Physics – Uspekhi **46** (6) 645 – 655 (2003)

METHODOLOGICAL NOTES

PACS numbers: **03.50.+p**, 41.60.Cr, **41.75.–i**

Special features of motion of particles in an electromagnetic wave

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DOI: 10.1070/PU2003v046n06ABEH001349

Contents

Abstract. The behavior of a charged particle in the field of a monochromatic electromagnetic wave is considered. The motion of a particle is determined not only by the wave field but also by the initial conditions. The trajectories of particles are calculated both by using the exact solution and by employing perturbation theory in the parameter $\eta = eE/(mc\omega)$, the ratio of the energy the field transfers to a particle over a wavelength, to the particle's rest energy. Two kinds of electromagnetic waves, those with coordinate independent amplitudes (uniform waves) and those with coordinate dependent amplitudes (non-uniform waves) are treated. The motion of particles that either are at rest or move with prescribed velocity at the initial time, is investigated. It is shown that a charged particle performs not only an oscillatory motion but also a systematic drift in the field of a wave. In a non-uniform wave, accelerating ponderomotive forces also act on a particle.

1. Introduction

The interaction of electromagnetic fields of various kinds with charged particles lies at the heart of many physical phenomena. The motion of particles in static fields is fairly well studied and is discussed in virtually every electrodynamics textbook. Problems of motion of particles in alternating fields, on the other hand, are much less addressed in the literature — even though these problems are of no less interest from both theoretical and applied points of view. Of particular interest are the problem of a charged particle moving in the field of an electromagnetic wave and the applications of this problem in the physics of accelerators, the theory of free electron lasers, and a number of other domains.

The problem of a motion of electron in the field of a plane electromagnetic field has an analytical solution. In 1935 D M Volkov obtained a rigorous solution of the Dirac equation for a motion of electron in the field of a plane electromagnetic wave [1]. The classic analogue of this solution is given in *Field Theory* by L D Landau and E M Lifshitz [2]. At the same time some aspects of charged particle behavior in an electromagnetic wave field are insufficiently covered in the literature. Besides, the solutions of Refs [1, 2] do not apply to the case of a motion of charged particle in a non-uniform wave (for example, a waveguide eigenmode or a wave produced by two plane waves traveling at an angle to each other). The reason is that the equation of motion of a particle in a wave field is nonlinear. Therefore if the external field acting on the particle is the sum of several plane waves, the solution of the equation of motion can not be presented as a sum of solutions obtained for each wave separately. In this case approximate methods such as perturbation theory are fully justifiably used. Note here that perturbation theory may even prove useful for the case of a single particle in the field of a single plane wave.

Some qualitative aspects of particle motion in the field of a wave are fairly well known [2]. For example, in the field of a circularly polarized wave a particle moves along the circumference of a circle in the plane perpendicular to the wave propagation direction. In a uniform, linearly polarized wave field a charge moves along a figure eight curve. The figure eight has its longitudinal axis directed along the wave's electric field strength, and its transverse axis along its propagation direction. The particle oscillates at a wave frequency ω in the direction of the electric field, and at a twice this frequency, 2ω, in the propagation direction.

This behavior can be understood based on simple qualitative considerations. Consider a plane electromagnetic wave whose electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ are described by the respective expressions

$$\mathbf{E} = \mathbf{E}_0 \exp\left[\mathrm{i}(\mathbf{kr} - \omega t)\right], \qquad \mathbf{H} = \mathbf{H}_0 \exp\left[\mathrm{i}(\mathbf{kr} - \omega t)\right], \quad (1)$$

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Received 23 October 2002, revised 26 March 2003
Uspekhi Fizicheskikh Nauk **173** (6) 667 – 678 (2003)
Translated by E G Strel'chenko; edited by A V Leonidov

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are the electric and magnetic field amplitudes, $\mathbf{k}$ the wave vector, and ω the wave frequency. $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ form a right-hand triple of vectors in this wave. Consider a charged particle in the field of this plane wave. Denote its charge by e. The equation of motion of such a particle has the form

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\mathbf{E} + \frac{e}{c}\,[\mathbf{vH}]\,, \tag{2}$$

where

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-(v/c)^2}} = m\mathbf{v}\gamma \tag{3}$$

m is the particle's rest mass, $\mathbf{v}$ is its velocity, and $\gamma = [1-(v/c)^2]^{-1/2}$ is the reduced energy.

Let us introduce a Cartesian coordinate system with the x, y, and z axes along the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, and the wave vector $\mathbf{k}$, respectively. In this system the equation of motion can be written for three components separately as follows:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = eE_x - \frac{e}{c}\,v_z H_y\,, \qquad \frac{\mathrm{d}p_y}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}p_z}{\mathrm{d}t} = \frac{e}{c}\,v_x H_y\,. \tag{4}$$

From these equations it can be seen, in particular, that the projection of the particle momentum onto the magnetic field vector is a constant quantity. Suppose at the initial time the particle is at rest at the origin of the coordinate system. From Eqns (4) it follows that the trajectory of the particle lies in a plane containing the vectors $\mathbf{E}$ and $\mathbf{k}$. Consider first the equation for the projection of the momentum onto the electric field vector. The right-hand side of this equation oscillates at the wave frequency ω. Hence so does the quantity p_x. If, as we have assumed, the charge was initially at rest, then its velocity will also oscillate with the frequency ω.

Now consider the equation for p_z. Its right-hand side is the product of two functions oscillating at the wave frequency and thus contains a periodic function oscillating at 2ω. Hence the component p_z also contains a term oscillating at 2ω. It is easily seen that in this case the particle follows a figure-eight-like curve in the xz plane.

A point to bear in mind is that solutions to the equations for p_x, p_y, and p_z may also contain constant — time-independent — terms corresponding to the particle's systematic motion (drift) in the field of a plane electromagnetic wave. The solutions given in Ref. [2] (see problems to Section 48) are obtained for the reference frame in which the particle is on the average at rest. This eliminates, in a sense, the question of particle drift because the parameters of the drift become parameters that determine the motion of the reference frame. In some cases, however, the motion of a particle in the laboratory frame is of interest. In these cases drift motion parameters enter explicitly into the expression for the particle trajectory. Below, it is the laboratory frame in which the motion of a particle under the influence of the wave field will be considered. Clearly, the trajectory of a particle in a wave field is determined by the initial conditions, i.e., by the position and velocity of the particle and by the phase of the wave at the particle's initial position. For example, for a particle injected into a wave field, the coordinates of the point of injection, the initial particle velocity, and the initial phase of the wave at the point of injection must be specified. We will consider this problem for various initial conditions in what follows.

As shown in Ref. [4], where motion in an alternating electric field alone is considered, a particle, while oscillating, also performs a drift motion whose velocity and direction depend on the initial conditions. The study of Ref. [4] neglected the effect of the wave's magnetic field, which is valid if the particle velocity is small compared to the speed of light. Below we consider special features of motion in an electromagnetic wave taking into account the magnetic field. In this case it will be shown that a systematic drift in the direction of the electric field is maintained, that the particle also performs a systematic drift in the direction of the wave, and that the drift in the electric field direction is qualitatively different from that along the wave direction.

The field of a plane electromagnetic wave has the property of being the same at any point on the plane normal to the wave vector. Therefore such a wave is sometimes called a plane uniform electromagnetic wave. The value of the field in such a wave is determined only by the phase $\varphi = kz - \omega t$ (recall that we are considering a z-propagating wave) and does not depend on the transverse coordinates x and y. In the case where the field depends on x and y, additional systematic forces acting on the particle appear. These will also be considered in a number of simple examples below.

2. The exact solution

Consider a particle of charge e in an external field described by a vector potential $\mathbf{A}$. Let $p_{0\mu}$ be an arbitrary four-dimensional vector,

$$p_{0\mu} = (p_{00}, p_{0x}, p_{0y}, p_{0z})\,.$$

The components of this vector satisfy the relation

$$p_{00}^2 - p_{0x}^2 - p_{0y}^2 - p_{0z}^2 = m^2c^2\,.$$

Clearly, in this case the components of the vector $p_{0\mu}$ can be considered as those of the four-dimensional momentum of a particle of mass m. The zeroth component is then proportional to the energy of the particle. It is sometimes said that the vector $p_{0\mu}$ is specified on the mass surface. The solution of Eqn (2) is expressed in terms of the components of the vectors $p_{0\mu}$ and A as follows:

$$p_\mu = p_{0\mu} - \frac{e}{c}\,A_\mu + k_\mu\left(\frac{e}{c}\frac{pA}{kp} - \frac{e^2}{c^2}\frac{A^2}{2kp}\right), \tag{5}$$

where k_μ is a four-dimensional vector $(\omega/c, k_x, k_y, k_z)$, and $A_\mu = (A_0, A_x, A_y, A_z)$. It is assumed that the components A_μ of the vector potential $\mathbf{A}$ depend on the argument

$$\varphi = k_\mu x_\mu = k_0 x_0 - k_x x - k_y y - k_z z\,.$$

This argument determines the phase of the plane wave. The phase maintains a constant value in the plane normal to the wave vector $\mathbf{k} = (k_x, k_y, k_z)$. The expressions for pA and kp are the scalar products of four-dimensional vectors:

$$pA = p_0 A_0 - p_x A_x - p_y A_y - p_z A_z\,,$$
$$kp = k_0\,p_0 - k_x\,p_x - k_y\,p_y - k_z\,p_z\,.$$

It is in this form that the solution was given in Ref. [3]. Formula (5) gives the kinetic momentum of the particle in the field A_μ. The components of the vector $p_{0\mu}$ can be considered as certain constants which can be chosen in such a way as to satisfy the initial conditions. At the same time the vector $p_{0\mu}$ determines the value of the momentum p_μ at the moments in time when $A_\mu = 0$.

Expressions for the solution (5) simplify in the case when the wave travels along the axis z and the vector potential $\mathbf{A}$ has one nonzero component, A_x. The wave then has its electric field vector directed along x, and its magnetic field vector along y, and the solution (5) becomes

$$p_x = p_{0x} - \frac{e}{c} A_x , \tag{6}$$

$$p_y = p_{0y} , \tag{7}$$

$$p_z = p_{0z} + k_z \left(-\frac{e}{c} \frac{c p_{0x} A_x}{\omega(p_{00} - p_{0z})} + \frac{e^2}{c^2} \frac{c A_x^2}{2\omega(p_{00} - p_{0z})} \right) , \tag{8}$$

$$p_0 = p_{00} + k_0 \left(-\frac{e}{c} \frac{c p_{0x} A_x}{\omega(p_{00} - p_{0z})} - \frac{e^2}{c^2} \frac{c A_x^2}{2\omega(p_{00} - p_{0z})} \right) . \tag{9}$$

Inspecting the solutions for p_0 and p_z we see that the difference between the values of p_0 and p_z is a constant,

$$p_0 - p_z = p_{00} - p_{0z} = \text{const} . \tag{10}$$

Formulas (6)–(10) permit qualitative conclusions about the behavior of a charged particle in the field of a plane electromagnetic wave. Let us specify the vector potential to be of the form

$$\mathbf{A} = \mathbf{A}_0 \cos\psi , \tag{11}$$

where

$$\psi = \frac{\omega}{c} z - \omega t + \varphi ,$$

and the vector $\mathbf{A}$ has only an x component. The average values of both the electric and magnetic fields are zero.

Let us see what form the solutions (6)–(10) take if the initial value of the momentum is zero. Let the particle be at rest at the origin at the initial time instance $t = 0$. For this case, as Eqn (6) suggests, it is expedient to write

$$p_{0x} = \frac{e}{c} A_x(x = 0,\ t = 0) = \frac{e}{c} A_0 \cos\varphi . \tag{12}$$

The expression for p_x then takes the form

$$p_x = \frac{e}{c} A_0 (\cos\varphi - \cos\psi) . \tag{13}$$

From this formula it is seen that a particle in the field of an electromagnetic wave performs a systematic drift in the direction of the electromagnetic field (the x axis) and that the average value of the particle momentum along x is

$$\langle p_x \rangle = \frac{e}{c} A_0 \cos\varphi . \tag{14}$$

Depending on the initial phase φ, the particle drifts either in the positive or negative x direction. For $\varphi = \pi/2 + n\pi$, no drift is present, and the particle oscillates about its initial position. If the initial value of p_x is different from zero, the average value of $\langle p_x \rangle$ is the sum of this initial value plus the quantity (14).

The fact that the time average of the field is zero and yet the particle performs a systematic drift in such a field is not obvious. Some authors maintain that a particle in such a field oscillates around a fixed point.

As already noted, the projection p_y of the particle momentum onto the wave magnetic field does not change in time.

Let us now consider the expression (8) for the component of momentum along the wave vector of the wave. Turning to the formula (8), we note that the denominators of the fractions occurring in this formula contain the factor $(p_0 - p_z)$. From Eqn (10) it follows that this quantity is invariant and that for the initial conditions chosen $p_0 - p_z = mc$. We can therefore rewrite the expression for p_z as

$$p_z = p_{0z} + k_z \left(-\frac{e p_{0x} A_x(\psi)}{\omega m c} + \frac{e^2 A_x^2(\psi)}{2\omega m c^2} \right) . \tag{15}$$

From the requirement that the component p_z vanish at the initial time, and making use of Eqn (11), the constant p_{0z} is

$$p_{0z} = \frac{1}{2} \frac{e^2 A_0^2 \cos^2\varphi}{m c^3} . \tag{16}$$

Taking into account this equality, the formula for p_z becomes

$$p_z = \frac{1}{2} \frac{e^2 A_0^2 \cos^2\varphi}{m c^3} + \left(-\frac{e^2 A_0 \cos\varphi A_x(\psi)}{m c^3} + \frac{e^2 A_x^2(\psi)}{2 m c^3} \right) . \tag{17}$$

The first term in the expression for p_z does not depend on time. The terms in parentheses do depend on time, and the average value of the first term in parentheses is zero. Taking the average, we obtain the average value of the z-component of the particle momentum,

$$\langle p_z \rangle = \frac{e^2 A_0^2}{4 m c^3} (\cos 2\varphi + 2) . \tag{18}$$

As can be seen from this formula, the particle performs a systematic drift along the direction of the wave. The average value of the momentum of the drifting particle depends on the initial phase φ.

Thus, a particle in the field of an electromagnetic wave drifts in two directions: along the electric field (the x axis) and along the wave vector (the z axis). Note that the drift along x may occur in both the positive and negative directions depending on the initial phase, whereas the drift along z is always in the same direction as the wave vector (i.e., in the direction of propagation of the wave). The sign of the drift velocity along the z axis is always the same, though its magnitude depends on the initial phase. The drift velocity along the x axis depends linearly on the wave amplitude, and that along z is proportional to the amplitude squared.

Consider now the formula (9) for the energy of a particle in the field of the wave. Recall that $p_0 = W/c$, where W is the particle energy. From Eqn (10) the value of p_0 for a particle in a wave differs from the z-component of momentum, p_z, by a constant quantity. From Eqns (8) and (9) it can be seen that either of the quantities p_0 and p_z is the sum of three terms, the first of which is a constant, the second of which is proportional to A_x, and the third of which is proportional to A_x^2. Because the vector potential A_x oscillates at a frequency ω, the term proportional to A_x^2 oscillates at 2ω. But it must be kept in mind that the argument of the function A_x in

Eqns (6) – (9) is the parameter

$$\psi = \frac{\omega}{c}\, z - \omega t + \varphi\,,$$

where z is in turn a function of time, $z = z(t)$. Thus

$$\psi = \frac{\omega}{c}\, z(t) - \omega t + \varphi\,,$$

where $z(t)$ determines the position of the particle at time t.

Let us consider the simplest special case in which the particle moves at constant velocity v along the z axis. If the velocity v is close to the speed of light c, it may be assumed that the wave field has little effect on this velocity. Then, crudely, we can consider that the velocity of the particle in the wave is also close to v, and that

$$z(t) = vt\,, \qquad \psi = \frac{\omega}{c}\, vt - \omega t + \varphi = \omega t\left(\frac{v}{c} - 1\right) + \varphi\,.$$

From this relation it can be seen that even though the field oscillates at a frequency ω, the field acting on the particle oscillates at the frequency

$$\omega' = \omega(1-\beta) = \frac{\omega}{2\gamma^2}\,,$$

where $\beta = v/c$, and $\gamma = (1-\beta^2)^{-1/2}$. Therefore in our case (with a particle traveling in the direction of the wave) the field acting on the particle changes much more slowly than the field at a fixed point in space. Hence, if the temporal period of the field is $T = 2\pi/\omega$, then the alternating force the wave exerts on the particle has a period of the order of magnitude $T' = 4\pi\gamma^2/\omega$. If the particle velocity is at an angle θ to the wave propagation direction, then the period of the alternating force is of the order of

$$T' = \frac{2\pi}{\omega(1-\beta\cos\theta)}\,.$$

Returning to Eqns (8) and (9) for p_z and p_0, we note that the momentum and energy of a particle in a wave do not change monotonically but rather oscillate around certain values obtained by averaging p_z and p_0 over time. Over the time interval T' the particle energy increases and over the next interval of the same order of magnitude it decreases. The values of p_z and p_0 around which the oscillations occur depend on the wave phase φ. Recall that, in Eqn (11), φ is the phase of the wave at $z = 0$, $t = 0$.

The fact that the particle energy does not increase monotonically but rather increases and decreases alternately does not mean that the particle cannot be accelerated in the field of the wave. For this to occur it is necessary to choose a suitable period of time during which the wave transfers its energy to the particle.

Note that the exact solution we have examined — one for determining the behavior of a particle in a plane wave field — was obtained by neglecting the radiation the particle emits. In reality, though, because the particle moves non-uniformly in the wave field, it becomes a source of radiation, which can be interpreted as being scattered by the particle. Let us discuss some of the properties of radiation emitted by a charged particle in the field of a plane electromagnetic wave. The field of radiation depends on how the particle moves along its trajectory. Reference [2] presents a parametric equation of the trajectory in the frame of reference where the particle is on average at rest. In this frame the particle performs a periodic motion along a closed figure eight curve. The curve is in the plane in which the electric field vector and the wave vector lie. The trajectory is symmetric, its axis of symmetry being parallel to the electric field.

From the results mentioned above one can draw conclusions about the trajectory of the particle in the laboratory frame. Let the coordinate frame, in which the particle is on average at rest, move with velocity $\mathbf{v}$ relative to the laboratory frame. Suppose that an observer in the laboratory frame measures the particle's period of rotation using his clock and finds it to be T. In the laboratory frame the particle's trajectory is no longer closed but is a certain spatially periodic curve. The equation of motion of the particle in the laboratory frame has the form $\mathbf{r} = \mathbf{r}(t)$, where the function $\mathbf{r}(t)$ satisfies

$$\mathbf{r}(t+T) = \mathbf{r}(t) + \mathbf{v}T\,. \tag{19}$$

This implies that the velocity of the particle in the laboratory frame is a periodic function of time with a period T:

$$\mathbf{v}(t+T) = \frac{\mathrm{d}\mathbf{r}(t+T)}{\mathrm{d}t} = \mathbf{v}(t)\,. \tag{20}$$

The last two properties of the particle trajectory reveal some characteristic features of the radiation from a charged particle moving in a plane wave field.

If the trajectory of a charged particle is specified by the relation $\mathbf{r} = \mathbf{r}(t)$, then radiation at frequency ω is described by the Fourier component $\mathbf{A}_\omega(\mathbf{r})$ of the vector potential $\mathbf{A}$

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{q}{2\pi c}\,\frac{\exp(\mathrm{i}kr)}{r}\int_{-\infty}^{\infty}\mathbf{v}(t)\exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\}\mathrm{d}t\,, \tag{21}$$

where

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\,, \qquad k = \frac{\omega}{c}\,\mathbf{n}\,, \qquad \mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}\,; \tag{22}$$

$\mathbf{n}$ is the unit vector in the direction from the region of particle motion to the point of observation $\mathbf{r}$. It is assumed that the observer is far enough from the former region.

Let us subdivide the interval of integration into segments of length T. Then the integral over t can be represented as the sum of integrals:

$$\int_{-\infty}^{\infty}\ldots\,\mathrm{d}t = \sum_n\int_{nT}^{(n+1)T}\ldots\,\mathrm{d}t\,, \tag{23}$$

where n runs over all integers.

Consider one term in this sum,

$$I_n = \int_{nT}^{(n+1)T}\mathbf{v}(t)\exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\}\mathrm{d}t\,.$$

By Eqns (19) and (20),

$$\mathbf{r}(t+nT) = \mathbf{r}(t) + n\mathbf{v}T\,, \qquad \mathbf{v}(t+nT) = \mathbf{v}(t)\,. \tag{24}$$

Therefore by a simple change of variables the selected integral can be reduced to

$$I_n = \exp\left[\mathrm{i}n(\omega - \mathbf{k}\mathbf{v})t\right]\int_0^T\mathbf{v}(t)\exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\}\mathrm{d}t\,.$$

Hence the integral in Eqn (21) can written in the form

$$\int_{-\infty}^{\infty} \mathbf{v}(t) \exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\} \mathrm{d}t$$
$$= \sum_n \exp\left[\mathrm{i}n(\omega - \mathbf{kv})T\right] \int_0^T \mathbf{v}(t) \exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\} \mathrm{d}t\,. \tag{25}$$

The sum of exponentials occurring in front of the integral on the right-hand side of this equation can be expressed in terms of delta function as follows:

$$\sum_n \exp\left[\mathrm{i}n(\omega - \mathbf{kv})T\right] = \sum_s \delta\left[(\omega - \mathbf{kv})T - 2\pi s\right], \tag{26}$$

where s is any integer.

Finally, the vector potential $\mathbf{A}$ for radiation from a particle in the field of an electromagnetic wave can be written

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{q}{2\pi c}\,\frac{\exp(\mathrm{i}kr)}{r} \sum_s \delta\left[(\omega - \mathbf{kv})T - 2\pi s\right]$$
$$\times \int_0^T \mathbf{v}(t) \exp\left\{\mathrm{i}\left[\omega t - \mathbf{k}\,\mathbf{r}(t)\right]\right\} \mathrm{d}t \tag{27}$$

implying that the particle emits only waves for which the arguments of the delta functions vanish, i.e.,

$$(\omega - \mathbf{kv})T - 2\pi s = 0\,. \tag{28}$$

Using Eqn (22) this can be rewritten

$$\omega = \frac{(2\pi/T)s}{1 - (v/c)\cos\theta}\,, \tag{29}$$

where θ is the angle between the velocity of particle displacement $\mathbf{v}$ and the direction of the radiation $\mathbf{n}$.

Equation (29) is a typical formula for the frequency of radiation emitted by a moving oscillator whose period is T but which is not harmonic. Therefore in the numerator of this formula, all the multiple frequencies can also occur along with the fundamental one $2\pi/T$. The denominator yields the Doppler shift in the frequency. The intensity of the radiation depends on the value of the integral over the period which enters the expression (27).

In our discussion, neither this secondary radiation nor its feedback effect on particle motion has been considered. These questions have been analyzed both in classical and quantum theories elsewhere [3].

3. Approximate methods

In many cases the motion of a particle in the field of a wave is conveniently described by perturbation theory, by expanding the unknown quantities in powers of a small parameter. This may even prove convenient when an exact solution is available — for example, for the case of particle motion in the field of a plane wave. If, however, a particle moves in a field composed of several plane waves — for example, in the field of a spatially non-uniform wave — for such a case perturbation theory yields a rather accurate picture of the motion.

The examples below illustrate the perturbation theory approach. Let us first rewrite Eqn (2) — the equation of motion for a particle in a wave field — in a more convenient form. As it stands, Eqn (2) is inconvenient in that its left-hand side contains the particle momentum $\mathbf{p}$, whereas the right-hand side contains its velocity $\mathbf{v}$. We can rewrite this equation is such a way that either side will contain only $\mathbf{v}$. For this let us use the equality obtained by multiplying both sides of Eqn (2) by the particle velocity $\mathbf{v}$:

$$\mathbf{v}\,\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\mathbf{Ev} = mc^2\,\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,. \tag{30}$$

Taking into account Eqn (30), the left-hand side of Eqn (3) can be written in the from

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\,\frac{\mathrm{d}\gamma\mathbf{v}}{\mathrm{d}t} = m\left(\gamma\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{e}{mc^2}\,\mathbf{v}(\mathbf{Ev})\right), \tag{31}$$

which, when substituted into Eqn (2), shows that the motion of the particle in a prescribed electromagnetic field is described by the equation

$$\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{\mathbf{E} + [\boldsymbol{\beta}\mathbf{H}] - \boldsymbol{\beta}(\boldsymbol{\beta}\mathbf{E})\right\}, \tag{32}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = [1 - (v/c)^2]^{-1/2}$ are the reduced velocity and reduced energy of the particle (in units of c and mc^2, respectively).

The vector equation (32) is equivalent to the following three equations for the three projections of the vector $\boldsymbol{\beta}$:

$$\frac{\mathrm{d}\beta_x}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{E_x + \beta_y H_z - \beta_z H_y - \beta_x(\beta_x E_x + \beta_y E_y + \beta_z E_z)\right\}, \tag{33}$$

$$\frac{\mathrm{d}\beta_y}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{E_y + \beta_z H_x - \beta_x H_z - \beta_y(\beta_x E_x + \beta_y E_y + \beta_z E_z)\right\}, \tag{34}$$

$$\frac{\mathrm{d}\beta_z}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{E_z + \beta_x H_y - \beta_y H_x - \beta_z(\beta_x E_x + \beta_y E_y + \beta_z E_z)\right\}. \tag{35}$$

Note that Eqns (2), (3), and (30)–(35) govern the motion of a charged particle in an arbitrary electromagnetic field. Below we will apply them to the case in which the field acting on the charge is that of an electromagnetic wave.

The system of equations (33)–(35) can be solved by the method of successive approximations. Let β_0 be the relative velocity of the particle at the initial time. For nonrelativistic particles β_0 is much less than unity, while for relativistic particles it is close to unity. In most practical cases the velocity a particle acquires in a wave field is much less than the speed of light. The relative magnitude of this velocity is of the order of

$$\eta = \frac{v_\sim}{c} = \beta_\sim = \frac{eE}{mc\omega}\,.$$

For the electron $\eta \simeq E\lambda \times 10^{-4}$ Oe cm. It can be seen that over a wide range of field strengths E and wavelengths λ, the parameter η is much less than unity. Using η as a small parameter, we represent the velocities and displacements of particles in series form as

$$\beta_x = \beta_x^{(0)} + \beta_x^{(1)} + \ldots\,, \qquad \beta_y = \beta_y^{(0)} + \beta_y^{(1)} + \ldots\,,$$
$$\beta_z = \beta_z^{(0)} + \beta_z^{(1)} + \ldots\,,$$
$$x = x^{(0)} + x^{(1)} + \ldots\,, \qquad y = y^{(0)} + y^{(1)} + \ldots\,,$$
$$z = z^{(0)} + z^{(1)} + \ldots\,,$$

where $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ and $\beta_x^{(0)}$, $\beta_y^{(0)}$, $\beta_z^{(0)}$ are the coordinates and velocities at the initial time.

The small parameter η has a simple physical meaning. Indeed, we may write this parameter in the form $\eta = eE\lambda/(2\pi mc^2)$. The expression in the numerator is of the order of the energy which the particle gains over the wavelength λ. The denominator contains the rest energy of the particle. Hence, the small value of η suggests that the energy gain over the wavelength is small compared to the particle's rest energy.

4. Motion in a plane wave

Let us consider the motion of a charged particle in the field of a plane, monochromatic electromagnetic wave traveling in the direction of the z axis. As before, we will assume that the electric field, the magnetic field, and the wave vector of the electromagnetic wave are along the x, y, and z axes, respectively. Suppose that at the initial time $t = 0$ the particle is at the point $x = y = z = 0$ and has a zero velocity. In the present case $\gamma = 1$. The fields in the uniform plane wave are given by the expressions

$$E_x = E\sin(\omega t - kz + \varphi)\,, \quad H_y = H\sin(\omega t - kz + \varphi)\,, \tag{36}$$

where φ is the phase of the field at the initial time.

Let us substitute the expression for E_x into the right-hand side of Eqn (33). Under the conditions of the problem we find that in the first approximation the motion along the x axis is described by the equation

$$\frac{\mathrm{d}\beta_x^{(1)}}{\mathrm{d}t} = \frac{e}{mc}\,E_x\,. \tag{37}$$

The solution of this equation has the form

$$\beta_x^{(1)} = -\left(\frac{eE}{mc\omega}\right)\cos(\omega t - kz + \varphi) + C\,, \tag{38}$$

where the constant C is determined from the initial conditions. Taking into account the initial conditions, we obtain

$$\beta_x^{(1)} = -\left(\frac{eE}{mc\omega}\right)\left[\cos(\omega t - kz + \varphi) - \cos\varphi\right]. \tag{39}$$

It can be seen that the second term in square brackets does not depend on time. This indicates that the particle not only oscillates at the frequency of the field but also performs a systematic directed motion (or drifts) along the vector of the wave's electric field. In the general case the velocity of the drift depends on the phase of the field at the initial time. If the drift velocity is denoted by $v_{x\mathrm{d}}$, it follows from Eqn (39) that

$$\beta_{x\mathrm{d}} = \left(\frac{eE}{mc\omega}\right)\cos\varphi\,. \tag{40}$$

It can be seen that systematic motion is absent only for $\cos\varphi = 0$, i.e., only for the specific values of the initial phase. For all other values of φ the drift velocity is different from zero, and a particle which was initially at rest travels in a systematic manner parallel to the electric field. Clearly, depending on the initial phase φ, the particle drifts either in the positive or in the negative x direction. If in the field of the wave there is a source continuously generating charged particles and if all values of the initial phase can occur with equal probability, then the φ-averaged drift velocity is zero, and there are two groups of particles drifting in opposite directions.

To describe the motion of the particle relative the z axis we substitute the velocity (39) into the second term on the right-hand side of Eqn (35). The remaining terms on the right are either zero or give higher-order contributions:

$$\frac{\mathrm{d}\beta_z}{\mathrm{d}t} = \frac{e}{mc}\left\{-\frac{1}{\omega}\left(\frac{eE}{mc}\right)\left[\cos(\omega t - kz + \varphi) - \cos\varphi\right]\right\} \times E\sin(\omega t - kz + \varphi)\,. \tag{41}$$

Here we have used the fact that $E = H$ in a plane wave. Integrating and taking into account the initial conditions we obtain a relation for the particle velocity along the z axis:

$$\beta_z = \frac{1}{4}\left(\frac{eE}{m\omega c}\right)^2\left[\cos 2(\omega t - kz + \varphi) - 4\cos\varphi\cos(\omega t - kz + \varphi) + 2 + \cos 2\varphi\right]. \tag{42}$$

From this equation it follows that, analogous to the motion along the x axis, the particle performs a systematic motion along the z axis. Averaging Eqn (42) over the period of the wave yields the particle drift velocity,

$$\beta_{z\mathrm{d}} = \frac{1}{4}\left(\frac{eE}{m\omega c}\right)^2[2 + \cos 2\varphi]\,. \tag{43}$$

The drift velocity is a second-order quantity and ranges from

$$\beta_{z\mathrm{d}} = \frac{1}{4}\left(\frac{eE}{mc\omega}\right)^2$$

for particles that appeared in the wave at phases $\varphi = (\pi/2) \pm \pi n$, to

$$\beta_{z\mathrm{d}} = \frac{3}{4}\left(\frac{eE}{mc\omega}\right)^2$$

for particles that appeared at phases $\varphi = \pm\pi n$. But, unlike the motion along x, the second-order drift velocity has the same sign for all particles, i.e., whatever the initial phase, the particles move in the same — wave propagation — direction.

Thus, if the initial velocity of the particle is zero, its interaction with a plane electromagnetic wave causes it not only to oscillate but also to execute a systematic drift along the electric field in the wave direction. The magnitude of the drift velocity in both directions depends on the initial phase. The velocity in the direction of the magnetic field remains zero.

Equations (40) and (43) give the same drift velocity values as Eqns (14) and (18) obtained from the exact solution. Note, however, that the perturbation theory formulas (40) and (43) hold only for small corrections to the initial velocities.

The distribution of charged particle in the field of a wave was investigated by numerically solving the equations of motion [5]. The results of these calculations are shown in Fig. 1. The calculations assumed that at the origin of the coordinate system there is a source which emits particles with negligibly small velocities. It may be said that particles

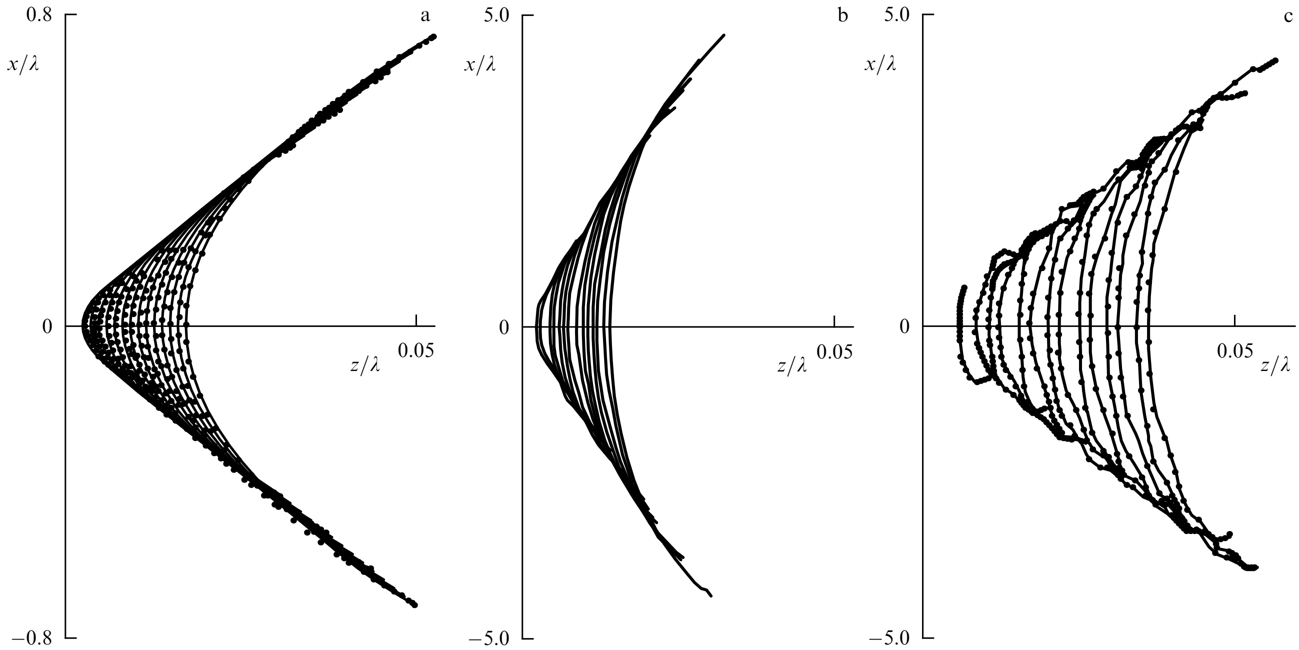

Figure 1. The arrangement of charged particles injected into a plane, linearly polarized electromagnetic wave. Particles start their motion with zero initial velocities at the origin of the coordinate system. The interval between the times successive particles start moving is 0.01 times the wave period: (a) $\eta = 0.1$; (b) $\eta = 1$; (c) $\eta = 2$.

emerging at the origin have zero velocity. The particles start moving due to the force of the wave's electric field and, once in motion, are also influenced by its magnetic field. The motion of the particles is determined both by the magnitude of the fields and by the initial phase, i.e., by the time at which the particle was injected into the wave.

In carrying out the calculations, the length of the wave period was subdivided into one hundred segments, and successive particles were made to start their motion at these one-hundredth-of-the-period intervals. The figures show the distribution of the particles injected over seven periods. The wave was assumed to travel in the z direction and the electric field was directed along the x axis. The figures show the spatial distribution of the particles, the abscissa and ordinate units being x/λ and z/λ, where λ is the wavelength of the wave, in which the particle moves. The plots in the figure are for different values of the parameter $\eta = eE\lambda/(2\pi mc^2)$. To the values $\eta = 0.1$, 1, and 2 there correspond Figs 1a, b, and c, respectively.

The case $\eta = 0.1$ can be calculated analytically using the successive approximation method described above. The figures show that as the parameter η increases (the field strength of the wave field increases), the drift in the direction of the wave plays an increasingly important role. As seen in Fig. 1a, for $\eta = 0.1$ particles move mainly normal to the wave direction (the scales of the x and z axes differ by a factor of 16 in Fig. 1a); in Fig. 1b, the drift along the z axis becomes comparable with that along x; and in Fig. 1c the drift along z is even more clearly evident.

Let us consider the effect of the initial velocity on particle dynamics in the field of a plane electromagnetic wave. Assume that at the initial time the particle has velocity $\beta_x = \beta_{0x}$ along the wave's electric field and velocity $\beta_y = \beta_{0y}$ in the direction coinciding with that of the wave's magnetic field. Under the assumed conditions, $\gamma = (1 - \beta_{0x}^2 - \beta_{0y}^2)^{-1/2}$. The equations describing the motion of the particle along all three axes have the form

$$\frac{d\beta_x^{(1)}}{dt} = \frac{e}{m\gamma c}\left\{(1 - \beta_{0x}^2)E_x\right\}, \tag{44}$$

$$\frac{d\beta_y^{(1)}}{dt} = \frac{e}{m\gamma c}\left\{-\beta_{0y}\beta_{0x}E_x\right\}, \tag{45}$$

$$\frac{d\beta_z^{(1)}}{dt} = \frac{e}{m\gamma c}\left\{\beta_{0x}H_y\right\}. \tag{46}$$

Let us substitute the expression for E_x into the right-hand sides of these equations. Upon integration, the following first-order expression for the particle velocities is obtained for the initial conditions adopted:

$$\beta_x = \beta_x^{(0)} + \beta_x^{(1)} = \beta_{0x} - (1 - \beta_{0x}^2)\left(\frac{eE}{m\gamma c\omega}\right) \times \left[\cos(\omega t - kz + \varphi) - \cos\varphi\right], \tag{47}$$

$$\beta_y = \beta_y^{(0)} + \beta_y^{(1)} = \beta_{0y} + \beta_{0y}\beta_{0x}\left(\frac{eE}{m\gamma c\omega}\right) \times \left[\cos(\omega t - kz + \varphi) - \cos\varphi\right], \tag{48}$$

$$\beta_z = \beta_z^{(0)} + \beta_z^{(1)} = -\beta_{0x}\left(\frac{eE}{m\gamma c\omega}\right) \times \left[\cos(\omega t - kz + \varphi) - \cos\varphi\right]. \tag{49}$$

Averaging Eqns (47)–(49) over the field period yields expressions for the particle drift velocities along the x, y, and z axes:

$$\beta_{xd} = (1 - \beta_{0x}^2)\left(\frac{eE}{m\gamma c\omega}\right)\cos\varphi, \tag{50}$$

$$\beta_{yd} = -\beta_{0y}\beta_{0x}\left(\frac{eE}{m\gamma c\omega}\right)\cos\varphi, \tag{51}$$

$$\beta_{zd} = \beta_{0x}\left(\frac{eE}{m\gamma c\omega}\right)\cos\varphi. \tag{52}$$

These equations show that the initial velocity of the particle considerably affects the character of its motion in a plane electromagnetic wave. First, the change in the particle velocity along the wave vector is a quantity of first — not second — order in η. Second, the velocity changes only with wave frequency, and there are no terms with the double frequency 2ω — unlike a particle which is at rest initially. And third, the velocity along the magnetic field — i.e., the non-Lorentz-force direction — is found to be modulated.

In the discussion above two cases have been considered. In the first, the particle was at rest at the initial time, while in the second its initial velocity was perpendicular to the wave direction. If the initial velocity is parallel to the wave vector, it is expedient to transform to a reference frame in which the particle is at rest. Then the field amplitudes and the wave frequency will be Lorentz transformed, and the problem will be reduced to one of a charge with zero initial velocity.

Whereas the above approximate results for a particle in a plane wave could be compared with an exact solution, for a particle in the field of a standing wave no exact solution exists. Below we treat this case using perturbation theory.

Suppose a standing wave is made up of two identical counter-propagating waves. In the first wave the electric field $E^{(1)}$ and the magnetic field $H^{(1)}$ are directed along the x and y axes, respectively, and the wave travels in the positive z direction. In the second wave $E^{(2)}$ and $H^{(2)}$ are also directed along x and y, respectively, but the wave travels in the negative z direction. The resulting field is a standing wave of the form

$$\begin{aligned} E_x &= E^{(1)} + E^{(2)} = E\sin(\omega t - kz + \varphi) \\ &\quad + E\sin(\omega t + kz + \varphi) = 2E\sin(\omega t + \varphi)\cos kz\,, \\ H_y &= H^{(1)} + H^{(2)} = H\sin(\omega t - kz + \varphi) \\ &\quad - H\sin(\omega t + kz + \varphi) = -2H\cos(\omega t + \varphi)\sin kz\,. \end{aligned} \tag{53}$$

For simplicity, we consider the case in which the phases of the waves are the same at the origin ($z = 0$). Let us substitute the expression for E into Eqn (37). As before, we seek the solution satisfying the initial condition $\beta_x = 0$ for $t = 0$. The solution is

$$\beta_x = \left(\frac{2eE}{mc\omega}\right)\big[\cos\varphi - \cos(\omega t + \varphi)\big]\cos kz\,. \tag{54}$$

From this solution it can be seen that, in a manner similar to the traveling wave case, a particle in a standing wave field performs a systematic drift motion in the electric field x-direction. But, unlike a traveling wave, the drift velocity depends not only on the phase of the wave but also on the coordinate z. If the particle is initially at a distance $\lambda/4$ from the origin (i.e., in the node of the standing wave), its drift velocity is zero. This is also the case for the particles at distances equal to multiples of $\lambda/2$.

To describe the motion of the particle relative to the axis z, substitute the velocity (54) into the second term on the right-hand side of Eqn (35), giving

$$\begin{aligned} \frac{\mathrm{d}\beta_z}{\mathrm{d}t} &= \frac{e}{mc}\,\beta_x H_y = -\frac{1}{2\omega}\left(\frac{2eE}{mc}\right)^2 \\ &\quad\times\big[\cos\varphi\cos(\omega t + \varphi) - \cos^2(\omega t + \varphi)\big]\sin 2kz\,. \end{aligned} \tag{55}$$

Averaging over time gives the time-average value of the particle acceleration at the point z,

$$\left\langle\frac{\mathrm{d}\beta_z}{\mathrm{d}t}\right\rangle = \frac{1}{\omega}\left(\frac{eE}{mc}\right)^2\sin 2kz\,. \tag{56}$$

From this relation it can be seen that in a standing wave field the particle is acted upon by a systematic force directed along the z axis. If we present this force as the gradient of the potential Φ,

$$\langle F_z\rangle = mc\left\langle\frac{\mathrm{d}\beta_z}{\mathrm{d}t}\right\rangle = -\mathrm{grad}\,\Phi\,, \tag{57}$$

the potential can be written

$$\Phi = \frac{e^2E^2}{m\omega^2}\cos^2 kz\,. \tag{58}$$

This force is directed toward lower amplitudes of the standing wave's electric field. Thus, particles group together at the nodes of the electric field — the behavior characteristic of particles in a high-frequency electromagnetic field with energy density non-uniform in space. The force that results from motion in such a field is sometimes called the Gaponov – Miller force [6]. Below one more example of this force — the situation of a non-uniform traveling wave — is discussed.

5. Non-uniform electromagnetic waves

So far we have discussed the motion of particles in the field of a plane, linearly polarized wave. The simplest non-uniform waves form in space when the fields of two plane, linearly polarized, identical-frequency waves, propagating at an angle with each other, are added together.

Let us derive expressions for these electromagnetic fields. Consider first waves with the electric field along the x axis and the magnetic field and the wave vector in the yz plane. We will assume that the propagation directions of both waves make an angle α with the axis z (Fig. 2a). The fields of the first wave are described by the equations

$$\begin{aligned} E_x^{(1)} &= E\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha)\,, \\ H_y^{(1)} &= E\cos\alpha\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha)\,, \\ H_z^{(1)} &= -E\sin\alpha\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha)\,, \end{aligned} \tag{59}$$

and those of the second, by the equations

$$\begin{aligned} E_x^{(2)} &= E\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha)\,, \\ H_y^{(2)} &= E\cos\alpha\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha)\,, \\ H_z^{(2)} &= E\sin\alpha\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha)\,. \end{aligned} \tag{60}$$

The components of the fields E_y, E_z, and H_x are zero. The combined field is given by the expressions

$$\begin{aligned} E_x^{(1+2)} &= 2E\cos(yk\sin\alpha)\sin(\omega t + \varphi - zk\cos\alpha)\,, \\ H_y^{(1+2)} &= 2E\cos\alpha\cos(yk\sin\alpha)\sin(\omega t + \varphi - zk\cos\alpha)\,, \\ H_z^{(1+2)} &= 2E\sin\alpha\sin(yk\sin\alpha)\cos(\omega t + \varphi - zk\cos\alpha)\,. \end{aligned} \tag{61}$$

Formulas (61) describe a wave traveling along the z axis. For example, the expression for $E_x^{(1+2)}$ contains a factor

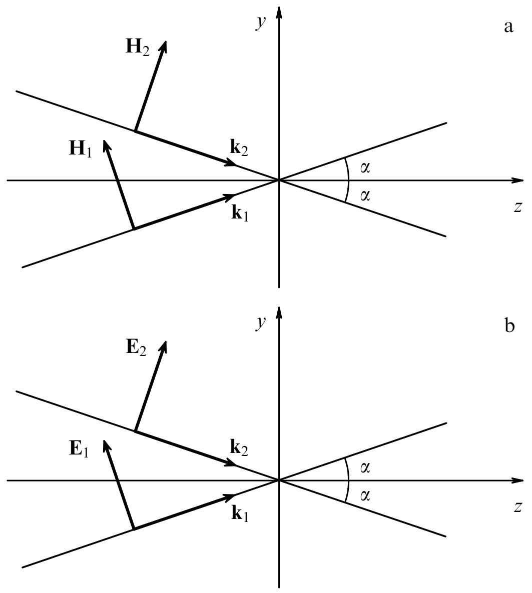

Figure 2. The orientation of the wave vectors of two plane waves forming one non-uniform wave. The wave vectors lie in the yz plane symmetrically about the z axis. (a) the electric field is perpendicular to the yz plane; (b) the magnetic field is perpendicular to the yz plane.

$\sin(\omega t + \varphi - zk\cos\alpha)$ characteristic of a traveling wave. The amplitude of this wave is $2E\cos(yk\sin\alpha)$, i.e., depends on the coordinate y transverse to the direction of the wave. The same may be said of the remaining wave components, described by Eqns (61). The dependence on the coordinate y means that the wave (61) is non-uniform.

Because the electric field is proportional to $\cos(yk\sin\alpha)$, it follows that at points satisfying the condition

$$yk\sin\alpha = \pi\left(\frac{1}{2} + n\right), \tag{62}$$

where $n = 0, \pm 1, \pm 2, \ldots$, the electric field of the wave is zero. The last relation is the equation of zero-electric-field planes. These are parallel to the xz plane and are at distances

$$y = \frac{\lambda}{2\sin\alpha}\left(\frac{1}{2} + n\right)$$

from it. Let us denote by a the spacing between two neighboring planes. Then $\sin\alpha$ and $\cos\alpha$ can be written in the respective forms

$$\sin\alpha = \frac{\lambda}{2a}, \qquad \cos\alpha = \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2},$$

which, when substituted into Eqns (61), give the final expressions for the electromagnetic field:

$$\begin{aligned}
E_x &= 2E\cos\left(\frac{\pi y}{a}\right)\sin(\omega t + \varphi - zk_z),\\
H_y &= 2E\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}\cos\left(\frac{\pi y}{a}\right)\sin(\omega t + \varphi - zk_z),\\
H_z &= 2E\left(\frac{\lambda}{2a}\right)\sin\left(\frac{\pi y}{a}\right)\cos(\omega t + \varphi - zk_z),\\
E_y &= E_z = H_x = 0,
\end{aligned} \tag{63}$$

where $\omega = 2\pi c/\lambda$, λ is the wavelength, $k_z = k\cos\alpha = k[1 - (\lambda/(2a))^2]^{1/2}$, and $k = 2\pi/\lambda$.

The expressions (63) describe a transverse electric wave because its field has its longitudinal electric component E_z equal to zero and because the magnetic field, along with a transverse component, has a longitudinal component H_z. Nowhere is the magnetic field of the wave zero. The wave travels in the z direction with a phase velocity

$$v_{\mathrm{ph}} = \frac{c}{\cos\alpha} = \frac{c}{\sqrt{1 - \left(\lambda/(2a)\right)^2}}. \tag{64}$$

As seen from Eqn (64), the phase velocity of the resulting wave in the z direction exceeds the speed of light. The wave's electric field has only one component, perpendicular to the propagation direction, and the field distribution is non-uniform along y and has a standing wave form. Between two neighboring zero-electric-field planes, the field of interest is identical to that of the TE wave in a plane waveguide.

A transverse magnetic wave is formed when two plane, uniform electromagnetic waves traveling in free space have their magnetic fields aligned in the same direction (Fig. 2b).

Suppose the fields of the waves are described by the expressions

$$\begin{aligned}
E_y^{(1)} &= E\cos\alpha\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha),\\
E_z^{(1)} &= -E\sin\alpha\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha),\\
H_x^{(1)} &= E\sin(\omega t + \varphi - yk\sin\alpha - zk\cos\alpha),
\end{aligned} \tag{65}$$

$$\begin{aligned}
E_y^{(2)} &= E\cos\alpha\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha),\\
E_z^{(2)} &= E\sin\alpha\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha),\\
H_x^{(2)} &= E\sin(\omega t + \varphi + yk\sin\alpha - zk\cos\alpha).
\end{aligned} \tag{66}$$

Then for the combined field we obtain

$$\begin{aligned}
E_y &= 2E\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}\cos\left(\frac{\pi y}{a}\right)\sin(\omega t + \varphi - zk_z),\\
E_z &= 2E\left(\frac{\lambda}{2a}\right)\sin\left(\frac{\pi y}{a}\right)\cos(\omega t + \varphi - zk_z),\\
H_x &= 2E\cos\left(\frac{\pi y}{a}\right)\sin(\omega t + \varphi - zk_z),\\
E_x &= H_y = H_z = 0.
\end{aligned} \tag{67}$$

From the last expressions it can be seen that the wave has, along with a transverse electric field component, a longitudinal component E_z. The transverse-magnetic wave (67), like the transverse-electric wave (63), is non-uniform in the y direction but, unlike it, has zero-magnetic-field (not zero-electric-field) planes. For both types of wave, the spacing between zero-field planes is a, i.e., the spatial period of transverse non-uniformity depends on the wavelength λ and on the direction of the plane uniform waves, i.e., on the angle α.

6. Motion in a transverse-electric field

Let us discuss the special features of particle dynamics in a non-homogeneous wave. For this purpose we will derive relations for the motion of particles in a transverse-electric wave whose fields are given by Eqns (63). It should be noted that this wave is non-uniform along only one of the transverse directions, namely along the y axis, and is uniform along the x axis.

We assume that the particle is at rest at the initial time. Then $\gamma = 1$, and in the first approximation the motion of the particle along the x axis is described by the equation

$$\frac{d\beta_x^{(1)}}{dt} = \frac{e}{mc} E_x \, . \tag{68}$$

Upon integrating, taking into account the initial conditions, we have

$$\beta_x^{(1)} = -\frac{2eE}{mc\omega} \cos\left(\frac{\pi y}{a}\right)\left[\cos\left(\omega t + \varphi - k_z z\right) - \cos\varphi\right] . \tag{69}$$

Substituting this into Eqns (34) and (35) we obtain equations describing the motion along the wave non-uniformity y-direction,

$$\frac{d\beta_y^{(2)}}{dt} = \frac{e}{mc}(-\beta_x H_z) = \frac{\lambda}{2a\omega}\left(\frac{2eE}{mc}\right)^2 \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$
$$\times \left[\cos\left(\omega t + \varphi - k_z z\right) - \cos\varphi\right] \cos\left(\omega t + \varphi - k_z z\right) , \tag{70}$$

and along the wave direction,

$$\frac{d\beta_z^{(2)}}{dt} = \frac{e}{mc}(\beta_x H_y) = -\frac{\chi}{\omega}\left(\frac{2eE}{mc}\right)^2 \cos^2\left(\frac{\pi y}{a}\right)$$
$$\times \left[\cos\left(\omega t + \varphi - k_z z\right) - \cos\varphi\right] \sin\left(\omega t + \varphi - k_z z\right) , \tag{71}$$

where

$$\chi = \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2} \, .$$

Because the left-hand sides of Eqns (70) and (71) contain accelerations, their right-hand sides are proportional to the corresponding forces. The equations suggest that the forces acting on the particle along the y and z axes are significantly different. The force along the direction of the wave, Eqn (71), is periodic and its time average is zero. The force along y has a constant term. Averaging Eqn (70) over time we obtain

$$\left\langle \frac{d\beta_y^{(2)}}{dt} \right\rangle = \frac{\lambda}{a\omega}\left(\frac{eE}{mc}\right)^2 \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right) , \tag{72}$$

from which it follows that the particle is acted upon by a force directed along y, and that the average value of this force differs from zero.

As shown in Ref. [6], a particle in a non-uniform high-frequency field

$$\mathbf{E}(x, y, z) \exp\left(i\omega t\right)$$

experiences a force proportional to the gradient of the square of the electrical field at the given point,

$$\mathbf{F} = -\frac{e^2}{4m\omega^2} \operatorname{grad} \left|E(x, y, z)\right|^2 . \tag{73}$$

The force $\mathbf{F}$, due to the non-uniform field, is sometimes called the Gaponov – Miller force or, because of its direction, the gradient force. In our case the acceleration $d\beta_y/dt$ is also caused by a certain force F_y, whose average value is found from Eqn (72) to be

$$F_y = mc\left\langle \frac{d\beta_y^{(2)}}{dt} \right\rangle = \frac{\lambda}{a\omega}\left(\frac{e^2E^2}{mc}\right) \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$
$$= -\frac{e^2}{4m\omega^2} \frac{\partial\langle E_x^2\rangle}{\partial y} \, . \tag{74}$$

It is easily seen that the force F_y given by Eqn (74) is identical to the gradient force given by Eqn (73).

Integrating Eqns (70) and (71) and taking into account the initial conditions we obtain the expression for the particle velocities along the y and z axes (recall that the electric field of the wave is along x in our case):

$$\beta_y^{(2)} = \frac{\lambda}{2a}\left(\frac{eE}{mc\omega}\right)^2 \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$
$$\times \left[2\omega t + \sin 2(\omega t + \varphi - k_z z)\right.$$
$$\left. - 4\cos\varphi \sin\left(\omega t + \varphi - k_z z\right) + \sin 2\varphi\right] , \tag{75}$$

$$\beta_z^{(2)} = \chi\left(\frac{eE}{mc\omega}\right)^2 \cos^2\left(\frac{\pi y}{a}\right)\left[\cos 2(\omega t + \varphi - k_z z)\right.$$
$$\left. - 4\cos\varphi \cos\left(\omega t + \varphi - k_z z\right) + 2 + \cos 2\varphi\right] . \tag{76}$$

Formula (75) for the y velocity of the particle contains a term which varies linearly in time. This term describes the acceleration the gradient force (74) imparts to the particle. The values of the coordinate y occurring in the argument of the sine and cosine in Eqns (75) and (76) should be considered close to the initial value y_0. In reality y varies as the particle moves, and so does the value of gradient force. Therefore the term linear in time in Eqn (75) is correct if the coordinate y of the moving particle changes by a small amount compared to a. Formula (75) suggests that motion along the y axis involves a drift along with the acceleration. The drift velocity is

$$\beta_{yd} = \frac{\lambda}{2a}\left(\frac{eE}{mc\omega}\right)^2 \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi y}{a}\right) \sin 2\varphi \, . \tag{77}$$

As seen from Eqn (76), drift also takes place along the z axis, i.e., in the direction of the wave vector. From Eqns (75) and (76) it can be seen that as a tends to infinity the particle's y velocity tends to zero, and the z velocity tends to the particle velocity in a uniform wave. Equations (75) and (76) also indicate that in a non-uniform wave, drift velocity depends not only on the initial phase of the wave but also on the initial coordinates of the particles. In a manner similar to the uniform wave case, particles drift both in positive and negative y directions, whereas the drift along the z axis occurs only in the positive direction, i.e., only in the wave direction.

Note here that the motion in the wave direction is drift, not a motion under the action of a certain averaged force. As Eqn (71) follows from, the averaged force along the z axis is zero to the second order. Recall that the particle was at rest at the initial time.

Let us now show that in a transverse-electric wave (63) a particle having a certain velocity at the initial time — with components both in the wave non-uniformity y-direction and along the electric field (the x axis) — will experience a nonzero average force in the direction of the wave vector.

Suppose at the initial time the particle has a velocity $\beta_x = \beta_{0x}$ along the wave electric field, and also has a velocity $\beta_y = \beta_{0y}$ whose direction coincides with the direction of the wave magnetic field. Under these conditions $\gamma = (1 - \beta_{0x}^2 - \beta_{0y}^2)^{-1/2}$. The equations of first order in η for the motion of the particle are

$$\frac{\mathrm{d}\beta_x^{(1)}}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{E_x + \beta_{0y}H_z - \beta_{0x}^2 E_x\right\}, \tag{78}$$

$$\frac{\mathrm{d}\beta_y^{(1)}}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{-\beta_{0x}H_z - \beta_{0y}\beta_{0x}E_x\right\}, \tag{79}$$

$$\frac{\mathrm{d}\beta_z^{(1)}}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{\beta_{0x}H_y\right\}. \tag{80}$$

Substituting the expressions for the fields of the transverse-electric wave, Eqn (63), into the right-hand sides of Eqns (78) and (80), we obtain

$$\frac{\mathrm{d}\beta_x^{(1)}}{\mathrm{d}t} = \left(\frac{2eE}{mc\gamma}\right)\left\{(1 - \beta_{0x}^2)\cos\Omega t \sin(\omega t + \varphi - k_z z) + \beta_{0y}\left(\frac{\lambda}{2a}\right)\sin\Omega t\cos(\omega t + \varphi - k_z z)\right\}, \tag{81}$$

$$\frac{\mathrm{d}\beta_z^{(1)}}{\mathrm{d}t} = \beta_{0x}\chi\left(\frac{2eE}{mc\gamma}\right)\cos\Omega t\sin(\omega t + \varphi - k_z z), \tag{82}$$

where $\Omega = \pi c\beta_{0y}/a$, with $y^{(0)} = c\beta_{0y}t$. Upon integrating, taking into account the initial conditions, we have

$$\begin{aligned}\beta_x^{(1)} = &-\left(\frac{2eE}{mc\gamma\omega}\right)\cos\Omega t\left[\cos(\omega t + \varphi - k_z z) - \cos\varphi\right] \\ &+ \beta_{0x}^2\left(\frac{2eE}{mc\gamma\omega}\right)\beta_{0y}\left(\frac{\lambda}{2a}\right) \\ &\times\left\{\cos\Omega t\left[\cos(\omega t + \varphi - k_z z) - \cos\varphi\right]\right. \\ &\left.+ \sin\Omega t\sin(\omega t + \varphi - k_z z)\right\},\end{aligned} \tag{83}$$

$$\begin{aligned}\beta_z^{(1)} = &-\beta_{0x}\chi\left(\frac{2eE}{mc\gamma\omega}\right)\left\{\cos\Omega t\left[\cos(\omega t + \varphi - k_z z) - \cos\varphi\right]\right. \\ &\left.+ \beta_{0y}\left(\frac{\lambda}{2a}\right)\sin\Omega t\left[\sin(\omega t + \varphi - k_z z) - \sin\varphi\right]\right\}.\end{aligned} \tag{84}$$

The equation of second order in η for particle motion along the wave vector has the form

$$\frac{\mathrm{d}\beta_z^{(2)}}{\mathrm{d}t} = \frac{e}{mc\gamma}\left\{\beta_x^{(1)}H_y - \beta_z^{(1)}\beta_{0x}E_x\right\}. \tag{85}$$

Let us substitute Eqns (83), (84), and (63) into Eqn (85). Averaging the result over the wave period yields the average acceleration along the z axis. After the averaging procedure, given that the electric field E_x and the magnetic field H_y are proportional to $\sin(\omega t + \varphi - k_z z)$, only terms proportional to $\sin^2(\omega t + \varphi - k_z z)$ will give a nonzero contribution. Such terms will result from multiplying the expression for H_y by the last term of Eqn (83) and the expression for E_x by the last term of Eqn (84). Taking the average yields for the average acceleration of the particle

$$\left\langle\frac{\mathrm{d}\beta_z^{(2)}}{\mathrm{d}t}\right\rangle = \frac{2c\pi\beta_{0y}\beta_{0x}^2\chi}{a}\left(\frac{eE}{mc\gamma\omega}\right)^2\sin 2\Omega t. \tag{86}$$

This equation shows that the ponderomotive force acting along the wave vector is proportional to the initial velocity of the particle in the direction of the wave field non-uniformity, β_{0y}, and to the square of the initial velocity in the electric field direction, β_{0x}. Therefore, in the case in which one of the components of the initial velocity (either β_{0x} or β_{0y}) is zero, the force is also zero in the second approximation. The ponderomotive force acts against the wave propagation direction when the particle moves towards the region of a strong field, and along the propagation direction for motion towards the weak field region. The force acts only in a non-uniform wave because its magnitude is inversely proportional to the distance a characterizing the spatial non-uniformity of the wave. When a tends to infinity, the force tends to zero.

In the case in which the initial velocity of the particle along the electric field is zero, the averaged force along the wave direction still exists, but represents a higher (fourth) order effect [7]. This force acts on the particle only if at the initial time the particle has a velocity along the wave non-uniformity direction (the y axis in our case).

The averaged ponderomotive force proportional to E^2 and acting in the direction of the wave results from the intersection of waves with a more complex spatial distribution of electromagnetic field — waves that are non-uniform in both transverse directions [8]. Also in waves of this type a force acts only on particles that traverse the wave, i.e., those with initial velocity perpendicular to the wave vector.

7. Conclusion

The fields E and H in the electromagnetic wave (1) are periodic, alternating sign fields, with their average values zero. One might expect that such fields should exert alternating-sign influence on a charged particle and that the resulting displacement should also average to zero. This is not the case, however. A particle in the field of a plane, uniform electromagnetic wave performs a systematic drift in the direction of the electric field as well as a drift in the wave propagation direction. In the field of a non-uniform wave additional forces appear which either push the particle out of the strong field region ('gradient forces') or act in the direction of the wave vector. These forces affect considerably the trajectory of a particle in an electromagnetic wave.

Acknowledgements
The authors are grateful to A I Nikishov and V I Ritus for their helpful suggestions and discussions.

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