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Higher spin gauge theory

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1. Standard gauge theories

The aim of the present talk is to present the key ideas and results of the higher spin gauge theory without delving into technical details of the setup. In essence, we shall consider the construction of a field theory model with maximally high gauge symmetry. It is expected that such theories allow a new vision of superstring theory which is presently thought to be the main candidate for the theory of fundamental interactions.

As usually, the gauge symmetries are those whose parameters are arbitrary functions of space-time coordinates x^{ν}. Historically the first gauge theory was that of electromagnetism suggested by Maxwell. In this case the gauge field is identified with the vector potential A_{ν} that generates the field strength

$$F_{\nu\mu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}, \qquad \partial_{\nu} = \frac{\partial}{\partial x^{\nu}}, \qquad \nu = 0, 1, 2, 3, \tag{1}$$

invariant under the gauge (gradient) transformations

$$\delta A_{\nu} = \partial_{\nu} \varepsilon \tag{2}$$

with an arbitrary gauge parameter $\varepsilon(x)$. The gauge-invariant Maxwell's action

$$S = -\frac{1}{4} \int \mathrm{d}^4 x \, F_{\nu\mu} F^{\nu\mu}, \qquad \delta S = 0 \tag{3}$$

is known to describe massless particles of spin 1, the photons.

Maxwell theory can be generalized to Yang–Mills theory by introducing a system of mutually charged spin 1 particles described by the matrix-valued potential $A^{j}_{\nu i}$ which takes values in some Lie algebra h. The corresponding strengths, gauge transformations, and action have the forms

$$G_{\nu\mu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} + g\,[A_{\nu},\, A_{\mu}]\,, \tag{4}$$

$$\delta A_{\nu} = \partial_{\nu} \varepsilon + g\,[A_{\nu}, \varepsilon]\,, \tag{5}$$

$$S = -\frac{1}{4} \int \mathrm{d}^4 x \, \mathrm{tr}(G_{\nu\mu} G^{\nu\mu})\,, \tag{6}$$

respectively. The Yang–Mills theory can be understood as the theory of interaction of massless spin 1 particles. Indeed, by imposing natural conditions bounding the orders of derivatives, the gauge symmetry principle fixes interactions of spin 1 fields unambiguously to within an arbitrary choice of the gauge group.

At first glance, the pure Yang–Mills theory seems poorly adapted to describe the real physics since quanta of the Yang–Mills fields are massless, at least perturbatively. At one time, it was this point that prevented Pauli from publishing the results he obtained where essentially the Yang–Mills theory was discovered. Later on, owing to discovery of the Higgs phenomenon in the phase with spontaneously broken symmetry, this difficulty was found to be apparent. From the symmetry viewpoint, the characteristic feature of this phenomenon is the appearance of the Higgs field Φ^A with the gauge transformation law in the form

$$\delta \Phi^A = \varepsilon^A(X) + \ldots\,, \tag{7}$$

where $\varepsilon^A(X)$ are some combinations of the gauge parameters $\varepsilon(X)$ and the ellipsis denotes higher-order terms. Such a transformation law of the Higgs field allows a partial fixation of the gauge freedom by choosing the gauge $\Phi^A = 0$. The remaining freedom is generated by those gauge parameters $\varepsilon(X)$ that do not contribute to $\varepsilon^A(X)$. Presently, the Yang–Mills theory is the foundation for the theory of strong and electroweak interactions. In particular, introducing Yang–Mills fields for the gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$ as carriers of the electroweak interaction has enabled one to surmount difficulties of Fermi theory of weak interactions.

The next textbook example of the gauge theory is general relativity. Here the metric tensor $g_{\mu\nu}$ plays the role of the gauge field and gauge transformations are identified with the coordinate ones

$$\delta g_{\nu\mu} = \partial_{\nu}(\varepsilon^{\rho}) g_{\rho\mu} + \partial_{\mu}(\varepsilon^{\rho}) g_{\rho\nu} + \varepsilon^{\rho} \partial_{\rho}(g_{\nu\mu})\,, \tag{8}$$

where $\varepsilon^{\rho}(x)$ are infinitesimal parameters. The gauge invariance principle is identified with Einstein's equivalence principle. The invariant Einstein–Gilbert action

$$S = -\frac{1}{4\varkappa^2} \int \sqrt{-\det|\,g\,|}\,(R + \Lambda) \tag{9}$$

contains two independent coupling constants: the gravitational constant $\varkappa$ and the cosmological constant Λ. To interpret this theory in terms of particles, one should make the expansion $g_{\nu\mu} = \eta_{\nu\mu} + \varkappa h_{\nu\mu}$ in some fixed background metrics $\eta_{\nu\mu}$ (flat for $\Lambda = 0$ or (anti) de Sitter for $\Lambda \neq 0$) where $h_{\nu\mu}$ describes dynamical fluctuations. For the flat space ($\Lambda = 0$) Fierz and Pauli showed that the linearized action S describes free massless particles of spin 2, the gravitons. Again, by imposing some natural conditions the Einstein–Gilbert action is found to be the only consistent (gauge invariant) action for a self-interacting massless field of spin 2.

In four dimensions, the only non-trivial modification of the gauge theory of spin 1 and spin 2 is supergravity — the theory in which, in addition to the spin 1 and spin 2 gauge fields, a massless gauge field of spin 3/2 appears, the gravitino, which is responsible for local supersymmetry transformations with the spinor gauge parameters $\varepsilon_{\alpha}(x)$. A new feature of supergravity is that it unifies particles carrying different spins, in particular bosons and fermions, into the same supermultiplets.

So usual gauge theories are based upon gauge fields of spin 1 with the scalar gauge parameters $\varepsilon(x)$, of spin 3/2 with the spinor gauge parameters $\varepsilon_\alpha(x)$, and of spin 2 with the vector gauge parameters $\varepsilon^\rho(x)$. Needless to say how important these theories are. It is also worth noting that within the supergravity framework, higher supersymmetries of the theory provide a softer quantum behavior due to cancellation of divergences. The natural question arises: are there other possibilities related to gauge fields of higher spins ($s > 2$) and higher tensors as gauge parameters that also lead to fruitful physical models and give hope for creating the theory of quantum gravity?

2. Free massless fields of higher spins

The theory of massless fields of all spins in four-dimensional space-time has been elaborated in detail due to the efforts of many authors (see, for example, [1, 2]). All free fields with $s \geqslant 1$ were found to be Abelian gauge fields. In particular, massless gauge fields of integer spins s can be described by totally symmetric tensors $\varphi_{\nu_1 \ldots \nu_s}$ subject to the double-tracelessness condition [1] $\varphi^\rho{}_\rho{}^\eta{}_{\eta\nu_5 \ldots \nu_s} = 0$ which becomes non-trivial for $s \geqslant 4$. The quadratic action S_s for free fields of higher spins [1] is fixed unambiguously by requiring gauge invariance under the Abelian transformations

$$\delta\varphi_{\nu_1 \ldots \nu_s} = \partial_{\{\nu_1} \varepsilon_{\nu_2 \ldots \nu_s\}_\nu} \tag{10}$$

with parameters $\varepsilon_{\nu_1 \ldots \nu_{s-1}}$ which are totally symmetric traceless rank-$(s-1)$ tensors, $\varepsilon^\rho{}_{\rho\nu_3 \ldots \nu_{s-1}} = 0$

$$\begin{aligned} S_s = \frac{1}{2}(-1)^s \int \mathrm{d}^4 x \, \{ & \partial_\nu \varphi_{\mu_1 \ldots \mu_s} \partial^\nu \varphi^{\mu_1 \ldots \mu_s} \\ & - \frac{1}{2} s(s-1) \, \partial_\nu \varphi^\rho{}_{\rho\mu_1 \ldots \mu_{s-2}} \partial^\nu \varphi^\sigma{}_\sigma{}^{\mu_1 \ldots \mu_{s-2}} \\ & + s(s-1) \, \partial_\nu \varphi^\rho_{\rho\mu_1 \ldots \mu_{s-2}} \partial_\sigma \varphi^{\nu\sigma\mu_1 \ldots \mu_{s-2}} \\ & - s \, \partial_\nu \varphi^\nu_{\mu_1 \ldots \mu_{s-1}} \partial_\rho \, \varphi^{\rho\mu_1 \ldots \mu_{s-1}} \\ & - \frac{1}{4} s(s-1)(s-2) \, \partial_\nu \varphi^\rho{}_\rho{}^\nu{}_{\mu_1 \ldots \mu_{s-3}} \partial_\sigma \varphi^\eta{}_\eta{}^{\sigma\mu_1 \ldots \mu_{s-3}} \} \, . \end{aligned} \tag{11}$$

For $s \geqslant 1$ this action describes massless particles of spin s having two independent degrees of freedom in $d = 3 + 1$. The quantization of this action leads to a unitary theory which is free from negative-norm states. For $s = 0, 1, 2$ S_s reduces to the standard actions of the lower-spin fields.

Massless gauge fields of half-integer spins are analogously described in terms of the totally symmetric rank-$(s - 1/2)$ spin-tensors [1].

3. Motivation

After we have shown that the theory of higher-spin free gauge fields is well determined, the next question is how to construct their consistent interactions. The consistency of gauge theories of higher spins means that they reduce to some combination of free systems of higher spins at the linearized level and that the number of gauge symmetries remains the same for free and interacting theories, i.e., the interactions are allowed to deform the Abelian gauge symmetries of free fields like they do in the Yang–Mills and Einstein theories, but not to reduce their number. The analysis of this problem is interesting from several points of view.

The well known fact that theories of supergravity allow at most 32 supersymmetries implies that these theories do not contain fields of spins higher than two. [1] Thus the analysis of the existence of higher-spin gauge fields can be interpreted as the analysis of the possibility of going beyond the framework of the maximal eleven-dimensional supergravity model.

Superstring theory contains infinite systems of all spin states. Excluding some fields of spin $s \leqslant 2$, all the higher-spin fields in this theory have large masses exceeding the energy scale of modern colliders. It can be expected that, in the same way as $W^\pm$- and Z-bosons in the electroweak theory, they acquire mass due to spontaneous symmetry breaking. This is manifested by superstring field theory which is based on the so called Stueckelberg symmetries with the following structure:

$$\delta\varphi_{\nu_1,\nu_2,\nu_3 \ldots}(x) = \partial_{\nu_1} \varepsilon_{\nu_2,\nu_3 \ldots}(x) + \ldots \, , \tag{12}$$

$$\delta\Phi_{\nu_1,\nu_2,\ldots}(x) = \varepsilon_{\nu_1,\nu_2 \ldots}(x) + \ldots \, , \tag{13}$$

where $\varepsilon_{\nu_1,\nu_2 \ldots}(x)$ are gauge parameters of the Stueckelberg symmetries which are, just like higher-spin gauge parameters, higher-rank Lorentz tensors. Fields $\varphi_{\nu_1,\nu_2,\nu_3 \ldots}(x)$ transform as higher-spin fields and the Stueckelberg fields $\Phi_{\nu_1,\nu_2,\ldots}(x)$ transform as the corresponding Higgs fields analogous to (7). Such a structure corresponds to some broken higher spin symmetries. In other words, the field formulation of superstring theory unambiguously indicates that this theory is a spontaneously broken phase of some higher-spin theory.

4. Obstacles

Irrespective of the specific motivation, the problem of higher spins reduces to finding a non-trivial non-linear theory describing interacting massless fields of spins $s > 2$. Such a problem setting may seem not very restrictive. This is not so in reality. Over decades, the opinion has dominated that the higher-spin problem has no solution at all. This was based on arguments of two types.

The S-matrix arguments by Coleman–Mandula and Haag–Lopuszanski–Sohnius [3] stated that if symmetries of the S-matrix of some relativistic theory in the Minkowski space go beyond the framework of ordinary internal (isotopic) symmetries associated with spin 1 gauge fields, space-time symmetries associated with spin 2 gauge fields (gravity), as well as (possibly) supersymmetries associated with spin 3/2 gauge fields (gravitino), the S-matrix of such a theory is trivial. In other words, if higher-spin symmetries are those of the S-matrix, the scattering and hence the real interaction is absent.

The analysis of gravitational interaction of higher-spin gauge fields first carried out by Aragon and Deser in 1979 [4] for a spin 5/2 field proved to be equally frustrating. Technically, the problem is sufficiently simple: in order to introduce an interaction with gravity possessing general coordinate invariance, the ordinary derivatives should be substituted by covariant ones: $\partial \to \mathrm{D} = \partial - \Gamma$. This breaks the invariance with respect to higher-spin gauge transformations, since in proving the invariance of action S_s one should commute derivatives and the commutator of the derivatives is

[1] This follows from the analysis of supermultiplets. If the number of supersymmetries exceeds 32, any supermultiplet contains states of spins higher than two.

proportional to the Riemann tensor $[\mathrm{D}\ldots, \mathrm{D}\ldots] = R\ldots$. As a result, the gauge variation of the covariantized action S_s^{cov} with respect to covariantized transformations of higher spins has the following structure:

$$\delta S_s^{\mathrm{cov}} = R_{\ldots}(\varepsilon_{\ldots}\mathrm{D}\varphi_{\ldots}) \neq 0\,. \tag{14}$$

For spins $s > 2$ this variation contains the traceless part of the Riemann tensor (the Weyl tensor) which does not permit such terms to be compensated by some change in the action and/or transformation laws, as was possible for the case $s = 3/2$, which opened the way to supergravity. So it seemed that the frustrating conclusion [4] is that the gauge fields of higher spins do not allow for a consistent description in the framework of Einstein's gravity. Due to the universal role of gravity, the existence of the consistent gravitational interaction of higher-spin gauge fields is of principal importance.

It is not surprising that all this hindered the development of a constructive attitude to the higher-spin gauge field theory.

5. Higher spin currents

Nevertheless, consistent cubic interactions of higher spins first constructed in [5] gave significant evidence that gauge theories of higher spins really exist. Although these interactions did not include gravitational interaction of massless fields, the very fact of their existence is extremely remarkable. In particular, interactions with conserved currents well known from the theory of electromagnetism and (super)gravity are of this type.

Usual internal symmetries are related by the Noether theorem to a spin 1 conserved current which can be constructed from different matter fields. For example, the electric current

$$J^{\nu} = \bar{\phi}\partial^{\nu}\phi - \partial^{\nu}\bar{\phi}\phi\,, \tag{15}$$

constructed from a complex scalar field is conserved on the solutions to the scalar field equations

$$\partial_{\nu}J^{\nu} = \bar{\phi}(\Box + m^2)\,\phi - (\Box + m^2)\,\bar{\phi}\phi\,. \tag{16}$$

Translational symmetry is related to a current $T^{\mu\nu}$ of spin 2 called the energy-momentum tensor. For a scalar matter field it has the form

$$T^{\mu\nu} = \partial^{\mu}\phi\partial^{\nu}\phi - \frac{1}{2}\,\eta^{\mu\nu}\left(\partial_{\rho}\phi\,\partial^{\rho}\phi - m^2\phi^2\right). \tag{17}$$

Supersymmetry is related to a conserved current called supercurrent. It obeys the fermion statistics and is constructed from bosons and fermions. For massless scalars ϕ and massless spinors ψ_{α} it has the form

$$J^{\nu}{}_{\alpha} = \partial_{\mu}\phi(\gamma^{\mu}\gamma^{\nu}\psi)_{\alpha}\,, \tag{18}$$

where $\gamma^{\nu}{}_{\alpha}{}^{\beta}$ are Dirac's matrices in d dimensions.

Conserved currents associated with Lorentz rotations can be constructed from the symmetric energy-momentum tensor:

$$S^{\nu;\mu\rho} = T^{\nu\mu}x^{\rho} - T^{\nu\rho}x^{\mu}\,, \qquad T^{\nu\mu} = T^{\mu\nu}\,. \tag{19}$$

It is less known that one can construct conserved currents of arbitrary spin. For example, currents of arbitrary integer spin constructed from a massless scalar field $\Box\phi^i = 0$ can be chosen in the form [6]

$$\begin{aligned} T^{\nu_1\ldots\nu_{2k}} &= \partial^{\nu_1}\ldots\partial^{\nu_k}\phi\,\partial^{\nu_{k+1}}\ldots\partial^{\nu_{2k}}\,\phi - \\ &- \frac{k}{2}\,\eta^{\nu_1\nu_2}\,\partial^{\nu_3}\ldots\partial^{\nu_{k+1}}\,\partial_{\mu}\phi\,\partial^{\nu_{k+2}}\ldots\partial^{\nu_{2k}}\,\partial^{\mu}\phi \end{aligned} \tag{20}$$

for even spins and

$$T^{\nu_1\ldots\nu_{2k+1}} = \partial^{\nu_1}\ldots\partial^{\nu_{k+1}}\bar{\phi}\,\partial^{\nu_{k+2}}\ldots\partial^{\nu_{2k+1}}\phi - \bar{\phi} \leftrightarrow \phi \tag{21}$$

for odd spins. The fact that currents of higher spins are conserved,

$$\partial_{\nu_1}T^{\nu_1\ldots\nu_s} \sim 0 \tag{22}$$

(~ 0 denotes equality to zero on the equations of motion of matter fields) allows us to construct the so called Noether interactions of higher spins in the form

$$\int \mathrm{d}^4x\,\phi_{\nu_1\ldots\nu_s}T^{\nu_1\ldots\nu_s}\,, \tag{23}$$

which are invariant under higher spin transformation (10), at least up to higher orders in fields. As the currents of higher spins contain higher derivatives, interactions of higher-spin fields also contain higher derivatives with the order of derivatives being proportional to the spin.

6. The role of anti-de Sitter geometry

All these results undeniably pointed to the existence of some non-trivial gauge theory of higher-spin fields though they did not resolve the problem with their gravitational interaction. The solution of this problem, found in the P N Lebedev Physical Institute of RAS [7], turned out to be fairly unexpected. It was shown that the consistent cubic gravitational interaction of higher spins can be constructed if the problem is considered within the framework of the expansion near the (anti) de Sitter background. In other words, gauge-invariant and general-coordinate-covariant gravitational interactions of higher spins contain some terms proportional to negative powers of the cosmological constant diverging in the flat limit. Schematically, the modification of the action leading to the desired result has the following form:

$$S \to S + \Delta S\,, \tag{24}$$

$$\Delta S = \sum_{p,q} \Lambda^{(2-p-q)/2}\mathrm{D}^p\phi\mathrm{D}^q\phi\mathcal{R}\,, \tag{25}$$

where ϕ denotes higher-spin fields and $\mathcal{R}$ describes deviations of the Riemann tensor from the background curvature of the anti-de Sitter (AdS) space:

$$R_{\mu\nu,\,\rho\sigma} = -\Lambda(g_{\mu\rho}\,g_{\nu\sigma} - g_{\nu\rho}\,g_{\mu\sigma}) + \mathcal{R}_{\mu\nu,\,\rho\sigma}\,. \tag{26}$$

Negative powers of Λ make terms with higher field derivatives in (25) dimensionless; the variation of these terms leads to a 'miraculous' cancellation of the terms in the variation of the initial action S. (Note that the order of derivatives of fields with fixed spins is finite and proportional to the spin).

It is important that this result agrees with conclusions of Ref. [4] where the problem was implicitly assumed to admit an analysis within the framework of expansion in powers of the

Riemann tensor. The point is that such expansions are admissible only if the Riemann tensor is sufficiently small, i.e., the geometry is almost flat, while action (24) explicitly contains negative powers of Λ and does not permit a flat limit.

Simultaneously, it became possible to get around the Coleman–Mandula–Haag–Lopuszanski–Sohnius no-go theorem [3] since analysis of the S-matrix in the AdS space is meaningless because the S-matrix can not exist in this space-time.

Gauge theories of higher spins require a non-zero and even large cosmological constant as these theories predict the following simple relation between the gauge constant g, the gravitational constant $\varkappa$, and the cosmological constant Λ:

$$g^2 \sim \Lambda \varkappa^2 \,. \tag{27}$$

At first glance, this circumstance could be considered as a difficulty for the theory. However, this property follows from the requirement that gauge symmetries of higher spins be unbroken and apparently has as remote a relation to the phase of the theory describing the reality as the Yang–Mills fields being massless has to the theory of electroweak interactions. It is expected that gauge symmetries of higher spins will be broken in the physical phase, which makes initially massless fields massive and simultaneously changes the value of the cosmological constant.

Note that unbroken symmetries of higher spins requiring a non-zero cosmological constant can serve as an explanation as to why the symmetric phase of superstring theory has not been found so far. The point is that the formulation of quantum theory of superstrings in the AdS space is a non-trivial task which has not as yet been solved.

Nevertheless, the fact that gauge theories of higher spins require a non-zero cosmological constant was regarded until recently only as a strange feature of the theory. The situation significantly changed after the so called 'AdS/CFT correspondence' (CFT — conformal field theory) had been discovered.

7. AdS/CFT correspondence

We recall that d-dimensional de Sitter and anti-de Sitter spaces can be realized as d-dimensional hyperboloids,

$$X^A X^B \eta_{AB} = \omega R^2 \,, \tag{28}$$

embedded in $d+1$-dimensional spaces with coordinates X^A and metrics η_{AB} with the signature $+ - - \cdots - \omega$ where $\omega = 1$ for anti-de Sitter and $\omega = -1$ for de Sitter space. The parameter R is called the radius of the (anti) de Sitter space. The cosmological constant Λ is proportional to R^{-2} so the flat limit of the Minkowski space corresponds to $\Lambda \to 0$ ($R \to \infty$).

This realization immediately implies that anti-de Sitter space AdS_d has the symmetry group $\mathrm{O}(d-1, 2)$, and de Sitter space dS_d has the symmetry group $\mathrm{O}(d, 1)$. In the flat limit, each of these groups transforms to the group of motions of the Minkowski space usually called the Poincare group. An important feature is that the group of motions of $d+1$-dimensional AdS space $\mathrm{O}(d, 2)$ coincides with the conformal group of d-dimensional Minkowski space. (We recall that the conformal group is an extension of the Poincare group and acts on certain, scale-invariant systems such as systems of massless fields).

AdS space has no boundary. However, it is possible to analyze the behavior of fields in AdS space in the asymptotic regime when coordinates and time tend to infinity. For example, such coordinates can be chosen in the form

$$t = X^0 \,, \qquad x^i = X^i \,, \quad i = 1, \ldots, d-1 \,. \tag{29}$$

At $t, x^i \to \infty$ fields exhibit the asymptotic behavior

$$\phi(\mu t, \mu x^i) = \mu^{\Delta} \phi(t, x^i) \,, \tag{30}$$

where the parameter Δ characterizes the field ϕ. So asymptotic values of fields ϕ in d-dimensional AdS space are characterized by a function of $d-1$ coordinates. By identifying the conformal infinity of AdS_5 with four-dimensional Minkowski space, the hypothesis first put forward by Maldacena [8] establishes the correspondence of correlators of fields in the AdS space with correlators of currents of a conformal theory on its boundary. More precisely, this means the correspondence of a type IIB superstring in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space with $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with gauge group $\mathrm{SU}(N)$ in four-dimensional Minkowski space if the following theory parameters are identified:

$$\frac{1}{(l_{\mathrm{str}})^4 \Lambda^2} = g_{\mathrm{YM}}^2 N \,, \tag{31}$$

where l_{str} is the string length scale, Λ is the cosmological constant in AdS_5, and g_{YM}^2 is the coupling constant of the Yang–Mills theory on the boundary.

The limit originally considered by Maldacena [8]

$$g_{\mathrm{YM}}^2 N \to \infty \,, \qquad \Lambda \to 0 \,, \tag{32}$$

relates the strong coupling limit in the Yang–Mills theory to the low-energy limit of the superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ described by the classical theory of IIB supergravity. In this case calculations in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are relatively simple and can be used to analyze the strong coupling regime in the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. However, recently Sandborg and Witten [9] suggested considering the opposite limit

$$g_{\mathrm{YM}}^2 N \to 0 \,, \qquad (l_{\mathrm{str}})^2 \Lambda \to \infty \,, \tag{33}$$

relating the weak coupling regime in $\mathcal{N} = 4$ 'super-Yang–Mills' theory to a very non-trivial superstring limit in $\mathrm{AdS}_5 \times \mathrm{S}^5$. The key observation is that in this limit the $\mathcal{N} = 4$ 'super-Yang–Mills' theory becomes free and hence has the infinite-dimensional symmetry which should be identified with some (conformal) symmetry of higher spins (the corresponding charges are obtained by constructing conserved currents analogous to currents of higher spins (20) and (21) constructed from a free scalar field). So the theory of superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in limit (33) should possess the same symmetry. Hence, it must be some gauge theory of higher spins in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Since little is known about superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, the meaning of this result lies in the hope of obtaining an explicit description of superstring in its maximally symmetric (using the language of higher-spin theory) phase, which makes the problem of higher spins even more actual. The higher-spin symmetry algebra corresponding to the Sandborg–Witten limit was explicitly constructed in [10].

8. Higher spin symmetries

The main properties of the theory of gauge fields of higher spins are determined by the higher spin global symmetry algebra which has the structure of the algebra of oscillators $\hat{y}_\alpha$, $\alpha = 1, \ldots, M$ satisfying the commutation relations

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2C_{\alpha\beta}\,, \tag{34}$$

where $C_{\alpha\beta}$ is some non-degenerate matrix [2]. Let us consider all possible functions of oscillators:

$$\hat{P}(\hat{y}) = \sum_{n=0}^{\infty} \frac{1}{2\,n!}\, P^{\alpha_1 \ldots \alpha_n}\, \hat{y}_{\alpha_1} \ldots \hat{y}_{\alpha_n} \tag{35}$$

(it is convenient to assume all coefficients $P^{\alpha_1 \ldots \alpha_n}$ to be totally symmetric with respect to permutations of indexes). Clearly, the product of two operator functions of this kind will again allow such a representation. The simplest (super)algebras of higher spins are obtained from (anti)commuting of elements (35). Needless to say, everyone who has made calculations with harmonic oscillators using the secondary quantization formalism has met these algebras. It is worth noting that originally the higher-spin algebras were found in [11] without using convenient oscillator realization (34) and (35) which was discovered later in [12].

Oscillator indexes $\alpha, \beta, \ldots$ are interpreted as spinor ones. In particular, in the case of four-dimensional space-time, spinors have four components ($M = 4$). The dynamics of higher-spin fields is described in terms of gauge fields of the higher-spin algebra $\hat{\omega}_\nu(\hat{y}|x)$. The spin of a field is determined by the power of operator variables

$$\hat{\omega}_\nu(\mu\hat{y}|x) = \mu^{2(s-1)}\hat{\omega}_\nu(\hat{y}|x)\,. \tag{36}$$

The spin 2 gravitational field is described by fields $\hat{\omega}_\nu(\hat{y}|x)$ bilinear in spinor variables that correspond to Einstein's gravity formulation in the Cartan formalism. The spin 1 gauge fields are described by potentials $\hat{\omega}_\nu(0|x)$ which are independent of auxiliary operator variables. To include non-Abelian Yang – Mills groups, it is sufficient to put matrix indexes on fields $\hat{\omega}_\nu(\hat{y}|x)$. As shown in [13], this allows one to construct higher-spin gauge theories with Yang – Mills gauge groups in the spin 1 sector $\mathrm{U}(n) \times \mathrm{U}(m)$, $\mathrm{O}(n) \times \mathrm{O}(m)$, and $\mathrm{USp}(n) \times \mathrm{USp}(m)$ with different n and m. Formula (36) also immediately implies that the presence of some spin $s > 2$ gauge field in the theory leads to the presence of a tower of fields with unlimitedly increasing spins. Indeed, the commutator of two polynomials in oscillators of power n_1 and n_2 yields a power $n_1 + n_2 - 2$ polynomial. In other words, a commutator of spin 3 symmetries yields a spin 4 symmetry, a commutator of spin 4 symmetries yields a spin 6 symmetry, etc. Usual symmetries emerging in field theory models with spins $s \leqslant 2$ and consistent with S-matrix no-go theorems turn out to be finite-dimensional sub-algebras of higher-spin symmetries generated by not higher than bilinear combinations of oscillators $\hat{y}_\alpha$ (or by their limits — in a way similar to how the Poincare group is the limiting case of the AdS group).

Note that full gauge theories of higher spins also contain fields with lower spins $s = 0$ and $1/2$ which are described by other generating functions. The minimal gauge theory of higher spins contains fields of all even spins (one field of each type).

As to theories in a different number of spatial dimensions, the list of results obtained up to the present time contains the full description of the non-linear dynamics of higher spins in four-dimensional AdS space at the level of the equations of motion [14], as well as partial results at the level of the action, which solve the problem of higher-spin gravitational interaction [7]. The case of AdS_5 most interesting from the AdS/CFT-correspondence viewpoint, was recently considered beyond the framework of free theory in [15], where cubic vertexes of higher-spin interaction with gravity were constructed. Interesting results obtained in Ref. [16] for free fields in AdS_7, indicate that the general approach, applied previously in four and five dimensions, may also bring success in the case of AdS_7. We can also mention the construction of full nonlinear invariant equations with higher-spin symmetries in three-dimensional AdS [17], although three-dimensional theory of higher spins is dynamically less interesting since higher-spin gauge fields in three dimensions do not carry their degrees of freedom. More detail on the formalism used for the formulation of higher-spin theory and more complete list of references can be found in [6].

9. Geometry of higher spins

The initial setting of the problem of higher spins is based on the standard concept of space-time. One of the most remarkable consequences of this theory is [18] that the geometry adequate for the theory of higher spins and possibly for the theory of fundamental interactions can prove to be more interesting. It is necessary to emphasize that this does not imply rejecting general relativity but some more general approach promising a higher degree of unification of different physical phenomena at the level of fundamental theory. The closest analogy is in passing to the Minkowski space-time geometry as a consequence of Maxwell's equations being invariant with respect to the Lorentz transformations. Under the Lorentz transformations, electric and magnetic fields form different components of the unique relativistic field strength tensor (1).

The key observation is that infinite systems of massless fields arising in four-dimensional theory of higher spins transforms according to the extension of usual relativistic symmetries to the $\mathrm{Sp}(8|R)$ group. In a way similar to how electric and magnetic fields transform independently under spatial rotations but are mixed by the Lorentzian boosts, each of the massless fields transforms through itself under relativistic transformations [conformal group SU(2,2)] but different massless fields are mixed under other symmetries from $\mathrm{Sp}(8|R)$. This observation raises the question of the nature of the geometry ensuring the geometrical character of the action of this more broad symmetry, and of how the dynamics of massless fields is described in the language of this geometry. The answer is as follows [10]. The relevant generalized space-time $\mathcal{M}_M$ is described by coordinates $X^{\alpha\beta} = X^{\beta\alpha}$ which are symmetric bispinors. In the case of four-dimensional space-time $M = 4$, the generalized space-time turns out to be ten-dimensional. The equations of motion for massless fields take the remarkably simple form:

$$\left(\frac{\partial^2}{\partial X^{\alpha\beta}\partial X^{\gamma\delta}} - \frac{\partial^2}{\partial X^{\alpha\gamma}\partial X^{\beta\delta}}\right) b(X) = 0 \tag{37}$$

[2] The canonical choice of the matrix $C_{\alpha\beta}$ corresponds to explicit decomposition of oscillators $\hat{y}_\alpha$ into creation and annihilation operators. In four-dimensional theory $C_{\alpha\beta}$ coincides with the charge conjugation matrix.

for the scalar field $b(X)$ and

$$\frac{\partial}{\partial X^{\alpha\beta}} f_\gamma(X) - \frac{\partial}{\partial X^{\alpha\gamma}} f_\beta(X) = 0 \qquad (38)$$

for the fermion field $f_\alpha(X)$. Here all massless fields of even spins in four-dimensional Minkowski space are described by one scalar field $b(X)$ and all massless fields of half-integer spins in four-dimensional Minkowski space are described by one fermion field $f_\alpha(X)$ in the generalized space-time $\mathcal{M}_4$.

The original Minkowski space-time appears as a subspace of $\mathcal{M}_4$ which allows the description in terms of local events. It is the accurate analysis of the notion of local events in the $\mathcal{M}_4$ space-time that has the decisive role in establishing the relation to the geometry in the Minkowski space-time. One can say that the dynamics of relativistic systems in $\mathcal{M}_4$ is such that the Minkowski space-time turns out to be a visualization of $\mathcal{M}_4$ via signals described by equations (37) and (38), which can be focused (i.e., allow delta-function-like initial conditions) in not more than three directions. On the other hand, any visualization of $\mathcal{M}_4$ destroys the explicit character of some of the Sp(8) symmetries. A detailed analysis of the peculiarities of the relativistic dynamics in $\mathcal{M}_4$ was carried out in [18] showing that its description in $\mathcal{M}_4$ is consistent with the principles of classical and quantum field theories. Note that this approach allows in particular a geometrical interpretation of electromagnetic duality as a specific Sp(8) transformation.

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