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Zero-point oscillations, zero-point fluctuations, and fluctuations of zero-point oscillations

F Ya Khalili

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Contents

Abstract. Several physical effects and methodological issues relating to the ground state of an oscillator are considered. Even in the simplest case of an ideal lossless harmonic oscillator, its ground state exhibits properties that are unusual from the classical point of view. In particular, the mean value of the product of two non-negative observables, kinetic and potential energies, is negative in the ground state. It is shown that semiclassical and rigorous quantum approaches yield substantially different results for the ground state energy fluctuations of an oscillator with finite losses. The dependence of zero-point fluctuations on the boundary conditions is considered. Using this dependence, it is possible to transmit information without emitting electromagnetic quanta. Fluctuations of electromagnetic pressure of zero-point oscillations are analyzed, and the corresponding mechanical friction is considered. This friction can be viewed as the most fundamental mechanism limiting the quality factor of mechanical oscillators. Observation of these effects exceeds the possibilities of contemporary experimental physics but almost undoubtedly will be possible in the near future.

F Ya Khalili M V Lomonosov Moscow State University,
Physics Department,
Vorob'evy Gory, 119899 Moscow, Russian Federation
Tel. (7-095) 939 12 24. Fax (7-095) 932 88 20
E-mail: farid@hbar.phys.msu.su

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1. Introduction

These methodological notes are concerned with several physical effects and methodological issues pertaining to the ground state of a harmonic oscillator. Despite the fact that this topic is not new, it still attracts considerable interest and even raises debate. In particular, with the aid of the ground state, which can be regarded as the most 'quantum' of all oscillator states, it is possible to clearly demonstrate the hazard of applying semiclassical analysis techniques to quantum systems.

The terms 'zero-point oscillations' and 'zero-point fluctuations' are quite often used synonymously in the literature. It is obvious, however, that the *zero-point oscillations* of an isolated harmonic oscillator are not the same as the *zero-point fluctuations* of, for instance, a thermostat kept at zero temperature. The former possess some regularity. They can be interpreted as harmonic oscillations with a given frequency and a given amplitude, but with an unknown phase, which is constant in time. The latter are a stochastic process characterized by a spectral density that is continuous in frequency.

The sections of this paper are arranged in order of increasing complexity of the systems under review. The ground state of a one-dimensional harmonic oscillator is considered in Section 2. Despite its extreme simplicity, even this object can exhibit unusual and, from a classical viewpoint, paradoxical properties.

The ground-state energy fluctuations for an oscillator with a finite Q factor are considered in Section 3. This example shows clearly that uncritical recourse to semiclassical calculation techniques in problems involving the ground state may lead to qualitatively incorrect results. Also

discussed in Section 3 is the problem of the exact form of the noise spectral density for dissipative elements, which has recently been the subject of some controversy in the literature.

Section 4 is concerned with the dependence of zero-point oscillations in electromagnetic resonators on boundary conditions. In principle, by taking advantage of this dependence, it is possible to transfer information without the emission of photons. The term 'fluctuations of zero-point oscillations' applies to precisely this class of phenomena.

Section 5 is devoted to the fluctuations of zero-point oscillation pressure force and their attendant mechanical friction, which can be viewed as the most basic mechanism of Q-factor limitation in mechanical oscillators.

The Appendices primarily detail the most cumbersome calculations, which are of interest only from the standpoint of the calculation techniques involved. The sole exception is Appendix 7.1, which outlines the derivation of the formula for the noise of a dissipative element from the most general considerations, without invoking any assumptions of its internal arrangement.

The following notation and normalization are employed in this paper. To unify the form of the formulas for oscillators of a different nature, in lieu of mass m and rigidity K, inductance L and capacitance C, etc. it is expedient to use the eigenfrequency ω_0 and the wave impedance ρ_0, which are equal to

$$\omega_0 = \sqrt{\frac{K}{m}}, \qquad \rho_0 = \sqrt{Km}, \tag{1.1}$$

for a mechanical oscillator and to

$$\omega_0 = \frac{1}{\sqrt{LC}}, \qquad \rho_0 = \sqrt{\frac{L}{C}}, \tag{1.2}$$

for an LC circuit.

Time-dependent variables and their spectra are denoted by the same letters: for instance, $q(t)$ is a coordinate and $q(\omega)$ is its spectrum. This should not give rise to confusion, since the argument is always specified explicitly.

Advantage is taken of the same spectral density normalization as in the book in Ref. [1]. Under this normalization, the spectral density of some random process is

$$S(\omega) = \int_{-\infty}^{\infty} B(t) \exp(-\mathrm{i}\omega t)\, \mathrm{d}t, \tag{1.3}$$

where $B(t)$ is the correlation function of this process. This definition [1] gives a spectral density value two times lower than that commonly encountered in radiophysics. In particular, the spectral density of random voltage across the resistor R in this case is

$$S(\omega) = \hbar\omega R \coth\frac{\hbar\omega}{\kappa T}, \tag{1.4}$$

where κ is the Boltzmann constant and T is the temperature (the Callen – Welton formula [2]).

2. Ground state of an oscillator

The presence of some finite energy in the ground state of finite systems follows directly from the very foundations of quantum theory. Indeed, the absence of this energy would imply the exact definition of both the momentum of an object (equal to zero) and its coordinate (corresponding to the point of minimal potential energy). Clearly, this is impossible on the strength of the uncertainty relation. It is also evident that for any finite system there is bound to exist a quantum state $|0\rangle$ (at least one) corresponding to a minimum of the functional of the total energy of the system.

It is well known that the uncertainty relation allows us to estimate, to within a numerical factor, the ground-state energy of any finite quantum system (for instance, an atom). Furthermore, for a harmonic oscillator it is possible to find the exact energy value of the ground state with the use of the uncertainty relation owing to the simple symmetric structure of its Hamiltonian [3]. Indeed, the average energy of a harmonic oscillator is

$$\begin{aligned}\langle\mathcal{E}\rangle &= \frac{\omega_0}{2}\left(\frac{\langle p^2\rangle}{\rho_0} + \rho_0\langle q^2\rangle\right)\\ &= \frac{\omega_0}{2}\left\{\frac{\langle p\rangle^2 + (\Delta p)^2}{\rho_0} + \rho_0\left[\langle q\rangle^2 + (\Delta q)^2\right]\right\},\end{aligned} \tag{2.1}$$

where q and p are the coordinate and momentum of the oscillator. For the energy to be minimal it is evidently necessary that the average values of coordinate and momentum, $\langle q\rangle$ and $\langle p\rangle$, be equal to zero. In this case, the energy will be determined only by the uncertainties of these quantities Δq and Δp. Meanwhile, their possible values are limited by the Heisenberg inequality

$$\Delta q\, \Delta p \geqslant \frac{\hbar}{2}. \tag{2.2}$$

Consequently,

$$\langle\mathcal{E}\rangle \geqslant \frac{\omega_0}{2}\left[\frac{\hbar^2}{4\rho_0(\Delta q)^2} + \rho_0(\Delta q)^2\right]. \tag{2.3}$$

For the right-hand side of inequality (2.3), the minimum is attained for $\Delta q = q_0/\sqrt{2}$, where $q_0 = \sqrt{\hbar/\rho_0}$ is the so-called amplitude of zero-point oscillations, and is equal to $\hbar\omega_0/2$, as expected.

By solving the corresponding variational problem one can also easily show that the wave function that corresponds to the minimum in inequality (2.2) is Gaussian in the coordinate representation. Hence there follows an explicit expression for the wave function of the oscillator in the ground state:

$$\langle q|0\rangle = \frac{1}{(\sqrt{\pi}q_0)^{1/2}}\exp\left(-\frac{q^2}{2q_0^2}\right). \tag{2.4}$$

The wave function in the momentum representation, obtained from the wave function (2.4) by the Fourier transform, is also Gaussian:

$$\begin{aligned}\langle p|0\rangle &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\langle q|0\rangle\exp\left(-\frac{\mathrm{i}px}{\hbar}\right)\mathrm{d}q\\ &= \frac{1}{(\sqrt{\pi}\rho_0 q_0)^{1/2}}\exp\left(-\frac{p^2}{2(\rho_0 q_0)^2}\right).\end{aligned} \tag{2.5}$$

This is also true of any linear combinations of coordinate and momentum of the form

$$q(\theta) = q\cos\theta + \frac{p}{\rho_0}\sin\theta, \tag{2.6}$$

[1] Relation (1.3) is generally referred to as the Wiener – Khintchin theorem.

where θ is an arbitrary phase angle. Indeed, the observable $q(\theta)$ is obtained from q as a result of the free evolution of the oscillator for a time $t = \theta/\omega_0$; meanwhile, the ground state, as a state with a given energy, is stationary and its form is therefore invariable in the course of free evolution.

Hence there arises a temptation to ascribe to the ground state a combined probability distribution for the coordinate and the momentum

$$W(q,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{q^2+(p/\rho_0)^2}{q_0^2}\right), \tag{2.7}$$

the more so as this is precisely the form of the Wigner function [4] for the ground state, which is quite often used with the purpose of clearly illustrating different quantum states as a 'substitute' for the combined probability distribution for the coordinate and the momentum. The fallacy in this approach is evident if only because of the following fact: to the classical two-dimensional probability distribution (2.7) there corresponds an exponential probability distribution for the oscillator energy

$$W(\mathcal{E}) = \frac{1}{\hbar\omega_0/2}\exp\left(-\frac{\mathcal{E}}{\hbar\omega_0/2}\right). \tag{2.8}$$

The average energy value for this distribution is $\hbar\omega_0/2$, which is correct, but the variance is $(\hbar\omega_0/2)^2$, which is wrong, since the ground state is a state with a given energy.

While the total energy in the ground state is precisely specified, its components — the kinetic $\mathcal{E}_k$ and potential $\mathcal{E}_p$ energies individually — have nonzero variances in the ground state equal to

$$(\Delta\mathcal{E}_k)^2 = (\Delta\mathcal{E}_p)^2 = 2\left(\frac{\hbar\omega_0}{4}\right)^2. \tag{2.9}$$

This implies that their standard deviations $\Delta\mathcal{E}_k$ and $\Delta\mathcal{E}_p$ exceed their average values $\hbar\omega_0/4$, which is in itself quite unusual for suchlike immanently nonnegative observables.

For the total energy of the ground state to have zero variance in this case, the potential and kinetic energies should be fully anticorrelated: their correlation coefficient should be equal to the minimal possible value -1. A direct calculation shows that this is indeed the case.

Vorontsov and Rembovsky [5] noted that so strong an anticorrelation along with the high variance values (2.9) leads to a result paradoxical from the classical viewpoint: the cross-average value of the two nonnegative observables — the kinetic and potential oscillator energies — is negative in the ground state:

$$\langle\Lambda_2\rangle \equiv \frac{1}{2}\langle\hat{\mathcal{E}}_k\hat{\mathcal{E}}_p + \hat{\mathcal{E}}_p\hat{\mathcal{E}}_k\rangle = -\left(\frac{\hbar\omega_0}{4}\right)^2. \tag{2.10}$$

A note is appropriate at this point. If the coordinate and momentum of an oscillator are measured simultaneously and then squared to yield the calculated potential and kinetic energies, both resultant values will naturally prove to be positive. However, this measurement would fail to disprove formula (2.10), since the coordinate and momentum operators are noncommutative and therefore cannot be measured simultaneously with sufficient accuracy. Meanwhile, as indicated by Vorontsov and Rembovsky [5], the $\hat{\Lambda}_2$ operator itself cannot be represented as a square of some Hermitian operator to which there corresponds some observable that allows an exact measurement to yield, on being squared, the calculated value of Λ_2.

To verify formula (2.10) requires employing a measurement that yields information on the observable Λ_2 and in doing so yields no information on the observables that do not commute with it. Measurements of this kind are referred to as quantum nondemolition [6]. A condition sufficient for the realization of this procedure is the proportionality of the meter–object (oscillator) interaction Hamiltonian to the operator of the observable to be measured ($\hat{\Lambda}_2$ in this case).

3. Ground-state energy fluctuations in an oscillator with a finite Q factor

3.1 Semiclassical (*incorrect*!) calculation

When the oscillator Q factor is finite, the energy of its ground state is no longer an exactly specified value and fluctuates in time. These fluctuations in the quantum case, when the thermostat temperature T is close to zero, and in the classical case, when $\kappa T \gg \hbar\omega_0$, are essentially different in nature.

Meanwhile, a simple semiclassical analysis technique is quite often employed in radiophysics and optics. It involves the solution of a classical problem while the 'quantumness' is introduced by invoking the fluctuation-dissipative Callen–Welton theorem in lieu of the classical Nyquist formula to specify the noise spectral density. When $\kappa T \ll \hbar\omega_0$, this technique may lead to qualitatively incorrect results.

Let there be a harmonic oscillator with a finite Q factor (for specificity, we shall consider an ordinary LC circuit). We calculate the spectral density of energy fluctuations in this circuit at absolute zero temperature.

We first consider the semiclassical calculation technique. The equation of motion for the charge on the capacitor of the LC circuit is of the form

$$\ddot{q}(t) + 2\gamma\dot{q}(t) + \omega_0^2 q(t) = \frac{\omega_0}{\rho_0}U(t), \tag{3.1}$$

where $U(t)$ is the noise produced by the loss resistance R and

$$\gamma = \frac{\omega_0 R}{2\rho_0} \tag{3.2}$$

is the damping constant. The noise $U(t)$ spectral density is derived from formula (1.4) for $T = 0$:

$$S_0(\omega) = \hbar|\omega|R. \tag{3.3}$$

The solution of Eqn (3.1) can be represented as

$$q(t) = \frac{\omega_0}{\rho_0}\int_{-\infty}^{\infty}\frac{U(\omega)\exp(\mathrm{i}\omega t)}{\mathcal{L}(\omega)}\frac{\mathrm{d}\omega}{2\pi}, \tag{3.4}$$

where

$$U(\omega) = \int_{-\infty}^{\infty}U(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t \tag{3.5}$$

is the spectrum of the random process $U(t)$ and

$$\mathcal{L}(\omega) = -\omega^2 + 2\mathrm{i}\gamma\omega + \omega_0^2. \tag{3.6}$$

Accordingly, the LC-circuit energy is

$$\mathcal{E}(t) = \frac{\rho_0}{2\omega_0}\left(\dot{q}^2(t) + \omega_0^2 q^2(t)\right) = \frac{\omega_0}{2\rho_0}\int_{-\infty}^{\infty} \frac{\omega_0^2 - \omega\omega'}{\mathcal{L}(\omega)\mathcal{L}(\omega')}\, U(\omega)U(\omega') \times \exp\left[\mathrm{i}(\omega+\omega')t\right] \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\,. \tag{3.7}$$

The average value of this quantity is calculated with the aid of the equality

$$\left\langle U(\omega)U(\omega')\right\rangle = 2\pi\delta(\omega+\omega')S_{\mathrm{FDT}}(\omega)\,. \tag{3.8}$$

The average value is

$$\langle\mathcal{E}\rangle = \hbar\gamma\int_{-\infty}^{\infty} \frac{|\omega|(\omega_0^2+\omega^2)}{\left|\mathcal{L}(\omega)\right|^2}\frac{\mathrm{d}\omega}{2\pi}\,. \tag{3.9}$$

Moving ahead, we note that this is precisely the value of average energy given by rigorous quantum calculations. At the same time, it is not quite consistent with what is intuitively expected: the value of $\langle\mathcal{E}\rangle$ in formula (3.9) is not equal to $\hbar\omega_0/2$; moreover, the integral which appears in this formula diverges logarithmically at high frequencies. This problem will be treated in greater detail in Section 3.3.

To determine the spectral density of energy fluctuations, we first find the correlation function for these fluctuations,

$$B_{\mathcal{E}}(t) = \left\langle\mathcal{E}(0)\mathcal{E}(t)\right\rangle - \langle\mathcal{E}\rangle^2 = \left(\frac{\omega_0}{2\rho_0}\right)^2\int_{-\infty}^{\infty}\frac{(\omega_0^2-\omega\omega')(\omega_0^2-\omega_1\omega_1')}{\mathcal{L}(\omega)\mathcal{L}(\omega')\mathcal{L}(\omega_1)\mathcal{L}(\omega_1')} \times \left\langle U(\omega)U(\omega')U(\omega_1)U(\omega_1')\right\rangle \exp\left[\mathrm{i}(\omega_1+\omega_1')t\right] \times \frac{\mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}\omega_1\,\mathrm{d}\omega_1'}{(2\pi)^4} - \langle\mathcal{E}\rangle^2\,. \tag{3.10}$$

Taking into account that the random process $U(t)$ is normal, the forth-order cross average involved in this formula can be expressed in terms of second-order moments:

$$\left\langle U(\omega)U(\omega')U(\omega_1)U(\omega_1')\right\rangle = (2\pi)^2\delta(\omega+\omega')\,\delta(\omega_1+\omega_1')\,S_0(\omega)S_0(\omega_1) + (2\pi)^2\left(\delta(\omega+\omega_1)\,\delta(\omega'+\omega_1') + \delta(\omega+\omega_1')\,\delta(\omega'+\omega_1)\right)S_0(\omega)S_0(\omega')\,. \tag{3.11}$$

It then follows that

$$B_{\mathcal{E}}(t) = 2(\hbar\gamma)^2\int_{-\infty}^{\infty}\frac{|\omega\omega'|(\omega_0^2-\omega\omega')^2}{\left|\mathcal{L}(\omega)\right|^2\left|\mathcal{L}(\omega')\right|^2} \times \exp\left[\mathrm{i}(\omega_1+\omega')t\right]\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\,. \tag{3.12}$$

The spectral fluctuation density is now easily determined with the aid of formula (1.3):

$$S_{\mathcal{E}}(\Omega) = 2(\hbar\gamma)^2\int_{-\infty}^{\infty}\frac{\left|\omega(\Omega-\omega)\right|\left[\omega_0^2-\omega(\Omega-\omega)\right]^2}{\left|\mathcal{L}(\omega)\right|^2\left|\mathcal{L}(\Omega-\omega)\right|^2}\frac{\mathrm{d}\omega}{2\pi}\,. \tag{3.13}$$

This expression is significantly simplified when the observation frequency Ω is low in comparison with the circuit eigenfrequency ω_0. In this case, the integrand has two narrow peaks for ω values close to ω_0, which allows us to rewrite the last formula as

$$S_{\mathcal{E}}(\Omega) \approx \frac{(\hbar\gamma\omega_0)^2}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}v}{\left[v^2+\gamma^2\right]\left[(v-\Omega)^2+\gamma^2\right]} = \frac{(\hbar\omega_0)^2\gamma}{\Omega^2+(2\gamma)^2}\,. \tag{3.14}$$

We have obtained fluctuations whose main energy is concentrated near the zero frequency. The correlation time of these fluctuations is equal to the circuit relaxation time $\tau^* = 1/2\gamma$ and their full swing (the square root of the variance) is equal to the energy of zero-point oscillations:

$$\Delta\mathcal{E} = \sqrt{\int_{-\infty}^{\infty}S_{\mathcal{E}}(\Omega)\,\mathrm{d}\Omega} = \frac{\hbar\omega_0}{2}\,. \tag{3.15}$$

Let us recall that the calculation technique employed in this Section is inappropriate and the results arrived at are wrong. They are merely intended to serve as an illustration of the incorrectness of the semiclassical approach.

3.2 Quantum calculation

For linear quantum systems, the equations of motion for linear observables (coordinates, momenta, and their linear combinations) in the Heisenberg picture are known to be of precisely the same form as the corresponding classical equations of motion. That is why the operator of charge on the LC-circuit capacitor obeys the equation

$$\ddot{\hat{q}}(t) + 2\gamma\dot{\hat{q}}(t) + \omega_0^2\hat{q}(t) = \frac{\omega_0}{\rho_0}\,\hat{U}(t)\,, \tag{3.16}$$

which does not differ in form from the classical Eqn (3.1). In the quantum case, however, account should be taken of not only the correlation properties of the random process $U(t)$, but its operator properties as well, namely, the autocommutator $[\hat{U}(t), \hat{U}(t')]$ and its action on the ground state $\hat{U}(t)|0\rangle$. As shown in Appendix 7.1, in the quantum case the fluctuation electromotive force can be represented as

$$\hat{U}(t) = \int_0^{\infty}\sqrt{2\hbar\omega R}\left(\hat{a}(\omega)\exp\left(-\mathrm{i}\omega t\right) + \hat{a}^{\dagger}(\omega)\exp\left(\mathrm{i}\omega t\right)\right)\frac{\mathrm{d}\omega}{2\pi}\,, \tag{3.17}$$

where $\hat{a}(\omega)$ and $\hat{a}^{\dagger}(\omega)$ are the annihilation and creation operators, which satisfy the relations

$$\left[\hat{a}(\omega), \hat{a}^{\dagger}(\omega')\right] = 2\pi\delta(\omega-\omega') \tag{3.18}$$

and

$$\hat{a}(\omega)|0\rangle = 0\,. \tag{3.19}$$

The solution of Eqn (3.16) is of the form

$$\hat{q}(t) = \frac{\omega_0}{\rho_0}\int_0^{\infty}\sqrt{2\hbar\omega R} \times\left(\frac{\hat{a}(\omega)\exp\left(-\mathrm{i}\omega t\right)}{\mathcal{L}^*(\omega)} + \frac{\hat{a}^{\dagger}(\omega)\exp\left(\mathrm{i}\omega t\right)}{\mathcal{L}(\omega)}\right)\frac{\mathrm{d}\omega}{2\pi}\,, \tag{3.20}$$

and the LC-circuit energy is

$$\hat{\mathcal{E}}(t) = \frac{\rho_0}{2\omega_0}\left(\dot{\hat{q}}^2(t) + \omega_0^2\hat{q}^2(t)\right) = 2\hbar\gamma\int_0^\infty \sqrt{\omega\omega'} \times \left(\frac{(\omega_0^2 - \omega\omega')}{\mathcal{L}^*(\omega)\mathcal{L}^*(\omega')}\,\hat{a}(\omega)\hat{a}(\omega')\exp\left[-\mathrm{i}(\omega+\omega')t\right] + \frac{(\omega_0^2 + \omega\omega')}{\mathcal{L}^*(\omega)\mathcal{L}(\omega')}\,\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[\mathrm{i}(\omega-\omega')t\right]\right)\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} + \text{h.c.}, \tag{3.21}$$

where h.c. denotes a term which is a Hermitian conjugate to the previous one.

The average value and the spectral fluctuation density for observables of the (3.21) type are calculated in Appendix 7.2. From formulas (7.35) and (7.41) obtained in that Appendix it follows that the average value of energy in the circuit is

$$\langle\mathcal{E}\rangle = 2\hbar\gamma\int_0^\infty \frac{\omega(\omega_0^2+\omega^2)}{\left|\mathcal{L}(\omega)\right|^2}\,\frac{\mathrm{d}\omega}{2\pi}, \tag{3.22}$$

and the spectral density of energy fluctuations is

$$S_\mathcal{E}(\Omega) = (2\hbar\gamma)^2\int_0^{|\Omega|} \frac{\omega\left(|\Omega|-\omega\right)\left[\omega_0^2 - \omega\left(|\Omega|-\omega\right)\right]^2}{\left|\mathcal{L}(\omega)\right|^2\left|\mathcal{L}\left(|\Omega|-\omega\right)\right|^2}\,\frac{\mathrm{d}\omega}{2\pi}. \tag{3.23}$$

By comparing these formulas with formulas (3.9) and (3.13) obtained in Section 3.1, one can readily see that the expressions for the spectral densities are qualitatively different, whereas the expressions for the average values coincide, being different only in notation. The functions (3.13) and (3.23) for an oscillator with a quality factor $Q = 10$ are plotted in Fig. 1.

The difference is most striking at low frequencies, $\Omega \ll \omega_0$. For this frequency range, it is easy to derive an explicit expression for $S_\mathcal{E}(\Omega)$. Since the range of integration in formula (3.23) limits the values of the integration variable ω to the low-frequency range, $0 \leqslant \omega \leqslant \Omega \ll \omega_0$, it can be assumed that $\mathcal{L}(\omega) \approx \omega_0^2$, and

$$S_\mathcal{E}(\Omega) \approx \left(\frac{2\hbar\gamma}{\omega_0^2}\right)^2\int_0^{|\Omega|}\omega\left(|\Omega|-\omega\right)\frac{\mathrm{d}\omega}{2\pi} = \frac{2\hbar^2\gamma^2|\Omega|^3}{3\omega_0^4} = \frac{\hbar^2|\Omega|^3}{6Q^2\omega_0^2}, \tag{3.24}$$

where $Q = \omega_0/2\gamma$ is the circuit Q-factor. For $\Omega \simeq \gamma$, this quantity is approximately Q^6 times smaller than that obtained by the semiclassical calculation.

The reason for the disparity becomes evident when attention is given to the only difference in formulas (3.13) and (3.23) — the limits of integration. In formula (3.13), the main contribution to the integral for $\Omega \ll \omega_0$ is made by the values of frequency ω close to the eigenfrequency ω_0. The energy fluctuations in this case are produced primarily due to the beats of oscillations with frequencies close to ω_0, on the down-conversion principle: $\Omega = \omega - \omega'$. This mechanism does take place for thermal oscillations, when the thermostat temperature $T \gg \hbar\omega_0/\kappa$.

Meanwhile, the near-resonant frequencies in the quantum case in the integrand in formula (3.23) are cut off by the limits of integration. As a result, the energy fluctuations in the purely quantum case, when the thermostat temperature is equal to zero, are produced only due to the beats of low-frequency oscillations $\omega \ll \omega_0$, which are remote from the resonance and are therefore many times weaker, on the principle of frequency up-conversion: $\Omega = \omega + \omega'$. This accounts for the sharp decrease in spectral density at low frequencies.

This behavior of the spectral density at low frequencies can also be compared to the behavior of energy in an isolated oscillator. The energy of zero-point oscillations in an isolated oscillator is an exactly specified quantity equal to $\hbar\omega_0/2$; the low-frequency fluctuations of zero-point oscillation energy are nonzero, but are small and rapidly tend to zero with an increasing Q factor. On the other hand, in the classical treatment the uncertainty in equilibrium oscillator energy is equal to the average energy, which is in complete agreement with formula (3.15) obtained by the semiclassical technique.

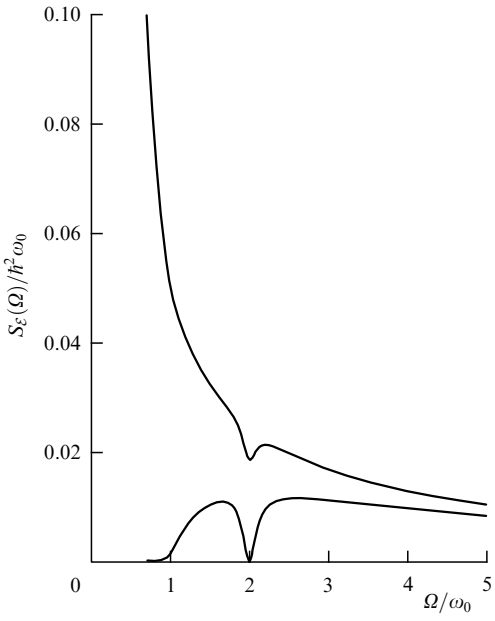

Figure 1. Spectral densities of energy fluctuations in an LC-circuit with a quality factor $Q = 10$ obtained by semiclassical (upper curve) and rigorous quantum (lower curve) calculations.

3.3 Average energy of zero-point oscillations in an oscillator with a finite Q factor

As already noted, the average energy value in an oscillator with dissipation is not equal to $\hbar\omega_0/2$ and, what is more, diverges. If we limit the integration in formula (3.22) by some cutoff frequency $\omega_{\max}$, we obtain the function $\mathcal{E}(\omega_{\max})$ plotted in Fig. 2. One can see that this function exhibits an 'almost' correct behavior, acquiring a value close to $\hbar\omega_0/2$ as its argument changes between $\omega_0 - \gamma$ and $\omega_0 + \gamma$. But then, for $\omega_{\max} \gg \omega_0$, it crosses the $\hbar\omega_0/2$ line to logarithmically tend to infinity.

To eliminate this divergence and restore the average energy to the 'correct' value $\hbar\omega_0/2$, Klimontovich [7–9] proposed that formula (1.4) should be modified by putting into it the spectral fluctuation density

$$S(\omega) = \hbar\omega_0 R\coth\frac{\hbar\omega_0}{2\kappa T}. \tag{3.25}$$

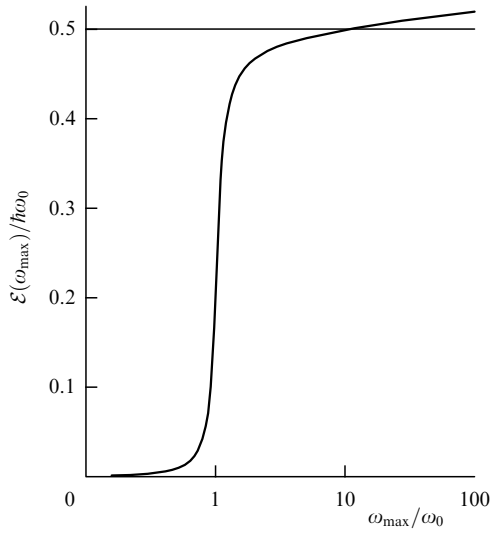

Figure 2. Average energy of zero-point oscillations in an oscillator with a finite Q factor as a function of the cutoff frequency of integration (the quality factor $Q = 10$).

However, formula (3.25) evidently violates the inequality (7.17) at high frequencies. Moreover, as fairly noted by Tatarskiĭ [10], the noise spectral density of a thermostat should not depend on any specific parameters of the system to which the thermostat is connected (see also the discussion in Ref. [11]).

To elucidate the reasons for the (apparent) disagreement of formula (3.22) with the statement that the energy of an oscillator (oscillation mode) in the ground state is exactly equal to $\hbar\omega_0/2$, we will enlarge on the notion of an oscillation mode.

In the rigorous mathematical sense, by modes are meant the eigen-solutions of the system of equations of motion for a given object. Therefore, a *mode* is a part of the compound system which (i) is one-dimensional, i.e., is characterized by a second-order equation of motion, and (ii) is totally independent of the remaining system. In this sense, the oscillator with the dissipation considered above is not a mode, since it is coupled to a thermostat. In this case we will resort to the term *degree of freedom*, using it in reference to a part of the compound system which can also be described by a second-order equation of motion, but which is conceivably coupled to the remaining system[2].

In this notation, the eigen-solutions for a closed electromagnetic resonator with *perfectly reflecting* mirrors are modes as well as degrees of freedom for the electromagnetic field inside it. Meanwhile, if the walls are made partly absorbent or transparent, these former modes become coupled to phonons in the resonator walls and/or to the electromagnetic degrees of freedom of the external space. The notion of *degree of freedom* applies to them as before, but the true *modes* of this system now penetrate into the walls and the external space.

From this viewpoint it comes as no surprise that the energy of an oscillator with a finite Q factor differs from $\hbar\omega_0/2$ and even diverges. This oscillator is the seat of localization for a part of the energy of all modes of the coupled 'oscillator + thermostat' system. For those modes whose frequency is close to the partial oscillator frequency ω_0, the bulk of their energy concentrates in the oscillator, while for the remaining modes it concentrates in the thermostat. Nevertheless, the 'tails' of high-frequency modes (with frequencies $\omega \gg \omega_0$) prove to be sufficient to ensure the divergence of oscillator energy. Appendix 7.3 is concerned with the problem of the eigenmodes of the 'oscillator + thermostat' system for a simple thermostat model (a long twin-wire line) and gives the derivation of the expression for the average energy in the oscillator, which coincides with formula (3.22).

[2] The author does not want to delve into a terminological debate as to what is a mode and what is a degree of freedom; at the same time two somehow different terms are required here. It is therefore suggested that we consider the above definitions as being 'locally' adopted for this section.

4. Dependence of zero-point oscillations on the boundary conditions

4.1 Simple example

Apart from the uncertainty of zero-point oscillation energy due to a *linear* coupling of the oscillator to the external world (the thermostat), which was considered in the previous section, there also exists another, *nonlinear* mechanism as a manifestation of this uncertainty. It stems from the fact that the energy of zero-point oscillations depends on the eigenfrequency of the oscillator, and this eigenfrequency is somewhat dependent on external conditions [12]. For instance, the frequency in a microwave cavity depends on its geometric parameters, which are the coordinates of some mechanical oscillators (the elastic modes of cavity walls). As a consequence, they always have their immanent uncertainties, both thermal and fundamentally quantum. This has the effect that the frequency of the resonator also becomes uncertain as does the energy of its zero-point oscillations. We emphasize that this effect can take place even in conservative systems — they need only consist of several nonlinearly coupled degrees of freedom.

Consider a simple example of a system of this sort. Let there be an LC circuit (Fig. 3) whose inductance and

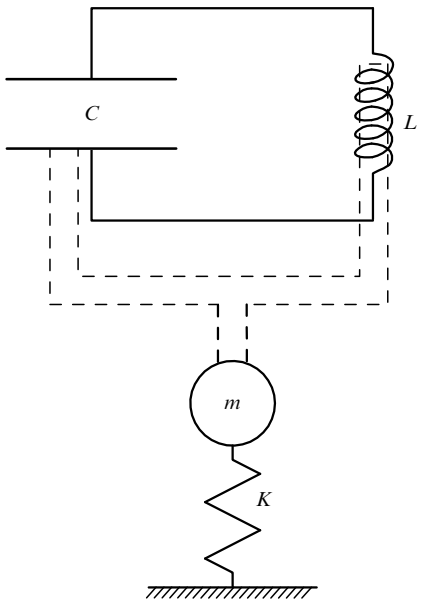

Figure 3. Electromagnetic resonator parametrically coupled to a mechanical oscillator.

capacitance depend on the coordinate x of a mechanical oscillator:

$$L = \frac{L_0}{1 - x/d}, \qquad C = \frac{C_0}{1 - x/d}, \tag{4.1}$$

where d is the coupling parameter. The eigenfrequency of this oscillator ω_m is assumed to be low in comparison with the LC-circuit frequency.

In this case, the eigenfrequency, and hence the ground-state energy of the LC circuit, will also depend on x:

$$\mathcal{E}_0^e = \frac{\hbar\omega_e}{2}\left(1 - \frac{x}{d}\right), \tag{4.2}$$

where $\omega_e = 1/\sqrt{L_0 C_0}$. If the mechanical oscillator is in the ground state, the uncertainty of its x coordinate is equal to $\sqrt{\hbar/2m\omega_m}$. In this case, the uncertainty of the ground-state energy of the LC circuit is

$$\Delta\mathcal{E}_0 = \frac{\hbar\omega_e}{2d}\sqrt{\frac{\hbar}{2m\omega_m}}. \tag{4.3}$$

At the same time, for a complete coupled system there is bound to exist a state with a minimal energy, and the total energy in this state should be exactly specified. The LC-circuit and mechanical oscillator energies should therefore be anticorrelated in this state, and the uncertainty of mechanical oscillator energy should also be defined by formula (4.3).

Appendix 7.4 solves the problem of states with a given energy of the system involved. From the results obtained in this Appendix it follows that the minimal energy is[3,4]

$$\mathcal{E}_{00} = \frac{\hbar\omega_e}{2} + \frac{\hbar\omega_m}{2} - \frac{m\omega_m^2 X_0^2}{2}, \tag{4.4}$$

and the state with the minimal energy is of the form

$$|\Psi_{00}\rangle = \hat{\mathcal{D}}(X_0)|0_m\rangle|0_e\rangle, \tag{4.5}$$

where $|0_e\rangle$ and $|0_m\rangle$ are the ground states of the LC circuit and the mechanical oscillator, respectively, $\hat{\mathcal{D}}$ is the x-coordinate displacement operator [see formula (7.62)], and

$$X_0 = \frac{\hbar\omega_e}{2m\omega_m^2 d} \tag{4.6}$$

is the shift of the equilibrium position of the mechanical oscillator under the attractive force induced by the zero-point oscillations in the LC circuit.

It is easily shown that the average energy values of the LC circuit and the mechanical oscillator in the state (4.5) are

$$\langle\mathcal{E}_0^e\rangle = \langle\Psi_{00}|\hbar\omega_e\left(\hat{n}_e + \frac{1}{2}\right)\left(1 - \frac{\hat{x}}{d}\right)|\Psi_{00}\rangle = \frac{\hbar\omega_e}{2}\left(1 - \frac{X_0}{d}\right), \tag{4.7}$$

$$\langle\mathcal{E}_0^m\rangle = \langle\Psi_{00}|\left(\frac{\hat{p}^2}{2m} + \frac{m\omega_m^2\hat{x}^2}{2}\right)|\Psi_{00}\rangle = \frac{\hbar\omega_m}{2} + \frac{m\omega_m^2 X_0^2}{2}, \tag{4.8}$$

while the uncertainties of these energies coincide and are given by formula (4.3), in agreement with our preliminary considerations.

4.2 Information transfer by means of modulation of zero-point oscillation energy density

Taking advantage of the dependence of zero-point oscillation energy density on the boundary conditions it is possible, at least in principle, to transfer information without the emission of electromagnetic quanta. The principle of the operation of this communication channel consists in the modulation of the boundary condition at one end of the transmission line and the measurement of its attendant variations of zero-point electromagnetic energy at the other end of the same line [12].

Let us imagine that a capacitor C with a plate separation d is connected to a transmission line of length $l = v\tau$ with a wave impedance ρ and a velocity of signal propagation v (Fig. 4). It is significant that the transmission band of the line should not be bounded below, i.e., that it may be, for instance, a coaxial cable or a twisted pair, but not a hollow waveguide.

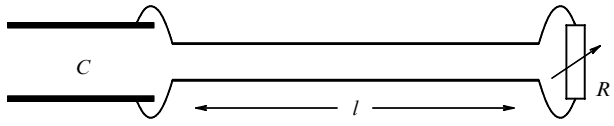

Figure 4. Information transfer scheme without the emission of photons.

The zero-point oscillations of the eigenmodes of the system produce the attractive force between the capacitor plates. The structure of these modes essentially depends on the form of the boundary condition at the distant (with respect to the capacitor) end of the line, and therefore by manipulating this boundary condition it is possible to modulate the attractive force.

When the distant end of the line is closed, the lowest eigenmode of the system corresponds to the LC circuit made up of the capacitor C and the distributed line inductance $\rho\tau$. The capacitor harbors a substantial part of the energy of this mode (about half the energy when the line is not too long). Meanwhile, the 'semiconcentrated' mode of this sort is missing when the line end is open. It would therefore appear reasonable that the attractive force would be strongest for a closed end and weakest for an open end.

Let us assume that the distant line end is connected to a variable resistance R, which, in particular, can be equal to zero or infinity. The fluctuation voltage across this resistance is [see expression (7.24)]

$$\hat{U}(t) = \int_0^\infty \sqrt{2\hbar\omega R}\,\hat{a}(\omega)\exp(-i\omega t)\,\frac{d\omega}{2\pi} + \text{h.c.} \tag{4.9}$$

Then, the fluctuation charge on the capacitor is

$$\hat{q}(t) = C\int_0^\infty \frac{\sqrt{2\hbar\omega R}\,\hat{a}(\omega)\exp(-i\omega t)}{z(\omega)}\,\frac{d\omega}{2\pi} + \text{h.c.}, \tag{4.10}$$

where

$$z(\omega) = \cos\omega\tau - \omega\tau_0\sin\omega\tau - i\,\frac{R}{\rho}\,(\sin\omega\tau + \omega\tau_0\cos\omega\tau) \tag{4.11}$$

[3] We do not consider the obviously unrealistic case wherein the last term in formula (4.4) exceeds the previous one and the capacitor plates 'collapse' under the attractive force produced by zero-point oscillations.

[4] An unfortunate misprint was made in Ref. [12]: the last term in the formula for the minimal energy appeared with a '+' sign instead of a '−' sign.

and $\tau_0 = \rho C$. Accordingly, the average attractive force between the capacitor plates is

$$F(R) = \frac{\langle 0|\hat{q}^2(t)|0\rangle}{2Cd} = \frac{\hbar RC}{d}\int_0^\infty \frac{\omega}{|z(\omega)|^2}\,\frac{d\omega}{2\pi}\,. \qquad (4.12)$$

This integral diverges logarithmically as $\omega \to \infty$, for the same reasons as the average energy of zero-point oscillations in the LC circuit (see Section 3.3). However, it is evident that the model of a concentrated oscillator and a one-dimensional line employed ceases to be true at sufficiently high frequencies, if for no other reason than that the metal it is made of ceases to be a good reflector. At the same time, we are concerned not with the magnitude of the force F, but with its dependence on the boundary conditions at the distant line end (of the magnitude R), i.e., with the difference $F(R) - F(R \to \infty)$. And this quantity proves to be finite.

The numerically calculated difference between the attractive forces for $R = 0$ and $R \to \infty$ is plotted in Fig. 5. One can see that its maximum value is attained for a capacitor capacitance $C = C_{\rm opt} \approx 2.4\tau/\rho$ and the magnitude of force modulation in this case is approximately $0.07\hbar/\tau d$. In addition, the function $F(R) - F(R \to \infty)$ is plotted in Fig. 6 for $C = C_{\rm opt}$.

This force is quite weak: for instance, for a velocity v equal to the velocity of light, $d = 10^{-4}$ cm, and $l = 10^2$ cm it is equal to 2×10^{-16} dyn. The ratio between this force and the Casimir force (5.1) per area d^2 for the same plate separation d is $\approx 1.7d/l \ll 1$. Nevertheless, the fact that the force (4.12) can be 'colored', i.e., can be transformed from a constant force to a force varying according to a given (for instance, harmonic) law by periodic modulation of resistance R, makes the task of its experimental detection not absolutely hopeless. The 2×10^{-16} dyn value corresponds, for instance, to the standard quantum limit [13] for the resonance force acting on a mechanical oscillator with a mass of 10^{-3} g and an eigenfrequency of 1 s^{-1} for a measurement time of 5 s. The same 'coloring' may enable us to single out the force (4.12) against the background of constant forces of another origin, for instance those caused by dissipation in the transmission line.

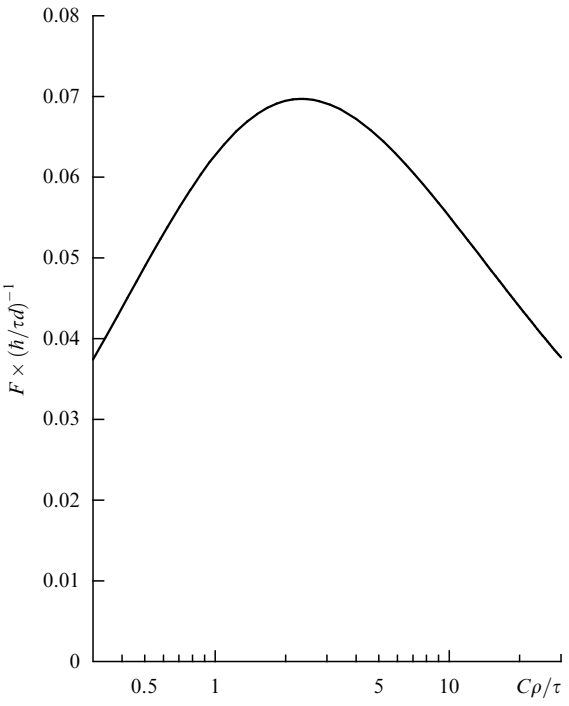

Figure 5. Attractive force between the plates of a capacitor connected to a transmission line as a function of capacitor capacitance.

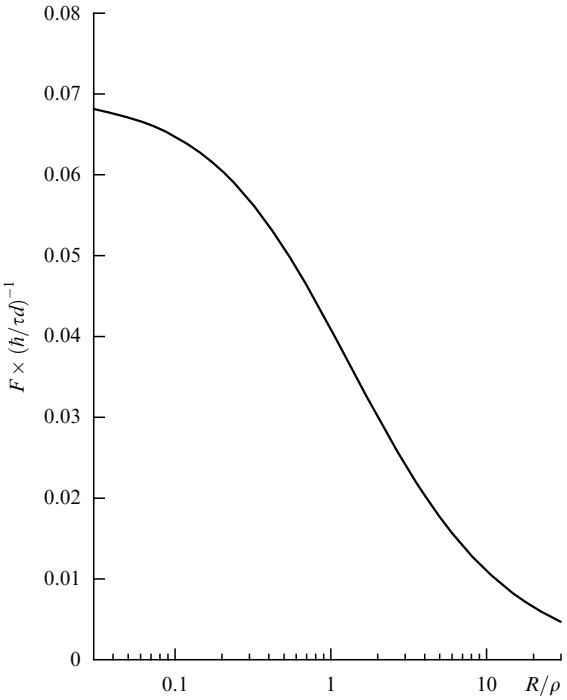

Figure 6. Attractive force between the plates of a capacitor connected to a transmission line as a function of the magnitude of resistance at the line end.

5. Fluctuations and friction produced by zero-point oscillations

5.1 Fluctuations and friction in free space

One of the best known manifestations of zero-point electromagnetic oscillations is the Casimir force [14]. Predicted more than half a century ago and discovered experimentally a few years later [15], it emerges for the same reason as the effects considered in the previous section — due to the dependence of the energy density of zero-point oscillations on the boundary conditions. In particular, the energy density between two metal plates of area $\mathcal{A}$ spaced at a distance d is lower than in free space due to low-frequency components with wavelengths $\lambda \sim d$. As a result there occurs an attractive force between the plates equal to

$$F = \frac{\pi^2\hbar c}{240 d^4}\,\mathcal{A}\,. \qquad (5.1)$$

This force can be regarded as the difference of the ponderomotive pressure forces of electromagnetic zero-point oscillations, which act on either plate from both sides and are much greater in magnitude than F but nearly cancel each other.

About 10 years ago G Barton noted that this compensation takes place only for the average values of pressure forces, while the fluctuation components of the pressure forces of zero-point oscillations, which act on the reflecting surface

from both sides, are independent and therefore do not cancel out. In Refs [16, 17] he solved the problem of the pressure $\hat{p}(t,\mathbf{r})$ which electromagnetic zero-point oscillations exert on an infinite perfectly reflecting surface and showed that the pressure fluctuations are described by the spatio-temporal correlation function of the form

$$B(t-t',\mathbf{r}-\mathbf{r}') \equiv \frac{1}{2}\left\langle \hat{p}(t,\mathbf{r})\,\hat{p}(t',\mathbf{r}') + \hat{p}(t',\mathbf{r}')\,\hat{p}(t,\mathbf{r})\right\rangle = \int_{-\infty}^{\infty} \Gamma(\Omega,\mathbf{K}) \cos\left[\Omega(t-t') - \mathbf{K}(\mathbf{r}-\mathbf{r}')\right] \frac{\mathrm{d}\Omega\,\mathrm{d}^2\mathbf{K}}{(2\pi)^3}\,, \quad (5.2)$$

where $\mathbf{r}$ and $\mathbf{r}'$ are two-dimensional radius vectors on the surface, $\mathbf{K}$ is a two-dimensional wave vector, and

$$\Gamma(\Omega,\mathbf{K}) = \begin{cases} \dfrac{\hbar^2}{60\pi^2 c^4}(\Omega^2 - c^2K^2)^{5/2}\,, & |\Omega| \geqslant cK, \\ 0\,, & |\Omega| < cK \end{cases} \quad (5.3)$$

is the spatio-temporal spectral density.

We consider a perfectly conducting thin plate with a dimension much greater than c/Ω, where Ω is the observation frequency. In this case it is possible to neglect the diffraction effects at the plate edges. Braginsky and Khalili [18] showed that each side of this plate will experience a random pressure force with a spectral density

$$S(\Omega) = \frac{\hbar^2|\Omega|^5}{120\pi^4 c^4}\,\mathcal{A}\,, \quad (5.4)$$

where $\mathcal{A}$ is the plate area. This fluctuation force transfers the energy of electromagnetic zero-point oscillations to the mechanical degrees of freedom. Clearly, there is bound to exist a reverse energy flux as well, otherwise it would be possible to accrue energy from zero-point oscillations. In other words, mechanical friction should be introduced into the motion of the surface plate. The underlying physical mechanism of this friction is evident: the parametric excitation of electromagnetic oscillations by the oscillations of the plate surface. While the frequency doubles in the excitation of mechanical oscillations by the electromagnetic ones (because the ponderomotive pressure force is proportional to the field intensity squared), in the inverse parametric process it, on the contrary, is halved. In both cases, the electromagnetic wave with a frequency ω proves to be coupled to mechanical oscillations with a 2ω frequency.

In accordance with the fluctuation-dissipative theorem, a friction

$$H(\Omega) = \frac{S(\Omega)}{\hbar|\Omega|} = \frac{\hbar\Omega^4\mathcal{A}}{120c^4} \quad (5.5)$$

corresponds to the spectral density (5.4).

It is significant that the magnitude of $H(\Omega)$ is proportional to Ω^4 and hence the friction force is proportional to the fifth derivative of the coordinate with respect to time. Consequently, the occurrence of this friction against 'free space' does not contradict the relativity principle, because for a constant velocity of motion the friction force is equal to zero.

This friction is quite small in magnitude. To show this, we estimate the Q factor of the lowest mode of elastic oscillations corresponding to this friction for a plate of thickness a. The frequency of this mode is $\Omega_\mathrm{m} = \pi v/a$, where v is the sound velocity, and the Q factor is

$$Q_\mathrm{m} = \frac{m\Omega_\mathrm{m}}{2H(\Omega_\mathrm{m})} = \frac{60\pi\rho c^4 a^4}{\hbar v^3}\,, \quad (5.6)$$

where ρ is the plate material density. For parameter values typical of a solid, Q factors of $10^{37}-10^{38}$ are obtained even for micrometer-thick plates. This effect can scarcely be detected employing currently available experimental techniques.

5.2 Fluctuations and friction under resonance conditions

As noted by Braginskii and Khalili [18], the fluctuation pressure force of zero-point oscillations and its attendant friction become much stronger under resonance conditions, for instance in an LC circuit weakly coupled to the surrounding electromagnetic vacuum.

The attractive force between the capacitor plates can be calculated using formula (3.20). It is equal to

$$\hat{F}(t) = \frac{\hat{q}^2(t)}{2Cd} = \frac{2\hbar\gamma\omega_0^2}{d}\int_0^\infty \sqrt{\omega\omega'}\left(\frac{\hat{a}(\omega)\hat{a}(\omega')\exp\left[-\mathrm{i}(\omega+\omega')t\right]}{\mathcal{L}^*(\omega)\mathcal{L}^*(\omega')} + \frac{\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[\mathrm{i}(\omega-\omega')t\right]}{\mathcal{L}^*(\omega)\mathcal{L}(\omega')}\right)\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} + \mathrm{h.c.}\,, \quad (5.7)$$

where d is the width of the capacitor gap.

In accordance with formula (7.35), the average value of attractive force is

$$\langle F\rangle = \frac{2\hbar\gamma\omega_0^2}{d}\int_0^\infty \frac{\omega}{\left|\mathcal{L}(\omega)\right|^2}\frac{\mathrm{d}\omega}{2\pi} = \frac{\hbar\omega_0}{4d}\,\frac{Q_\mathrm{e}}{\sqrt{4Q_\mathrm{e}^2-1}}\left(1+\frac{2}{\pi}\arctan\frac{Q_\mathrm{e}(2Q_\mathrm{e}^2-1)}{\sqrt{4Q_\mathrm{e}^2-1}}\right), \quad (5.8)$$

where Q_e is the Q factor of the circuit. For large Q_e this quantity tends rapidly to $\hbar\omega_0/4d$.

As follows from formula (7.41), the spectral density of the fluctuations of attractive force is

$$S_F(\Omega) = \left(\frac{2\hbar\gamma\omega_0^2}{d}\right)^2 \int_0^{|\Omega|} \frac{\omega\left(|\Omega|-\omega\right)}{\left|\mathcal{L}(\omega)\right|^2\left|\mathcal{L}\left(|\Omega|-\omega\right)\right|^2}\frac{\mathrm{d}\omega}{2\pi}\,. \quad (5.9)$$

This formula can be simplified if the observation frequency is close to twice the resonance frequency, $\Omega = 2(\omega_0+\Delta)$, $|\Delta| \ll \omega_0$, and the Q factor of the circuit is large, $\gamma \ll \omega_0$. In this case,

$$S_F(\Omega) \approx \left(\frac{\hbar\omega_0}{4d}\right)^2 \frac{\gamma}{\Delta^2+\gamma^2}\,. \quad (5.10)$$

To find the friction introduced by zero-point oscillations, it is certainly possible to take advantage of the fluctuation-dissipative theorem. However, for methodological considerations there is good reason to derive it directly, which will allow one to ascertain the validity of formula (7.27) in this specific case.

Let us assume that one of the capacitor plates is mobile and its displacement is specified by the x-coordinate, so that the total capacitor gap is $d+x$. In this case, the equation of

motion for the charge on the capacitor capacitance takes on the form

$$\ddot{\hat{q}}(t) + 2\gamma\dot{\hat{q}}(t) + \left(1 + \frac{\hat{x}(t)}{d}\right)\omega_0^2\,\hat{q}(t) = \frac{\omega_0}{\rho_0}\int_0^\infty \sqrt{2\hbar\omega R}\left(\hat{a}(\omega)\exp(-\mathrm{i}\omega t) + \hat{a}^\dagger(\omega)\exp(\mathrm{i}\omega t)\right)\frac{\mathrm{d}\omega}{2\pi} \tag{5.11}$$

(the values of parameters ω_0 and ρ_0 correspond to the initial value of the capacitor gap, for $x = 0$).

Since we are concerned with only that part of the solution which is linear in x (for small x values), the solution will be sought by the method of sequential approximations. To the zero approximation $\hat{q}_0$ (for $x = 0$) there corresponds formula (3.20). Its substitution into formula (5.11) yields the first-approximation equation:

$$\ddot{\hat{q}}_1(t) + 2\gamma\dot{\hat{q}}_1(t) + \omega_0^2\hat{q}_1(t) = -\frac{\hat{x}(t)}{d}\,\omega_0^2\,\hat{q}_0(t)\,.$$

The part of attractive force proportional to x is

$$\hat{F}_1(t) = -\frac{\omega_0\rho_0}{2d}\left[\hat{q}_0(t)\,\hat{q}_1(t) + \hat{q}_1(t)\,\hat{q}_0(t)\right]. \tag{5.13}$$

We solve Eqn (5.12), substitute its solution $\hat{q}_1(t)$ into formula (5.13), and average the resultant expression over the ground state of electromagnetic degrees of freedom (because our concern now is only with the regular part of the force) to represent it in the form

$$\left\langle 0\middle|\hat{F}_1(t)\middle|0\right\rangle = \int_{-\infty}^{\infty}\chi(\Omega)\hat{x}(\Omega)\exp(\mathrm{i}\omega t)\,\frac{\mathrm{d}\Omega}{2\pi}\,, \tag{5.14}$$

where $\hat{x}(\Omega)$ is the spectrum of $\hat{x}(t)$, and

$$\chi(\Omega) = \frac{2\hbar\gamma\omega_0^4}{d^2}\int_0^\infty \frac{\omega}{\left|\mathcal{L}(\omega)\right|^2}\left(\frac{1}{\mathcal{L}(\Omega-\omega)} + \frac{1}{\mathcal{L}(\Omega+\omega)}\right)\frac{\mathrm{d}\omega}{2\pi} \tag{5.15}$$

is the generalized susceptibility associated with the dynamic action of zero-point oscillations. By making a change in the integration variable, the imaginary part of this expression can be brought into the form

$$\operatorname{Im}\chi(\Omega) = -\frac{2\hbar\gamma^2\omega_0^4}{d^2}\int_0^{|\Omega|}\frac{\omega\left(|\Omega|-\omega\right)}{\left|\mathcal{L}(\omega)\right|^2\left|\mathcal{L}\left(|\Omega|-\omega\right)\right|^2}\,\frac{\mathrm{d}\omega}{2\pi}\,. \tag{5.16}$$

By comparing this formula with the formula for the spectral density of force fluctuations (5.9) it is easy to verify that the fluctuation-dissipative theorem is valid in this case:

$$S_F(\Omega) = \hbar\left|\operatorname{Im}\chi(\Omega)\right|. \tag{5.17}$$

We estimate the feasibility of experimentally recording the friction considered above. From the structure of formula (5.16) it follows that the friction factor has a sharp peak near the $2\omega_0$ frequency. When the circuit Q factor Q_e is high enough and the observation frequency Ω is close to $2\omega_0$, the friction is

$$\left|\chi(\Omega)\right| \approx \frac{\hbar\omega_0 Q_\mathrm{e}}{8d^2}\,. \tag{5.18}$$

Let us imagine that the mobile capacitor plate of mass m is a part of a mechanical oscillator with a frequency ω_m (Fig. 7).

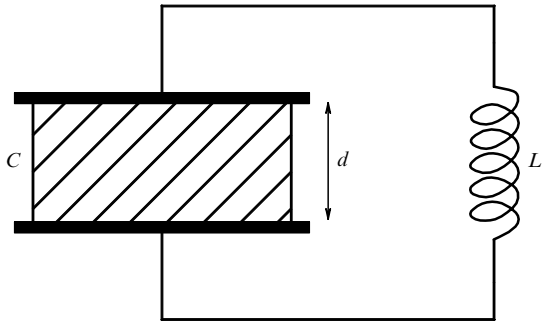

Figure 7. Possible scheme for the observation of friction introduced by zero-point electromagnetic oscillations.

Then, the friction discussed above will limit the Q factor of this oscillator by a value

$$Q_\mathrm{m} \approx \frac{m\omega_\mathrm{m}^2}{2\left|\chi(\Omega)\right|} \approx \frac{16m\omega_\mathrm{m}d^2}{\hbar Q_\mathrm{e}} \tag{5.19}$$

(this formula is valid for $Q_\mathrm{m} \gg Q_\mathrm{e}$).

Let, for instance, a dielectric plate of thickness d whose frequency of lowest-mode transverse mechanical oscillations $\pi v/d$ is equal to $2\omega_0$ be placed between the oscillator plates (see Fig. 7). In this case, formula (5.19) is rearranged to give

$$Q_\mathrm{m} \approx \frac{4\pi^2 YV}{\hbar\omega_0 Q_\mathrm{e}}\,, \tag{5.20}$$

where Y is the Young modulus of the plate material and V is the plate volume. If $d = 10^{-4}$ cm, $V = 10^{-7}$ cm^3, and $\omega_0 = 10^{10}$ s^{-1}, then

$$Q_\mathrm{m}Q_\mathrm{e} \approx 10^{24}\,. \tag{5.21}$$

It is worth noting that Q factors of the order of 10^{11} have already been attained for electromagnetic resonators of the microwave range [19] and those exceeding 10^8 for mechanical resonators [20].

6. Conclusions

These methodological notes have by no means touched upon all the effects caused by zero-point oscillations. The effects related to the Casimir force were considered in detail in review Ref. [21]. Also given therein is an extensive bibliography on this topic.

Among the effects caused by the energy fluctuations of zero-point oscillations, additional mention can be made of the fluctuations and friction introduced by the electromagnetic vacuum into charged particle motion, which were considered by Braginskiĭ, Khalili, and Sazhin [22]. They pointed out an intriguing circumstance: to detect this friction requires expending energy of the order of $\hbar/\tau$, where τ is the measurement time, which will be radiated into the vacuum in the form of one or several photons with a frequency of the order of τ^{-1}.

As regards the experimental observation of the effects discussed above, for the majority of them it should be remarked that such experiments nowadays are hardly feasible or are too expensive. However, they cannot be treated as hopelessly difficult, either, and in view of the

rapid progress of experimental physics seen during the last few decades (see, for instance, a review Ref. [23]) their implementation will most likely become possible in the foreseeable future.

The author expresses his appreciation to V B Braginskiĭ and S P Vyatchanin for helpful discussions on this paper.

7. Appendices

7.1 Operator properties of thermostat noise

7.1.1 Arbitrary linear two-pole system. A thermostat, irrespective of how complex its internal structure is, is a linear two-pole system (as is, for instance, an ordinary resistor) from the viewpoint of an 'external interface'. This system is characterized by a linear relation between an external action $q(t)$ and the response to it $\hat{\mathcal{U}}(t)$:

$$\hat{\mathcal{U}}(t) = \hat{U}(t) + \int_{-\infty}^{t} \chi(t,t')q(t')\,\mathrm{d}t'\,, \tag{7.1}$$

where $\chi(t,t')$ is the generalized susceptibility function and $\hat{U}(t)$ is the value of $\hat{\mathcal{U}}(t)$ in the absence of external action, i.e., the intrinsic fluctuations of the system.

On the other hand, the system under our consideration can be characterized by a Hamiltonian of the form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - q(t)\hat{\mathcal{U}}\,, \tag{7.2}$$

where $\hat{\mathcal{H}}_0$ is the unperturbed Hamiltonian. Proceeding from this Hamiltonian, the solution of the equation of motion for the $\hat{\mathcal{U}}(t)$ operator in the Heisenberg picture of evolution can be represented as a perturbation theory series:

$$\hat{\mathcal{U}}(t) = \hat{U}(t) - \frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{d}t'\,\big[\hat{U}(t),\hat{U}(t')\big]\,q(t') + \frac{1}{(\mathrm{i}\hbar)^2}\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}t''\big[\big[\hat{U}(t),\hat{U}(t')\big],\hat{U}(t'')\big]q(t')q(t'') + \ldots \tag{7.3}$$

A comparison of formulas (7.3) and (7.1) shows readily that the $\hat{U}(t)$ operator should be secondary autocommutative:

$$\big[\big[\hat{U}(t),\hat{U}(t')\big],\hat{U}(t'')\big] \equiv 0 \qquad \forall\ \ t,t',t''\,, \tag{7.4}$$

thereby terminating the perturbation theory series with the second term. It also follows from formulas (7.3) and (7.1) that the generalized susceptibility is expressed in terms of the autocommutator of $\hat{U}(t)$ [1]:

$$\chi(t,t') = \begin{cases} \dfrac{\mathrm{i}}{\hbar}\,\big[\hat{U}(t)\,,\hat{U}(t')\big]\,, & t \geqslant t'\,, \\ 0\,, & t < t' \text{ (due to the causality principle)}\,. \end{cases} \tag{7.5}$$

This equality can also be written as

$$\big[\hat{U}(t),\hat{U}(t')\big] = \begin{cases} -\mathrm{i}\hbar\chi(t,t')\,, & t \geqslant t'\,, \\ \mathrm{i}\hbar\chi(t',t)\,, & t \leqslant t'\,, \end{cases} \tag{7.6}$$

or

$$\big[\hat{U}(t),\hat{U}(t')\big] = \mathrm{i}\hbar\big(\chi(t',t) - \chi(t,t')\big)\,. \tag{7.7}$$

7.1.2 Stationary linear two-pole system. In the subsequent discussion we restrict our consideration to those systems which are stationary as regards both dynamic properties and noise, i.e., those whose generalized susceptibility $\chi(t,t')$ as well as correlation function for the noise $\hat{U}(t)$,

$$B(t,t') = \frac{1}{2}\,\big\langle \hat{U}(t)\hat{U}(t') + \hat{U}(t')\hat{U}(t)\big\rangle\,, \tag{7.8}$$

depend explicitly only on the difference between the points in time involved, t and t'. It is evident that a thermostat satisfies this requirement.

For stationary systems it would be reasonable to take advantage of the spectral representation, wherein

$$\chi(t,t') = \int_{-\infty}^{\infty} \chi(\omega)\exp\big[\mathrm{i}\omega(t-t')\big]\,\frac{\mathrm{d}\omega}{2\pi}\,, \tag{7.9}$$

$$B(t,t') = \int_{-\infty}^{\infty} S(\omega)\exp\big[\mathrm{i}\omega(t-t')\big]\,\frac{\mathrm{d}\omega}{2\pi}\,, \tag{7.10}$$

where $S(\omega)$ is the spectral density of the noise $\hat{U}(t)$. Formula (7.1) in the spectral representation takes the form

$$\hat{\mathcal{U}}(\omega) = \hat{U}(\omega) + \chi(\omega)q(\omega)\,. \tag{7.11}$$

Below we will also need the expressions for the autocommutator of the $\hat{U}(\omega)$ operator,

$$\begin{aligned} &\big[\hat{U}^{\dagger}(\omega),\hat{U}(\omega')\big] \\ &\quad= \int_{-\infty}^{\infty}\big[\hat{U}(t),\hat{U}(t')\big]\exp\big[\mathrm{i}(\omega t - \omega' t')\big]\,\mathrm{d}t\,\mathrm{d}t' \\ &\quad= 2\pi\mathrm{i}\hbar\big(\chi(\omega) - \chi(-\omega)\big)\,\delta(\omega-\omega') \\ &\quad= -4\pi\,\mathrm{Im}\,\chi(\omega)\,\delta(\omega-\omega')\,, \end{aligned} \tag{7.12}$$

as well as for the autocorrelator of this operator,

$$\begin{aligned} &\frac{1}{2}\big\langle\hat{U}^{\dagger}(\omega)\hat{U}(\omega') + \hat{U}(\omega')\hat{U}^{\dagger}(\omega)\big\rangle \\ &\quad= \int_{-\infty}^{\infty} B(t,t')\exp\big[\mathrm{i}(\omega t - \omega' t')\big]\,\mathrm{d}t\,\mathrm{d}t' \\ &\quad= 2\pi S(\omega)\,\delta(\omega-\omega')\,. \end{aligned} \tag{7.13}$$

Let us consider an operator of the form

$$\hat{\mathcal{Q}} = \int_{-\infty}^{\infty} Q(\omega)\hat{U}(\omega)\,\frac{\mathrm{d}\omega}{2\pi}\,, \tag{7.14}$$

where $Q(\omega)$ is some function. We calculate the average value of the operator $\hat{\mathcal{Q}}^{\dagger}\hat{\mathcal{Q}}$:

$$\begin{aligned} \langle\hat{\mathcal{Q}}^{\dagger}\hat{\mathcal{Q}}\rangle &= \int_{-\infty}^{\infty} Q^{*}(\omega)Q(\omega')\big\langle\hat{U}^{\dagger}(\omega)\hat{U}(\omega')\big\rangle\,\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} Q^{*}(\omega)Q(\omega')\big\langle\hat{U}^{\dagger}(\omega)\hat{U}(\omega') \\ &\quad+ \hat{U}(\omega')\hat{U}^{\dagger}(\omega)\big\rangle\,\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} \\ &\quad+ \frac{1}{2}\int_{-\infty}^{\infty} Q^{*}(\omega)Q(\omega')\big[\hat{U}^{\dagger}(\omega),\hat{U}(\omega')\big]\,\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} \\ &= \int_{-\infty}^{\infty}\big|Q(\omega)\big|^2\big(S(\omega) - \hbar\,\mathrm{Im}\,\chi(\omega)\big)\,\frac{\mathrm{d}\omega}{2\pi}\,. \end{aligned} \tag{7.15}$$

The $\hat{\mathcal{Q}}^\dagger\hat{\mathcal{Q}}$ operator is a positive semidefinite, and therefore its average value cannot be lower than zero. Hence it follows, in view of the arbitrariness of the $Q(\omega)$ function, that the inequality

$$S(\omega) \geqslant \hbar \operatorname{Im}\chi(\omega) \tag{7.16}$$

should be fulfilled for arbitrary values of ω.

It is easily seen that this inequality can be strengthened: its left-hand side is a symmetric function of frequency and its right-hand side an antisymmetric one, whence it follows that it may be rewritten as

$$S(\omega) \geqslant \hbar\left|\operatorname{Im}\chi(\omega)\right| \tag{7.17}$$

(this formula is a special case of the relations obtained in Ref. [13] for arbitrary quantum 2N-pole systems).

Formula (7.17) bears strong resemblance to the fluctuation-dissipative theorem. The main difference is that the latter implies a state of thermal equilibrium for the system involved. As for relation (7.17), it was obtained under much more general assumptions and is valid for any quantum state of the system. The 'price' to be paid for this generality is that this relation is in the form of an inequality.

7.1.3 Thermostat in the ground state. We now assume that the system under discussion is in the ground state, i.e., its energy is as low as permitted by the uncertainty relation. In this case, the fluctuations at its 'external interface' are also minimal, and therefore inequality (7.16) transforms into an equality either for positive or for negative values of frequency ω, depending on the sign of the imaginary part of $\chi(\omega)$. The sign of the imaginary part of $\chi(\omega)$ in its turn is determined from the condition that the system under consideration should introduce positive dissipation.

We complement the thermostat Hamiltonian (7.2) with the Hamiltonian of the probing oscillator connected to it, for which q is a generalized coordinate:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - \hat{q}(t)\hat{\mathcal{U}} + \hat{\mathcal{H}}_{\text{osc}}\,. \tag{7.18}$$

In this case, the equation of motion for the $\hat{q}(t)$ operator is of the form

$$\ddot{\hat{q}}(t) + \omega_0^2\hat{q}(t) = \frac{\omega_0}{\rho_0}\,\hat{\mathcal{U}}(t)\,. \tag{7.19}$$

We go over to the spectral representation and combine the last equation with relation (7.11) to obtain

$$\left(-\omega^2 - \frac{\omega_0}{\rho_0}\,\chi(\omega) + \omega_0^2\right)\hat{q}(\omega) = \frac{\omega_0}{\rho_0}\,\hat{U}(\omega)\,. \tag{7.20}$$

The imaginary part of $\chi(\omega)$, which is proportional to $\mathrm{i}\omega$, describes energy dissipation [because the expression $\mathrm{i}\omega q(\omega)$ crosses over into $\dot{q}(t)$ in the temporal representation]. The reason why a dissipative term appears in the equation of motion for a purely Hamiltonian 'oscillator + thermostat' system has been repeatedly discussed in the literature (see, for instance, Ref. [10] or the book in Ref. [24]), and this subject therefore will not be discussed here.

In order for the dissipation to be positive, the imaginary part of $\chi(\omega)$ should be nonpositive for $\omega \geqslant 0$ and nonnegative for $\omega \leqslant 0$. Hence it follows, when the thermostat is in the ground state, that

$$S(\omega) - \hbar \operatorname{Im}\chi(\omega) = \begin{cases} 2\hbar\left|\operatorname{Im}\chi(\omega)\right|, & \omega \geqslant 0\,, \\ 0\,, & \omega \leqslant 0\,. \end{cases} \tag{7.21}$$

We revert to formula (7.15) and put $Q(\omega) = \delta(\omega - \omega_1)$ into it, where ω_1 is some arbitrary *negative* frequency. From relation (7.21) it follows that in this case

$$\langle 0|\hat{\mathcal{Q}}^\dagger\hat{\mathcal{Q}}|0\rangle = 0\,, \tag{7.22}$$

i.e., $\hat{\mathcal{Q}}|0\rangle$ is a vector of zero length. In other words, for any $\omega < 0$ we have

$$\hat{U}(\omega)|0\rangle = 0\,. \tag{7.23}$$

Hence it follows, in view of formula (7.12), that the noise $\hat{U}(t)$ can be represented as

$$\hat{U}(t) = \int_0^\infty \sqrt{2\hbar\left|\chi(\omega)\right|}\,\left(\hat{a}(\omega)\exp\left(-\mathrm{i}\omega t\right)\right.$$
$$\left. + \hat{a}^\dagger(\omega)\exp\left(\mathrm{i}\omega t\right)\right)\frac{\mathrm{d}\omega}{2\pi}\,, \tag{7.24}$$

where $a(\omega)$ and $a^\dagger(\omega)$ are some operators for which

$$\left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = 2\pi\delta(\omega - \omega') \tag{7.25}$$

and

$$\hat{a}(\omega)|0\rangle = 0\,. \tag{7.26}$$

As is readily shown, the spectral density of this noise is

$$S(\omega) = \hbar\left|\operatorname{Im}\chi(\omega)\right|. \tag{7.27}$$

One can easily see that the algebraic properties of the operators $a(\omega)$ and $a^\dagger(\omega)$ coincide precisely with those of the operators of annihilation and creation of quanta. A thermostat (for any internal structure) can therefore be treated as an infinite set of harmonic oscillators, for which the $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ operators introduced by formula (7.24) are the annihilation and creation operators.

7.2 Calculation of average values and fluctuation spectral densities for quantities like energy or power

In this appendix we calculate the average value and spectral density of fluctuations in the ground state for an operator of the form

$$\hat{\mathcal{Q}}(t) = \int_0^\infty \Big(Q_1(\omega,\omega')\hat{a}(\omega)\hat{a}(\omega')\exp\left[-\mathrm{i}(\omega+\omega')t\right]$$
$$+ Q_2(\omega,\omega')\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[\mathrm{i}(\omega'-\omega)t\right]$$
$$+ Q_2^*(\omega,\omega')\hat{a}(\omega)\hat{a}^\dagger(\omega')\exp\left[\mathrm{i}(\omega-\omega')t\right]$$
$$+ Q_1^*(\omega,\omega')\hat{a}^\dagger(\omega)\hat{a}^\dagger(\omega')\exp\left[\mathrm{i}(\omega+\omega')t\right]\Big)\frac{\mathrm{d}\omega}{2\pi}\,, \tag{7.28}$$

where $Q_{1,2}(\omega,\omega')$ are some functions.

This operator is assumed to be Hermitian:

$$\hat{\mathcal{Q}}^\dagger(t) = \hat{\mathcal{Q}}(t)\,. \tag{7.29}$$

Hence, it follows that

$$\int_0^\infty \Big(Q_2(\omega,\omega')\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[\mathrm{i}(\omega'-\omega)t\right] + Q_2^*(\omega,\omega')\hat{a}(\omega)\hat{a}^\dagger(\omega')\exp\left[\mathrm{i}(\omega-\omega')t\right]\Big)\frac{\mathrm{d}\omega}{2\pi} = \int_0^\infty \Big(Q_2^*(\omega,\omega')\hat{a}^\dagger(\omega')\hat{a}(\omega)\exp\left[\mathrm{i}(\omega-\omega')t\right] + Q_2(\omega,\omega')\hat{a}(\omega')\hat{a}^\dagger(\omega)\exp\left[\mathrm{i}(\omega'-\omega)t\right]\Big)\frac{\mathrm{d}\omega}{2\pi}\,. \quad (7.30)$$

This formula is simplified with the aid of the commutator (7.25) to eventually give a condition on the Q_2 function:

$$\int_0^\infty Q_2(\omega,\omega)\,\frac{\mathrm{d}\omega}{2\pi} = \int_0^\infty Q_2^*(\omega,\omega)\,\frac{\mathrm{d}\omega}{2\pi}\,. \quad (7.31)$$

From the form of formula (7.28) it also follows that the Q_1 function can be considered to be symmetric regarding the permutation of its arguments:

$$Q_1(\omega,\omega') = Q_1(\omega',\omega)\,, \quad (7.32)$$

since the integral containing the antisymmetric part is identically equal to zero.

The $\hat{\mathcal{Q}}(t)$ operator action on the ground state gives

$$\hat{\mathcal{Q}}(t)|0\rangle = \int_0^\infty \Big(Q_1^*(\omega,\omega')\hat{a}^\dagger(\omega)\hat{a}^\dagger(\omega')\exp\left[\mathrm{i}(\omega+\omega')t\right] + Q_2^*(\omega,\omega')\hat{a}(\omega)\hat{a}^\dagger(\omega')\exp\left[\mathrm{i}(\omega'-\omega)t\right]\Big)\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}|0\rangle\,. \quad (7.33)$$

With the aid of the commutator (7.25) and an obvious equality $\hat{a}(\omega)|0\rangle = 0$ this formula is rearranged to give

$$\hat{\mathcal{Q}}(t)|0\rangle = \left(\int_0^\infty Q_1^*(\omega,\omega')\hat{a}^\dagger(\omega)\hat{a}^\dagger(\omega') \times \exp\left[\mathrm{i}(\omega+\omega')t\right]\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} + \int_0^\infty Q_2^*(\omega,\omega)\,\frac{\mathrm{d}\omega}{2\pi}\right)|0\rangle\,. \quad (7.34)$$

Hence, the average value of $\mathcal{Q}$ is

$$\langle\hat{\mathcal{Q}}\rangle = \langle 0|\hat{\mathcal{Q}}(t)|0\rangle = \int_0^\infty Q_2(\omega,\omega)\,\frac{\mathrm{d}\omega}{2\pi}\,. \quad (7.35)$$

To calculate the fluctuation spectral density for Q, we first determine its *nonsymmetrized* correlation function

$$B'(t) = \langle 0|\big(\hat{\mathcal{Q}}(0)-\langle\hat{\mathcal{Q}}\rangle\big)\big(\hat{\mathcal{Q}}(t)-\langle\hat{\mathcal{Q}}\rangle\big)|0\rangle\,. \quad (7.36)$$

We use formula (7.33) once again to obtain

$$B'(t) = \int_0^\infty Q_1(\omega,\omega')Q_1^*(\omega_1,\omega_1') \times \langle 0|\hat{a}(\omega)\hat{a}(\omega')\hat{a}^\dagger(\omega_1)\hat{a}^\dagger(\omega_1')|0\rangle \times \exp\left[\mathrm{i}(\omega_1+\omega_1')t\right]\frac{\mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}\omega_1\,\mathrm{d}\omega_1'}{(2\pi)^4}\,. \quad (7.37)$$

The matrix element entering into this formula is calculated by way of the systematic carry-over of the $\hat{a}$ operators to the right and the $a^\dagger$ operators to the left, as well as by making use of the conditions that $\hat{a}(\omega)|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger(\omega) = 0$:

$$\langle 0|\hat{a}(\omega)\hat{a}(\omega')\hat{a}^\dagger(\omega_1)\hat{a}^\dagger(\omega_1')|0\rangle = (2\pi)^2\big(\delta(\omega-\omega_1)\delta(\omega'-\omega_1') + \delta(\omega'-\omega)\delta(\omega-\omega_1')\big)\,. \quad (7.38)$$

Consequently,

$$B'(t) = 2\int_0^\infty \big|Q_1(\omega,\omega')\big|^2\exp\left[\mathrm{i}(\omega+\omega')t\right]\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\,. \quad (7.39)$$

The correlation function itself is obtained by way of the symmetrization of $B'(t)$:

$$B(t) = \frac{1}{2}\Big\{\langle 0|\big(\hat{\mathcal{Q}}(0)-\langle\hat{\mathcal{Q}}\rangle\big)\big(\hat{\mathcal{Q}}(t)-\langle\hat{\mathcal{Q}}\rangle\big)|0\rangle + \langle 0|\big(\hat{\mathcal{Q}}(t)-\langle\hat{\mathcal{Q}}\rangle\big)\big(\hat{\mathcal{Q}}(0)-\langle\hat{\mathcal{Q}}\rangle\big)|0\rangle\Big\} = 2\int_0^\infty \big|Q_1(\omega,\omega')\big|^2\cos(\omega+\omega')t\,\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\,. \quad (7.40)$$

The spectral fluctuation density is derived from here by taking advantage of the Wiener–Khintchin theorem (1.3):

$$S(\Omega) = \int_0^\infty \big|Q_1(\omega,\omega')\big|^2\big(\delta(\omega+\omega'-\Omega) + \delta(\omega+\omega'+\Omega)\big)\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{2\pi} = \int_0^{|\Omega|}\big|Q_1\big(\omega,|\Omega|-\omega\big)\big|^2\,\frac{\mathrm{d}\omega}{2\pi}\,. \quad (7.41)$$

7.3 Oscillation modes for the 'oscillator + thermostat' type system

It is known from radiophysics that a semi-infinite twin-wire line is in principle undistinguishable from an ordinary resistor if measurements are conducted only at its end. We therefore take it as a thermostat model. We first consider a line of finite length l and then go over to the limiting case $l\to\infty$.

The combined equations of motion for an LC circuit connected to a line with a wave impedance R and a velocity of signal propagation v is of the form

$$\frac{\partial\hat{U}(t,x)}{\partial t} = -Rv\,\frac{\partial\hat{I}(t,x)}{\partial x}\,, \quad (7.42a)$$

$$\frac{\partial\hat{I}(t,x)}{\partial t} = -\frac{v}{R}\,\frac{\partial\hat{U}(t,x)}{\partial x}\,, \quad (7.42b)$$

$$\ddot{\hat{q}}(t)+\omega_0^2\hat{q}(t) = \frac{\omega_0}{\rho_0}\,\hat{U}(t,0)\,, \quad (7.42c)$$

$$\dot{\hat{q}}(t) = -\hat{I}(t,0)\,, \quad (7.42d)$$

where $U(x,t)$ and $I(x,t)$ respectively are the voltage and current in the line and $q(t)$ is the charge on the circuit capacity.

These equations should be complemented with the boundary condition at the distant (from the LC circuit) line end. The specific form of this boundary condition is of no significance. It will be assumed, for instance, that the distant

end of the line is open:

$$\hat{I}(l,t)=0\,. \tag{7.42e}$$

We will seek the solution of these equations in the form

$$\hat{U}(t,x)=\sum_n U_n(x)\hat{a}_n\exp(-\mathrm{i}\omega_n t)+\mathrm{h.c.}\,, \tag{7.43a}$$

$$\hat{I}(t,x)=\sum_n I_n(x)\hat{a}_n\exp(-\mathrm{i}\omega_n t)+\mathrm{h.c.}\,, \tag{7.43b}$$

$$\hat{q}(t)=\sum_n q_n\hat{a}_n\exp(-\mathrm{i}\omega_n t)+\mathrm{h.c.}\,, \tag{7.43c}$$

where ω_n are the eigenfrequencies of the modes of the system, $\hat{a}_n$ are the annihilation operators for these modes, and the summation is performed over all modes.

Substitution of these expressions into Eqns (7.42) gives:

$$\mathrm{i}\omega_n U_n(x)=Rv\,\frac{\partial I_n(x)}{\partial x}\,, \tag{7.44a}$$

$$\mathrm{i}\omega_n I_n(x)=\frac{v}{R}\,\frac{\partial U_n(x)}{\partial x}\,, \tag{7.44b}$$

$$(\omega_0^2-\omega_n^2)q_n=\frac{\omega_0}{\rho_0}\,\hat{U}_n(0)\,, \tag{7.44c}$$

$$\mathrm{i}\omega_n q_n=I_n(0)\,, \tag{7.44d}$$

$$I_n(l)=0\,. \tag{7.44e}$$

From these formulas it follows that

$$U_n(x)=A_n\cos\frac{\omega_n}{v}\,(x-l)\,, \tag{7.45a}$$

$$I_n(x)=\frac{\mathrm{i}A_n}{R}\sin\frac{\omega_n}{v}\,(x-l)\,, \tag{7.45b}$$

$$q_n=-\frac{A_n}{R}\sin\omega_n\tau\,, \tag{7.45c}$$

where $\tau=l/v$ and the eigenfrequencies ω_n are determined from the equation

$$\tan\omega_n\tau=-\frac{2\gamma\omega_n}{\omega_0^2-\omega_n^2}\,, \tag{7.46}$$

$$\gamma=\frac{\omega_0 R}{2\rho_0}\,. \tag{7.47}$$

The amplitudes A_n should be determined from the condition that the total energy in each mode in the ground state is equal to $\hbar\omega_n/2$. This energy can be represented as the following sum:

$$\mathcal{E}_n=\mathcal{E}_n^{\mathrm{LC}}+\mathcal{E}_n^{\mathrm{line}}\,, \tag{7.48}$$

where

$$\mathcal{E}_n^{\mathrm{line}}=\frac{1}{2v}\int_0^l\left(\frac{|U_n|^2}{R}+R|I_n|^2\right)\mathrm{d}x=\frac{\tau}{2R}\,|A_n|^2 \tag{7.49}$$

is the part of the energy residing in the line and

$$\mathcal{E}_n^{\mathrm{LC}}=\frac{\rho_0}{2\omega_0}\,(\omega_n^2+\omega_0^2)|q_n|^2=\frac{\gamma(\omega_n^2+\omega_0^2)}{R\bigl|\mathcal{L}(\omega_n)\bigr|^2}\,|A_n|^2 \tag{7.50}$$

is the part of the energy residing in the LC circuit. Consequently,

$$|A_n|^2=\frac{\hbar\omega_n R}{\tau+2\gamma(\omega_n^2+\omega_0^2)/\bigl|\mathcal{L}(\omega_n)\bigr|^2}\,. \tag{7.51}$$

We now can calculate the total energy in the LC circuit:

$$\mathcal{E}_{\mathrm{LC}}=\sum_n\mathcal{E}_n^{\mathrm{LC}}=\hbar\gamma\sum_n\frac{\omega_n(\omega_n^2+\omega_0^2)}{\tau\bigl|\mathcal{L}(\omega_n)\bigr|^2+2\gamma(\omega_n^2+\omega_0)}\,. \tag{7.52}$$

Even from this expression one can clearly see that the logarithmic divergence at high frequencies does take place, and the underlying reason is indeed the energy of zero-point oscillations of the high-frequency modes of the system.

We now pass over to the limiting case of an infinite line. Eqn (7.46) is rewritten as

$$\omega_n=\frac{\pi n}{\tau}-\frac{1}{\tau}\arctan\frac{2\gamma\omega_n}{\omega_0^2-\omega_n^2}\,. \tag{7.53}$$

Hence it is possible to calculate the mode density:

$$D(\omega)=\left(\frac{\mathrm{d}\omega_n}{\mathrm{d}n}\right)^{-1}=\frac{\tau}{\pi}\left(1+\frac{1}{\tau}\,\frac{\mathrm{d}}{\mathrm{d}\omega}\arctan\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

$$=\frac{\tau\bigl|\mathcal{L}(\omega)\bigr|^2+2\gamma(\omega_0^2+\omega_n^2)}{\pi\bigl|\mathcal{L}(\omega)\bigr|^2}\,. \tag{7.54}$$

We replace the summation in the last formula with integration according to the rule

$$\sum_n\to\int_0^\infty\mathrm{d}\omega\,D(\omega)\,, \tag{7.55}$$

to derive the equation which exactly coincides with formula (3.22).

7.4 Eigenstates of an LC circuit parametrically coupled to a mechanical oscillator

The Hamiltonian of the system schematized in Fig. 3 is

$$\hat{\mathcal{H}}=\hbar\omega_{\mathrm{e}}\left(\hat{n}_{\mathrm{e}}+\frac{1}{2}\right)\left(1-\frac{\hat{x}}{d}\right)+\frac{\hat{p}^2}{2m}+\frac{m\omega_{\mathrm{m}}^2\hat{x}^2}{2}\,, \tag{7.56}$$

where $\hat{n}_{\mathrm{e}}$ is the quantum number operator of the LC circuit, p is the momentum of the mechanical oscillator, and m is its mass. We determine the eigenvalues $\mathcal{E}$ and eigenstates $|\psi\rangle$ for this Hamiltonian:

$$\hat{\mathcal{H}}|\Psi\rangle=\mathcal{E}|\Psi\rangle\,. \tag{7.57}$$

The eigenstates are sought in the form

$$|\Psi\rangle=\bigl|\psi(n_{\mathrm{e}})\bigr\rangle|n_{\mathrm{e}}\rangle\,, \tag{7.58}$$

where $|n_{\mathrm{e}}\rangle$ are the states with a given number of quanta n_{e} for the LC circuit and $|\psi(n_{\mathrm{e}})\rangle$ is the wave function of the mechanical oscillator (which depends on n_{e} as a parameter). We substitute $|\Psi\rangle$ into Eqn (7.57) to obtain the

equation for $|\psi(n_e)\rangle$:

$$\left[-\hbar\omega_e\left(n_e+\frac{1}{2}\right)\frac{\hat{x}}{d}+\frac{\hat{p}^2}{2m}+\frac{m\omega_m^2\hat{x}^2}{2}\right]|\psi(n_e)\rangle = \left[\mathcal{E}-\hbar\omega_e\left(n_e+\frac{1}{2}\right)\right]|\psi(n_e)\rangle\,. \tag{7.59}$$

It can also be written as

$$\hat{\mathcal{D}}(X_{n_e})\left[\frac{\hat{p}^2}{2m}+\frac{m\omega_m^2\hat{x}^2}{2}-\frac{m\omega_m^2X_{n_e}^2}{2}\right]\hat{\mathcal{D}}^\dagger(X_{n_e})|\psi(n_e)\rangle = \left[\mathcal{E}-\hbar\omega_e\left(n_e+\frac{1}{2}\right)\right]|\psi(n_e)\rangle\,, \tag{7.60}$$

where

$$X_{n_e}=\frac{\hbar\omega_e}{m\omega_m^2 d}\left(n_e+\frac{1}{2}\right) \tag{7.61}$$

is the magnitude of displacement of the equilibrium of the mechanical oscillator by action of the attractive force caused by the LC circuit and

$$\hat{\mathcal{D}}(\xi)=\exp\left(-\frac{i\hat{p}\xi}{\hbar}\right) \tag{7.62}$$

is the unitary displacement operator, whose main property is expressed by the formula

$$\hat{\mathcal{D}}^\dagger(\xi)\hat{x}\hat{\mathcal{D}}(\xi)=\hat{x}+\xi\,. \tag{7.63}$$

The eigen-solutions of Eqn (7.60) are the eigenstates of the mechanical oscillator $|n_m\rangle$ modified by the action of the displacement operator:

$$|\psi_{n_m}(n_e)\rangle=\hat{\mathcal{D}}(X_{n_e})|n_m\rangle\,. \tag{7.64}$$

Consequently, the energy eigenvalues for the total system are

$$\mathcal{E}_{n_e n_m}=\hbar\omega_e\left(n_e+\frac{1}{2}\right)+\hbar\omega_m\left(n_m+\frac{1}{2}\right)-\frac{m\omega_m^2X_{n_e}^2}{2}\,, \tag{7.65}$$

and the eigenstates are

$$|\Psi_{n_e n_m}\rangle=\hat{\mathcal{D}}(X_{n_e})|n_m\rangle|n_e\rangle\,. \tag{7.66}$$

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