Physics – Uspekhi **46** (7) 667 – 688 (2003)

REVIEWS OF TOPICAL PROBLEMS

PACS numbers: **02.50.–r**, **05.40.–a**, **05.45.–a**

Clustering and diffusion of particles and passive tracer density in random hydrodynamic flows

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DOI: 10.1070/PU2003v046n07ABEH001600

Contents

Abstract. The diffusion of particles and conservative, passive tracer density fields in random hydrodynamic flows is considered. The crucial feature of this diffusion in a divergent hydrodynamic flow is the clustering of the conservative, passive tracer density field (in the Euler description) and occasionally of the particles themselves (in the Lagrange description) — a coherent phenomenon which occurs with probability unity and should arise in almost all dynamic scenarios of the process. In the present paper, statistical clustering parameters are described in statistical topography terms. Because of their inertial properties, particles and their concentration field can also cluster in random divergence-free velocity fields, the divergence of the particle velocity field itself being a crucial aspect of such a diffusion. The delta-correlated in time velocity field for fluctuating flow (as, e.g., in the Fokker – Planck diffusion equation for low-inertia particles) is in principle an invalid approximation for the statistical description of particle dynamics, and the diffusion approximation accounting for the finite time correlation radius should instead be used for this purpose.

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Received 3 March 2003
Uspekhi Fizicheskikh Nauk **173** (7) 689 – 710 (2003)
Translated by Yu V Morozov; edited by A Radzig

1. Introduction

The question of the propagation of particles and the passive tracer in random hydrodynamic flows is one of the problems of statistical hydrodynamics and has important implications for the solution of environmental problems associated with tracer diffusion in the Earth's atmosphere and oceans [1 – 5], diffusion in porous media [6], and large-scale mass distribution at the final stage of the formation of the Universe [7, 8]. The problem has been extensively studied since the pioneering works [9 – 11] were first published. Later, many researchers derived a variety of equations to describe statistical characteristics of a tracer field in both Eulerian and Lagrangian descriptions (see, for instance, Refs [12 – 15]). The derivation of such equations for different models of fluctuating parameters in various approximation schemes (for both the moment functions of tracer density field and tracer probability density) and their analysis has been underway over the last decades (see, for instance, Refs [16 – 28]).

Recently, the attention of both theorists and experimentalists has been drawn to the relationship between the dynamics of averaged characteristics of the problem solution and its behavior in individual realizations. This issue is of great interest for the geophysics of oceans and atmosphere where, generally speaking, there is no appropriate averaging ensemble and experimenters have to deal with individual realizations.

A solution to dynamic problems for concrete realizations of medium parameters is virtually hopeless because they are very complicated mathematically. At the same time, researchers are interested in the main characteristics of the phenomena being occurred rather than in their peculiar features, hence, the attractiveness of the idea of using the well-developed mathematical apparatus of random processes and fields — that is, to consider statistical averaging over the entire ensemble of possible realizations rather than individual realizations of the processes under study. Suffice it to say that presently the approach to almost all problems concerning the physics of the oceans and the atmosphere is to a greater or lesser extent based on statistical analysis.

Introduction of randomness into parameters of the medium gives rise to stochasticity of physical fields themselves. Individual realizations of, say, two-dimensional scalar fields

$$\rho(\mathbf{r}, t), \ \mathbf{r} = (x, y)$$

resemble a complex mountain terrain with randomly distributed peaks, ridges, valleys, passes, etc. Methods usually employed for statistical averaging (i.e., computation of averages like the mean value $\langle \rho(\mathbf{r}, t) \rangle$, space – time correlation function $\langle \rho(\mathbf{r}, t)\, \rho(\mathbf{r}', t') \rangle$, etc., where $\langle \ldots \rangle$ indicates ensemble averaging over realizations of random parameters) tend to smooth qualitative peculiarities of individual realizations; as a result, statistical characteristics thus obtained not infrequently have little to do with the behavior of different realizations or, at first glance, contradict one another. By way of example, statistical averaging over all realizations smooths out the mean passive tracer field in a random velocity field, while each of its individual realizations tends to be increasingly rugged in space due to the mixing of regions with significantly different concentrations.

Therefore, statistical averages of this type normally characterize 'global' spatial – temporal scales of a region in which stochastic processes occur but tell nothing about details of progressing these processes inside the region. In the meantime, such details for the given example strongly depend on the velocity field that can be either divergent or divergence-free. Then, in the former case, certain realizations lead (with probability unity) to the formation of *clusters* — that is, compact regions of elevated tracer concentration (see, for instance, Refs [29, 30]) surrounded by vast low-density 'voids'. In this case, however, all statistical moments of the interparticle spacing exponentially grow in time, which implies statistical dispersion of particles in the mean (see, for instance, Refs [31 – 33]).

It is natural to call physical processes and phenomena that occur with probability unity *coherent* (see paper [34] and books [36] where this issue is discussed at greater length). Such 'statistical coherence' may be viewed as a mode of organization of a complex dynamical system, while identification of its *statistically stable characteristics* is analogous to the notion of *coherence* as the *self-organization* of multicomponent systems arising from chaotic interactions between their constituent elements (see, for instance, Ref. [37]).

Doubtless, complete statistics (for example, a totality of space – time n-point moment functions) contain all the information about a dynamical system. In practice, however, it is possible to study only the simplest statistical characteristics largely associated with simultaneous and one-point probability distributions. The problem then is how to deduce principal qualitative and quantitative peculiarities of the behavior of individual realizations of the system from its statistical characteristics and specific features and to describe such physical phenomena as clustering of particles and passive tracer density fields in hydrodynamic flows.

A possible solution to the problem should be sought by methods of statistical topography (see, for instance, paper [35] and books [36]) that permit the revision of the 'philosophy' of statistical analysis of stochastic dynamical systems and may be used by experimenters when planning statistical treatment of experimental materials.

Starting from the classical work of G Stokes published in 1851 [38] (see also books [39, 40]), investigations into dynamics and diffusion of inertial particles in hydrodynamic flows have been the focus of attention of many scientists due to their numerous practical applications (see, for instance, books [41, 42] and papers [43 – 47] containing a comprehensive bibliography). It is worthwhile to note that one of the earliest works [45] was the first to emphasize that the velocity field of inertial particles in the divergence-free velocity field of a hydrodynamic flow is divergent, unlike that of inertialess passive particles. This fact was widely employed in papers [48, 49] when analyzing a multiplicity of applications to hydrodynamics, geophysics, and astrophysics. Divergent character of the velocity field of inertial particles implies clustering of such particles and the passive tracer field formed by them, even in a divergence-free hydrodynamic flow. The principal task is to evaluate the main parameters that characterize such clustering [50].

2. Formulation of the problem

2.1 Low-inertia particles and low-inertia tracer field

Diffusion of the number density field $n(\mathbf{r}, t)$ of inertial particles travelling in random hydrodynamic flows that are described by the Euler velocity field $\mathbf{U}(\mathbf{r}, t)$ satisfies the continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\, \mathbf{V}(\mathbf{r}, t)\right) n(\mathbf{r}, t) = 0\,, \qquad n(\mathbf{r}, 0) = n_0(\mathbf{r})\,. \tag{1}$$

Here, $\mathbf{V}(\mathbf{r}, t)$ denotes the velocity field of particles in a hydrodynamic flow.

The total number of particles is conserved in the course of evolution, i.e., one finds

$$N_0 = \int n(\mathbf{r}, t)\, \mathrm{d}\mathbf{r} = \int n_0(\mathbf{r})\, \mathrm{d}\mathbf{r} = \mathrm{const}\,.$$

If the particle density is ρ_0, the evolution of the density field $\rho(\mathbf{r}, t) = \rho_0 n(\mathbf{r}, t)$ of a passive tracer moving in a hydrodynamic flow is also described by the continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\, \mathbf{V}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t) = 0\,, \qquad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,,$$

which can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t)\, \frac{\partial}{\partial \mathbf{r}}\right) \rho(\mathbf{r}, t) + \frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial \mathbf{r}}\, \rho(\mathbf{r}, t) = 0\,. \tag{2}$$

We do not take into consideration the effect of molecular diffusion that is correct at the initial stages of the development of diffusion, and then the total tracer mass remains unaltered during evolution, i.e., we have

$$M = M(t) = \int \rho(\mathbf{r}, t)\, \mathrm{d}\mathbf{r} = \int \rho_0(\mathbf{r})\, \mathrm{d}\mathbf{r} = \mathrm{const}\,.$$

Velocity field $\mathbf{V}(\mathbf{r}, t)$ of particles in a hydrodynamic flow $\mathbf{U}(\mathbf{r}, t)$ for low-inertia particles can be described by a partial derivative quasi-linear equation (see, for instance, Refs [43–47])

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\,\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)=-\lambda\bigl[\mathbf{V}(\mathbf{r},t)-\mathbf{U}(\mathbf{r},t)\bigr]\,,\qquad(3)$$

which we shall regard as a phenomenological one. In the general case, the nonuniqueness of the solution of Eqn (3), discontinuities, etc. are possible. However, in an asymptotic case of low-inertia particles (parameter $\lambda\to\infty$), which is of special interest to us, there exists a unique solution over a reasonable time interval. It should be noted that the term $\mathbf{F}(\mathbf{r},t)=\lambda\mathbf{V}(\mathbf{r},t)$ on the right-hand side of Eqn (3), linear in the velocity field $\mathbf{V}(\mathbf{r},t)$, is the well-known *Stokes formula* for a resistive force acting on a slowly moving particle. If the particle is approximated by a ball of radius a, then one obtains

$$\lambda=\frac{6\pi a\eta}{m_{\mathrm{p}}}\,,$$

where η is the coefficient of dynamic viscosity, and m_{p} is the particle's mass (see, for instance, Refs [39, 40]).

In the general case, the hydrodynamic Eulerian velocity field has the form

$$\mathbf{U}(\mathbf{r},t)=\mathbf{u}_0(\mathbf{r},t)+\mathbf{u}(\mathbf{r},t)\,,$$

where $\mathbf{u}_0(\mathbf{r},t)$ is the deterministic component of the velocity field (mean flow), and $\mathbf{u}(\mathbf{r},t)$ is the random component. Random field $\mathbf{u}(\mathbf{r},t)$ may have both solenoidal [for which $\operatorname{div}\mathbf{u}(\mathbf{r},t)=0$] and potential [for which $\operatorname{div}\mathbf{u}(\mathbf{r},t)\neq0$] components.

As mentioned above, Eqns (1)–(3) provide the *Eulerian description* of the evolution of the number density field of low-inertia particles and the density field of a passive tracer. These equations are actually first-order partial derivative equations and can be solved by the method of characteristics.

Introduction of characteristic curves $\mathbf{r}(t)$, $\mathbf{V}(t)$ that describe particle motion with the help of equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t)=\mathbf{V}\bigl(\mathbf{r}(t),t\bigr)\,,\qquad \mathbf{r}(0)=\mathbf{r}_0\,,\\&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{V}(t)=-\lambda\bigl[\mathbf{V}(t)-\mathbf{U}\bigl(\mathbf{r}(t),t\bigr)\bigr]\,,\qquad \mathbf{V}(0)=\mathbf{V}_0(\mathbf{r}_0)\end{aligned}\qquad(4)$$

allows for the passage from Eqns (1) and (2) to ordinary differential equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,n(t)=-n(t)\,\frac{\partial\mathbf{V}\bigl(\mathbf{r}(t),t\bigr)}{\partial\mathbf{r}}\,,\qquad n(0)=n_0(\mathbf{r}_0)\,,\\&\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t)=-\rho(t)\,\frac{\partial\mathbf{V}\bigl(\mathbf{r}(t),t\bigr)}{\partial\mathbf{r}}\,,\qquad \rho(0)=\rho_0(\mathbf{r}_0)\,.\end{aligned}\qquad(5)$$

It should be emphasized that Eqns (4) are ordinary Newton equations for the dynamics of a particle with the linear frictional force described by the Stokes force $\mathbf{F}(t)=-\lambda\mathbf{V}(\mathbf{r}(t),t)$, under the effect of a random force $\mathbf{f}(t)=\lambda\mathbf{U}(\mathbf{r}(t),t)$ induced by the hydrodynamic flow.

The solutions for Eqns (5) have a clear geometric interpretation. They describe the evolution of particle number and passive tracer densities in the vicinity of a fixed particle, the trajectory of which is determined by the solution $\mathbf{r}=\mathbf{r}(t)$ to the system of equations (4). It follows from equations (5) that particle number and passive tracer densities in divergent flows vary; specifically, they are greater in the regions of compression, and smaller in the regions of rarefaction of the medium.

2.2 Inertialess particles and inertialess tracer field

For inertialess particles, the parameter $\lambda\to\infty$ and, as follows from Eqn (3), we arrive at

$$\mathbf{V}(\mathbf{r},t)=\mathbf{U}(\mathbf{r},t)\,.$$

In this case, the particle's trajectory and number density in a hydrodynamic flow with the velocity field $\mathbf{U}(\mathbf{r},t)$ are described by the equations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t)=\mathbf{U}\bigl(\mathbf{r}(t),t\bigr)\,,\qquad \mathbf{r}(0)=\mathbf{r}_0\,,\\&\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t)=-\rho(t)\,\frac{\partial\mathbf{U}\bigl(\mathbf{r}(t),t\bigr)}{\partial\mathbf{r}}\,,\qquad \rho(0)=\rho_0(\mathbf{r}_0)\,,\end{aligned}\qquad(6)$$

and the Euler density field satisfies the equation

$$\left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{r},t)\,\frac{\partial}{\partial\mathbf{r}}\right)\rho(\mathbf{r},t)+\frac{\partial\mathbf{U}(\mathbf{r},t)}{\partial\mathbf{r}}\,\rho(\mathbf{r},t)=0\,.\qquad(7)$$

Thus, the problem of determining trajectories of inertialess particles in a hydrodynamic flow is a purely kinematic one.

Let us now consider stochastic peculiarities of the solution of problem (6) for a system of particles in the absence of average flow ($\mathbf{u}_0(\mathbf{r},t)=0$). According to Eqn (6), each particle travels independently. However, if the random field $\mathbf{u}(\mathbf{r},t)$ has a finite spatial correlation radius l_{cor}, the particles spaced less than l_{cor} are all located in the zone of influence of the random field $\mathbf{u}(\mathbf{r},t)$; therefore, new collective features are likely to appear in the dynamics of such a system of particles.

For a stationary velocity field $\mathbf{u}(\mathbf{r},t)\equiv\mathbf{u}(\mathbf{r})$, Eqn (6) is simplified and acquires the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t)=\mathbf{u}(\mathbf{r})\,,\qquad \mathbf{r}(0)=\mathbf{r}_0\,.$$

It follows, thence, that the stationary points $\tilde{\mathbf{r}}$ at which $\mathbf{u}(\tilde{\mathbf{r}})=0$ remain fixed. Depending on whether these points are stable or unstable, they either attract or repel particles in their neighborhood. By virtue of the stochastic nature of the function $\mathbf{u}(\mathbf{r})$, the positions of points $\tilde{\mathbf{r}}$ are random too. A similar situation ought to persist in the general case of random space–time velocity field $\mathbf{u}(\mathbf{r},t)$.If some points $\tilde{\mathbf{r}}$ remain stable during a sufficiently long period, then in certain realizations of the random field $\mathbf{u}(\mathbf{r},t)$ cluster regions of particles ought to form in their neighborhood (i.e., compact regions of increased particle concentration located mostly in rarefied zones). If, however, the stable points become unstable soon enough and the particles have no time to rearrange, cluster regions do not form.

Numerical simulation (see Refs [27, 51, 52]) shows that the dynamic behavior of a system of particles differs considerably depending on whether the random velocity field is divergent or divergence-free. By way of example, Fig. 1a illustrating a concrete realization of a stationary divergence-free velocity field $\mathbf{u}(\mathbf{r})$ schematically depicts evolution of a system of particles (two-dimensional case) uniformly distributed within a circle in dimensionless time related to the statistical parameters of the field $\mathbf{u}(\mathbf{r})$. In this case, the area enclosed by the contour is conserved, and the particles more or less uniformly fill the region of a space bounded by the distorted

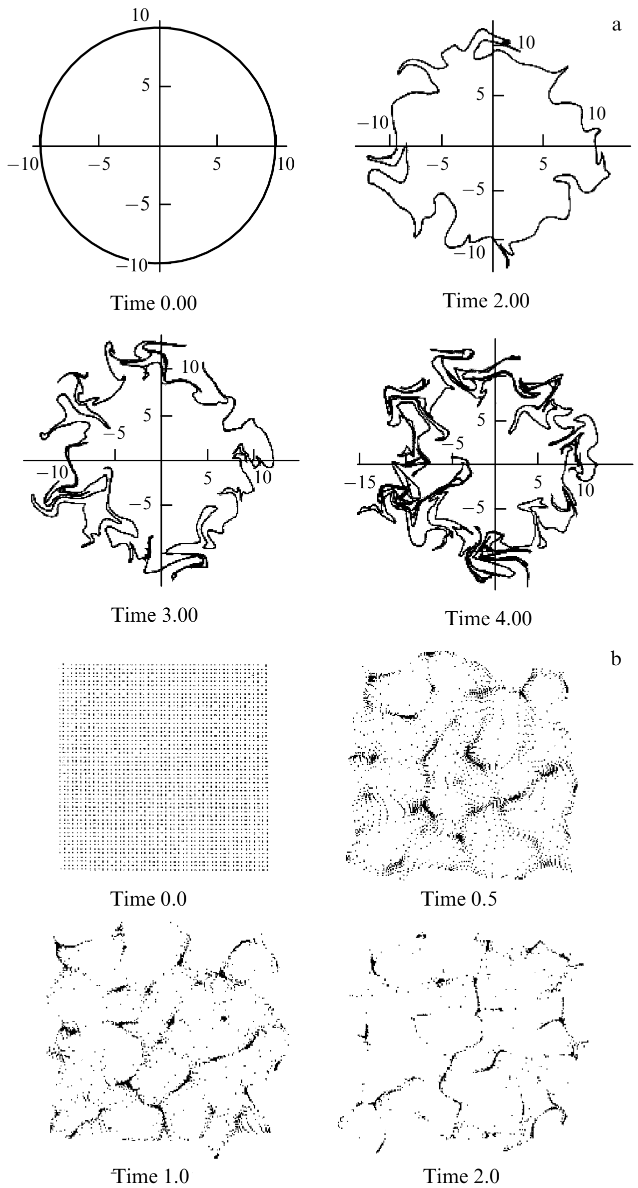

Figure 1. Results of the numerical simulation of particle's diffusion in solenoidal (a) and divergent (b) random velocity fields $\mathbf{u}(\mathbf{r})$.

contour. Only the strong unevenness of the contour in a fractal-like fashion makes itself evident. In the case of a divergent velocity field $\mathbf{u}(\mathbf{r})$, the particles that were initially uniformly distributed inside a square eventually huddle in clusters in the course of time evolution. The results of numerical simulation for this case are presented in Fig. 1b. It should be emphasized once again that cluster formation in this case is a purely kinematic phenomenon. Evidently, this feature of particle dynamics disappears upon ensemble averaging of random velocity field realizations.

The simplest example of particle clustering is that in a random velocity field $\mathbf{u}(\mathbf{r}, t)$ having the structure [53]

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{v}(t) \sin(2\mathbf{kr}) , \tag{8}$$

where $\mathbf{v}(t)$ is the random vector process. Such a form of the function $\mathbf{u}(\mathbf{r}, t)$ corresponds to the first term of a series expansion in terms of harmonic components; it is usually used in numerical simulation of the problem [51, 52].

In this case, Eqn (6) can be written down as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{v}(t) \sin(2\mathbf{kr}) , \qquad \mathbf{r}(0) = \mathbf{r}_0 .$$

For such a model, the motion of a particle in the direction of the vector $\mathbf{k}$ and in the perpendicular direction can be resolved. If the x-axis is aligned with the direction of the vector $\mathbf{k}$, then the equations take the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = v_x(t) \sin(2kx) , \qquad x(0) = x_0 , \\ &\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{R}(t) = \mathbf{v}_{\mathbf{R}}(t) \sin(2kx) , \qquad \mathbf{R}(0) = \mathbf{R}_0 . \end{aligned} \tag{9}$$

The solution to the first equation in set (9) has the form

$$x(t) = \frac{1}{k} \arctan\left\{\exp\left[T(t)\right] \tan(kx_0)\right\} , \tag{10}$$

where

$$T(t) = 2k \int_0^t v_x(\tau)\,\mathrm{d}\tau . \tag{11}$$

Taking into consideration the equality that ensues from formula (10):

$$\sin(2kx) = \sin(2kx_0)\,\frac{1}{\exp\left[-T(t)\right]\cos^2(kx_0) + \exp\left[T(t)\right]\sin^2(kx_0)} ,$$

the last equation in set (9) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{R}(t|\mathbf{r}_0) = \sin(2kx_0)\,\frac{\mathbf{v}_{\mathbf{R}}(t)}{\exp\left[-T(t)\right]\cos^2(kx_0) + \exp\left[T(t)\right]\sin^2(kx_0)} .$$

Hence, we obtain

$$\mathbf{R}(t|\mathbf{r}_0) = \mathbf{R}_0 + \sin(2kx_0) \times \int_0^t \frac{\mathbf{v}_{\mathbf{R}}(\tau)}{\exp\left[-T(\tau)\right]\cos^2(kx_0) + \exp\left[T(\tau)\right]\sin^2(kx_0)}\,\mathrm{d}\tau . \tag{12}$$

Thus, the initial location x_0 of the particle being such that

$$kx_0 = n\,\frac{\pi}{2} , \tag{13}$$

where $n = 0, \pm 1, \ldots$, the particle will be fixed, and $\mathbf{r}(t) \equiv \mathbf{r}_0$.

Equalities (13) define planes and points in the general and one-dimensional cases, respectively. They correspond to zeroes of the velocity field. However, the stability of these points depends on the sign of the function $\mathbf{v}(t)$ that changes in the course of evolution. As a result, the particles can be expected to crowd in the vicinity of these points if $v_x(t) \neq 0$, which corresponds to their clustering.

For a divergence-free velocity field, when $v_x(t) = 0$ (and, hence, $T(t) \equiv 0$), one finds

$$x(t|x_0) \equiv x_0 , \qquad \mathbf{R}(t|\mathbf{r}_0) = \mathbf{R}_0 + \sin(2kx_0) \int_0^t \mathbf{v}_{\mathbf{R}}(\tau)\,\mathrm{d}\tau ,$$

so that no clustering occurs.

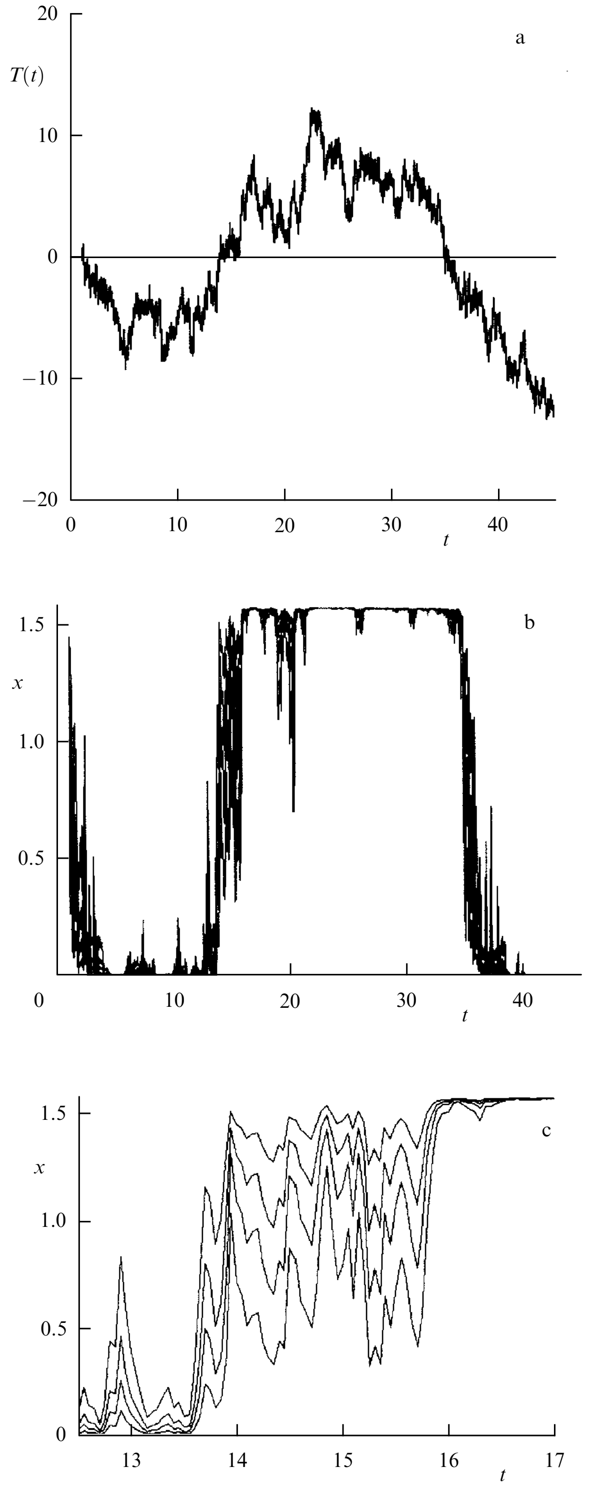

Figure 2. Fragment of realization of the random process $T(t)$ (a) obtained by numerical integration of equality (11) for a single realization of the random process $v_x(t)$ and used to calculate time evolution of the x-coordinates of four particles (b, c).

Figure 2a displays a fragment of the realization of the random process $T(t)$, obtained by numerical integration of equality (11) for one particular realization of the random process $v_x(t)$ and used for numerical simulation of the time evolution of the coordinates $x(t)$ $(x \in (0, \pi/2)$ of four particles having the initial coordinates $x_0(i) = (\pi/2)(i/5)$ $(i = 1, 2, 3, 4)$ (see Fig. 2b). It can be seen that at a dimensionless moment of time $t \approx 4$ (see Ref. [53]), the particles form a cluster in the vicinity of point $x = 0$. Furthermore, at the instant of time $t \approx 16$, this cluster disappears and a new cluster forms in the neighborhood of point $x = \pi/2$. At the instant $t \approx 40$, the cluster is formed anew in the vicinity of point $x = 0$, and so forth. The particles in these clusters remember their history and throughout each transient period they become widely separated (Fig. 2c).

Thus, in this example, the cluster as an entity does not move from one region in space to another but breaks down, accompanied by the generation of a new one. As this takes place, the lifetime of the clusters is much greater than the transient time. This is apparently a specific property of the given velocity field model related to the stationary position of the points (13).

The diffusion of particles along the y-axis is not associated with the formation of clusters.

It is also possible to follow up cluster formation in the Eulerian description by taking the random velocity field of the form (8) as an example. In this case, the density field $\rho(\mathbf{r}, t)$ is described by the expression [53]

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}_0)\, \frac{1}{\exp\left[T(t)\right]\cos^2(kx) + \exp\left[-T(t)\right]\sin^2(kx)}\,, \tag{14}$$

where the function $T(t)$ is given by formula (11), and the parameter $\mathbf{r}_0$ is found from expressions (10) and (12).

For a divergence-free velocity field, when $v_x(t) = 0$, $T(t) \equiv 0$, one finds

$$\rho(\mathbf{r}, t) = \rho_0\left(\mathbf{r} - \sin(2kx)\int_0^t \mathbf{v}(\tau)\,\mathrm{d}\tau\right).$$

In a particular case when the initial density distribution is independent of $\mathbf{r}$, i.e., $\rho_0(\mathbf{r}) = \rho_0$, the equality (14) is simplified, taking the form

$$\frac{\rho(\mathbf{r}, t)}{\rho_0} = \frac{1}{\exp\left[T(t)\right]\cos^2(kx) + \exp\left[-T(t)\right]\sin^2(kx)}\,. \tag{15}$$

Figure 3 illustrates space–time evolution of an Eulerian density field $1 + \rho(\mathbf{r}, t)/\rho_0$, calculated with the use of formula (15) in dimensionless space–time variables (unity is added to avoid problems with logarithms at near-zero density values). These figures clearly show sequential flow of the density field to the narrow vicinities of points $x \approx 0$ and $x \approx \pi/2$, which implies the formation of clusters. For example, Figs 3a and 3b show the time sequence ($t = 1-10$) of cluster formation in the neighborhood of point $x \approx 0$. Figures 3c and 3d show the time sequence ($t = 16-25$) of density field flow from the neighborhood of point $x \approx 0$ to the neighborhood of point $x \approx \pi/2$ — that is, cluster disintegration about a point $x \approx 0$ and the formation of a new cluster in the vicinity of $x \approx \pi/2$. This process is further repeated in time. It appears from the figures that the 'lifetime' of such clusters in the model under consideration is of the same order as the 'time of their formation'.

Thus, we have considered the simplest model of tracer (particles and Eulerian density field) diffusion in a random velocity field that clearly discloses the process of cluster formation. A specific feature of this model is the presence of fixed points where clusters form. Evidently, this somewhat compromises the value of the model.

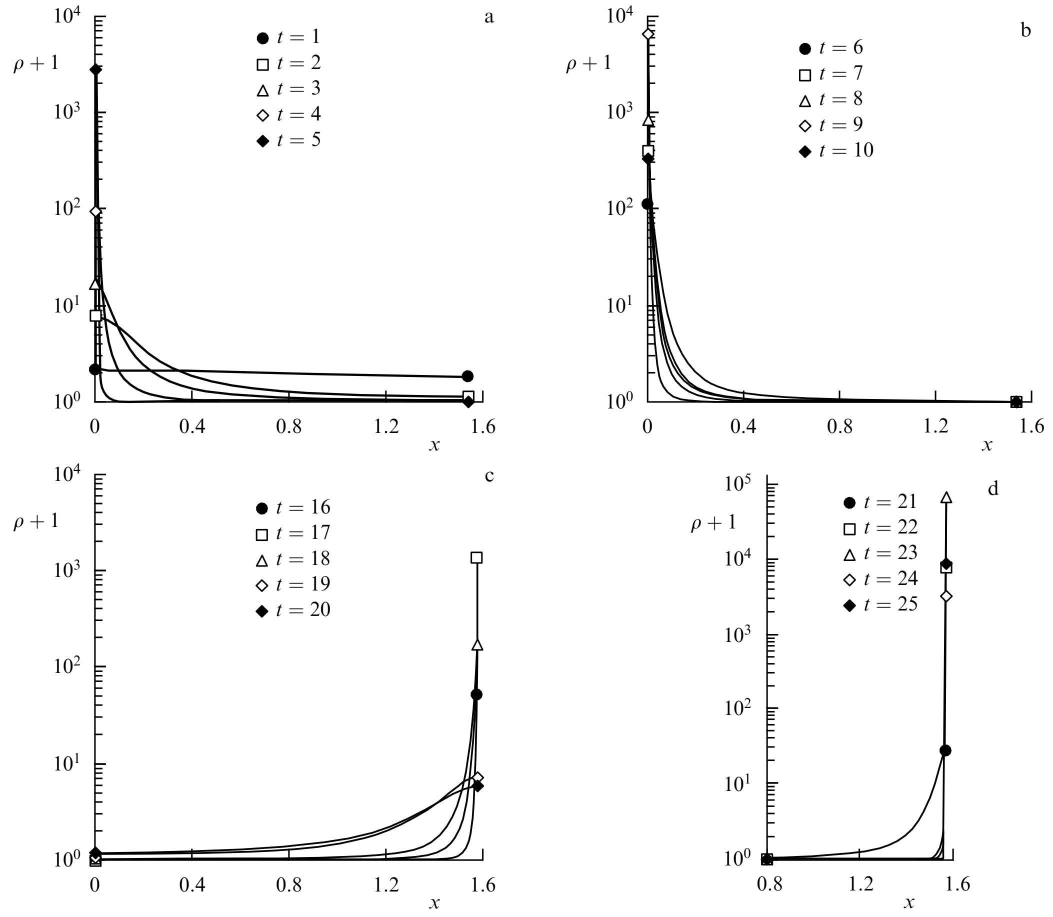

Figure 3. Space – time evolution of the Eulerian density field.

However, this model permits us to understand the basic difference between particle's diffusion in divergent and divergence-free velocity fields. In divergence-free (incompressible) velocity fields, particles (hence, the density field) do not have enough time to drift to stable attraction centers, while the latter still exist and only slightly fluctuate about their initial positions. In divergent (compressible) velocity fields, the same lifetime of the stable attraction centers proves sufficient for the particles to be drawn to them because the process of particle attraction is exponentially accelerated as follows from formula (15).

It is worth noting that such clustering for a system of particles and a tracer field was first reported in Refs [54, 55] where numerical modeling of the so-called Eole experiment was undertaken based on the simplest equations of atmospheric dynamics. In the framework of this global experiment, 500 balloons of constant density were launched in Argentina in 1970 – 1971 and spread over the entire Southern hemisphere at an altitude of roughly 12 km.

The results of statistical processing of relative distances between particles, examined experimentally, can be found, for instance, in Refs [56 – 58].

Solution to the system of equations (6) depends on the characteristic parameter $\mathbf{r}_0$, i.e., the particle's initial coordinate (denoted by the vertical bar):

$$\mathbf{r}(t) = \mathbf{r}(t|\mathbf{r}_0)\,, \qquad \rho(t) = \rho(t|\mathbf{r}_0)\,. \tag{16}$$

The components of vector $\mathbf{r}_0$ that unambiguously determine the position of an arbitrary particle are known as its *Lagrangian coordinates*. Then, equations (6) correspond to the *Lagrangian description* of the evolution of the number density field of particles and active tracer density. The linkage between Eulerian and Lagrangian descriptions is given by the first of equalities (16). Its solution with respect to $\mathbf{r}_0$ yields a relation that expresses the Lagrangian coordinates in terms of the Eulerian ones:

$$\mathbf{r}_0 = \mathbf{r}_0(\mathbf{r}, t)\,. \tag{17}$$

Subsequent elimination of $\mathbf{r}_0$-dependence in the remaining equality (16) using Eqn (17) leads back to the Eulerian description of the passive tracer density field:

$$\rho(\mathbf{r}, t) = \rho\big(t|\mathbf{r}_0(\mathbf{r}, t)\big) = \int \rho(t|\mathbf{r}_0)\, j(t|\mathbf{r}_0)\, \delta\big(\mathbf{r}(t|\mathbf{r}_0) - \mathbf{r}\big)\, \mathrm{d}\mathbf{r}_0\,, \tag{18}$$

where a new function called *divergence* was introduced:

$$j(t|\mathbf{r}_0) = \det \| j_{ik}(t|\mathbf{r}_0) \| = \det \left\| \frac{\partial r_i(t|\mathbf{r}_0)}{\partial r_{0k}} \right\| .$$

Differentiation of Eqn (6) with respect to components of the vector $\mathbf{r}_0$ yields equations for elements of the Jacobian matrix $j_{ik}(t|\mathbf{r}_0)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, j_{ik}(t|\mathbf{r}_0) = \frac{\partial U_i(\mathbf{r}, t)}{\partial r_l}\, j_{lk}(t|\mathbf{r}_0)\,, \qquad j_{ik}(0|\mathbf{r}_0) = \delta_{ik}\,.$$

It follows herefrom that the determinant of this matrix is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, j(t|\mathbf{r}_0) = \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}}\, j(t|\mathbf{r}_0)\,, \qquad j(0|\mathbf{r}_0) = 1\,, \tag{19}$$

with $j(t|\mathbf{r}_0)$ being a quantitative measure of the degree of compression or expansion of physically infinitely small liquid particles. Comparison of Eqns (6) and (19) shows that

$$\rho(t|\mathbf{r}_0) = \frac{\rho_0(\mathbf{r}_0)}{j(t|\mathbf{r}_0)}\,. \tag{20}$$

Therefore, expression (18) can be rewritten as the equality

$$\rho(\mathbf{r},t) = \int \rho_0(\mathbf{r}_0)\,\delta\big(\mathbf{r}(t|\mathbf{r}_0) - \mathbf{r}\big)\,\mathrm{d}\mathbf{r}_0\,, \tag{21}$$

which relates the Lagrangian and Eulerian characteristics. The delta-function on the right-hand side of Eqn (21) is an *indicator function* for the position of a Lagrangian particle. Hence, averaging equality (21) over the ensemble of realizations of a random velocity field yields the well-known relation between the mean passive tracer density in the Eulerian description and simultaneous probability density

$$P(t;\mathbf{r}|\mathbf{r}_0) = \big\langle \delta\big(\mathbf{r}(t|\mathbf{r}_0) - \mathbf{r}\big)\big\rangle$$

of the position of an inertialess Lagrangian particle (see, for instance, Ref. [1]):

$$\big\langle \rho(\mathbf{r},t)\big\rangle = \int \rho_0(\mathbf{r}_0)\, P(t,\mathbf{r}|\mathbf{r}_0)\,\mathrm{d}\mathbf{r}_0\,.$$

Evidently, this equality also holds true for low-inertia particles described by Eqns (2).

Thus, the behavior of inertialess particles and passive tracer density is described in the Lagrangian representation by ordinary differential equations (4), (19). It is easy to pass from them to the linear Liouville equation for indicator functions in the corresponding phase space (see, for instance, Refs [36, 61]). For this purpose, the following indicator function should be introduced:

$$\Phi_{\mathrm{Lag}}(t;\mathbf{r},\rho,j|\mathbf{r}_0) = \delta\big(\mathbf{r}(t|\mathbf{r}_0) - \mathbf{r}\big)\,\delta\big(\rho(t|\mathbf{r}_0) - \rho\big)\,\delta\big(j(t|\mathbf{r}_0) - j\big)\,, \tag{22}$$

the form of which explicitly takes into account that the solution of the input dynamic equations depends on the Lagrangian coordinates $\mathbf{r}_0$. Differentiation of Eqn (22) with respect to time and the use of Eqns (4), (5), and (19) lead to the Liouville equation equivalent to the original problem:

$$\begin{gathered}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{U}(\mathbf{r},t)\right)\Phi_{\mathrm{Lag}}(t;\mathbf{r},\rho,j|\mathbf{r}_0) \\ = \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}}\left(\frac{\partial}{\partial \rho}\,\rho - \frac{\partial}{\partial j}\,j\right)\Phi_{\mathrm{Lag}}(t;\mathbf{r},\rho,j|\mathbf{r}_0)\,, \\ \Phi_{\mathrm{Lag}}(0;\mathbf{r},\rho,j|\mathbf{r}_0) = \delta(\mathbf{r}_0 - \mathbf{r})\,\delta\big(\rho_0(\mathbf{r}_0) - \rho\big)\,\delta(j-1)\,.\end{gathered} \tag{23}$$

The simultaneous probability density for the solution of statistical equations (4) and (19) coincides with the indicator function averaged over the ensemble of realizations:

$$P(t;\mathbf{r},\rho,j|\mathbf{r}_0) = \big\langle \Phi_{\mathrm{Lag}}(t;\mathbf{r},\rho,j|\mathbf{r}_0)\big\rangle\,.$$

For the description of the density field in the Eulerian representation, an indicator function analogous to Eqn (22) is introduced:

$$\Phi(t,\mathbf{r};\rho) = \delta\big(\rho(\mathbf{r},t) - \rho\big)\,, \tag{24}$$

which is localized on the surface $\rho(\mathbf{r},t) = \rho = \mathrm{const}$ in the three-dimensional case or on the contour in the case of two dimensions. This function is related to the Lagrangian indicator function by an explicit equality

$$\Phi(t,\mathbf{r};\rho) = \int \mathrm{d}\mathbf{r}_0 \int_0^\infty \mathrm{d}j\, j\,\Phi_{\mathrm{Lag}}(t;\mathbf{r},\rho,j|\mathbf{r}_0)$$

and, therefore, satisfies the equation

$$\begin{gathered}\left(\frac{\partial}{\partial t} + \mathbf{U}(\mathbf{r},t)\,\frac{\partial}{\partial \mathbf{r}}\right)\Phi(t,\mathbf{r};\rho) = \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}}\,\frac{\partial}{\partial \rho}\big[\rho\Phi(t,\mathbf{r};\rho)\big]\,, \\ \Phi(0,\mathbf{r};\rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,,\end{gathered} \tag{25}$$

suggesting that salient peculiarities arise only for the divergent velocity field $\mathbf{U}(\mathbf{r},t)$. Certainly, Eqn (25) can be obtained directly from dynamic equation (7) (see, for instance, Refs [36, 61]).

In this case, the one-point probability density for the solution of dynamic equation (2) coincides with the indicator function averaged over the ensemble of realizations:

$$P(t,\mathbf{r};\rho) = \big\langle \delta\big(\rho(\mathbf{r},t) - \rho\big)\big\rangle\,.$$

It means that one-point probability density of a density field in the Eulerian description is related to simultaneous probability density in the Lagrangian description by the equality

$$P(t,\mathbf{r};\rho) = \int \mathrm{d}\mathbf{r}_0 \int_0^\infty \mathrm{d}j\, j P(t;\mathbf{r},\rho,j|\mathbf{r}_0)\,. \tag{26}$$

Besides, the Euler indicator function gives a wealth of qualitative and quantitative information about the geometric structure of the random field $\rho(\mathbf{r},t)$ (*statistical topography*). The main object of statistical topography, as in the conventional topography of mountain massifs, is a set of contours — that is, level lines (in the two-dimensional case) or surfaces (in the three-dimensional case) of constant values defined by the equality

$$\rho(\mathbf{r},t) = \rho = \mathrm{const}\,.$$

For the analysis of a set of contours (for simplicity, we discuss the two-dimensional case), it is convenient to introduce the singular indicator function (24) located on them and serving as the functional of the parameters specifying the medium.

In terms of function (24), it is possible to express such quantities as the total area of the regions enclosed by the level lines, where $\rho(\mathbf{r},t) > \rho$:

$$S(t,\rho) = \int_\rho^\infty \mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\,\Phi(t,\mathbf{r};\tilde{\rho})\,, \tag{27}$$

and the field's total 'mass' confined within these regions:

$$M(t,\rho) = \int_\rho^\infty \tilde{\rho}\,\mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\,\Phi(t,\mathbf{r};\tilde{\rho})\,. \tag{28}$$

By way of example, for passive tracer dynamics described by the Liouville equation (25), differentiation of equalities (27) and (28) with respect to time yields the expressions

$$\frac{\partial}{\partial t}\, S(t,\rho) = \int \mathrm{d}\mathbf{r} \int_{\rho}^{\infty} \mathrm{d}\tilde{\rho}\; \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}} \left(\frac{\partial}{\partial \tilde{\rho}}\, \tilde{\rho} + 1 \right) \Phi(t,\mathbf{r};\tilde{\rho})\,,$$

$$\frac{\partial}{\partial t}\, M(t,\rho) = \int \mathrm{d}\mathbf{r} \int_{\rho}^{\infty} \mathrm{d}\tilde{\rho}\; \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}}\, \tilde{\rho} \left(\frac{\partial}{\partial \tilde{\rho}}\, \tilde{\rho} + 1 \right) \Phi(t,\mathbf{r};\tilde{\rho})\,.$$

Thus, the size of the area enclosed by the contour $\rho(\mathbf{r},t) = \rho = \mathrm{const}$, and the total mass concentrated in this area are conserved for a divergence-free velocity field. Evidently, in this case, the number of closed contours is also conserved since they can neither arise nor disappear in the medium; rather, they undergo evolution in time depending on their spatial distribution patterns at an initial moment, defined by the equality $\rho_0(\mathbf{r}) = \rho = \mathrm{const}$.

If the velocity field has a potential component, all these quantities undergo time evolution.

Values of expressions (27) and (28) averaged over the ensemble of realizations are directly defined by the one-point probability density.

Additional information on the detailed structure of the field $\rho(\mathbf{r},t)$ can be obtained by considering its spatial gradient $\mathbf{p}(\mathbf{r},t) = \nabla\rho(\mathbf{r},t)$. For example, the quantity

$$l(t,\rho) = \int \left|\mathbf{p}(\mathbf{r},t)\right| \delta\big(\rho(\mathbf{r},t) - \rho\big)\, \mathrm{d}\mathbf{r} = \oint \mathrm{d}l \qquad (29)$$

describes the total length of contours $\rho(\mathbf{r},t) = \rho = \mathrm{const}$.

3. Statistical analysis of diffusion of inertialess particles and tracer density field

Let us now consider the problem of the statistical description of the diffusion of inertialess particles and the density field of a passive tracer in a random velocity field in the absence of an average flow ($\mathbf{u}_0(\mathbf{r},t) = 0$).

The random component of the velocity field is assumed in the general case to be divergent ($\operatorname{div}\mathbf{u}(\mathbf{r},t) \neq 0$) and, at the same time, approximated by a statistically homogeneous, isotropic in space, and stationary random Gaussian field with correlation and spectral tensors ($\langle \mathbf{u}(\mathbf{r},t)\rangle = 0$)

$$\begin{aligned} B_{ij}(\mathbf{r}-\mathbf{r}',t-t') &= \left\langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\right\rangle \\ &= \int E_{ij}(\mathbf{k},t-t') \exp\left[\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}')\right] \mathrm{d}\mathbf{k}\,, \\ E_{ij}(\mathbf{k},t) &= E^{\mathrm{s}}_{ij}(\mathbf{k},t) + E^{\mathrm{p}}_{ij}(\mathbf{k},t)\,, \end{aligned} \qquad (30)$$

where spectral components of the velocity field tensor have the structure

$$\begin{aligned} E^{\mathrm{s}}_{ij}(\mathbf{k},t) &= E^{\mathrm{s}}(k,t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \\ E^{\mathrm{p}}_{ij}(\mathbf{k},t) &= E^{\mathrm{p}}(k,t)\, \frac{k_i k_j}{k^2}\,. \end{aligned} \qquad (31)$$

Here, $E^{\mathrm{s}}(k,t)$ and $E^{\mathrm{p}}(k,t)$ denote solenoidal and potential components of the spectral density of the velocity field, respectively.

The time correlation radius of the field $\mathbf{u}(\mathbf{r},t)$ is defined by the equalities

$$\begin{aligned} \tau_0\sigma_{\mathbf{u}}^2 &= \int_0^{\infty} B_{ii}(0,\tau)\,\mathrm{d}\tau = \int_0^{\infty} \left\langle \mathbf{u}(\mathbf{r},t+\tau)\mathbf{u}(\mathbf{r},t)\right\rangle \mathrm{d}\tau \\ &= \int_0^{\infty} \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\, E_{ii}(\mathbf{k},\tau) \\ &= \int_0^{\infty} \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\, \left[(d-1)\,E^{\mathrm{s}}(k,\tau) + E^{\mathrm{p}}(k,\tau)\right], \end{aligned}$$

where the velocity field dispersion

$$\sigma_{\mathbf{u}}^2 = B_{ii}(0,0) = \left\langle \mathbf{u}^2(\mathbf{r},t)\right\rangle,$$

and the parameter d stands for the space dimension.

The following cases are of immediate practical interest:

• A purely divergence-free hydrodynamic flow for which $\operatorname{div}\mathbf{u}(\mathbf{r},t) = 0$ ($E^{\mathrm{p}}(k,t) = 0$).

• The case of a purely potential velocity field ($E^{\mathrm{s}}(k,t) = 0$). Such a case is realized, for instance, during tracer diffusion in random wave fields.

• A mixed case realized, for instance, during diffusion of a floating tracer.

Because the velocity field $\mathbf{u}(\mathbf{r},t)$ is homogeneous and isotropic, the following equalities hold true:

$$\begin{aligned} &B_{kl}(0,\tau) = D_0(\tau)\delta_{kl}\,, \qquad \frac{\partial}{\partial r_i}\, B_{kl}(0,\tau) = 0\,, \\ &-\frac{\partial^2}{\partial r_i\,\partial r_j}\, B_{kl}(0,\tau) = \frac{D^{\mathrm{s}}(\tau)}{d(d+2)}\left[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\right] \\ &\qquad\qquad + \frac{D^{\mathrm{p}}(\tau)}{d(d+2)}\left[\delta_{kl}\delta_{ij} + \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}\right]. \end{aligned} \qquad (32)$$

Here, d is the space dimension, the summation over repeating indices is implied as usual, and the following notations are introduced:

$$\begin{aligned} D_0(\tau) &= \frac{1}{d}\left\langle \mathbf{u}(\mathbf{r},t+\tau)\mathbf{u}(\mathbf{r},t)\right\rangle \\ &= \frac{1}{d}\int \left[(d-1)E^{\mathrm{s}}(k,\tau) + E^{\mathrm{p}}(k,\tau)\right] \mathrm{d}\mathbf{k}\,, \\ D^{\mathrm{s}}(\tau) &= \int k^2 E^{\mathrm{s}}(k,\tau)\,\mathrm{d}\mathbf{k}\,, \\ D^{\mathrm{p}}(\tau) &= \int k^2 E^{\mathrm{p}}(k,\tau)\,\mathrm{d}\mathbf{k} = \left\langle \frac{\partial \mathbf{u}(\mathbf{r},t+\tau)}{\partial \mathbf{r}}\, \frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}} \right\rangle. \end{aligned} \qquad (33)$$

Note that the integrals of coefficients (33) with respect to time are described by the expressions

$$\begin{aligned} D_0 &= \int_0^{\infty} D_0(\tau)\,\mathrm{d}\tau = \frac{1}{d}\,\sigma_{\mathbf{u}}^2\tau_0\,, \\ D^{\mathrm{s}} &= \int_0^{\infty} D^{\mathrm{s}}(\tau)\,\mathrm{d}\tau = \int_0^{\infty} \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\; k^2 E^{\mathrm{s}}(k,\tau)\,, \\ D^{\mathrm{p}} &= \int_0^{\infty} D^{\mathrm{p}}(\tau)\,\mathrm{d}\tau = \tau_{\operatorname{div}\mathbf{u}} \left\langle \left(\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}} \right)^2 \right\rangle, \end{aligned} \qquad (34)$$

where $\tau_{\operatorname{div}\mathbf{u}}$ is the time correlation radius of the field $\operatorname{div}\mathbf{u}(\mathbf{r},t)$, and $\sigma^2_{\operatorname{div}\mathbf{u}} = \langle(\partial\mathbf{u}(\mathbf{r},t)/\partial\mathbf{r})^2\rangle$ is the correlation dispersion.

For a divergence-free velocity field ($E^{\mathrm{p}} = 0$), equalities (32) and (33) are simplified and the quantity

$$D^{\mathrm{s}} = \int_0^{\infty} \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\, k^2 E^{\mathrm{s}}(k,\tau) = -\frac{1}{d-1} \int_0^{\infty} \big\langle \mathbf{u}(\mathbf{r}, t+\tau)\, \Delta\mathbf{u}(\mathbf{r}, t) \big\rangle \,\mathrm{d}\tau \qquad (35)$$

is related to the vortical structure of the random divergence-free field $\mathbf{u}(\mathbf{r}, t)$.

The random field $\mathbf{u}(\mathbf{r}, t)$ correlates with the solutions of Eqns (23), (25) which are the functionals of the field $\mathbf{u}(\mathbf{r}, t)$. Correlation splitting for the Gaussian field $\mathbf{u}(\mathbf{r}, t)$ is governed by the Furutsu–Novikov formula

$$\big\langle u_k(\mathbf{r}, t)\, R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big] \big\rangle = \int \mathrm{d}\mathbf{r}' \int_0^t \mathrm{d}t'\, B_{kl}(\mathbf{r}-\mathbf{r}', t-t') \left\langle \frac{\delta R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]}{\delta u_l(\mathbf{r}', t')} \right\rangle, \qquad (36)$$

which holds for the random Gaussian field $\mathbf{u}(\mathbf{r}, t)$ with the zero mean value and its arbitrary functional $R[t; \mathbf{u}(\mathbf{r}, \tau)]$, $0 \leqslant \tau \leqslant t$ (see, for instance, Refs [59, 60] and also [36, 61]).

3.1 Approximation of delta-correlated in time velocity field

The statistical properties of the diffusion of inertialess particles and the density field can be computed using the approximation of the delta-correlated in time velocity field $\mathbf{u}(\mathbf{r}, t)$, in the framework of which the correlation tensor (30) is approximated by the expression

$$B_{ij}(\mathbf{r}, \tau) = 2B_{ij}^{\mathrm{eff}}(\mathbf{r})\, \delta(\tau)\,, \qquad (37)$$

where

$$B_{ij}^{\mathrm{eff}}(\mathbf{r}) = \frac{1}{2} \int_{-\infty}^{\infty} B_{ij}(\mathbf{r}, \tau)\, \mathrm{d}\tau = \int_0^{\infty} B_{ij}(\mathbf{r}, \tau)\, \mathrm{d}\tau\,.$$

In this case, equalities (32) are replaced by the equalities

$$B_{kl}^{\mathrm{eff}}(0) = D_0 \delta_{kl}\,, \qquad \frac{\partial}{\partial r_i} B_{kl}^{\mathrm{eff}}(0) = 0\,,$$
$$-\frac{\partial^2}{\partial r_i \partial r_j} B_{kl}^{\mathrm{eff}}(0) = \frac{D^{\mathrm{s}}}{d(d+2)} \big[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\big] + \frac{D^{\mathrm{p}}}{d(d+2)} \big[\delta_{kl}\delta_{ij} + \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}\big]\,, \qquad (38)$$

respectively, with the coefficients defined by expressions (34).

Tracer diffusion in a random velocity field is described in the Lagrangian representation by the Liouville equation (23), and by Eqn (25) in the Eulerian representation. Averaging these equations over the ensemble of realizations of the velocity field $\{\mathbf{u}\}$ yields equations for the simultaneous Lagrangian probability distribution $P(t; \mathbf{r}, \rho, j|\mathbf{r}_0)$ and the one-point Eulerian probability density distribution $P(t, \mathbf{r}; \rho)$.

Correlation splitting for the Gaussian field $\mathbf{u}(\mathbf{r}, t)$ with its functionals is based on the Furutsu–Novikov formula (36), which is simplified for the delta-correlated field $\mathbf{u}(\mathbf{r}, t)$ and takes the form (see also Refs [36, 61])

$$\big\langle u_k(\mathbf{r}, t)\, R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big] \big\rangle = \int B_{kl}^{\mathrm{eff}}(\mathbf{r}-\mathbf{r}') \left\langle \frac{\delta R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]}{\delta u_l(\mathbf{r}', t-0)} \right\rangle \mathrm{d}\mathbf{r}'\,, \qquad (39)$$

where $0 \leqslant \tau \leqslant t$.

3.1.1 Lagrangian description (particle diffusion).

Averaging Eqn (23) over the ensemble of realizations of the random field $\mathbf{u}(\mathbf{r}, t)$ with the help of the Furutsu–Novikov formula (39) and taking into account the equality

$$\frac{\delta}{\delta u_\beta(\mathbf{r}', t-0)}\, \Phi_{\mathrm{Lag}}(t; \mathbf{r}, \rho, j|\mathbf{r}_0) = \left\{ -\frac{\partial}{\partial r_\beta}\, \delta(\mathbf{r}-\mathbf{r}') + \frac{\partial \delta(\mathbf{r}-\mathbf{r}')}{\partial r_\beta} \left(\frac{\partial}{\partial \rho}\, \rho - \frac{\partial}{\partial j}\, j \right) \right\} \times \Phi_{\mathrm{Lag}}(t; \mathbf{r}, \rho, j|\mathbf{r}_0)$$

and relations (38) lead to the Fokker–Planck equation for the simultaneous Lagrangian probability density $P(t; \mathbf{r}, \rho, j|\mathbf{r}_0)$ of the particle's coordinate $\mathbf{r}(t|\mathbf{r}_0)$, density $\rho(t|\mathbf{r}_0)$, and divergence $j(t|\mathbf{r}_0)$:

$$\left(\frac{\partial}{\partial t} - D_0 \varDelta \right) P(t; \mathbf{r}, \rho, j|\mathbf{r}_0) = D^{\mathrm{p}} \left(\frac{\partial}{\partial \rho}\, \rho^2\, \frac{\partial}{\partial \rho} - 2\, \frac{\partial^2}{\partial \rho\, \partial j}\, \rho j + \frac{\partial^2}{\partial j^2}\, j^2 \right) P(t; \mathbf{r}, \rho, j|\mathbf{r}_0)\,,$$
$$P(0; \mathbf{r}, \rho, j|\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0)\, \delta\big(\rho_0(\mathbf{r}_0) - \rho\big)\, \delta(j-1)\,. \qquad (40)$$

Equation (40) has the solution

$$P(t; \mathbf{r}, \rho, j|\mathbf{r}_0) = P(t; \mathbf{r}|\mathbf{r}_0)\, P(t; j|\mathbf{r}_0)\, \delta\left(\rho - \frac{\rho_0(\mathbf{r}_0)}{j} \right), \qquad (41)$$

where

$$P(t; \mathbf{r}|\mathbf{r}') = \exp\,(D_0 t \varDelta)\, \delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(4\pi D_0 t)^{d/2}} \exp\left\{ \frac{(\mathbf{r}-\mathbf{r}')^2}{4D_0 t} \right\} \qquad (42)$$

is the probability distribution of the coordinates of the passive tracer particle, and

$$P(t; j|\mathbf{r}_0) = \exp\left(D^{\mathrm{p}} t\, \frac{\partial^2}{\partial j^2}\, j^2 \right) \delta(j-1) = \frac{1}{2j\sqrt{\pi\tau}} \exp\left\{ -\frac{\ln^2(j \exp\tau)}{4\tau} \right\} \qquad (43)$$

is the probability distribution of the field of divergence in its neighborhood. Here [Eqn (43)] and in what follows, the dimensionless time $\tau = D^{\mathrm{p}} t$ is used. It should be emphasized that solution (41) implies statistical independence of coordinates $\mathbf{r}(t|\mathbf{r}_0)$ and divergence $j(t|\mathbf{r}_0)$ in the vicinity of a particle with the Lagrangian coordinates $\mathbf{r}_0$. The logarithmically normal distribution (43) means that the quantity $\chi(t|\mathbf{r}_0) = \ln j(t|\mathbf{r}_0)$ is distributed according to the Gaussian law with the parameters

$$\big\langle \chi(t|\mathbf{r}_0) \big\rangle = -\tau\,, \qquad \sigma_\chi^2(t) = 2\tau\,. \qquad (44)$$

Specifically, the following expressions for the moments of a random field of divergence ensue from Eqn (43) and directly from Eqn (40):

$$\big\langle j^n(t|\mathbf{r}_0) \big\rangle = \exp\big[n(n-1)\tau\big]\,, \qquad n = \pm 1, \pm 2, \ldots \qquad (45)$$

It should be emphasized that the mean divergence is constant: $\langle j(t|\mathbf{r}_0) \rangle = 1$, while its highest moments exponentially increase with time.

Also, it is worthwhile to note that, in accordance with Eqns (20) and (45), the following expression holds for the Lagrangian density moments:

$$\left\langle \rho^n(t|\mathbf{r}_0)\right\rangle = \rho_0^n(\mathbf{r}_0)\exp\left[n(n+1)\tau\right].$$

In particular, it suggests the exponential growth of the mean density and its highest moments in the Lagrangian representation. Probability density for the particle's density here takes the form

$$P(t;\rho|\mathbf{r}_0) = \frac{1}{2\rho\sqrt{\pi\tau}}\exp\left\{-\frac{\ln^2\left(\rho\exp(-\tau)/\rho_0(\mathbf{r}_0)\right)}{4\tau}\right\}. \quad (46)$$

It can also be obtained as the solution of the Fokker – Planck equation following from Eqns (40):

$$\frac{\partial}{\partial t}P(t;\rho|\mathbf{r}_0) = D^{\mathrm{p}}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}P(t;\rho|\mathbf{r}_0),$$

$$P(0;\rho|\mathbf{r}_0) = \delta\big(\rho_0(\mathbf{r}_0)-\rho\big).$$

The above paradoxical behavior of statistical characteristics of divergence and density, manifested as the simultaneous increase of their moments in time, finds an explanation in terms of the log normal probability distribution (see Refs [34, 36, 62]). Thus, the typical realization of a random divergence is given as the exponentially decaying curve

$$j^*(t) = \exp(-\tau).$$

It should be recalled that we mean by the curve of typical realization of the random process $z(t)$ the deterministic curve $z^*(t)$ possessing the following property: for any time interval (t_1, t_2), the random process $z(t)$ proceeds as if it 'winds around' the curve $z^*(t)$, so that the mean time during which the inequality $z(t) > z^*(t)$ is fulfilled coincides with the mean time during which the inverse inequality $z(t) < z^*(t)$ is fulfilled, i.e., one finds

$$\left\langle T_{z(t)>z^*(t)}\right\rangle = \left\langle T_{z(t)<z^*(t)}\right\rangle = \frac{1}{2}(t_2-t_1). \quad (47)$$

Moreover, realizations of a log normal process also admit majorant estimates, for example, with probability $p = 1/2$, one has

$$j(t|\mathbf{r}_0) < 4\exp\left(-\frac{\tau}{2}\right)$$

over the entire time interval $t \in (t_1, t_2)$.

Similarly, for density realizations, there are a typical realization curve and minorant estimate:

$$\rho^*(t) = \rho_0\exp(\tau), \qquad \rho(t|\mathbf{r}_0) > \frac{\rho_0}{4}\exp\left(\frac{\tau}{2}\right).$$

It should be emphasized that the above-studied Lagrangian properties of a particle in flows containing a random potential component are qualitatively different from its statistical properties in divergence-free flows, where $j(t|\mathbf{r}_0) \equiv 1$ and the density in the vicinity of the fixed particle is conserved: $\rho(t|\mathbf{r}_0) = \rho_0(\mathbf{r}_0) = \mathrm{const}$. The above statistical estimates for the particle indicate that statistics of the random processes $j(t|\mathbf{r}_0)$ and $\rho(t|\mathbf{r}_0)$ are shaped by outliers of their realizations in relation to typical realizations.

At the same time, probability distributions of the particles' coordinates in the cases of both divergent and divergence-free velocity fields are essentially the same.

Let us now consider the joint dynamics of two particles in the absence of a mean flow. In this situation, the indicator function

$$\Phi(t;\mathbf{r}_1,\mathbf{r}_2) = \delta\big(\mathbf{r}_1(t)-\mathbf{r}_1\big)\,\delta\big(\mathbf{r}_2(t)-\mathbf{r}_2\big)$$

is described by the Liouville equation

$$\frac{\partial}{\partial t}\Phi(t;\mathbf{r}_1,\mathbf{r}_2) = -\left[\frac{\partial}{\partial\mathbf{r}_1}\mathbf{u}_1(\mathbf{r},t)+\frac{\partial}{\partial\mathbf{r}_2}\mathbf{u}_2(\mathbf{r},t)\right]\Phi(t;\mathbf{r}_1,\mathbf{r}_2).$$

Averaging this function over the ensemble of $\mathbf{u}(\mathbf{r},t)$-field realizations and taking into account the Furutsu – Novikov formula (36) and the equality

$$\frac{\delta}{\delta u_j(\mathbf{r}',t-0)}\Phi(t;\mathbf{r}_1,\mathbf{r}_2)$$
$$= -\left[\frac{\partial}{\partial r_{1j}}\delta(\mathbf{r}_1-\mathbf{r}')+\frac{\partial}{\partial r_{2j}}\delta(\mathbf{r}_2-\mathbf{r}')\right]\Phi(t;\mathbf{r}_1,\mathbf{r}_2)$$

leads, for the joint probability density of the position of the two particles

$$P(t;\mathbf{r}_1,\mathbf{r}_2) = \big\langle\Phi(t;\mathbf{r}_1,\mathbf{r}_2)\big\rangle,$$

to the Fokker – Planck equation

$$\frac{\partial}{\partial t}P(t;\mathbf{r}_1,\mathbf{r}_2) = \left[\frac{\partial^2}{\partial r_{1i}\,\partial r_{1j}}+\frac{\partial^2}{\partial r_{2i}\,\partial r_{2j}}\right]B_{ij}^{\mathrm{eff}}(0)\,P(t;\mathbf{r}_1,\mathbf{r}_2)$$
$$+\,2\frac{\partial^2}{\partial r_{1i}\,\partial r_{2j}}B_{ij}^{\mathrm{eff}}(\mathbf{r}_1-\mathbf{r}_2)\,P(t;\mathbf{r}_1,\mathbf{r}_2). \quad (48)$$

Multiplying Eqn (48) by the function $\delta(\mathbf{r}_1-\mathbf{r}_2-\mathbf{l})$ and integrating with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ yields, for the probability density of relative diffusion of the two particles

$$P(t;\mathbf{l}) = \big\langle\delta\big(\mathbf{r}_1(t)-\mathbf{r}_2(t)-\mathbf{l}\big)\big\rangle,$$

the Fokker – Planck equation

$$\frac{\partial}{\partial t}P(t;\mathbf{l}) = \frac{\partial^2}{\partial l_\alpha\,\partial l_\beta}D_{\alpha\beta}(\mathbf{l})\,P(t;\mathbf{l}), \qquad P(0;\mathbf{l}) = \delta(\mathbf{l}-\mathbf{l}_0), \quad (49)$$

where

$$D_{\alpha\beta}(\mathbf{l}) = 2\big[B_{\alpha\beta}^{\mathrm{eff}}(0)-B_{\alpha\beta}^{\mathrm{eff}}(\mathbf{l})\big]$$

is the structural matrix of the vector field $\mathbf{u}(\mathbf{r},t)$, and $\mathbf{l}_0$ is the initial distance between the particles.

In the general case, Eqn (49) is insoluble. However, the initial distance between the particles being $l_0 \ll l_{\mathrm{cor}}$ [where l_{cor} is the spatial correlation radius of the velocity field $\mathbf{u}(\mathbf{r},t)$], the function $D_{\alpha\beta}(\mathbf{l})$ can be expanded in a Taylor series to give the first-order approximation

$$D_{\alpha\beta}(\mathbf{l}) = -\left.\frac{\partial^2 B_{\alpha\beta}^{\mathrm{eff}}(\mathbf{l})}{\partial l_i\,\partial l_j}\right|_{\mathbf{l}=0} l_i l_j.$$

With the use of the representation (31) – (33), the diffusion tensor becomes simplified and can be written down in the form

$$D_{\alpha\beta}(\mathbf{l}) = \frac{1}{d(d+2)}\left[\left(D^{\mathrm{s}}(d+1) + D^{\mathrm{p}}\right)\delta_{\alpha\beta}\, l^2 - 2(D^{\mathrm{s}} - D^{\mathrm{p}})\, l_\alpha l_\beta\right], \qquad (50)$$

where d is the space dimension.

Substituting Eqn (50) into Eqn (49), multiplying both sides of the resulting equation by l^n, and integrating with respect to $\mathbf{l}$ leads to the closed equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left\langle l^n(t)\right\rangle = \frac{1}{d(d+2)}\left[\left(D^{\mathrm{s}}(d+1) + D^{\mathrm{p}}\right)n(d+n-2) - 2(D^{\mathrm{s}} - D^{\mathrm{p}})\,n(n-1)\right],$$

the solution of which corresponds to exponentially growing in time functions for all moments ($n = 1, 2, \ldots$). In this case, the random process $l(t)/l_0$ has the log normal probability distribution. It is worthwhile to note that multiplying Eqn (49) by $\delta(l(t) - l)$ and integrating with respect to $\mathbf{l}$ readily leads to the equation for probability density of the modulus of the vector $\mathbf{l}(t)$:

$$P(t;l) = \left\langle\delta\left(\left|\mathbf{l}(t)\right| - l\right)\right\rangle = \int\delta\left(\left|\mathbf{l}(t)\right| - l\right)P(t,\mathbf{l})\,\mathrm{d}\mathbf{l}$$

having the form

$$\frac{\partial}{\partial t}P(t,l) = -\frac{\partial}{\partial l}\frac{D_{ii}(l)}{l}P(t,l) + \frac{\partial}{\partial l}N(l)\,P(t,l) + \frac{\partial^2}{\partial l^2}N(l)\,P(t,l),$$

where $N(l) = l_j l_i D_{ij}(\mathbf{l})/l^2$.

It can be concluded that the typical realization for the distance between the two particles is the time exponential function

$$l^*(t) = \exp\left\{\frac{1}{d(d+2)}\left\{D^{\mathrm{s}}d(d-1) - D^{\mathrm{p}}(4-d)\right\}t\right\}. \qquad (51)$$

Hence it follows that in the two-dimensional case ($d = 2$), the expression

$$l^*(t) = \exp\left\{\frac{1}{4}(D^{\mathrm{s}} - D^{\mathrm{p}})\,t\right\}$$

is essentially dependent on the sign of the difference $(D^{\mathrm{s}} - D^{\mathrm{p}})$. In particular, for a divergence-free velocity field ($D^{\mathrm{p}} = 0$), there is an exponentially growing typical realization corresponding to exponentially fast dispersion of closely spaced particles. This result holds for as long as

$$\frac{1}{4}D^{\mathrm{s}}t \ll \ln\left(\frac{l_{\mathrm{cor}}}{l_0}\right),$$

in which case expansion (50) remains valid. In another limiting case (potential velocity field: $D^{\mathrm{s}} = 0$), the typical realization is represented by an exponentially decaying curve that suggests a tendency towards particle 'coalescence'. Simultaneously, liquid particles undergo compression, leading to the formation of *clusters*, i.e., compact regions of high particle concentration interspersed to a greater extent within rarefied zones. This observation is consistent with the results of numerical simulation of the evolution of the realization of the initially uniform particle distribution in a random potential velocity field, depicted in Fig. 1b (although for a totally different statistical model of the velocity field). This means that clustering by itself is independent of the random velocity field model, even though statistical parameters characterizing this phenomenon can strongly depend on the model structure.

In the three-dimensional case ($d = 3$), it follows from Eqn (51) that

$$l^*(t) = \exp\left\{\frac{1}{15}(6D^{\mathrm{s}} - D^{\mathrm{p}})t\right\},$$

and the typical realization for the interparticle spacing exponentially decays in time if a more stringent (than in the two-dimensional case) condition is satisfied:

$$D^{\mathrm{p}} > 6D^{\mathrm{s}}.$$

In the one-dimensional case, one has

$$l^*(t) = \exp(-D^{\mathrm{p}}t),$$

so that the typical realization always decays in time because the velocity field in this situation is invariably divergent.

3.1.2 Eulerian description. A description of the behavior of realizations of a tracer field in the random velocity field implies the knowledge of the probability distribution for the tracer density. An equation for Eulerian probability density is easy to derive based on formula (28), multiplying Eqn (40) by j, and integrating it with respect to all possible values of j and $\mathbf{r}_0$. The resultant equation for probability density of the density field has the form

$$\left(\frac{\partial}{\partial t} - D_0\Delta\right)P(t,\mathbf{r};\rho) = D^{\mathrm{p}}\frac{\partial^2}{\partial\rho^2}\rho^2 P(t,\mathbf{r};\rho), \quad P(0,\mathbf{r};\rho) = \delta\left(\rho_0(\mathbf{r}) - \rho\right). \qquad (52)$$

Equation (52) can also be obtained directly by averaging Eqn (25) over the ensemble of realizations of the delta-correlated in time random field $\mathbf{u}(\mathbf{r},t)$ in the absence of a mean flow, using the Furutsu–Novikov formula (39) and the expression for the variational derivative

$$\frac{\delta\Phi(t,\mathbf{r};\rho)}{\delta u_\beta(\mathbf{r}',t-0)} = \left\{-\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_\beta} + \frac{\partial\delta(\mathbf{r}-\mathbf{r}')}{\partial r_\beta}\frac{\partial}{\partial\rho}\rho\right\}\Phi(t,\mathbf{r};\rho).$$

It follows from Eqn (52) that moment functions of the tracer density field are described by the equation

$$\left(\frac{\partial}{\partial t} - D_0\Delta\right)\left\langle\rho^n(\mathbf{r},t)\right\rangle = D^{\mathrm{p}}n(n-1)\left\langle\rho^n(\mathbf{r},t)\right\rangle, \quad \left\langle\rho^n(\mathbf{r},0)\right\rangle = \rho_0^n(\mathbf{r}). \qquad (53)$$

Its solution assumes the form

$$\left\langle\rho^n(\mathbf{r},t)\right\rangle = \exp\left[n(n-1)\tau\right]\int P(t;\mathbf{r}|\mathbf{r}')\,\rho_0^n(\mathbf{r}')\,\mathrm{d}\mathbf{r}', \qquad (54)$$

where the function $P(t;\mathbf{r}|\mathbf{r}')$ is described by equality (42).

In particular, the initial tracer density being the same throughout, $\rho_0(\mathbf{r}) = \rho_0 = \text{const}$, the probability density distribution is independent of $\mathbf{r}$ and logarithmically normal with the probability density

$$P(t;\rho) = \frac{1}{2\rho\sqrt{\pi\tau}} \exp\left\{-\frac{\ln^2\left(\rho\exp(\tau)/\rho_0\right)}{4\tau}\right\}. \qquad (55)$$

In this case, all moment functions starting from the second one exponentially grow in time as

$$\langle\rho(\mathbf{r},t)\rangle = \rho_0\,, \qquad \langle\rho^n(\mathbf{r},t)\rangle = \rho_0^n \exp\left[n(n-1)\tau\right], \qquad (56)$$

whereas the typical realization of the tracer density field at any fixed point in the space falls off exponentially with time as

$$\rho^*(t) = \rho_0 \exp(-\tau)\,. \qquad (57)$$

This suggests the cluster nature of medium density fluctuations in arbitrary divergent flows. Formation of the Eulerian statistics of the tracer density at any fixed point in the space comes about through density fluctuations about the curve.

So far, we have studied the one-point probability density distribution of a tracer in the Eulerian representation and have drawn a few conclusions concerning the behavior of realizations of the tracer density field in time at fixed points in the space. Now, we shall demonstrate that this distribution can also be used to elucidate certain characteristic peculiarities of the spatial – temporal structure of tracer density field realizations.

For clarity, we shall confine ourselves to the two-dimensional case. As mentioned above, important data about the spatial behavior of the field realizations are provided by the analysis of level lines defined by the equality

$$\rho(\mathbf{r},t) = \rho = \text{const}\,, \qquad (58)$$

and such functionals of the density field as the total area $S(t,\rho)$, where $\rho(\mathbf{r},t) > \rho$, and the total tracer mass $M(t,\rho)$ enclosed in these regions, the mean values of which are described by one-point probability density:

$$\begin{aligned} \langle S(t,\rho)\rangle &= \int_\rho^\infty \mathrm{d}\tilde\rho \int \mathrm{d}\mathbf{r}\, P(t,\mathbf{r};\tilde\rho)\,, \\ \langle M(t,\rho)\rangle &= \int_\rho^\infty \mathrm{d}\tilde\rho\;\tilde\rho \int \mathrm{d}\mathbf{r}\, P(t,\mathbf{r};\tilde\rho)\,. \end{aligned} \qquad (59)$$

By substitution of the solution of Eqn (52) and applying simple transformations, it is easy to derive explicit expressions for these quantities:

$$\begin{aligned} \langle S(t,\rho)\rangle &= \int \Phi\left(\frac{1}{2\sqrt{\tau}} \ln\left(\frac{\rho_0(\mathbf{r})\exp(-\tau)}{\rho}\right)\right) \mathrm{d}\mathbf{r}\,, \\ \langle M(t,\rho)\rangle &= \int \rho_0(\mathbf{r})\, \Phi\left(\frac{1}{2\sqrt{\tau}} \ln\left(\frac{\rho_0(\mathbf{r})\exp\tau}{\rho}\right)\right) \mathrm{d}\mathbf{r}\,, \end{aligned} \qquad (60)$$

where $\Phi(z)$ is the error integral:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{y^2}{2}\right\} \mathrm{d}y\,.$$

It is clear, in particular, that for $\tau \gg 1$ the mean area of the regions where the tracer density is above ρ contracts with time according to the law

$$\langle S(t,\rho)\rangle \approx \frac{1}{\sqrt{\pi\tau\rho}} \exp\left(-\frac{\tau}{4}\right) \int \sqrt{\rho_0(\mathbf{r})}\,\mathrm{d}\mathbf{r}\,, \qquad (61)$$

whereas the mean tracer mass enclosed within these regions, namely

$$\langle M(t,\rho)\rangle \approx M - \sqrt{\frac{\rho}{\pi\tau}} \exp\left(-\frac{\tau}{4}\right) \int \sqrt{\rho_0(\mathbf{r})}\,\mathrm{d}\mathbf{r}\,, \qquad (62)$$

tends monotonically toward the total mass of the system, $M = \int \rho_0(\mathbf{r})\,\mathrm{d}\mathbf{r}$. This confirms once again our earlier conclusion that in the course of time the tracer particles tend to coalesce into clusters, i.e., compact regions of an increased particle concentration amidst rarefied zones.

The dynamics of the cluster formation can be illustrated by an example in which the tracer is at first uniformly distributed over a plane: $\rho_0(\mathbf{r}) = \rho_0 = \text{const}$. In this case, the mean specific area of the regions in which $\rho(\mathbf{r},t) > \rho$ is equal to

$$s(t,\rho) = \int_\rho^\infty P(t;\tilde\rho)\,\mathrm{d}\tilde\rho = \Phi\left(\frac{\ln\left(\rho_0\exp(-\tau)/\rho\right)}{2\sqrt{\tau}}\right), \qquad (63)$$

where $P(t;\rho)$ is the $\mathbf{r}$-independent solution of Eqn (52), i.e., function (55), and the mean specific tracer mass (per unit area) falling within these regions is described by the expression

$$m(t,\rho) = \frac{1}{\rho_0} \int_\rho^\infty \tilde\rho\, P(t;\tilde\rho)\,\mathrm{d}\tilde\rho = \Phi\left(\frac{\ln\left(\rho_0\exp(\tau)/\rho\right)}{2\sqrt{\tau}}\right). \qquad (64)$$

It follows from Eqns (63) and (64) that the mean tracer specific area at large times decreases exponentially according to the law

$$s(t,\rho) = \Phi\left(-\frac{\sqrt{\tau}}{2}\right) \approx \frac{1}{\sqrt{\pi\tau}} \exp\left(-\frac{\tau}{4}\right) \qquad (65)$$

regardless of the ratio ρ/ρ_0, whereas almost all the tracer's mass aggregates inside this area:

$$m(t,\rho) = \Phi\left(\frac{\sqrt{\tau}}{2}\right) \approx 1 - \frac{1}{\sqrt{\pi\tau}} \exp\left(-\frac{\tau}{4}\right). \qquad (66)$$

The character of time evolution of the cluster structure formation essentially depends on the ratio ρ/ρ_0 (see Refs [34, 36]). For example, if $\rho/\rho_0 < 1$, then at the initial moment $s(0,\rho) = 1$ and $m(0,\rho) = 1$. Furthermore, tracer particles first tend to disperse, giving rise to small regions with $\rho(\mathbf{r},t) < \rho$, which contain a minor part of the total tracer mass. In the course of time, these regions rapidly grow in size, while the mass contained in them flows to the cluster regions, fairly rapidly reaching the asymptotic dependences (65), (66) (Fig. 4). At the instant of time $\tau^* = \ln(\rho/\rho_0)$, the specific area is equal to $s(t^*,\rho) = 1/2$.

In the opposite and more interesting case of $\rho/\rho_0 > 1$, the initial values are $s(0,\rho) = 0$ and $m(0,\rho) = 0$. The initial dispersion of particles results in small cluster regions with $\rho(\mathbf{r},t) > \rho$; these regions are virtually conserved in time and intensively 'suck in' a sizable portion of the total tracer mass. Thereafter, their areas contract, while the mass contained in them increases to reach the same asymptotic dependences (65) and (66) (Fig. 5).

As mentioned above, for a more detailed description of the tracer density field in a random velocity field its spatial

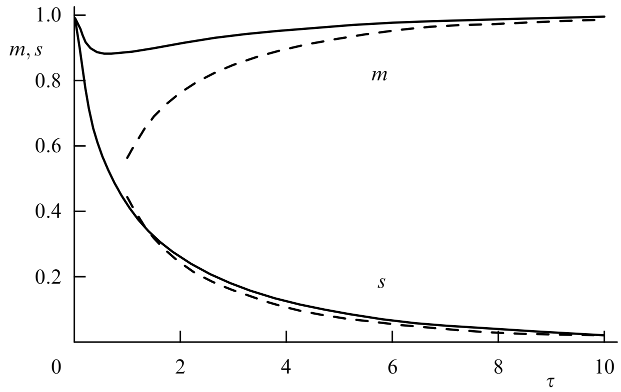

Figure 4. Dynamics of cluster formation for $\rho/\rho_0 = 0.5$.

gradient $\mathbf{p}(\mathbf{r}, t) = \nabla\rho(\mathbf{r}, t)$ and, generally speaking, higher-order derivatives need to be considered. For a divergence-free liquid flow, the mean density gradient of a tracer particle is conserved: $\langle \mathbf{p}(\mathbf{r}, t)\rangle = \mathbf{p}_0(\mathbf{r}_0)$. Moment functions of the density gradient modulus are described in this case by the equations (see, for instance, Refs [27, 34, 36])

$$\frac{\partial}{\partial t}\left\langle p^n(t|\mathbf{r}_0)\right\rangle = \frac{n(d+n)(d-1)}{d(d+2)}\, D^{\mathrm{s}}\left\langle p^n(t|\mathbf{r}_0)\right\rangle, \qquad \left\langle p^n(0|\mathbf{r}_0)\right\rangle = p_0^n(\mathbf{r}_0). \tag{67}$$

This means that the modulus of the density field gradient in the Lagrangian description is a log normal quantity, the typical realization and moment functions of which exponentially increase in time. Specifically, the first and second moments in the two-dimensional case are described by the corresponding equalities

$$\left\langle \left|\mathbf{p}(t|\mathbf{r}_0)\right|\right\rangle = \left|\mathbf{p}_0(\mathbf{r}_0)\right| \exp\left(\frac{3}{8}\, D^{\mathrm{s}} t\right), \qquad \left\langle \mathbf{p}^2(t|\mathbf{r}_0)\right\rangle = \mathbf{p}_0^2(\mathbf{r}_0)\exp\left(D^{\mathrm{s}} t\right). \tag{68}$$

It is worthwhile to note that the log normal distribution for the tracer's gradient modulus, first postulated in Ref. [63], is consistent with experimental findings for the atmosphere [64, 65].

Moreover, it follows from formula (29) that the total mean length of the contour $\rho(\mathbf{r}, t) = \rho = \text{const}$ (in the two-dimensional case) also grows exponentially with time as

$$\left\langle l(t, \rho)\right\rangle = l_0 \exp\left(D^{\mathrm{s}} t\right),$$

where l_0 is the initial contour length [26, 27]. It should be recalled that in this case a divergence-free velocity field conserves the number of contours that can neither appear nor disappear in the medium but only evolve in time starting from their initial distribution in space.

Thus, the initially smooth tracer distribution acquires an increasingly inhomogeneous spatial structure; its spatial gradients sharpen and the level lines undergo fractalization. Such a picture is presented in Fig. 1a based on numerical simulation (although for an altogether different model of velocity field fluctuations). This means that the aforementioned general behavioral patterns are insensitive to the type of the models.

It has been shown in the foregoing that, in the presence of the potential component of the velocity field, particles tend to coalesce into clusters depending on the relationship between the solenoidal and potential components of the velocity field. At the same time, clustering inevitably occurs in the Eulerian density field in the presence of the potential component. Alongside dynamic equation (7), of certain interest is the following equation for the nonconservative tracer transport (see, for instance, Ref. [40]):

$$\left(\frac{\partial}{\partial t} + \mathbf{U}(\mathbf{r}, t)\,\frac{\partial}{\partial \mathbf{r}}\right)\rho(\mathbf{r}, t) = 0\,, \qquad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,.$$

In this case, the equation for particle dynamics in the Lagrangian description coincides with Eqn (4); hence, the possibility of clustering at the particle level. It is easy to see,

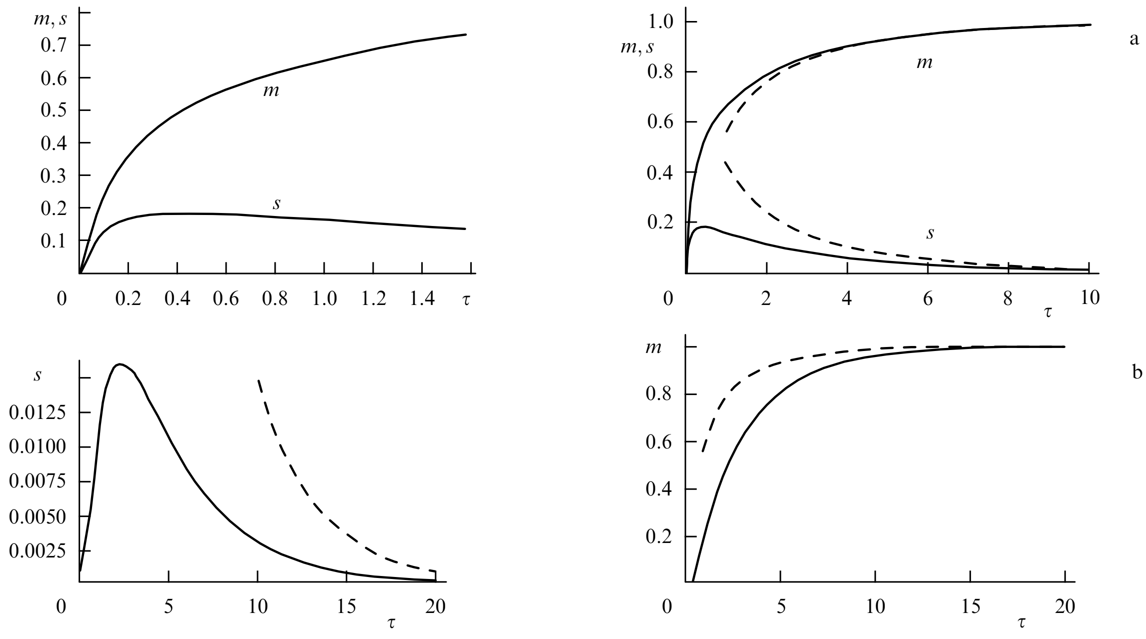

Figure 5. Dynamics of cluster formation for (a) $\rho/\rho_0 = 1.5$, and (b) $\rho/\rho_0 = 10$.

however, that the Eulerian density fields do not give rise to clustering. This case is similar to the divergence-free one in that it conserves the mean contour number, the mean area where $\rho(\mathbf{r}, t) > \rho$, and the mean tracer 'mass' $\int \rho(\mathbf{r}, t)\, \mathrm{d}S$ enclosed within these contours.

3.2 Conditions of validity of the delta-correlated approximation and diffusion approximation

A usability condition for the approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r}, t)$ (37) is the smallness of the time correlation radius τ_0 of $\mathbf{u}(\mathbf{r}, t)$ in comparison with all the time scales of the problem being considered, namely, $\tau_0 \ll \tau_1$. In the presence of a mean flow, $\tau_1 \sim L/v$ or $\tau_1 \sim L/\sqrt{\langle \mathbf{u}^2 \rangle}$, where the parameter L is the typical scale of length. This scale may depend on the mean flow properties (e.g., $L = v/|\nabla v|$ is the typical eddy size) or tracer density ($L = \rho/|\nabla \rho|$). In any case, these measures decrease with time due to the appearance of small-scale structures. As a result, the two time scales become comparable, and the delta-correlated approximation turns inapplicable. It must be taken into consideration that the time correlation radius τ_0 is finite. In the absence of a mean flow, the parameter $L = l_0$ coincides with the spatial correlation radius of the random field $\mathbf{u}(\mathbf{r}, t)$, and the usability condition for the delta-correlated approximation to the random field $\mathbf{u}(\mathbf{r}, t)$ is given by the conditions

$$t \gg \tau_0 \,, \qquad \frac{\sigma_{\mathbf{u}}^2 \tau_0^2}{l_0^2} \ll 1 \,. \tag{69}$$

The finiteness of the time correlation radius of the random field $\mathbf{u}(\mathbf{r}, t)$ can be taken into account in the framework of the diffusion approximation (see, for instance, the review [34] and books [36, 61]). This approximation is more demonstrable and physically meaningful than the formal mathematical approximation of the delta-correlated in time random velocity field. In the framework of this approximation, it is suggested that the effect of random factors on time scales of order τ_0 is insignificant — that is, the particles and tracer field evolve on these scales as free ones. The application of this approximation will be discussed at greater length in the next section devoted to the analysis of clustering of low-inertia tracer particles. Here, it is worth noting that, in the absence of a mean flow, the equations for the probability densities of both Lagrangian and Eulerian variables upon the condition $t \gg \tau_0$ totally coincide with the above equations derived in the approximation of the delta-correlated field $\mathbf{u}(\mathbf{r}, t)$. Usability conditions for the diffusion approximation are also given by inequalities (69).

The limiting case of the stationary random velocity field $\mathbf{u}(\mathbf{r})$, corresponding to the limiting case $\tau_0 \to \infty$, cannot be described in terms of the diffusion approximation. This case, convenient for numerical simulation, is very difficult for an analytical study even though some results have already been published (see, for instance, Refs [17, 22]).

3.3 Peculiarities of tracer diffusion in rapidly varying random wave fields

The motion of particles in the rapidly varying random velocity fields or under effect of rapidly varying random forces constitutes an important problem having numerous implications for mechanics, hydrodynamics, plasma physics, etc. It is well known that the stochastic transport in rapidly varying vibration and wave fields is associated with a number of important physical phenomena, such as the Fermi acceleration, stochastic plasma heating, etc. [66, 67].

In certain cases, diffusion coefficients in both the delta-correlated random field approximation and the diffusion approximation tend to vanish. For example, such a case takes place when particles travel in rapidly varying random wave velocity fields [68] (see also Ref. [69]).

In such a way, diffusion of inertialess particles is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathbf{r}(t) = \mathbf{u}(\mathbf{r}, t) \,, \qquad \mathbf{r}(0) = \mathbf{r}_0 \,, \tag{70}$$

where $\mathbf{u}(\mathbf{r}, t)$ is the random wave vector field, statistically homogeneous in space and steady in time, such that $\langle \mathbf{u}(\mathbf{r}, t) \rangle = 0$.

Now, let us introduce a new field $\tilde{\mathbf{u}}(\mathbf{r}, t)$ with the unit dispersion, such that

$$\mathbf{u}(\mathbf{r}, t) = \sigma_{\mathbf{u}} \tilde{\mathbf{u}}(\mathbf{r}, t) \,,$$

where the velocity field dispersion is given by

$$\sigma_{\mathbf{u}}^2 = B_{ii}(0, 0) \,.$$

Let us assume that this random field is of a wave origin; then, its correlation tensor has the structure

$$B_{ij}(\mathbf{r}, t) = \int F_{ij}(\mathbf{k}) \cos\left\{\mathbf{kr} - \omega(\mathbf{k}) t\right\} \mathrm{d}\mathbf{k} \,, \tag{71}$$

where the spectral function $F_{ij}(\mathbf{k})$ is such that $\int F_{ii}(\mathbf{k})\, \mathrm{d}\mathbf{k} = 1$, and $\omega = \omega(\mathbf{k}) > 0$ is the dispersion curve for wave motions. For example, for acoustic waves one has $\omega(\mathbf{k}) = ck$, where c is the velocity of sound; for gravitational waves at the surface of a deep fluid, $\omega(\mathbf{k}) = \sqrt{gk}$; for internal gravity waves in a stratified medium, $\omega(\mathbf{k}) = N\sqrt{k^2 - k_z^2}/k$, where N is the Brunt – Vaisala frequency; for the Rossby waves in the atmosphere and the ocean, $\omega(\mathbf{k}) = -\beta k_x/k^2$, where β is the gradient of the Coriolis force in the direction y, and so forth.

For traditional wave motions, the spectral function of the velocity field satisfies the condition $\Phi_{ij}(0) = 0$, where $\Phi_{ij}(\omega) = \int F_{ij}(\mathbf{k})\, \delta\left[\omega - \omega(\mathbf{k})\right] \mathrm{d}\mathbf{k}$, while the tensor diffusion coefficient in the corresponding Fokker – Planck equation vanishes, i.e., one obtains

$$D_{ij} = \int_0^\infty B_{ij}(0, t)\, \mathrm{d}t = 0 \,.$$

The same diffusion coefficient arises in the diffusion approximation on the condition that $t \gg \tau_0$, where τ_0 is the time correlation radius of the velocity field. Therefore, neither the delta-correlated velocity field approximation nor the diffusion approximation leads to a final result; the latter can be obtained by taking into account the terms of the higher order of smallness [68].

Let the maximum of the spectral function $F_{ij}(\mathbf{k})$ correspond to a certain wave number k_{m}, and that of the spectral function $\Phi_{ij}(\omega)$ to the frequency ω_{m}. Let us define spatial and temporal scales as $l = 2\pi/k_{\mathrm{m}}$ and $\tau_0 = 2\pi/\omega_{\mathrm{m}}$, respectively. Then, the quantity $\varepsilon = \sigma_{\mathbf{u}} \tau_0 / l$ for real wave fields is, as a rule, small and may be regarded as the main small parameter of the problem, i.e., $\varepsilon \ll 1$. Let us also assume that the inequality $\sigma_{\mathbf{u}} k \ll \omega(k)$ holds for the entire area in which the velocity field spectrum is known and is responsible for the absence of resonances between different components of the velocity field.

The existence of maxima of spectral functions $F_{ij}(\mathbf{k})$ and $\Phi_{ij}(\omega)$ does not necessarily mean the presence of a quasi-regular component in a random velocity field. The existence of these maxima is due to the fact that the velocity field itself results from differentiation (in space and time) of other auxiliary wave fields (e.g., field of potential for the potential velocity field or interface displacement field, etc.). Certainly, if the spectral functions are too 'narrow', that is if they are delta-shaped with respect to the central frequency (wave number), the problem can be preliminarily simplified by means of dynamic averaging over fast oscillations with a central frequency (wave number) of the input stochastic equations. However, such a situation cannot be realized for the majority of geophysical wave problems.

It should be noted that the hypothesis of statistical spatial homogeneity has, generally speaking, limited implications and does not hold for such phenomena as the waves in atmospheric or oceanic waveguides, the transport by bounded wave packets, etc. In what follows, we shall confine ourselves to the consideration of the statistically homogeneous Gaussian wave velocity field with a focus on the principal aspect of the problem. Concrete quantitative data can be obtained if statistical models of the wave field itself are considered up to the quadratic terms. Generally speaking, the mean transport (Stokes drift) takes place in this case. Particle diffusion in different concrete situations was considered, for example, in Refs [70–72] based on Taylor's approach [11]. Application of a more general and consistent approach to the class of problems under consideration that holds for waves of a different nature (proposed in Ref. [68] and based on the consecutive approximation technique for the solution of equations for variational derivatives) allows for certain generalizations of the transport theory built around the Fokker–Planck equation. This approach permits us to compute different statistical characteristics of the ensembles of particles transferred with wave currents and analyze the effects related to clustering and formation of coherent structures in tracer density fields using statistical topography methods.

3.3.1 Lagrangian description. By using spectral representation of the velocity field (71) and its properties, it is possible to compute diffusion coefficients in the second approximation and write down the equation for the probability density $P(\mathbf{r},t)$ of the particle's position at large time values ($t \gg \tau_0$) in the form [68]

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathbf{r},t) &= -\sigma_{\mathbf{u}}^2\int\frac{d\mathbf{k}}{\omega(\mathbf{k})}\,k_iF_{ki}(\mathbf{k})\,\frac{\partial}{\partial r_k}P(\mathbf{r},t)\\ &+\sigma_{\mathbf{u}}^4\,\frac{\pi}{2}\int d\mathbf{k}_1\int\frac{d\mathbf{k}_2}{\omega_2^2}\,k_{1l}k_{1j}F_{ki}(\mathbf{k}_1)F_{lj}(\mathbf{k}_2)\\ &\times\delta(\omega_1-\omega_2)\,\frac{\partial^2}{\partial r_k\,\partial r_i}P(\mathbf{r},t)\\ &+\sigma_{\mathbf{u}}^4\,\frac{\pi}{2}\int d\mathbf{k}_1\int\frac{d\mathbf{k}_2}{\omega_2^2}\,k_{1l}k_{2i}F_{ki}(\mathbf{k}_1)F_{lj}(\mathbf{k}_2)\\ &\times\delta(\omega_1-\omega_2)\,\frac{\partial^2}{\partial r_k\,\partial r_j}P(\mathbf{r},t)\,,\end{aligned}\tag{72}$$

where $\omega_1=\omega(\mathbf{k}_1)$, and $\omega_2=\omega(\mathbf{k}_2)$.

Equation (72) is a Fokker–Planck equation describing probability density of the position of a particle transferred by the statistically homogeneous Gaussian wave velocity field.

For isotropic fluctuations of the field $\tilde{\mathbf{u}}(\mathbf{r},t)$, Eqn (72) can be simplified to

$$\frac{\partial}{\partial t}P(\mathbf{r},t)=D_{\mathrm{d}}\,\frac{\partial^2}{\partial\mathbf{r}^2}P(\mathbf{r},t)\,,\tag{73}$$

corresponding to the Gaussian random vector process $\mathbf{r}(t)$ with the mean value $\langle\mathbf{r}(t)\rangle=\mathbf{r}_0$ and dispersion

$$\sigma_{\mathbf{r}}^2(t)=\left\langle\big(\mathbf{r}(t)-\mathbf{r}_0\big)^2\right\rangle=2dD_{\mathrm{d}}t\,,$$

where d is the space dimension, and the coefficient of diffusion

$$D_{\mathrm{d}}=\sigma_{\mathbf{u}}^4\,\frac{\pi}{2d}\int d\mathbf{k}_1\int\frac{d\mathbf{k}_2}{\omega_2^2}\,k_{1l}k_{1j}F_{ii}(\mathbf{k}_1)F_{lj}(\mathbf{k}_2)\,\delta(\omega_1-\omega_2)\,.\tag{74}$$

In this case, the spectral tensor of the wave velocity field has the structure

$$F_{ki}(\mathbf{k})=F^{\mathrm{s}}(k)\left(\delta_{ik}-\frac{k_ik_k}{k^2}\right)+F^{\mathrm{p}}(k)\,\frac{k_ik_k}{k^2}\,,\tag{75}$$

where $F^{\mathrm{s}}(k)$ and $F^{\mathrm{p}}(k)$ are the solenoidal and potential components of the spectral tensor, respectively, and $\omega(\mathbf{k})\equiv\omega(k)$. Hence the following expression for the diffusion coefficient:

$$\begin{aligned}D_{\mathrm{d}}&=\sigma_{\mathbf{u}}^4\,\frac{\pi}{2d}\int\frac{d\mathbf{k}_1}{\omega_1^2}\,k_1^2F_{ii}(\mathbf{k}_1)\int d\mathbf{k}_2\,F_{ll}(\mathbf{k}_2)\,\delta(\omega_1-\omega_2)\\ &=\sigma_{\mathbf{u}}^4\,\frac{\pi}{2d}\int\frac{d\mathbf{k}_1}{\omega_1^2}\,k_1^2\big[F^{\mathrm{s}}(k_1)(d-1)+F^{\mathrm{p}}(k_1)\big]^2\\ &\quad\times\int d\mathbf{k}_2\,\delta(\omega_1-\omega_2)\,.\end{aligned}\tag{76}$$

For an anisotropic medium, spatial asymmetry of the vector process $\mathbf{r}(t)$ arises. Its mean value and dispersion are described by the expressions

$$\begin{aligned}\langle r_m(t)\rangle&=r_{0m}+t\sigma_{\mathbf{u}}^2\int k_iF_{mi}(\mathbf{k})\,\frac{d\mathbf{k}}{\omega(\mathbf{k})}\,,\\ \sigma_{\mathbf{r}}^2(t)&=\left\langle\mathbf{r}^2(t)-\langle\mathbf{r}(t)\rangle^2\right\rangle\\ &=t\sigma_{\mathbf{u}}^4\pi\int d\mathbf{k}_1\int\frac{d\mathbf{k}_2}{\omega_2^2}\,k_{1l}k_{1j}F_{ii}(\mathbf{k}_1)F_{lj}(\mathbf{k}_2)\,\delta(\omega_1-\omega_2)\\ &+t\sigma_{\mathbf{u}}^4\pi\int d\mathbf{k}_1\int\frac{d\mathbf{k}_2}{\omega_2^2}\,k_{1l}k_{2i}F_{ki}(\mathbf{k}_1)F_{lk}(\mathbf{k}_2)\,\delta(\omega_1-\omega_2)\,.\end{aligned}\tag{77}$$

Evidently, the diffusion coefficient is proportional to the square of velocity field dispersion rather than velocity field dispersion itself, because the problem in question does not contain resonances of the 'wave–particle' type, which leads to a decrease of the order of dispersion of the particles' random drift velocity. The problem resembles that of Kapitza's pendulum oscillations or the eddy drift of charged particles in a rapidly varying electric field [66], where the main effect is also of quadratic order.

3.3.2 Eulerian description. Let us now turn to the statistical description of the Eulerian representation. For simplicity, the initial density field distribution is assumed to be constant, i.e., $\rho_0(\mathbf{r}) = \rho_0 = \text{const}$; hence, the random function $\rho(\mathbf{r}, t)$ is statistically homogeneous in space — that is, all its one-point statistical characteristics are independent of the spatial point $\mathbf{r}$.

By using the spectral representation (71) and calculating coefficients in the second approximation, it is possible to derive the equation [68]

$$\frac{\partial}{\partial t} P(t;\rho) = \widetilde{D}_{\mathrm{d}}^{(2)} \frac{\partial^2}{\partial \rho^2} \rho^2 P(t;\rho) + \widetilde{D}_{\mathrm{d}}^{(3)} \frac{\partial^2}{\partial \rho^2} \rho^2 \frac{\partial}{\partial \rho} \rho P(t;\rho), \tag{78}$$

where d is the space dimension, and

$$\begin{aligned} \widetilde{D}_{\mathrm{d}}^{(2)} &= \sigma_{\mathbf{u}}^4 \frac{\pi}{2} \int \mathrm{d}\mathbf{k}_1\, k_{1k} k_{1i} k_{1l} (k_{1j} - k_{2j}) F_{ki}(\mathbf{k}_1) \\ &\quad \times \int \frac{\mathrm{d}\mathbf{k}_2}{\omega_2^2} F_{lj}(\mathbf{k}_2)\, \delta(\omega_1 - \omega_2), \\ \widetilde{D}_{\mathrm{d}}^{(3)} &= -\sigma_{\mathbf{u}}^4 \frac{\pi}{2} \int \mathrm{d}\mathbf{k}_1\, k_{1k} k_{1i} k_{1l} k_{2j} F_{ki}(\mathbf{k}_1) \\ &\quad \times \int \frac{\mathrm{d}\mathbf{k}_2}{\omega_2^2} F_{lj}(\mathbf{k}_2)\, \delta(\omega_1 - \omega_2). \end{aligned} \tag{79}$$

Equation (78) holds for both isotropic and anisotropic velocity field fluctuations. Therefore, the probability distribution $P(t;\rho)$ in random isotropic compressible wave fields (in the approximation being considered) is a log normal one, and tracer field clustering must take place. Then, taking into account formula (75), the following expression holds for the coefficient $\widetilde{D}_{\mathrm{d}}^{(2)}$:

$$\begin{aligned} \widetilde{D}_{\mathrm{d}}^{(2)} &= \sigma_{\mathbf{u}}^4 \frac{\pi}{2d} \int \frac{\mathrm{d}\mathbf{k}_1}{\omega^2(k_1)} k_1^4 F^{\mathrm{p}}(k_1) \big[F^{\mathrm{s}}(k_1)(d-1) + F^{\mathrm{p}}(k_1) \big] \\ &\quad \times \int \mathrm{d}\mathbf{k}_2\, \delta(\omega_1 - \omega_2). \end{aligned}$$

In the case of anisotropic velocity fields, the solution of Eqn (78) is expressed through the Airy function of the density logarithm. Then in the range of small ρ, the solution becomes negative. However, the high-density region, and hence the moment functions of the field $\rho(\mathbf{r}, t)$, are described correctly. A certain change in the distribution function in the high-density region does not hamper a tracer field clustering.

Thus, taking into account the first nonvanishing corrections to the equation for the probability density of both diffusing particles and the conservative passive tracer field in random wave fields leads to nonzero transport coefficients. For compressible anisotropic wave velocity fields, there appear mean particle transport (Stokes drift) and anisotropy in the probability distribution of the Lagrangian particle positions. In this case too, clustering of the conservative passive tracer field takes place.

It is worthwhile to note, however, that these processes proceed on different spatial scales as expressed by different powers of wave vectors $\mathbf{k}_i$ in the diffusion coefficients entering Eqns (72) and (78). By way of example, small-scale fluctuations of the velocity field have a significantly greater effect on tracer clustering in the Eulerian description than on Lagrangian particle diffusion. Expressions (79) for the diffusion coefficients may be found divergent if the wave field has a sufficiently broad spectrum (e.g., undergoing power-like decay at sufficiently large wave numbers, as in the case of turbulence). Then, the contribution of resonance effects to the coefficient of diffusion (74) can also be calculated.

4. Clustering of a tracer in random divergence-free hydrodynamic flows

It appears from the foregoing that the velocity field of a hydrodynamic flow must be divergent if clustering of an inertialess tracer field is to occur. In many problems pertaining to the physics of the Earth's atmosphere and oceans, the medium is normally considered to be *incompressible*, i.e., described by a divergence-free velocity field. In this situation, however, clustering is still possible in certain cases, which are discussed below.

4.1 Floating tracer diffusion

Let us first of all consider the diffusion of a floating tracer following the papers [29, 34]. If a passive tracer moves with horizontal and vertical velocity components $\mathbf{u} = (\mathbf{U}, w)$ over the surface $z = 0$ in an incompressible medium ($\operatorname{div} \mathbf{u}(\mathbf{r}, t) = 0$) in the absence of a mean flow, then an effective compressible two-dimensional flow with two-dimensional divergence

$$\nabla_{\mathbf{R}} \mathbf{U}(\mathbf{R}, t) = -\frac{\partial w(\mathbf{r}, t)}{\partial z}\bigg|_{z=0}$$

is created on the surface. We assume that the spatial spectral tensor of the velocity field $\mathbf{u}(\mathbf{r}, t)$ has the form

$$E_{ij}(\mathbf{k}, t) = E(k, t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right).$$

Representation of the floating tracer field as

$$\rho(\mathbf{r}, t) = \rho(\mathbf{R}, t)\,\delta(z), \qquad \mathbf{r} = (\mathbf{R}, z), \qquad \mathbf{R} = (x, y),$$

substitution of this expression into Eqn (2), and integration with respect to z leads to the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{R}} \mathbf{U}(\mathbf{R}, t)\right) \rho(\mathbf{R}, t) = 0, \qquad \rho(\mathbf{R}, 0) = \rho_0(\mathbf{R}).$$

The resultant field $\mathbf{U}(\mathbf{R}, t)$ is Gaussian, uniform and isotropic, with the spectral tensor

$$E_{\alpha\beta}(\mathbf{k}_\perp, t) = \int_{-\infty}^{\infty} E(\mathbf{k}_\perp^2 + k_z^2, t)\left(\delta_{\alpha\beta} - \frac{k_{\perp\alpha} k_{\perp\beta}}{\mathbf{k}_\perp^2}\right) \mathrm{d}k_z, \tag{80}$$
$$\alpha, \beta = 1, 2.$$

Comparison of equation (80) with Eqns (30) and (31) yields an expression for solenoidal and potential components of the velocity $\mathbf{U}(\mathbf{R}, t)$ in the plane $z = 0$ [29]:

$$\begin{aligned} E^{\mathrm{s}}(\mathbf{k}_\perp, t) &= \int_{-\infty}^{\infty} E(\mathbf{k}_\perp^2 + k_z^2, t)\, \mathrm{d}k_z, \\ E^{\mathrm{p}}(\mathbf{k}_\perp, t) &= \int_{-\infty}^{\infty} E(\mathbf{k}_\perp^2 + k_z^2, t) \frac{k_z^2}{\mathbf{k}_\perp^2 + k_z^2}\, \mathrm{d}k_z. \end{aligned} \tag{81}$$

Therefore, the equation for the probability density of the density field $\rho(\mathbf{R}, t)$ will be described by a two-dimensional equation (52), namely

$$\left(\frac{\partial}{\partial t} - D_0\Delta\right) P(t,\mathbf{r};\rho) = D^{\mathrm{p}}\,\frac{\partial^2}{\partial\rho^2}\,\rho^2 P(t,\mathbf{r};\rho)\,, \qquad P(0,\mathbf{r};\rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,, \tag{82}$$

with the diffusion coefficients defined, in accordance with Eqns (33), (34), and (81), by the equalities

$$D_0 = 2\pi\int_0^\infty \mathrm{d}\tau \int_0^\infty k^2\,\mathrm{d}k\; E(k,\tau)\,,$$
$$D^{\mathrm{s}} = \frac{4\pi}{3}\int_0^\infty \mathrm{d}\tau \int_0^\infty k^4\,\mathrm{d}k\; E(k,\tau)\,, \tag{83}$$
$$D^{\mathrm{p}} = \frac{4\pi}{5}\int_0^\infty \mathrm{d}\tau \int_0^\infty k^4\,\mathrm{d}k\; E(k,\tau)\,.$$

This means that clustering of a density field in the Eulerian description is needed for density diffusion of an inertialess floating tracer. At the same time, no clustering occurs in the case of diffusion of inertialess floating particles as suggested by the inequality $D^{\mathrm{s}} > D^{\mathrm{p}}$ ensuing from formulas (83).

4.2 Diffusion of low-inertia particles and tracer field in random divergence-free hydrodynamic flows

Now, let us consider diffusion of low-inertia particles and the tracer field in random divergence-free hydrodynamic flows, following Refs [50, 73].

Diffusion of the number density field $n(\mathbf{r},t)$ of particles moving in a random hydrodynamic flow is described by the continuity equation (1):

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial\mathbf{r}}\,\mathbf{V}(\mathbf{r},t)\right) n(\mathbf{r},t) = 0\,, \qquad n(\mathbf{r},0) = n_0(\mathbf{r})\,, \tag{84}$$

where the Eulerian tracer velocity field $\mathbf{V}(\mathbf{r},t)$ in the absence of the mean flow velocity satisfies the equation (3):

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r},t)\,\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{V}(\mathbf{r},t) = -\lambda\big[\mathbf{V}(\mathbf{r},t) - \mathbf{u}(\mathbf{r},t)\big]. \tag{85}$$

Let us assume the velocity field $\mathbf{u}(\mathbf{r},t)$ to be a Gaussian random field that is divergence-free [i.e., $\operatorname{div}\mathbf{u}(\mathbf{r},t) = \partial\mathbf{u}(\mathbf{r},t)/\partial\mathbf{r} = 0$], homogeneous, isotropic in space, and steady in time, having a zero mean value and correlation tensor

$$B_{ij}(\mathbf{r}-\mathbf{r}', t-t') = \big\langle u_i(\mathbf{r},t)\,u_j(\mathbf{r}',t')\big\rangle\,.$$

For such a model, spatial spectral and spatial–temporal spectral functions of the field $\mathbf{u}(\mathbf{r},t)$ are defined as

$$B_{ij}(\mathbf{r},t) = \int E_{ij}(\mathbf{k},t)\exp(\mathrm{i}\mathbf{kr})\,\mathrm{d}\mathbf{k}\,,$$
$$B_{ij}(\mathbf{r},t) = \int \mathrm{d}\mathbf{k}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\Phi_{ij}(\mathbf{k},\omega)\exp(\mathrm{i}\mathbf{kr} + \mathrm{i}\omega t)\,,$$

where

$$E_{ij}(\mathbf{k},t) = E(k,t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \qquad \Phi_{ij}(\mathbf{k},\omega) = \Phi(k,\omega)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{86}$$

In this case, one obtains

$$B_{ij}(0,t) = \frac{d-1}{d}\int E(k,t)\delta_{ij}\,\mathrm{d}\mathbf{k}\,, \tag{87}$$

where d is the space dimension, and the fourth-order tensor $\partial^2 B_{ij}(0,\tau)/\partial r_k\,\partial r_l$ is represented as

$$-\frac{\partial^2 B_{ij}(0,\tau)}{\partial r_k\,\partial r_l} = \frac{D(\tau)}{d(d+2)}\big[(d+1)\delta_{kl}\,\delta_{ij} - \delta_{ki}\,\delta_{lj} - \delta_{kj}\,\delta_{li}\big]\,. \tag{88}$$

Coefficient $D(\tau)$ in Eqn (88) is defined as follows

$$D(\tau) = \int \mathbf{k}^2 E(k,\tau)\,\mathrm{d}\mathbf{k} = -\frac{1}{d-1}\big\langle \mathbf{u}(\mathbf{r},t+\tau)\Delta\mathbf{u}(\mathbf{r},t)\big\rangle\,,$$

and the quantity

$$D(0) = -\frac{1}{d-1}\big\langle \mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\big\rangle$$

is related to the vortical structure of the random divergence-free field $\mathbf{u}(\mathbf{r},t)$.

Characteristic curves $\mathbf{r}(t)$ and $\mathbf{V}(t)$ for Eqn (85), in accordance with equations (4), satisfy the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{V}(t)\,, \qquad \mathbf{r}(0) = \mathbf{r}_0\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{V}(t) = -\lambda\big[\mathbf{V}(t) - \mathbf{u}\big(\mathbf{r}(t),t\big)\big]\,, \qquad \mathbf{V}(0) = \mathbf{V}_0(\mathbf{r}_0)\,, \tag{89}$$

and describe particle dynamics.

For inertialess particles $\lambda \to \infty$, and, as follows from Eqn (89), the equality

$$\mathbf{V}(\mathbf{r},t) = \mathbf{u}(\mathbf{r},t) \tag{90}$$

is attained. Therefore, in this limiting case, the dispersion of the random field $\mathbf{V}(\mathbf{r},t)$ and its time correlation radius $\tau_{\mathbf{V}}$ are related to the dispersion of the random field $\mathbf{u}(\mathbf{r},t)$ and correlation time τ_0 by explicit equalities

$$\sigma_{\mathbf{V}}^2 = \sigma_{\mathbf{u}}^2\,, \qquad \tau_{\mathbf{V}} = \tau_0\,. \tag{91}$$

4.2.1 Particularities of low-inertia particle diffusion (Lagrangian description). To begin with, it should be noted that the approximation of the delta-correlated random field $\mathbf{u}(\mathbf{r},t)$ is inapplicable to the description of the diffusion of low-inertia particles. The same is true of the Fokker–Planck equation for the joint probability density of the particle's position and speed [73].

Indeed, let us introduce the indicator function for the solution of Eqn (89):

$$\phi(t;\mathbf{r},\mathbf{V}) = \delta\big(\mathbf{r}(t)-\mathbf{r}\big)\,\delta\big(\mathbf{V}(t)-\mathbf{V}\big)\,,$$

which is described by the Liouville equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\,\frac{\partial}{\partial\mathbf{r}} - \lambda\,\frac{\partial}{\partial\mathbf{V}}\,\mathbf{V}\right)\phi(t;\mathbf{r},\mathbf{V}) = -\lambda\mathbf{u}(\mathbf{r},t)\,\frac{\partial}{\partial\mathbf{V}}\,\phi(t;\mathbf{r},\mathbf{V})\,, \qquad \phi(0;\mathbf{r},\mathbf{V}) = \delta(\mathbf{r}-\mathbf{r}_0)\,\delta\big(\mathbf{V}-\mathbf{V}_0(\mathbf{r}_0)\big)\,. \tag{92}$$

The average value of the indicator function $\phi(\mathbf{r}, \mathbf{V}; t)$, taken over the ensemble of realizations of the random field $\mathbf{u}(\mathbf{r}, t)$, describes the joint simultaneous probability density of the particle's position and speed:

$$P(t; \mathbf{r}, \mathbf{V}) = \langle \phi(t; \mathbf{r}, \mathbf{V}) \rangle = \langle \delta(\mathbf{r}(t) - \mathbf{r})\, \delta(\mathbf{V}(t) - \mathbf{V}) \rangle_{\mathbf{u}} .$$

Assuming delta-correlation of random field $\mathbf{u}(\mathbf{r}, t)$ (37), averaging Eqn (92) over the ensemble of realizations of the random field $\mathbf{u}(\mathbf{r}, t)$, and taking into consideration the Furutsu – Novikov formula (39), the expression for variational derivative

$$\frac{\delta}{\delta u_l(\mathbf{r}', t-0)}\, \phi(t; \mathbf{r}, \mathbf{V}) = -\lambda \delta(\mathbf{r} - \mathbf{r}')\, \frac{\partial}{\partial V_l}\, \phi(t; \mathbf{r}, \mathbf{V}) ,$$

and equality (38) leads to the Fokker – Planck equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\, \frac{\partial}{\partial \mathbf{r}} - \lambda\, \frac{\partial}{\partial \mathbf{V}}\, \mathbf{V} \right) P(t; \mathbf{r}, \mathbf{V}) = \lambda^2 D_0\, \frac{\partial^2}{\partial \mathbf{V}^2}\, P(t; \mathbf{r}, \mathbf{V}) , \tag{93}$$
$$P(0; \mathbf{r}, \mathbf{V}) = \delta(\mathbf{r} - \mathbf{r}_0)\, \delta(\mathbf{V} - \mathbf{V}_0(\mathbf{r}_0)) ,$$

where the diffusion coefficient takes the form

$$D_0 = \frac{1}{d} \int_0^\infty \langle \mathbf{u}(\mathbf{r}, t+\tau) \mathbf{u}(\mathbf{r}, t) \rangle \, \mathrm{d}\tau = \frac{1}{d}\, \tau_0 \langle \mathbf{u}^2(\mathbf{r}, t) \rangle .$$

Here, as before, d is the space dimension, τ_0 is the time correlation radius of the random field $\mathbf{u}(\mathbf{r}, t)$, and $\sigma_{\mathbf{u}}^2 = \langle \mathbf{u}^2(\mathbf{r}, t) \rangle$ is its dispersion.

One usability condition for the approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r}, t)$ can be written in the form

$$\lambda \tau_0 \ll 1 . \tag{94}$$

It follows from Eqn (93) that functions $\mathbf{r}(t)$ and $\mathbf{V}(t)$ are Gaussian random processes; the system of equations for their moment functions is produced in the usual way:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle r_i(t) r_j(t) \rangle = 2 \langle r_i(t) V_j(t) \rangle ,$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda \right) \langle r_i(t) V_j(t) \rangle = \langle V_i(t) V_j(t) \rangle , \tag{95}$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\lambda \right) \langle V_i(t) V_j(t) \rangle = 2\lambda^2 D_0 \delta_{ij} .$$

As appears from the system of equations (95), the stationary values of all simultaneous correlations for $\lambda t \gg 1$ and $t/\tau_0 \gg 1$ are described by the expressions

$$\langle V_i(t) V_j(t) \rangle = \lambda D_0 \delta_{ij} , \qquad \langle r_i(t) V_j(t) \rangle = D_0 \delta_{ij} ,$$
$$\langle r_i(t) r_j(t) \rangle = 2t D_0 \delta_{ij} .$$

In particular, dispersion of the process $\mathbf{V}^2(t)$ and spatial diffusion coefficient

$$D = \frac{1}{2}\, \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{r}^2(t) \rangle$$

are described by the equalities

$$\sigma_{\mathbf{V}}^2 = \langle \mathbf{V}^2(t) \rangle = \lambda \int_0^\infty \langle \mathbf{u}(\mathbf{r}, t+\tau) \mathbf{u}(\mathbf{r}, t) \rangle \, \mathrm{d}\tau = \lambda \tau_0 \sigma_{\mathbf{u}}^2 , \tag{96}$$
$$D = \frac{1}{2}\, \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{r}^2(t) \rangle = d D_0 = \int_0^\infty \langle \mathbf{u}(\mathbf{r}, t+\tau) \mathbf{u}(\mathbf{r}, t) \rangle \, \mathrm{d}\tau = \tau_0 \sigma_{\mathbf{u}}^2 .$$

By analogy, it is easy to derive an expression for the time correlation radius $\tau_{\mathbf{V}}$ of a random process $\mathbf{V}(t)$ for the delta-correlated in time field $\mathbf{u}(\mathbf{r}, t)$ by considering the time correlation $\langle V_i(\mathbf{r}, t+\tau) V_j(\mathbf{r}, t) \rangle$. Specifically, one obtains [73]

$$\tau_{\mathbf{V}} = \frac{1}{\lambda} . \tag{97}$$

Comparison of equalities (91) with equalities (96) and (97) shows that they are incompatible; in other words, equality (90) is satisfied when not only conditions $\lambda t \gg 1$ and $t/\tau_0 \gg 1$ but also the condition

$$\lambda \tau_0 \gg 1 \tag{98}$$

are fulfilled, at variance with the condition (94) of validity of the approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r}, t)$. As regards the spatial diffusion coefficient D in Eqn (96), this quantity, as appears from Eqn (89), is given by

$$D = \frac{1}{2}\, \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{r}^2(t) \rangle = \int_0^\infty \langle \mathbf{V}(\mathbf{r}, t+\tau) \mathbf{V}(\mathbf{r}, t) \rangle \, \mathrm{d}\tau = \tau_{\mathbf{V}} \sigma_{\mathbf{V}}^2$$
$$= \tau_0 \sigma_{\mathbf{u}}^2 = \int_0^\infty \langle \mathbf{u}(\mathbf{r}, t+\tau) \mathbf{u}(\mathbf{r}, t) \rangle \, \mathrm{d}\tau ,$$

both in the delta-correlated approximation and in the approximation of the inertialess tracer field; moreover, it shows no dependence whatever on the parameter λ (provided, of course, that $\lambda t \gg 1$).

Therefore, the approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r}, t)$ in the case of a dynamical system (89) incorrectly describes statistics of the particle velocity and its correlation with the particle position upon passage to the inertialess particle approximation. At the same time, this approximation does not contradict the spatial diffusion of particles. It is worthwhile to note that distinguishing the spatial description of the particle's diffusion from its spatial – temporal description constitutes the so-called Kramers problem (see, for instance, the review [74]).

4.2.2 Diffusion of a low-inertia tracer in the Eulerian description. Given a random field $\mathbf{V}(\mathbf{r}, t)$ is Gaussian, statistically homogeneous, spatially isotropic, and steady in time, with a zero mean value and the correlation tensor

$$\langle V_i(\mathbf{r}, t) V_j(\mathbf{r}', t') \rangle = B_{ij}^{(\mathbf{V})}(\mathbf{r} - \mathbf{r}', t - t') ,$$

the one-point probability density $P(t, \mathbf{r}; n)$ for the solution of dynamic equation (84) in both the approximation of the delta-correlated in time field $\mathbf{V}(\mathbf{r}, t)$ and in the diffusion approximation is described by Eqn (52):

$$\left(\frac{\partial}{\partial t} - D_0\, \frac{\partial^2}{\partial \mathbf{r}^2} \right) P(t, \mathbf{r}; n) = D^{(\mathbf{V})}\, \frac{\partial^2}{\partial n^2}\, n^2 P(t, \mathbf{r}; n) , \tag{99}$$
$$P(0, \mathbf{r}; n) = \delta(n_0(\mathbf{r}) - n) ,$$

where the diffusion coefficients have the forms

$$D_0 = \frac{1}{d}\int_0^\infty \big\langle \mathbf{V}(\mathbf{r}, t+\tau)\,\mathbf{V}(\mathbf{r}, t)\big\rangle\,\mathrm{d}\tau = \frac{1}{d}\,\tau_{\mathbf{V}}\big\langle \mathbf{V}^2(\mathbf{r}, t)\big\rangle\,,$$

$$D^{(\mathbf{V})} = \int_0^\infty \left\langle \frac{\partial \mathbf{V}(\mathbf{r}, t+\tau)}{\partial \mathbf{r}}\,\frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial \mathbf{r}}\right\rangle \mathrm{d}\tau = \tau_{\mathrm{div}\,\mathbf{V}}\left\langle\left(\frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial \mathbf{r}}\right)^2\right\rangle, \tag{100}$$

describe spatial dispersion of the number density $n(\mathbf{r}, t)$ of particles, and the characteristic times $\tau_{\mathbf{V}}$ and $\tau_{\mathrm{div}\,\mathbf{V}}$ of cluster formation give time correlation radii for random fields $\mathbf{V}(\mathbf{r}, t)$ and $\partial \mathbf{V}(\mathbf{r}, t)/\partial \mathbf{r}$, while d stands for the space dimension.

Thus, the problem is reduced to the evaluation of diffusion coefficients (100) using the stochastic equation (85) — that is, to computing time correlation radii $\tau_{\mathbf{V}}$ and $\tau_{\mathrm{div}\,\mathbf{V}}$ of random fields $\mathbf{V}(\mathbf{r}, t)$ and $\partial \mathbf{V}(\mathbf{r}, t)/\partial \mathbf{r}$, their spatial correlation scales, and dispersions [50].

We assume that the random velocity field dispersion $\sigma_{\mathbf{u}}^2 = \big\langle \mathbf{u}^2(\mathbf{r}, t)\big\rangle$ in a hydrodynamic flow is sufficiently small and determines the main small parameter of the problem. For a large λ-parameter value (low inertia of the particles), it is possible to linearize equation (85) with respect to the function $\mathbf{V}(\mathbf{r}, t) \approx \mathbf{u}(\mathbf{r}, t)$ and pass to a simpler vector equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r}, t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r}, t) = -\left(\mathbf{V}(\mathbf{r}, t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}(\mathbf{r}, t) - \lambda\big[\mathbf{V}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)\big]\,. \tag{101}$$

In what follows, we shall calculate the statistical characteristics of the field $\mathbf{V}(\mathbf{r}, t)$ in the first nonvanishing order of smallness in the parameter $\sigma_{\mathbf{u}}^2$. It is noteworthy that statistics of the field $\mathbf{V}(\mathbf{r}, t)$ described by stochastic equations (85) and (101) is not Gaussian in the general case. However, it is easy to see that the highest field cumulants $\mathrm{div}\,\mathbf{V}(\mathbf{r}, t)$ are of a higher order of smallness than the second cumulant. It means that the approximation of the Gaussian field $\mathbf{V}(\mathbf{r}, t)$ can really be used to derive equation (99).

The above example of particle diffusion indicates that the approximation of the delta-correlated in time random field $\mathbf{u}(\mathbf{r}, t)$ is incorrect in the case of a low-inertia tracer. It is therefore necessary to do calculations using an arbitrary value of the parameter $\lambda\tau_0$. This can be made in the diffusion approximation.

Random field $\mathbf{u}(\mathbf{r}, t)$ correlates with the function $\mathbf{V}(\mathbf{r}, t)$, which is the functional of the field $\mathbf{u}(\mathbf{r}, t)$. Correlation splitting for the Gaussian field $\mathbf{u}(\mathbf{r}, t)$ is based on the Furutsu–Novikov formula (36) containing variational derivatives. Equations for the respective mean values in the diffusion approximation are written down exactly. The corresponding simplification of the problem is introduced at the level of the functional dependence of the problem's solution on fluctuating parameters (see, for instance, Ref. [36]); it is assumed that the influence of the field $\mathbf{u}(\mathbf{r}, t)$ is insignificant on time scales of order τ_0.

In the diffusion approximation, the equation

$$\left(\frac{\partial}{\partial t} + \lambda\right)\frac{\delta V_i(\mathbf{r}, t)}{\delta u_l(\mathbf{r}', t')} = 0$$

holds for variational derivatives, with the initial condition at $t = t'$:

$$\left.\frac{\delta V_i(\mathbf{r}, t)}{\delta u_l(\mathbf{r}', t')}\right|_{t=t'+0} = -\left[\delta(\mathbf{r}-\mathbf{r}')\,\frac{\partial V_i(\mathbf{r}, t')}{\partial r_l} + \delta_{il}\,\frac{\partial \delta(\mathbf{r}-\mathbf{r}')}{\partial r_k}\,V_k(\mathbf{r}, t')\right] + \delta(\mathbf{r}-\mathbf{r}')\,\lambda\delta_{il}\,,$$

which ensues from equation (101). The solution of this equation takes the form

$$\frac{\delta V_i(\mathbf{r}, t)}{\delta u_l(\mathbf{r}', t')} = \exp\big[-\lambda(t-t')\big]\left\{-\left[\delta(\mathbf{r}-\mathbf{r}')\,\frac{\partial V_i(\mathbf{r}, t')}{\partial r_l} + \frac{\partial \delta(\mathbf{r}-\mathbf{r}')}{\partial r_k}\,\delta_{il}V_k(\mathbf{r}, t')\right] + \delta(\mathbf{r}-\mathbf{r}')\,\lambda\delta_{il}\right\}.$$

The field $\mathbf{V}(\mathbf{r}, t)$ itself has the structure

$$\mathbf{V}(\mathbf{r}, t) = \exp\big[-\lambda(t-t')\big]\,\mathbf{V}(\mathbf{r}, t')$$

in the diffusion approximation, therefore one obtains

$$\mathbf{V}(\mathbf{r}, t') = \exp\big[\lambda(t-t')\big]\,\mathbf{V}(\mathbf{r}, t)\,.$$

Hence, the final expression for the variational derivative in the diffusion approximation assumes the form

$$\frac{\delta V_i(\mathbf{r}, t)}{\delta u_l(\mathbf{r}', t')} = -\left[\delta(\mathbf{r}-\mathbf{r}')\,\frac{\partial V_i(\mathbf{r}, t)}{\partial r_l} + \delta_{il}\,\frac{\partial \delta(\mathbf{r}-\mathbf{r}')}{\partial r_\mu}\,V_\mu(\mathbf{r}, t)\right] + \delta(\mathbf{r}-\mathbf{r}')\,\lambda\exp\big[-\lambda(t-t')\big]\,\delta_{il}\,. \tag{102}$$

Formulas (36) and (102) are sufficient for all necessary calculations that were made in the work [50] containing coefficients (100):

$$D_0 = \frac{1}{d}\,\tau_{\mathbf{V}}\big\langle \mathbf{V}^2(\mathbf{r}, t)\big\rangle = \frac{1}{d}\,\tau_0 B_{ii}(0,0) = \frac{d-1}{d}\,\tau_0\int E(k,0)\,\mathrm{d}\mathbf{k}\,, \tag{103}$$

$$D^{(\mathbf{V})} = \tau_{\mathrm{div}\,\mathbf{V}}\left\langle\left(\frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial \mathbf{r}}\right)^2\right\rangle = \frac{4}{\lambda}\,\frac{d^2-1}{d(d+2)}\,D_1 D_2(\lambda)\,,$$

where

$$D_1 = \int_0^\infty D(\tau)\,\mathrm{d}\tau = \int_0^\infty \mathrm{d}\tau\int \mathrm{d}\mathbf{k}\,\mathbf{k}^2 E(k,\tau)\,,$$

$$D_2(\lambda) = \int_0^\infty \exp(-\lambda\tau)\,D(\tau)\,\mathrm{d}\tau = \int_0^\infty \mathrm{d}\tau\,\exp(-\lambda\tau)\int \mathrm{d}\mathbf{k}\,\mathbf{k}^2 E(k,\tau)\,.$$

Specifically, in the three-dimensional case for low-inertia particles when $\lambda\tau_0 \gg 1$, we arrive at

$$D_0 = \frac{1}{3}\,\tau_{\mathbf{V}}\big\langle \mathbf{V}^2(\mathbf{r}, t)\big\rangle = \frac{1}{3}\,\tau_0 B_{ii}(0,0) = \frac{2}{3}\,\tau_0\int E(k,0)\,\mathrm{d}\mathbf{k}\,, \tag{104}$$

$$D^{(\mathbf{V})} = \tau_{\mathrm{div}\,\mathbf{V}}\left\langle\left(\frac{\partial \mathbf{V}(\mathbf{r}, t)}{\partial \mathbf{r}}\right)^2\right\rangle = \frac{8}{15}\,\frac{\tau_0}{\lambda^2}\big\langle \mathbf{u}(\mathbf{r}, t)\,\Delta\mathbf{u}(\mathbf{r}, t)\big\rangle^2\,.$$

In the two-dimensional case for $\lambda\tau_0 \gg 1$, we find

$$D_0 = \frac{1}{2}\,\tau_{\mathbf{V}}\langle \mathbf{V}^2(\mathbf{r},t)\rangle = \frac{1}{2}\,\tau_0 B_{ii}(0,0) = \tau_0 \int E(k,0)\,\mathrm{d}\mathbf{k}\,,$$
$$D^{(\mathbf{V})} = \tau_{\mathrm{div}\,\mathbf{V}}\left\langle\left(\frac{\partial \mathbf{V}(\mathbf{r},t)}{\partial \mathbf{r}}\right)^2\right\rangle = \frac{3}{2}\,\frac{\tau_0}{\lambda^2}\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle^2\,. \tag{105}$$

Thus, $D^{(\mathbf{V})} \sim \sigma_{\mathbf{u}}^4$ because the eddy component of the field $\mathbf{u}(\mathbf{r},t)$ first of all generates the eddy component of the field $\mathbf{V}(\mathbf{r},t)$ by a direct linear mechanism without advection; thereafter, the eddy component of the field $\mathbf{V}(\mathbf{r},t)$ gives rise to the divergent component of the field $\mathbf{V}(\mathbf{r},t)$ through the advection mechanism.

Evidently, an applicability condition for the above expressions can be written in the form

$$\frac{\sigma_{\mathbf{u}}^2\tau_0^2}{l_0^2} \ll 1\,,$$

where l_0 is the spatial correlation scale of the random field $\mathbf{u}(\mathbf{r},t)$.

Let us now discuss a two-dimensional hydrodynamic flow taking into account rotation, which is described by the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r},t)\,\frac{\partial}{\partial \mathbf{r}}\right) V_i(\mathbf{r},t)$$
$$= -\lambda\big[V_i(\mathbf{r},t) - u_i(\mathbf{r},t)\big] + 2\Omega\Gamma_{i\mu}\,V_\mu(\mathbf{r},t)\,,$$

where the matrix

$$\Gamma = \left\|\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right\|\,, \qquad \Gamma^2 = -E\,,$$

and E is the identity matrix. This equation can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r},t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t) = -\Lambda\big[\mathbf{V}(\mathbf{r},t) - \mathbf{U}(\mathbf{r},t)\big]\,, \tag{106}$$

where the matrix $\Lambda = (\lambda E - 2\Omega\Gamma)$, and the random velocity field $\mathbf{U}(\mathbf{r},t)$ has the form

$$\mathbf{U}(\mathbf{r},t) = \lambda\Lambda^{-1}\,\mathbf{u}(\mathbf{r},t)\,, \qquad \Lambda^{-1} = \frac{\lambda E + 2\Omega\Gamma}{\lambda^2 + 4\Omega^2}\,. \tag{107}$$

In the case of $\{\lambda \text{ or } \Omega\} \to \infty$, an approximate expression is obtained in the form

$$\mathbf{V}(\mathbf{r},t) \approx \mathbf{U}(\mathbf{r},t)\,. \tag{108}$$

It should be noted that the introduction of a new vector $\mathbf{W}(\mathbf{r},t) = \Gamma\mathbf{V}(\mathbf{r},t)$ leads to the quantity

$$\xi(\mathbf{r},t) = \frac{\partial W_i(\mathbf{r},t)}{\partial r_i} = \frac{\partial \mathbf{W}(\mathbf{r},t)}{\partial \mathbf{r}} = \frac{\partial V_2(\mathbf{r},t)}{\partial r_1} - \frac{\partial V_1(\mathbf{r},t)}{\partial r_2}$$

that describes the eddy component of the velocity field $\mathbf{V}(\mathbf{r},t)$.

Equation (106) differs from equation (85) by the tensor character of the parameter Λ. Moreover, the field $\mathbf{U}(\mathbf{r},t)$ in Eqn (106) is a divergent one, and for the divergence-free field $\mathbf{u}(\mathbf{r},t)$ the quantity

$$\mathrm{div}\,\mathbf{U}(\mathbf{r},t) = \frac{\partial \mathbf{U}(\mathbf{r},t)}{\partial \mathbf{r}} = \lambda\,\frac{\partial}{\partial r_k}\,\Lambda_{k\mu}^{-1}\,u_\mu(\mathbf{r},t)$$
$$= \frac{2\lambda\Omega}{\lambda^2 + 4\Omega^2}\,\Gamma_{k\mu}\,\frac{\partial u_\mu(\mathbf{r},t)}{\partial r_k}$$

is related to the eddy component of the field $\mathbf{u}(\mathbf{r},t)$.

Let us assume, as before, that dispersion $\sigma_{\mathbf{u}}^2 = \langle \mathbf{u}^2(\mathbf{r},t)\rangle$ is small and Eqn (106) can be linearized with respect to flow (108) for large parameters $\{\lambda,\ \Omega\}$. In this case, the coefficient D_0 of spatial diffusion in Eqn (100) shows no dependence on the parameter λ and is described by the formula [50]

$$D_0 = \frac{1}{2}\,\tau_{\mathbf{V}}\langle \mathbf{V}^2(\mathbf{r},t)\rangle = \frac{1}{2}\int_0^\infty B_{ii}(0,\tau)\cos(2\Omega\tau)\,\mathrm{d}\tau$$
$$= \frac{\pi}{2}\int \Phi(k,2\Omega)\,\mathrm{d}\mathbf{k}\,, \tag{109}$$

where $\Phi(k,\omega)$ is the space – time spectral function (86) of the field $\mathbf{u}(\mathbf{r},t)$. The following expression can be obtained for the diffusion coefficient $D^{(\mathbf{V})}$ [50]:

$$D^{(\mathbf{V})} = \frac{4\lambda^2\Omega^2}{(\lambda^2 + 4\Omega^2)^2}\int_0^\infty \exp(-\lambda\tau)\cos(2\Omega\tau)\,D(\tau)\,\mathrm{d}\tau\,. \tag{110}$$

If $\{\lambda,\Omega\}\tau_0 \gg 1$, then one finds

$$D^{(\mathbf{V})} = \frac{4\lambda^3\Omega^2 D(0)}{(\lambda^2 + 4\Omega^2)^3} = \begin{cases} \dfrac{4\Omega^2 D(0)}{\lambda^3}\,, & \text{if} \quad \lambda \gg \Omega\,, \\[2ex] \dfrac{\lambda^3 D(0)}{16\Omega^4}\,, & \text{if} \quad \lambda \ll \Omega\,, \end{cases} \tag{111}$$

where, as before, the notation is used:

$$D(0) = \int k^2 E(k,0)\,\mathrm{d}\mathbf{k} = -\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle\,.$$

Thus, conditions $\{\lambda,\Omega\}\tau_0 \gg 1$ being fulfilled, the process of generating the divergent part of the field $\mathbf{V}(\mathbf{r},t)$ in the problem under consideration is described by a linear equation without regard for advective terms. If, in addition, $\lambda \gg \Omega$, it is necessary to take into account the correction terms (105) of order $\sigma_{\mathbf{u}}^4$, which in certain cases can be compared with those in formula (111); hence the expression

$$D^{(\mathbf{V})} = \frac{3}{2}\,\frac{\tau_0}{\lambda^2}\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle^2 - \frac{4\Omega^2}{\lambda^3}\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle$$
$$= -\frac{4\Omega^2}{\lambda^3}\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle\left\{1 - \frac{3\lambda\tau_0}{2\Omega^2}\langle \mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\rangle\right\}. \tag{112}$$

In the foregoing, we have described a method to derive expressions for diffusion coefficients that characterize clustering of the number density of small-inertia particles in hydrodynamic flows under different asymptotic regimes. A study of these coefficients (hence, the phenomenon of clustering itself) for concrete geophysical and astrophysical problems was beyond the scope of the present work. These are totally independent problems that can be solved using the above-given expressions.

5. Conclusions

To conclude, the following remarks are in order:

• Statistical characteristics for the solution to the problem of diffusion of particles and conservative passive tracer density field in random divergent velocity fields may have little in common with the behavior of concrete realizations. The traditional approach to such problems, based on the moment function description, is of small informational value. What is necessary for their solution is a statistical description in terms of the probability density (at least one-point or simultaneous).

• However, problems pertaining to diffusion of particles and passive tracer concentration fields in random divergent velocity fields contain statistically coherent physical phenomena that occur with probability unity (clustering of particles and the conservative tracer field in a divergent velocity field). This means that a given phenomenon occurs in almost all random velocity field realizations.

• Coherent phenomena themselves are largely independent of the specific model of fluctuating parameters of a dynamical system. In the simplest case, their time-dependent dynamics can be described in terms of simultaneous and one-point probability distributions by the methods of statistical topography. Of course, concrete parameters characterizing this phenomenon (like typical times of the formation of cluster structures and their spatial characteristic scales) may show strong dependence on the type of the models.

• Clustering of low-inertia particles and their concentration field may also occur in random divergence-free velocity fields. Their statistical description is impossible, in principle, with the aid of the approximation of the delta-correlated in time velocity field of a fluctuating flow (e.g., the Fokker–Planck equation for the diffusion of low-inertia particles). It is therefore necessary to take into consideration the finiteness of its time correlation radius.

This work was supported by the Russian Foundation for Basic Research (projects Nos 01-05-64042 and 02-05-64375) and State Contract No. 40.020.1.1.1177.

References

1. Monin A S, Yaglom A M *Statisticheskaya Gidromekhanika* Pts 1, 2 (Statistical Fluid Mechanics) (Moscow: Nauka, 1965, 1967) [Translated into English (Cambridge, Mass.: MIT Press, 1971, 1975)]
2. Csanady G T *Turbulent Diffusion in the Environment* (Geophys. and Astrophys. Monographs, Vol. 3) (Dordrecht: D. Reidel Publ. Co., 1973)
3. Lesieur M *Turbulence in Fluids*: *Stochastic and Numerical Modelling* (Fluid Mechanics and Its Applications, Vol. 1) 2nd ed. (Dordrecht: Kluwer Acad. Publ., 1990)
4. McComb W D *The Physics of Fluid Turbulence* (Oxford Engineering Sci. Ser., Vol. 25) (Oxford: Clarendon Press, 1990)
5. Okubo A *Diffusion and Ecological Problems*: *Mathematical Models* (Biomathematics, Vol. 10) (Berlin: Springer-Verlag, 1980)
6. Dagan G *Annu. Rev. Fluid Mech.* **19** 183 (1987)
7. Gurbatov S N, Malakhov A N, Saichev A I *Nelineĭnye Sluchaĭnye Volny v Sredakh bez Dispersii* (Nonlinear Random Waves in Nondispersive Media) [Seriya Sovremennye Problemy Fiziki (Current Problems of Physics Series)] Issue 81 (Moscow: Nauka, 1990) [Translated into English: *Nonlinear Random Waves and Turbulence in Nondispersive Media*: *Waves, Rays, Particles* (Manchester: Manchester Univ. Press, 1991)]
8. Shandarin S F, Zeldovich Ya B *Rev. Mod. Phys.* **61** 185 (1989)
9. Batchelor G K *J. Fluid Mech.* **5** 113 (1959)
10. Batchelor G K, Howells I D, Townsend A A *J. Fluid Mech.* **5** 134 (1959)
11. Taylor G I *Proc. London Math. Soc. Ser. 2* **20** 196 (1923); *Proc. R. Soc. London Ser. A* **164** 476 (1938)
12. Roberts P H *J. Fluid Mech.* **11** 257 (1961)
13. Kraichnan R H *Phys. Fluids* **11** 945 (1968); **13** 22 (1970); *J. Fluid Mech.* **64** 737 (1974)
14. Saffman P G *Phys. Fluids* **12** 1786 (1969)
15. Chen H, Chen S, Kraichnan R H *Phys. Rev. Lett.* **63** 2657 (1989)
16. Sinai Ya G, Yakhot V *Phys. Rev. Lett.* **63** 1962 (1989)
17. Avellaneda M, Majda A J *Commun. Math. Phys.* **131** 381 (1990)
18. Gao F *Phys. Fluids A: Fluid Dyn.* **3** 511 (1991)
19. Lipscombe T C, Frenkel A L, ter Haar D *J. Stat. Phys.* **63** 305 (1991)
20. Careta A et al. *J. Stat. Phys.* **71** 235 (1993)
21. Kimura Y, Kraichnan R H *Phys. Fluids A*: *Fluid Dyn.* **5** 2264 (1993)
22. Majda A J *Phys. Fluids A: Fluid Dyn.* **5** 1963 (1993); *J. Stat. Phys.* **75** 1153 (1994)
23. Kraichnan R H *Phys. Rev. Lett.* **72** 1016 (1994)
24. Klyatskin V I *Usp. Fiz. Nauk* **164** 531 (1994) [*Phys. Usp.* **37** 501 (1994)]
25. Klyatskin V I, Woyczynski W A *Zh. Eksp. Teor. Fiz.* **108** 1403 (1995) [*JETP* **81** 770 (1995)]
26. Klyatskin V I, Woyczynski W A, Gurarie D, in *Stochastic Modelling in Physical Oceanography* (Progress in Probability, Vol. 39, Eds R J Adler, P Müller, B Razovskii) (Boston: Birkhäuser, 1996) p. 221; *J. Stat. Phys.* **84** 797 (1996)
27. Saichev A I, Woyczynski W A, in *Stochastic Models in Geosystems* (The IMA Volumes in Mathematics and Its Applications, Vol. 85, Eds S A Molchanov, W A Woyczynski) (New York: Springer-Verlag, 1997) p. 359
28. Gurarie D, Klyatskin V, in *Two-Dimensional Turbulence in Plasmas & Fluids* (AIP Conf. Proc., Vol. 414, Eds R L Dewar, R W Griffiths) (Woodbury, NY: American Institute of Physics, 1997) p. 213
29. Klyatskin V I, Saichev A I *Zh. Eksp. Teor. Fiz.* **111** 1297 (1997) [*JETP* **84** 716 (1997)]
30. Klyatskin V I *Izv. Ross. Akad. Nauk Ser. Fiz. Atm. Okeana* **36** 177 (2000) [*Izv., Atmos. Ocean. Phys.* **36** 162 (2000)]
31. Klyatskin V I *Statisticheskoe Opisanie Dinamicheskikh Sistem s Fluktuiruyushchimi Parametrami* (Statistical Description of Dynamic Systems with Fluctuating Parameters) (Moscow: Nauka, 1975)
32. Saichev A I *Dynamic Systems* **1** 1 (1993)
33. Gurbatov S N, Saichev A I *Chaos* **3** 333 (1993)
34. Klyatskin V I, Gurarie D *Usp. Fiz. Nauk* **169** 171 (1999) [*Phys. Usp.* **42** 165 (1999)]
35. Isichenko M B *Rev. Mod. Phys.* **64** 961 (1992)
36. Klyatskin V I *Stokhasticheskie Uravneniya Glazami Fizika. Osnovnye Polozheniya, Tochnye Rezul'taty i Asimptoticheskie Priblizheniya* (Stochastic Equations in the Physicist's Eyes: Fundamentals, Exact Results, and Asymptotic Approximations) (Moscow: Fizmatlit, 2001); *Dinamika Stokhasticheskikh Sistem.* Kurs Lektsiĭ (Dynamics of Stochastic Systems. A Course of Lectures) (Moscow: Fizmatlit, 2002)
37. Nicolis G, Prigogine I *Exploring Complexity*: *an Introduction* (New York: W.H. Freeman, 1989)
38. Stokes G G *Camb. Trans.* **9** 8 (1851)
39. Lamb H *Hydrodynamics* (Cambridge: Univ. Press, 1930) [Translated into Russian (Moscow: Gostekhizdat, 1947)]
40. Landau L D, Lifshitz E M *Gidrodinamika* (Fluid Mechanics) (Moscow: Nauka, 1986) [Translated into English (Oxford: Pergamon Press, 1987)]
41. Ungarish M *Hydrodynamics of Suspensions*: *Fundamentals of Centrifugal and Gravity Separation* (Berlin: Springer-Verlag, 1993)
42. Tory E M (Ed.) *Sedimentation of Small Particles in a Viscous Fluid* (Advances in Fluid Mechanics, Vol. 7) (Southampton, UK: Computational Mechanics Publ., 1996)
43. Maxey M R, Riley J J *Phys. Fluids* **26** 883 (1983)
44. Maxey M R, Corrsin S *J. Atmos. Sci.* **43** 1112 (1986)
45. Maxey M R *J. Fluid Mech.* **174** 441 (1987); *Philos. Trans. R. Soc. London Ser. A* **333** 289 (1990)
46. Wang L-P, Maxey M R *J. Fluid Mech.* **256** 27 (1993)
47. Maxey M R, Chang E J, Wang L-P *Exp. Therm. Fluid Sci.* **12** 417 (1996)
48. Elperin T, Kleeorin N, Rogachevskii I *Phys. Rev. E* **53** 3431 (1996); **58** 3113 (1998); *Phys. Rev. Lett.* **76** 224 (1996); **77** 5373 (1996); **81** 2898 (1998); *Atmos. Res.* **53** 117 (2000)
49. Balkovsky E, Falkovich G, Fouxon A *Phys. Rev. Lett.* **86** 2790 (2001)
50. Klyatskin V I, Elperin T *Zh. Eksp. Teor. Fiz.* **122** 238 (2002) [*JETP* **95** 282 (2002)]
51. Zirbel C L, Çinlar E, in *Stochastic Models in Geosystems* (The IMA Volumes in Mathematics and Its Applications, Vol. 85, Eds S A Molchanov, W A Woyczynski) (New York: Springer-Verlag, 1997) p. 459
52. Koshel' K V, Aleksandrova O V *Izv. Ross. Akad. Nauk Ser. Fiz. Atm. Okeana* **35** 638 (1999) [*Izv., Atmos. Ocean. Phys.* **35** 578 (1999)]
53. Klyatskin V I, Koshel' K V *Usp. Fiz. Nauk* **170** 771 (2000) [*Phys. Usp.* **43** 717 (2000)]

54. Mesinger F *J. Atmos. Sci.* **22** 479 (1965); *Mon. Weather Rev.* **99** 15 (1971)
55. Mesinger F, Mintz Y, Technical Report No. 4, No. 5 (Los Angeles, CA: Dep. Meteorology, Univ. of California, 1970)
56. Er-El J, Peskin R L *J. Atmos. Sci.* **38** 2264 (1981)
57. Mirabel' A P, Monin A S *Izv. Ross. Akad. Nauk Ser. Fiz. Atm. Okeana* **19** 202 (1983)
58. Ott S, Mann J *J. Fluid Mech.* **422** 207 (2000)
59. Furutsu K *J. Res. NBS D* **67** 303 (1963)
60. Novikov E A *Zh. Eksp. Teor. Fiz.* **47** 919 (1964) [*Sov. Phys. JETP* **20** 1290 (1965)]
61. Klyatskin V I *Stokhasticheskie Uravneniya i Volny v Sluchaĭno-Neodnorodnykh Sredakh* (Stochastic Equations and Waves in Randomly Inhomogeneous Media) (Moscow: Nauka, 1980); *Ondes et Équations Stochastiques dans les milieus Aléatoirement non Homogènes* (Besançon – Cedex: Les Éditions de Physique, 1985)
62. Klyatskin V I, Saichev A I *Usp. Fiz. Nauk* **162** (3) 161 (1992) [*Sov. Phys. Usp.* **35** 231 (1992)]
63. Gurvich A S, Yaglom A M *Phys. Fluids* **10** (Suppl.) S59 (1967)
64. Dahm W J A, Buch K A *Phys. Fluids A: Fluid Dyn.* **1** 1290 (1989)
65. Kerstein A R, McMurtry P A *Phys. Rev. E* **49** 474 (1994)
66. Zaslavskiĭ G M, Sagdeev R Z *Vvedenie v Nelineĭnuyu Fiziku: Ot Mayatnika do Turbulentnosti i Khaosa* (Introduction to Nonlinear Physics: From Pendulum to Turbulence and Chaos) (Moscow: Nauka, 1988); see also Sagdeev R Z, Usikov D A, Zaslavsky G M *Nonlinear Physics: from the Pendulum to Turbulence and Chaos* (Chur: Harwood Acad. Publ., 1988)
67. Lichtenberg A J, Lieberman M A *Regular and Stochastic Motion* (Applied Mathematical Sciences, Vol. 38) (New York: Springer-Verlag, 1983) [Translated into Russian (Moscow: Mir, 1984)]
68. Klyatskin V I, Yakushkin I G *Zh. Eksp. Teor. Fiz.* **118** 849 (2000) [*JETP* **91** 736 (2000)]
69. Weichman P B, Glazman R E *Phys. Rev. Lett.* **83** 5011 (1999); *J. Fluid Mech.* **420** 147 (2000); **453** 263 (2002)
70. Herterich K, Hasselmann R *J. Phys. Oceanogr.* **12** 704 (1982)
71. Sanderson B G, Okubo A *J. Geophys. Res.* **93** 3570 (1988)
72. Walterscheid R L, Hocking W K *J. Atmos. Sci.* **48** 2213 (1991)
73. Klyatskin V I, Elperin T *Izv. Ross. Akad. Nauk Ser. Fiz. Atm. Okeana* **38** 817 (2002) [*Izv., Atmos. Ocean. Phys.* **38** 725 (2002)]
74. Uhlenbeck G E *Usp. Fiz. Nauk* **103** 275 (1971)