows how broad the range of investigations is in the field of turbulent diffusion analysis on the basis of correlation ideas. The methods which employ 'renormalizations' and correlation analysis for these purposes are under steady improvement [55–69]. In particular, the percolation approach has progressed towards the inclusion of multiple scales [6]. In this approach, advantage has been taken of the λ scale hierarchy in lieu of $\Psi_0 \approx V_0\lambda$. We then obtain new scaling dependences, which employ the index H:

$$\Psi_\lambda = \Psi_0 \left|\frac{\lambda}{\lambda_0}\right|^H, \quad V_\lambda = V_0 \left|\frac{\lambda}{\lambda_0}\right|^{H-1} .$$

With the use of multiple scales it has been possible to consider from a general standpoint the Dreizin–Dykhne models generalized to the cases $d > 1$.

Percolation estimates were also made [82–85] of diffusion effects in a stochastic magnetic field [41, 42, 80, 81]. A modification of the percolation method was proposed for a nonzero average flow velocity [86, 87].

In active use is the approach involving the factorization of the Eulerian correlation function $C_{\mathrm{E}}(\Delta, t) \approx f(\Delta)\, g(t)$ [88–90]. Methods based on 'renormalization' of the equation for the correlation function are progressing [91].

Recently, considerable interest was attracted by the works on 'self-organized criticality' related to the description of anomalous transfer in plasmas [92–94]. An analysis of experimental data led to the employment of a simple scaling

$$x \propto t^H, \quad H \approx 0.62-0.72 .$$

Using simple estimates one can show that this leads to the power form of correlation functions and hence to the necessity of considering 'long-range non-Gaussian correlations':

$$C(t) \propto \frac{\mathrm{d}}{\mathrm{d}t}\, D \propto \frac{x^2}{t^2} \propto \frac{1}{t^\beta}, \quad H = 1 - \frac{\beta}{2} . \qquad (109)$$

It is evident that the quest and analysis of models wherein correlation effects are of first importance remain a topical problem.

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