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Physical fundamentals of the generalized Boltzmann kinetic theory of ionized gases

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Contents

Abstract. **In recent years, the kinetic and hydrodynamic descriptions of transport phenomena have been improved significantly through the extension of the Boltzmann physical kinetics. In this review the basic results of the generalized Boltzmann kinetic theory of partially and fully ionized gases are presented and some of its applications are discussed.**

1. Introduction

At the heart of the kinetic theory of neutral and ionized gases is the Boltzmann equation (BE) which describes how the one-particle distribution function f_1 changes over times of the order of the mean time between collisions and of the order of the gasdynamic flow time. Despite certain difficulties in the theory long-positioned as classic, the Boltzmann equation, now 130 years old [1], has had no alternatives until recently as the basis for physical kinetics.

A weak point of the classical Boltzmann kinetic theory is the way it treats the dynamic properties of interacting particles. On the one hand, as the so-called 'physical' derivation of the BE suggests [1, 2], Boltzmann particles are treated as material points; on the other hand, the collision integral in the BE brings into existence the cross sections for collisions between particles. A rigorous approach to the derivation of the kinetic equation for f_1 (KE_{f_1}) is based on the hierarchy of the Bogolyubov – Born – Green – Kirkwood – Yvon (BBGKY) equations. A KE_{f_1} obtained by the multiscale method turns into the BE if one ignores the change of the distribution function (DF) over a time of the order of the collision time (or, equivalently, over a length of the order of the particle interaction radius). It is important to note [3 – 5] that accounting for the third of the scales mentioned above has the consequence that, prior to introducing any approximations destined to break the Bogolyubov chain, additional terms, generally of the same order of magnitude, appear in the BE. If the method of correlation functions is used to derive KE_{f_1} from the BBGKY equations, then a passage to the BE implies the neglect of nonlocal and time delay effects.

Given the above difficulties of the Boltzmann kinetic theory (BKT), the following clearly interrelated questions arise. First, what is a physically infinitesimal volume and how does its introduction (and, as a consequence, the unavoidable smoothing out of the DF) affect the kinetic equation [3, 6]? And second, how does a systematic account for the proper diameter of the particle in the derivation of the KE_{f_1} affect the Boltzmann equation? In the theory we develop here, we will refer to the corresponding KE_{f_1} as the generalized Boltzmann equation, or GBE.

The derivation of the generalized Boltzmann equation and the applications of the generalized Boltzmann physical kinetics are presented, in particular, in Refs [3 – 5]. The review we offer the reader is a natural follow-up of our recent *Physics Uspekhi* paper [5] which outlines the basic concepts of the generalized Boltzmann kinetic theory as applied to neutral

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rarefied gases. In the same paper, a brief historical background and a general outline of the problem are given. Accordingly, our purpose in this introduction is first to explain the essence of the physical generalization of the BE and then to take a look at the specifics of the derivation of the GBE, when — as is the case in plasma physics — the self-consistent field of forces must of necessity be introduced.

As the Boltzmann equation is the centerpiece of the theory of transport processes (TPT), the introduction of an alternative KE_{f_1} leads in fact to an overhaul of the entire theory, including its macroscopic (for example, hydrodynamic) aspects. Conversely, a change in the macroscopic description will inevitably affect the kinetic level of description. Because of the complexity of the problem, this interrelation is not always easy to trace when solving a particular TPT problem. The important point to emphasize is that at issue here is not how to modify the classical equations of physical kinetics and hydrodynamics to include additional transport mechanisms (in reacting media, for example); rather we face a situation in which, those involved believe, we must go beyond the classical picture if we wish the revised theory to describe experiment adequately. The alternative TPTs can be grouped conventionally into the following categories: (1) theories that modify the macroscopic (hydrodynamic) description and neglect the possible changes of the kinetic description; (2) those changing the kinetic description at the KE_{f_1} level without bothering much whether these changes are consistent with the structure of the entire BBGKY chain, and (3) kinetic and hydrodynamic alternative theories consistent with the BBGKY hierarchy.

One of the pioneering efforts in the first line of research was a paper by Davydov [7], which stimulated a variety of studies (see, for instance, Refs [8 – 10]) on the hyperbolic equation of thermal conductivity. Introducing the second derivative of temperature with respect to time permitted a passage from the parabolic to the hyperbolic heat conduction equation, thus allowing for a finite heat propagation velocity. However, already in his 1935 paper B I Davydov points out that his method ''cannot be extended to the three-dimensional case'' and that ''here the assumption that all the particles move at the same velocity would separate out a five-dimensional manifold from the six-dimensional phase space, suggesting that the problem cannot be limited to the coordinate space alone''. We note, however, that also quasi-linear parabolic equations can produce wave solutions. Therefore, to hyperbolize the heat conduction equation phenomenologically [8] is not valid unless a rigorous kinetic justification is given. The hyperbolic heat conduction equation appears when the BE is solved by the Grad method [10] retaining a term which involves a derivative of the heat flow with respect to time and to which, in the context of the Chapman – Enskog method, no particular order of approximation can be ascribed.

Major difficulties arose when the question of existence and uniqueness of solutions of the Navier – Stokes equations was addressed. O A Ladyzhenskaya has shown for three-dimensional flows that under smooth initial conditions a unique solution is only possible over a finite time interval. Ladyzhenskaya even introduced a 'correction' into the Navier – Stokes equations in order that its unique solvability could be proved. It turned out that in this case the viscosity coefficient should be dependent on transverse flow-velocity gradients — with the result that the very idea of introducing kinetic coefficients should be overhauled.

G Uhlenbeck, in his review of the fundamental problems of statistical mechanics [11], examines in particular the Kramers equation [12] derived as a consequence of the Fokker – Planck equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = \beta\left[\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}f) + \frac{kT}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial f}{\partial \mathbf{v}}\right], \tag{1.1}$$

where $f(\mathbf{r}, \mathbf{v}, t)$ is the distribution function of Brownian particles, $\mathbf{a}$ is the acceleration due to an external field of forces, and $m\beta$ is the coefficient of friction for the motion of a colloid particle in the medium. What intrigues Uhlenbeck is how Kramers goes over from the Fokker – Planck equation (1.1) to the Einstein – Smoluchowski equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{\mathbf{a}}{\beta}\,\rho - \frac{kT}{m\beta}\frac{\partial \rho}{\partial \mathbf{r}}\right) = 0\,, \tag{1.2}$$

(ρ is the density) which has the character of the hydrodynamic continuity equation. In Uhlenbeck's words, ''the proof of this change-over is very interesting, it is a typical Kramers-style proof. It is in fact very simple but at the same time some tricks and subtleties it involves make it very hard to discuss''. The velocity distribution of colloid particles is assumed to be Maxwellian. The 'trick', however, is that Kramers integrated along the line $\mathbf{r} + \mathbf{v}/\beta = \mathbf{r}_0$, and the number density of particles turned out to be given by the formula

$$n(\mathbf{r}_0, t) = \int f\left(\mathbf{r}_0 - \frac{\mathbf{v}}{\beta}\,,\, \mathbf{v}, t\right)\mathrm{d}\mathbf{v}\,. \tag{1.3}$$

So what exactly did H Kramers do? Let us consider this change from the point of view of the generalized Boltzmann kinetic theory (GBKT) using, wherever possible, qualitative arguments to see things more clearly.

The structure of the KE_{f_1} is generally as follows

$$\frac{\mathrm{D}f_1}{\mathrm{D}t} = J^{\mathrm{B}} + J^{\mathrm{td}}\,, \tag{1.4}$$

where $\mathrm{D}/\mathrm{D}t$ is the substantial (particle) derivative, J^{B} is the (local) Boltzmann collision integral, and J^{td} is the nonlocal integral term incorporating the time delay effect. The generalized Boltzmann physical kinetics, in essence, involves a local approximation

$$J^{\mathrm{td}} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\tau\,\frac{\mathrm{D}f_1}{\mathrm{D}t}\right) \tag{1.5}$$

for the second collision integral, here τ being the mean time *between* the particle collisions. We can draw here an analogy with the Bhatnager – Gross – Krook (BGK) approximation for J^{B}:

$$J^{\mathrm{B}} = \frac{f_1^{(0)} - f_1}{\tau}\,, \tag{1.6}$$

whose popularity as a means to represent the Boltzmann collision integral is due to the huge simplifications it offers. The ratio of the second to the first term on the right-hand side of Eqn (1.4) is given to an order of magnitude as

$$\frac{J^{\mathrm{td}}}{J^{\mathrm{B}}} \approx O(\mathrm{Kn}^2)\,, \tag{1.7}$$

and at large Knudsen numbers these terms become of the same order of magnitude.

It would seem that at small Knudsen numbers answering to hydrodynamic description the contribution from the second term on the right-hand side of Eqn (1.4) is negligible. This is not the case, however. When one goes over to the hydrodynamic approximation (by multiplying the kinetic equation by collision invariants and then integrating over velocities), the Boltzmann integral part vanishes, and the second term on the right-hand side of Eqn (1.4) gives a single-order contribution in the generalized Navier – Stokes description. Mathematically, we cannot neglect a term with a small parameter in front of the higher derivative. Physically, the appearing additional terms are due to viscosity and they correspond to the small-scale Kolmogorov turbulence [3, 5]. The integral term J^{td}, thus, turns out to be important both at small and large Knudsen numbers in the theory of transport processes.

The important methodical question to be considered is how classical conservation laws fit into the GBE picture. Continuum mechanics conservation laws are derived on the macroscopic level by considering a certain reference volume within the medium, which is enclosed by an infinitesimally thin surface. Moving material points (gas particles) can be either within or outside the volume, and it is by writing down the corresponding balance equations for mass, momentum flux, and energy that the classical equations of continuity, motion, and energy are obtained. In particular, we obtain the continuity equation in the form

$$\frac{\partial \rho^{\mathrm{a}}}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\rho \mathbf{v}_0)^{\mathrm{a}} = 0\,, \tag{1.8}$$

where ρ^{a} is the gas density, $\mathbf{v}_0^{\mathrm{a}}$ is the hydrodynamical flow velocity, and $(\rho \mathbf{v}_0)^{\mathrm{a}}$ is the momentum flux density obtained by neglecting fluctuations. Thus, Boltzmann particles are fully 'packed' in the reference volume. It would appear that in continuum mechanics the idea of discreteness can be abandoned altogether and the medium under study be considered as a continuum in the literal sense of the word. Such an approach is of course possible and indeed leads to Euler equations in hydrodynamics. But when the viscosity and thermal conductivity effects are to be included, a totally different situation arises. As is well known, the dynamical viscosity is proportional to the mean time τ between the particle collisions, and a continuum medium in the Euler model with $\tau = 0$ implies that neither viscosity nor thermal conductivity are possible. The appearance of finite size particles in the reference contour leads to new effects.

Let a particle of finite radius be characterized, as before, by the position vector $\mathbf{r}$ and velocity $\mathbf{v}$ of its center of mass at some instant of time t. Then the fact that its center of mass is in the reference volume does not mean that all of the particle is there. In other words, at any given point in time there are always particles which are partly inside and partly outside of the reference surface, unavoidably leading to fluctuations in mass — and hence in other hydrodynamic quantities.

There are two important points to be made here. First, the fluctuations will be proportional to the mean time between the collisions (rather than the collision time). This fact is rigorously established in Refs [3 – 5], but it can also be made evident by means of quite simple arguments. Suppose we have a gas of hard spheres kept in a hard-wall cavity as shown in Fig. 1. Consider a reference contour drawn at a distance of the order of a particle diameter from the cavity wall. The

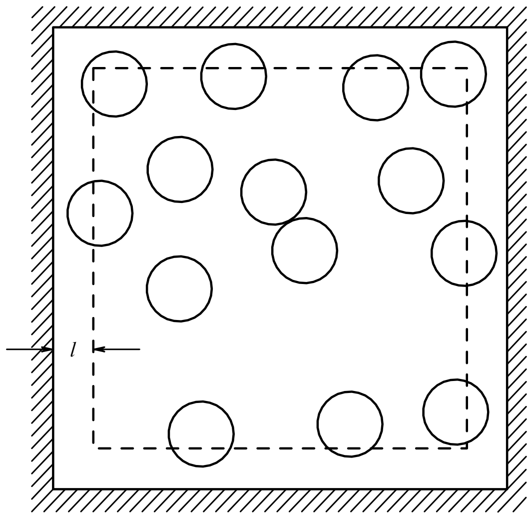

Figure 1. Closed cavity and the reference contour containing particles of a finite diameter.

mathematical expectation of the number of particles moving through the reference surface strictly perpendicular to the hard wall is zero. Therefore, in the first approximation, fluctuations will be proportional to the mean free path (or, equivalently, to the mean time between the collisions). As a result, the hydrodynamic equations will explicitly involve fluctuations proportional to τ. For example, the continuity equation changes its form and will contain terms proportional to viscosity [3]. On the other hand — and this is the second point to be made — if the reference volume extends over the whole of the cavity, then the classical conservation laws should be obeyed, and this is exactly what the paper [5] proves. However, we will here attempt to 'guess' the structure of the generalized continuity equation using the arguments outlined above.

Neglecting fluctuations, the continuity equation should have the classical form (1. 8) with

$$\rho^{\mathrm{a}} = \rho - \tau A\,, \tag{1.9}$$

$$(\rho \mathbf{v}_0)^{\mathrm{a}} = \rho \mathbf{v}_0 - \tau \mathbf{B}\,. \tag{1.10}$$

Strictly speaking, the factors A and $\mathbf{B}$ can be obtained from the generalized kinetic equation — in our case, from the GBE. Still, we can guess their form without appeal to the KE_{f_1}.

Indeed, let us write the generalized continuity equation

$$\frac{\partial}{\partial t}(\rho - \tau A) + \frac{\partial}{\partial \mathbf{r}} \cdot (\rho \mathbf{v}_0 - \tau \mathbf{B}) = 0 \tag{1.11}$$

in the dimensionless form using l, the distance from the reference contour to the hard wall (see Fig. 1), as a length scale. Then, instead of τ, the (already dimensionless) quantities A and $\mathbf{B}$ will have the Knudsen number $\mathrm{Kn}_l = \lambda / l$ as a coefficient. In the limit $l \to 0$, $\mathrm{Kn}_l \to \infty$, the contour embraces the entire cavity contained within hard walls, and there are no fluctuations on the walls. In other words, the classical equations of continuity and motion must be satisfied at the wall. Using hydrodynamic terminology, we note that the conditions

$$A = 0\,, \qquad \mathbf{B} = 0 \tag{1.12}$$

correspond to a laminar sublayer in a turbulent flow. Now if a local Maxwellian distribution is assumed, then the generalized equation of continuity in the Euler approximation is written as

$$\frac{\partial}{\partial t}\left\{\rho-\tau\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\cdot(\rho \mathbf{v}_0)\right]\right\}$$
$$+\frac{\partial}{\partial \mathbf{r}}\cdot\left\{\rho \mathbf{v}_0-\tau\left[\frac{\partial}{\partial t}(\rho \mathbf{v}_0)+\frac{\partial}{\partial \mathbf{r}}\cdot\rho \mathbf{v}_0\mathbf{v}_0+\overleftrightarrow{I}\cdot\frac{\partial p}{\partial \mathbf{r}}-\rho \mathbf{a}\right]\right\}=0\,. \tag{1.13}$$

In the hydrodynamic approximation, the mean time τ between the collisions is related to the dynamic viscosity η by $\tau p = \Pi\eta$, where the factor Π depends on the choice of a collision model and is $\Pi \approx 0.8$ for the particular case of neutral gas comprising hard spheres. The generalized equations of energy and motion are much more difficult to guess in this way, making the GBE indispensable. It is worthwhile though to say a few words about the treatment of the GBE (1.4) in the spirit of the fluctuation theory:

$$\frac{\mathrm{D}f^{\,\mathrm{a}}}{\mathrm{D}t}=J^{\mathrm{B}}(f)\,, \tag{1.14}$$

where $J^{\mathrm{B}}(f)$ is the Boltzmann collision integral, and

$$f^{\,\mathrm{a}}=f-\tau\,\frac{\mathrm{D}f}{\mathrm{D}t}\,. \tag{1.15}$$

Thus, $\tau\,\mathrm{D}f/\mathrm{D}t$ is the distribution function fluctuation, and writing Eqn (1.14) without taking into account Eqn (1.15) makes the BE nonclosed. From the viewpoint of the fluctuation theory, Boltzmann employed the simplest possible closure procedure:

$$f^{\,\mathrm{a}}=f. \tag{1.16}$$

Now, having in mind the Kramers method, let us compare the generalized continuity equation (1.13) and the Einstein – Smoluchowski equation (1.2). Equation (1.13) reduces to Eqn (1.2) if (a) the convective transfer corresponding to the hydrodynamical velocity $\mathbf{v}_0$ is neglected; (b) the temperature gradient is less important than the gradient of the number density of particles, $n\,\partial T/\partial \mathbf{r} \ll T\,\partial n/\partial \mathbf{r}$, and (c) the temporal part of the density fluctuations is left out of account. By integrating with respect to velocity v from $-\infty$ to $+\infty$ along the line $\mathbf{r}+\mathbf{v}/\beta=\mathbf{r}_0$, Kramers [see also Eqn (1.3)] introduced nonlocal collisions without accounting for the time delay effect. In our theory, the coefficient of friction $\beta=\tau^{-1}$, which corresponds to the binary collision approximation. If the simultaneous interaction with many particles is important and must be accounted for, additional difficulties associated with the definition of the coefficient of friction β arise, and Einstein – Smoluchowski theory becomes semiphenomenological. Overcoming these difficulties may require the use of the theory of non-Markov processes for describing Brownian motion [13].

Notice that the application of the above principles also leads to the modification of the system of Maxwell equations. While the traditional formulation of this system does not involve the continuity equation, its derivation explicitly employs the equation

$$\frac{\partial \rho^{\mathrm{a}}}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{j}^{\mathrm{a}}=0\,, \tag{1.17}$$

where ρ^{a} is the charge per unit volume, and $\mathbf{j}^{\mathrm{a}}$ the current density, both calculated without accounting for the fluctuations. As a result, the system of Maxwell equations written in the standard notation, namely

$$\begin{aligned}&\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{B}=0\,,\qquad \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{D}=\rho^{\mathrm{a}}\,,\\&\frac{\partial}{\partial \mathbf{r}}\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}\,,\qquad \frac{\partial}{\partial \mathbf{r}}\times\mathbf{H}=\mathbf{j}^{\mathrm{a}}+\frac{\partial \mathbf{D}}{\partial t}\,,\end{aligned} \tag{1.18}$$

contains $\rho^{\mathrm{a}}=\rho-\rho^{\mathrm{fl}}$, and $\mathbf{j}^{\mathrm{a}}=\mathbf{j}-\mathbf{j}^{\mathrm{fl}}$. The ρ^{fl}, $\mathbf{j}^{\mathrm{fl}}$ fluctuations calculated using the GBE are given, for example, in Ref. [3].

We shall now turn to approaches in which the KE_{f_1} can be changed in a way which is generally inconsistent with the BBGKY hierarchy.

It has been repeatedly pointed out that using a wrong distribution function for charged particles may have a catastrophic effect on the macroparameters of a weakly ionized gas. Let us have a look at some examples of this.

As is well known, the temperature dependence of the density of atoms ionized in plasma to various degrees was first studied by Saha [14] and Eggert [15]. For a system in thermodynamic equilibrium they obtained the equation

$$\frac{n_{j+1}n_{\mathrm{e}}}{n_j}=\frac{s_{j+1}}{s_j}\,\frac{(2\pi m_{\mathrm{e}}kT)^{3/2}}{h^3}\exp\left(-\frac{\varepsilon_j}{kT}\right), \tag{1.19}$$

where n_j is the number density of j-fold ionized atoms, n_{e} is the number density of free electrons, m_{e} is the electron mass, k the Boltzmann constant, h the Planck constant, s_j the statistical weight for a j-fold ionized atom [16], and ε_j the jth ionization potential. The Saha equation (1.19) is derived for the Maxwellian distribution and should necessarily be modified if another velocity distribution of particles exists in the plasma. This problem was studied in work [17], in which, for illustrative purposes, the values of $n_{j+1}n_{\mathrm{e}}/n_j$ calculated with the Maxwell distribution function are compared with those obtained with the Druyvesteyn distribution function, the average energies for both distributions being assumed equal. Let $T=10^4$ K, $n_{\mathrm{e}}=10^{14}\ \mathrm{cm}^{-3}$, $\varepsilon_j=10$ eV, the charge number $Z=1$, and $s_{j+1}/s_j=1$. Then one arrives at [17]

$$\frac{n_{j+1}n_{\mathrm{e}}}{n_j}=6\times10^2\ \text{(calculation using the Druyvesteyn distribution)},$$
$$\frac{n_{j+1}n_{\mathrm{e}}}{n_j}=4.53\times10^{16}\ \text{(calculation using the Maxwellian distribution function, by the Saha formula)}.$$

As E Dewan explained, ''the discrepancy in fourteen orders of magnitude obtained above is clearly due to the fact that, unlike Maxwellian distribution, the Druyvesteyn distribution does not have a 'tail''.

In our second example, two quantities — the ionization rate constant and the ionization cross section — were calculated by Gryzinski et al. [18] using the two above-mentioned distributions. The ionization cross section σ_{i} is defined by the following interpolation formula known to match satisfactorily the experimental data:

$$\sigma_{\mathrm{i}}=\frac{\sigma_0}{\varepsilon_{\mathrm{i}}^2}\,G_{\mathrm{i}}(\xi,\zeta)\,, \tag{1.20}$$

where $\sigma_0 = 6.56 \times 10^{-14}$ cm^2 (eV)^2, ε_i is the ionization potential of the atom, and ξ is a dimensionless parameter characterizing the atomic electron shell:

$$\xi = \frac{W}{\varepsilon_i} , \tag{1.21}$$

where W is the average kinetic energy of the atomic electrons, given by the formula

$$W = \frac{1}{N_e} \sum_{j=1}^{N_e} \varepsilon_j , \tag{1.22}$$

in which N_e is the number of electrons in the atom, and ε_j are the ionization potentials for the atom successively stripped of its electrons. The parameter ζ is defined by the expression

$$\zeta = \frac{U_e}{\varepsilon_i} , \tag{1.23}$$

where U_e is the energy of the electrons bombarding the atom. The neutral particle velocities are assumed to be much lower than the average electron velocity, and the plasma is taken to be uniform. The average value of the ionization cross section is then given by

$$\overline{\sigma_i} = \int_0^\infty \sigma_i(v_e) \, f(v_e) \, dv_e , \tag{1.24}$$

and the ionization rates are evaluated by the formula

$$\overline{\sigma_i v_e} = \int_0^\infty \sigma_i(v_e) \, v_e \, f(v_e) \, dv_e , \tag{1.25}$$

provided the function $G_i(\xi, \zeta)$ defined as [18]

$$G_i(\xi, \zeta) = (\zeta - 1)\left(1 + \frac{2}{3}\,\xi\right) \frac{1}{(\zeta + 1)(1 + \xi + \zeta)} \tag{1.26}$$

is known. Table 1 illustrates the calculated values of $\overline{\sigma_i}$ and $\overline{\sigma_i v_e}$ for $\xi = 1$ and various $\hat{T} = kT_e/\varepsilon_i$. It can be seen that the results obtained with different DFs can differ widely, indeed catastrophically so even for relatively small values of $\hat{T}$. Thus, the reliable computation of DFs remains a topic of intense current interest in plasma physics problems, the weak effect of the DF form on its moments being rather an exception than the rule.

Table 1. Comparison of ionization cross sections $\overline{\sigma_i}$ and ionization rates $\overline{\sigma_i v_e}$ calculated with the Maxwellian and Druyvesteyn DFs ($\xi = 1$).

$\hat{T}$	Maxwellian DF		Druyvesteyn DF	
	$\overline{\sigma_i}$	$\overline{\sigma_i v_e}$	$\overline{\sigma_i}$	$\overline{\sigma_i v_e}$
0.1	4.206×10^{-6}	1.184×10^{-5}	1.278×10^{-27}	4.077×10^{-27}
0.2	8.262×10^{-4}	1.184×10^{-3}	4.382×10^{-9}	1.011×10^{-8}
0.3	5.029×10^{-3}	9.251×10^{-3}	2.128×10^{-5}	4.135×10^{-5}
0.4	1.259×10^{-2}	2.103×10^{-2}	5.403×10^{-4}	9.405×10^{-4}
0.5	2.194×10^{-2}	3.415×10^{-2}	2.773×10^{-3}	4.466×10^{-3}
0.6	3.180×10^{-2}	4.687×10^{-2}	7.305×10^{-3}	1.110×10^{-2}
0.7	4.143×10^{-2}	5.842×10^{-2}	1.376×10^{-2}	1.998×10^{-2}
0.8	5.047×10^{-2}	6.857×10^{-2}	2.145×10^{-2}	3.001×10^{-2}
0.9	5.875×10^{-2}	7.733×10^{-2}	2.973×10^{-2}	4.033×10^{-2}
1	6.624×10^{-2}	8.482×10^{-2}	3.813×10^{-2}	5.039×10^{-2}
2	1.079×10^{-1}	1.171×10^{-1}	9.918×10^{-2}	1.132×10^{-1}
3	1.195×10^{-1}	1.190×10^{-1}	1.233×10^{-1}	1.312×10^{-1}
4	1.209×10^{-1}	1.137×10^{-1}	1.311×10^{-1}	1.717×10^{-1}
5	1.185×10^{-1}	1.069×10^{-1}	1.320×10^{-1}	1.298×10^{-1}
6	1.146×10^{-1}	9.992×10^{-2}	1.299×10^{-1}	1.243×10^{-1}
7	1.102×10^{-1}	9.326×10^{-2}	1.263×10^{-1}	1.184×10^{-1}
8	1.056×10^{-1}	8.704×10^{-2}	1.222×10^{-1}	1.125×10^{-1}
9	1.010×10^{-1}	8.123×10^{-2}	1.179×10^{-1}	1.069×10^{-1}
10	9.662×10^{-2}	7.589×10^{-2}	1.137×10^{-1}	1.017×10^{-1}

The use of collision cross sections which are 'self-consistent' with kinetic equations is also suggested by the well-known Enskog theory of moderately dense gases [19]. Enskog's idea was to describe the properties of such gases by separating the nonlocal part out of the essentially local Boltzmann collision integral. The transport coefficients obtained in this way for the hard-sphere model yielded an incorrect temperature dependence for the system's kinetic coefficients. To remedy this situation, the model of 'soft' spheres was introduced to fit the experimental data (see, for instance, Ref. [20]).

In the theory of the so-called kinetically consistent difference schemes [21], the DF is expanded in a power series of time, which corresponds to using an incomplete second approximation in the 'physical' derivation of the Boltzmann equation [5]. The result is that the difference schemes obtained contain only an artificial *ad hoc* viscosity chosen specially for the problem at hand. Some workers followed the steps of Davydov by adding the term $\partial^2 f/\partial t^2$ to the kinetic equations for fast processes. Bakaĭ and Sigov [22] suggest using such a term in the equation for describing DF fluctuations in a turbulent plasma. The so-called 'ordering parameter' they introduce alters the very type of the equation. To describe spatial nonlocality, Bakaĭ and Sigov complement the kinetic equation by the $\partial^2 f/\partial x^2$ term and higher derivatives, including mixed time – coordinate partial derivatives — a modification which can possibly describe non-Gaussian random sources in the Langevin equations [23]. It is interesting to note that the GBE also makes it possible to include higher derivatives of the DF (see the approximation (5.8) in Ref. [5]).

Clearly, approaches to the modification of the KE$_{f_1}$ must be based on certain principles, and it is appropriate to outline these in brief here. Of the approaches we have mentioned above, the most consistent one is the third, which clearly reveals the relation between alternative KE$_{f_1}$'s and the BBGKY hierarchy. There are general requirements to which the generalized KE$_{f_1}$ must satisfy.

(1) Because the artificial breaking of the BBGKY hierarchy is unavoidable in changing to a one-particle description, the generalized KE$_{f_1}$ should be obtainable with the known methods of the theory of kinetic equations, such as the multiscale approach, correlation function method, iterative methods, and so forth, or combinations of them. In each of these, some specific features of the particular alternative KE$_{f_1}$ are highlighted.

(2) There must be an explicit link between the KE$_{f_1}$ and the way we introduce the physically infinitesimal volume — and hence with the way the moments in the reference contour with transparent boundaries fluctuate due to the finite size of the particles.

(3) In the nonrelativistic case, the KE$_{f_1}$ must satisfy the Galileo transformation.

(4) The KE$_{f_1}$ must ensure a connection with the classical H-theorem and its generalizations.

(5) The KE$_{f_1}$ should not lead to unreasonable complexities in the theory.

The last requirement needs some commentary. The integral collision terms — in particular, the Boltzmann local integral, and especially the nonlocal integral with time delay — have a very complex structure. The 'caricature' the BGK approximation makes of the Boltzmann collision integral (to use Yu L Klimontovich's expressive word) has turned out to be a very successful approach, and this algebraically approximated Boltzmann collision integral is widely used in the kinetic theory of neutral and ionized gases. The generalized Boltzmann equation introduces a nonlocal differential approximation for the nonlocal collision integral with time delay. Here, we are faced in fact with the 'price – quality' problem familiar from economics. That is, what price — in terms of the increased complexity of the kinetic equation — are we ready to pay for the improved quality of the theory? An answer to this question is possible only through experience with practical problems.

A consistent theory meeting the above requirements is being developed, in particular, by Klimontovich [6, 24] and, the present author believes, within the GBE framework. One can recognize points of common ideology in the two approaches. However, whereas in Klimontovich's work the treatment of the physically infinitesimal volume is transferred to the 'upper echelon' of the BBGKY hierarchy and leads to a change in the Liouville equation, in the GBE theory it turns out that approximated nonlocal terms can even be introduced at the level of a one-particle description. The essential point to be made here is that GBE theory does well without specifying the smoothing procedure, whereas in Klimontovich's theory altering this procedure unavoidably modifies the alternative KE_{f_1}.

Vlasov [25] suggested that nonlocal effects could be described by introducing additional independent dynamic variables (derivatives of the velocity) into the one-particle distribution function. However — due primarily to the reasonable complexity requirement which should be met for a theory to be useful in practice — this approach is, in our view, too early to try until all traditional resources for describing the DF are exhausted.

From this perspective, fluctuation terms in the GBE are due to the fact that the reference volume as a measuring element for a system of finite-sized particles is introduced without changing the DF form used for describing point structureless particles.

The reader is referred to a monograph by Rudyak [26] for a detailed review of some other theories of transport properties (see also review [5]).

To conclude, it remains only to note that the effects listed above will always be relevant to a kinetic theory using a one-particle description — including, in particular, applications to liquids or plasmas, where self-consistent forces with appropriately cut-off radius of their action are introduced to expand the capabilities of the GBE.

2. Generalized Boltzmann equation including self-consistent forces

The dimensionless equation of the Bogolyubov – Born – Green – Kirkwood – Yvon (BBGKY) hierarchy for the s-particle distribution function f_s ($s = 1, \ldots, N$, N is the number of particles in the system) has the form

$$\frac{\partial \hat{f}_s}{\partial \hat{t}_{\mathrm{b}}} + \sum_{i=1}^{s} \hat{\mathbf{v}}_{i\mathrm{b}} \cdot \frac{\partial \hat{f}_s}{\partial \hat{\mathbf{r}}_{i\mathrm{b}}} + \sum_{ij=1}^{s} \hat{\mathbf{F}}_{ij} \cdot \frac{\partial \hat{f}_s}{\partial \hat{\mathbf{v}}_{i\mathrm{b}}} + \alpha \sum_{i=1}^{s} \hat{\mathbf{F}}_i \cdot \frac{\partial \hat{f}_s}{\partial \hat{\mathbf{v}}_{i\mathrm{b}}}$$
$$= -\varepsilon \frac{1}{N} \sum_{i=1}^{s} \sum_{j=s+1}^{N} \int \hat{\mathbf{F}}_{ij} \cdot \frac{\partial}{\partial \hat{\mathbf{v}}_{i\mathrm{b}}} \hat{f}_{s+1}(\hat{t}, \hat{\Omega}_1, \ldots, \hat{\Omega}_s, \hat{\Omega}_j)\, \mathrm{d}\hat{\Omega}_j\,, \tag{2.1}$$

where $\hat{f}_s = f_s v_{0\mathrm{b}}^{3s} n^{-s}$; $v_{0\mathrm{b}}$ is the characteristic collision velocity; n is the number density of particles; $\alpha = F_{0\lambda}/F_0$ is the ratio of the scales of the internal and external forces; $\mathrm{d}\Omega_j = \mathrm{d}\mathbf{r}_j\, \mathrm{d}\mathbf{v}_j$ is an elementary phase volume of the particle j, whose position is determined by the radius vector $\mathbf{r}_j$ and whose velocity is $\mathbf{v}_j$. We employ dot notation for a scalar product.

A spatial variable is nondimensionalized by introducing the interaction length r_{b}, and the characteristic time scale is set by $r_{\mathrm{b}} v_{0\mathrm{b}}^{-1}$; ε corresponds to the number of particles which is contained in the interaction volume v_{int} and serves as a small parameter in the kinetic theory of rarefied neutral gases. There are actually at least three groups of scales to consider in a rarefied gas. Apart from r_{b}, $v_{0\mathrm{b}}$, and $t_{0\mathrm{b}} = r_{\mathrm{b}}/v_{0\mathrm{b}}$, there exist 'mean free path' λ-scales (the mean free path λ, the mean free-flight velocity $v_{0\lambda}$, and the characteristic time scale $\lambda/v_{0\lambda}$) and L-parameters corresponding to hydrodynamic flow parameters (the characteristic hydrodynamic dimension L, the hydrodynamical velocity v_{0L}, and the hydrodynamic time L/v_{0L}).

The fundamental aspect of plasma physics is the presence of multiparticle interaction. The choice of the characteristic scales which determine the evolution of a plasma volume and are used in the multiscale method below is discussed in Appendix 1. Let us introduce a small parameter $\varepsilon = n r_{\mathrm{b}}^3 = v_{\mathrm{int}}$ assuming that the interaction energy per particle is much less than the particle's kinetic energy. We also assume that the plasma is nondegenerate and employ the multiscale approach. In the discussion to follow we shall concern ourselves with describing a physical system at the level of one-particle distribution function f_1 on the scales $r_{\mathrm{b}} \equiv l, \lambda, L$ (l, the Landau length; λ, the mean free path of a probe particle between two 'close' collisions, and L, hydrodynamic scale). Note that the mean free path of a plasma particle is introduced as

$$\lambda_n = \Lambda^{-1} \lambda\,, \tag{2.2}$$

with Λ being the Coulomb logarithm. The mean free path λ_n or the corresponding mean time between the collisions are involved in the definition of kinetic coefficients [27]. In the multiscale method [28, 29], $\hat{f}_s$ is expressed in the form of an asymptotic series

$$\hat{f}_s = \sum_{v=0}^{\infty} \hat{f}_s^{v}(\hat{t}_{\mathrm{b}}, \hat{\mathbf{r}}_{i\mathrm{b}}, \hat{\mathbf{v}}_{i\mathrm{b}}; \hat{t}_\lambda, \hat{\mathbf{r}}_{i\lambda}, \hat{\mathbf{v}}_{i\lambda}; \hat{t}_L, \hat{\mathbf{r}}_{iL}, \hat{\mathbf{v}}_{iL}) \varepsilon^{v} \tag{2.3}$$

in which the functions $\hat{f}_s^{v}$ depend on all the three types of variables.

From the above-written BBGKY equation, taking the derivatives of the composite functions on the left-hand side of this equation and then equating the coefficients of ε^0 and ε^1, we find that

$$\frac{\partial \hat{f}_1^0}{\partial \hat{t}_{\mathrm{b}}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} = 0\,, \tag{2.4}$$

$$\frac{\partial \hat{f}_1^1}{\partial \hat{t}_{\mathrm{b}}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \varepsilon_2 \frac{\partial \hat{f}_1^0}{\partial \hat{t}_\lambda} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1\lambda}}$$
$$+ \varepsilon_2 \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\lambda}} + \varepsilon_1 \varepsilon_2 \varepsilon_3 \frac{\partial \hat{f}_1^0}{\partial \hat{t}_L} + \varepsilon_1 \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1L}} + \frac{\varepsilon_2}{\varepsilon_3} \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1L}}$$
$$= -\sum_{\delta=1}^{\mu} \frac{N_\delta}{N} \int \hat{\mathbf{F}}_{1,\, j \in N_\delta} \cdot \frac{\partial}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} \hat{f}^0_{2,\, j \in N_\delta}\, \mathrm{d}\hat{\Omega}_{j \in N_\delta}\,, \tag{2.5}$$

where μ is the number of components in the mixture, N_δ is the number of particles of the δth kind, $\varepsilon_1 = \lambda/L$ (the Knudsen number), $\varepsilon_2 = v_{0\lambda}/v_{0\mathrm{b}}$, and $\varepsilon_3 = v_{0L}/v_{0\lambda}$. The integration in Eqn (2.5) is performed on the r_b scale. Importantly, no restriction is placed on the value of the Knudsen number. Equation (2.4) shows that the function $\hat{f}_1^0$ does not change along the phase trajectory on the r_b-scale — in other words, following the integration on the r_b-scale we have

$$\hat{f}_1^0 = \hat{f}_1^0(\hat{t}_\lambda, \hat{\mathbf{v}}_{1\lambda}, \hat{\mathbf{r}}_{1\lambda}; \hat{t}_L, \hat{\mathbf{v}}_{1L}, \hat{\mathbf{r}}_{1L})\,. \tag{2.6}$$

If the last function is known, $\hat{f}_1^1$ needs to be found from Eqn (2.5). This is possible if certain additional assumptions are posed on the function $\hat{f}_2^0$ entering the right-hand, integral part of the expression (2.5). Thus we see that the system of equations contains linked terms. In real life, the dependence (2.6) is unknown beforehand. Then Eqn (2.5) can serve to determine $\hat{f}_1^0$ on the λ- and L-scales, but in this case it becomes doubly linked, with respect to both the lower index '2' and the upper index '1'. As a result, the problem of breaking the equations arises.

Let us now write the analogue of Eqn (2.4) for the two-particle function $\hat{f}_2^0$ dependent on time and on the dynamic variables for the particles 1 and j:

$$\frac{\partial \hat{f}_2^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{r}}_{j\in N_\delta,\mathrm{b}}} + \hat{\mathbf{F}}_{1,j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}$$
$$+ \hat{\mathbf{F}}_{j\in N_\delta,1} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_{j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,1}} = 0\,. \tag{2.7}$$

Introducing the new variable $\hat{\mathbf{x}}_{1,j\in N_\delta} = \hat{\mathbf{r}}_{1\mathrm{b}} - \hat{\mathbf{r}}_{j\in N_\delta,\mathrm{b}}$, we find from Eqn (2.7) that

$$-\hat{\mathbf{F}}_{1,j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} = \frac{\partial \hat{f}_2^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}}$$
$$+ (\hat{\mathbf{v}}_{1\mathrm{b}} - \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}) \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{x}}_{1,j\in N_\delta}} + \hat{\mathbf{F}}_{j\in N_\delta,1} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}}$$
$$+ \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_{j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}}\,. \tag{2.8}$$

Using the last equation, we obtain the following representation for the integral in Eqn (2.5):

$$-\int \hat{\mathbf{F}}_{1,j\in N_\delta} \cdot \frac{\partial}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}\, \hat{f}_2^0(\hat{t}, \hat{\Omega}_1, \hat{\Omega}_{j\in N_\delta})\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$= \int (\hat{\mathbf{v}}_{1\mathrm{b}} - \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}) \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{x}}_{1,j\in N_\delta}}\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$+ \int \left(\frac{\partial \hat{f}_2^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_2^0}{\partial \mathbf{v}_{1\mathrm{b}}} \right.$$
$$\left. + \alpha \hat{\mathbf{F}}_{j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}} \right) \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$+ \int \hat{\mathbf{F}}_{j\in N_\delta,1} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}}\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}\,. \tag{2.9}$$

The last integral on the right-hand side of Eqn (2.9) can be written in the form

$$\int \hat{\mathbf{F}}_{j\in N_\delta,1} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}}\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$= \int \left[\int \frac{\partial}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}} \cdot (\hat{\mathbf{F}}_{j\in N_\delta,1}\, \hat{f}_2^0)\, \mathrm{d}\hat{\mathbf{v}}_{j\in N_\delta} \right] \mathrm{d}\hat{\mathbf{r}}_{j\in N_\delta}\,. \tag{2.10}$$

But the inner integral can be transformed by the Gauss theorem into an integral taken over an infinitely distant surface in the velocity space, which vanishes because $\hat{f}_2^0 \to 0$ for $\hat{v}_j \to \infty$.

Let us now introduce two-particle correlation functions $\hat{W}_2(\hat{t}, \hat{\Omega}_1, \hat{\Omega}_{j\in N_\delta})$ (hereinafter f_j is the one-particle function corresponding to the particles N_j):

$$\hat{f}_2^0(\hat{t}, \hat{\Omega}_1, \hat{\Omega}_j) = \hat{f}_1^0(\hat{t}, \hat{\Omega}_1)\, \hat{f}_{j\in N_\delta}(\hat{t}, \hat{\Omega}_{j\in N_\delta}) + \hat{W}_2^0(\hat{t}, \hat{\Omega}_1, \hat{\Omega}_{j\in N_\delta})\,. \tag{2.11}$$

The next to last integral in Eqn (2.9) then becomes

$$\int \left(\frac{\partial \hat{f}_2^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_{j\in N_\delta} \cdot \frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta}} \right) \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$= \int \left[\hat{f}_{j\in N_\delta}^0 \left(\frac{\partial \hat{f}_1^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} \right) \right] \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$+ \int \hat{f}_1^0\, \frac{\partial \hat{f}_{j\in N_\delta}^0}{\partial \hat{t}_\mathrm{b}}\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$+ \alpha \int \frac{\partial}{\partial \hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}} \cdot (\hat{\mathbf{F}}_{j\in N_\delta}\, \hat{f}_1^0\, \hat{f}_{j\in N_\delta}^0)\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}$$
$$+ \int \left(\frac{\partial \hat{W}_2^0}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{W}_2^0}{\partial \mathbf{r}_{1\mathrm{b}}} + \alpha \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{W}_2^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} \right.$$
$$\left. + \alpha \hat{\mathbf{F}}_{j\in N_\delta} \cdot \frac{\partial \hat{W}_2^0}{\partial \hat{\mathbf{v}}_{j\in N_\delta}} \right) \mathrm{d}\hat{\Omega}_{j\in N_\delta}\,. \tag{2.12}$$

In expression (2.12), the first integral on the right is zero because of the relation (2.4), and the third integral is zero for the same reasons as in Eqn (2.10). The situation with the second and fourth integrals, however, requires a more detailed treatment. Consider first the integral

$$A = \int \hat{f}_1^0\, \frac{\partial \hat{f}_{j\in N_\delta}^0}{\partial \hat{t}_\mathrm{b}}\, \mathrm{d}\hat{\Omega}_{j\in N_\delta}\,. \tag{2.13}$$

The dynamic variables determining the motion of the given trial particles 1 and j are correlated with one another in the collision of the particles, i.e., on the r_b-scale. In the center-of-mass system, the equations of motion for these particles are written as

$$\dot{\mathbf{v}}_{1\mathrm{b}} = \mathbf{F}_{1j}\,, \qquad \dot{\mathbf{v}}_{j\mathrm{b}} = \mathbf{F}_{j1}\,, \qquad \mathbf{p}_1 = -\mathbf{p}_j\,, \tag{2.14}$$

where a dot over denotes differentiation with respect to time, and $\mathbf{p}$ is the particle momentum.

Using equations (2.14) and integrating by parts, we arrive at the relation

$$A \cong -\hat{\mathbf{F}}_{1\delta}^\mathrm{a} \cdot \frac{\partial \hat{f}_1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}\,, \tag{2.15}$$

where $\hat{\mathbf{F}}^{\mathrm{a}}_{1\delta}$ is the average force acting on particle 1 during its collision with particle j which has an arbitrary velocity and an arbitrary position on the r_{b}-scale (particles j belong to the chemical component δ):

$$\hat{\mathbf{F}}^{\mathrm{a}}_{1\delta} = \int \hat{f}_j \hat{\mathbf{F}}_{1j} \,\mathrm{d}\hat{\mathbf{v}}_{j\in N_\delta}\,\mathrm{d}\hat{\mathbf{r}}_{j\in N_\delta}\,. \tag{2.16}$$

Thus, the integral A vanishes provided that the self-consistent force of internal nature can be neglected, in particular, in comparison with the external force acting on particle 1. We next transform the integral A further by using the series (2.2), to obtain

$$A \cong -\hat{\mathbf{F}}^{\mathrm{a}}_{1\delta}\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} - \varepsilon\hat{\mathbf{F}}^{\mathrm{a}}_{1\delta}\cdot\frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}\,. \tag{2.17}$$

The last term in Eqn (2.17) ensures, as we shall see below, that the generalized kinetic equation is written in a symmetrical form.

Now consider the fourth — the last — integral on the right-hand side of Eqn (2.12). To do this, we write down an equation for the two-particle function f_2 of the Bogolyubov chain, in which, in this case, we do not separate out groups of particles belonging to a certain chemical component. The two-particle function f_2 corresponds to the dynamical variables of particles N_1, N_2 and is written in the form $f_2 = f_2(1,2)$ for brevity. Thus, one finds

$$\begin{aligned}
&\frac{\partial f_2}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f_2}{\partial \mathbf{r}_1} + \mathbf{v}_2\cdot\frac{\partial f_2}{\partial \mathbf{r}_2} + \mathbf{F}_{12}\cdot\frac{\partial f_2}{\partial \mathbf{v}_1} + \mathbf{F}_{21}\cdot\frac{\partial f_2}{\partial \mathbf{v}_2}\\
&\quad + \mathbf{F}_1\cdot\frac{\partial f_2}{\partial \mathbf{v}_1} + \mathbf{F}_2\cdot\frac{\partial f_2}{\partial \mathbf{v}_2}\\
&\quad = -\int\Big\{\mathbf{F}_{13}\cdot\frac{\partial}{\partial \mathbf{v}_1}\big[f_1(1)f_1(2)f_1(3) + f_1(1)W_2(2,3)\\
&\quad + f_1(2)W_2(1,3) + f_1(3)W_2(1,2)\big]\\
&\quad + \mathbf{F}_{23}\cdot\frac{\partial}{\partial \mathbf{v}_2}\big[f_1(1)f_1(2)f_1(3) + f_1(1)W_2(2,3)\\
&\quad + f_1(2)W_2(1,3) + f_1(3)W_2(1,2)\big]\Big\}\,\mathrm{d}\Omega_3\,,
\end{aligned} \tag{2.18}$$

where the three-particle function f_3 has been approximated by using correlation functions as follows:

$$\begin{aligned}
&f_3(\Omega_1,\Omega_2,\Omega_3,t) = f_1(\Omega_1,t)\,f_1(\Omega_2,t)\,f_1(\Omega_3,t)\\
&\quad + f_1(\Omega_1,t)W_2(\Omega_2,\Omega_3,t) + f_1(\Omega_2,t)W_2(\Omega_1,\Omega_3,t)\\
&\quad + f_1(\Omega_3,t)W_2(\Omega_1,\Omega_2,t) + W_3(\Omega_1,\Omega_2,\Omega_3,t)\,.
\end{aligned} \tag{2.19}$$

The effect of the correlation function $W_3(\Omega_1,\Omega_2,\Omega_3,t)$ is here neglected.

Equation (2.18) written in the zeroth approximation in ε as an equation for finding f_2^0 is identical with Eqn (2.7) only when the zero-order correlation functions are zero, viz.

$$\begin{aligned}
&W_2^0(\Omega_2,\Omega_3,t) = 0\,, \qquad W_2^0(\Omega_1,\Omega_3,t) = 0\,,\\
&W_2^0(\Omega_1,\Omega_2,t) = 0\,, \qquad W_3^0(\Omega_1,\Omega_2,\Omega_3,t) = 0\,,
\end{aligned} \tag{2.20}$$

and when the interaction forces determining the effect of the third particle on the first and the second ones during their 'close' collision are small, i.e., $F_{13}\approx 0$, $F_{23}\approx 0$. Hence, in the multiscale approach, polarization terms on the right-hand side of Eqn (2.18) appear in the next, of order small $o(\varepsilon^2)$, approximation.

Thus, in the multiscale approach, the last integral on the right-hand side of Eqn (2.12) vanishes because of the condition (2.20). The integral relation (2.9) can be written in the following form

$$\begin{aligned}
&-\int \hat{\mathbf{F}}_{1,\,j\in N_\delta}\cdot\frac{\partial}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}\hat{f}_2^0(\hat{t},\hat{\Omega}_1,\hat{\Omega}_{j\in N_\delta})\,\mathrm{d}\hat{\Omega}_{j\in N_\delta}\\
&\quad = \int(\hat{\mathbf{v}}_{1\mathrm{b}} - \hat{\mathbf{v}}_{j\in N_\delta})\cdot\frac{\partial \hat{f}_2^0}{\partial \hat{\mathbf{x}}_{1,\,j\in N_\delta}}\,\mathrm{d}\hat{\Omega}_{j\in N_\delta}\\
&\quad - \hat{\mathbf{F}}^{\mathrm{a}}_{1\delta}\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} - \varepsilon\hat{\mathbf{F}}^{\mathrm{a}}_{1\delta}\cdot\frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}\,.
\end{aligned} \tag{2.21}$$

We now introduce the cylindrical coordinate system $\hat{l}, \hat{b}, \varphi$ with the origin at point $\mathbf{r}_1$ and $\hat{l}$-axis parallel to the vector of the relative velocity of the colliding particles 1 and j. Then, in terms of $\hat{b}$ (dimensionless impact parameter) and φ (azimuthal angle), the first term on the right-hand side of Eqn (2.21) is written as

$$\begin{aligned}
\hat{J}^{\,\mathrm{st},0} &= \sum_{\delta=1}^{\mu}\frac{N_\delta}{N}\int \hat{g}_{j\in N_\delta,1}\left[\int_{-\infty}^{+\infty}\frac{\partial \hat{f}_2^0}{\partial \hat{l}}\,\mathrm{d}\hat{l}\right]\hat{b}\,\mathrm{d}\hat{b}\,\mathrm{d}\varphi\,\mathrm{d}\hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}\\
&= \sum_{\delta=1}^{\mu}\frac{N_\delta}{N}\int\big[\hat{f}_2^0(+\infty) - \hat{f}_2^0(-\infty)\big]\,\hat{g}_{j\in N_\delta,1}\hat{b}\,\mathrm{d}\hat{b}\,\mathrm{d}\varphi\,\mathrm{d}\hat{\mathbf{v}}_{j\in N_\delta,\mathrm{b}}\,.
\end{aligned} \tag{2.22}$$

The integration in Eqn (2.22) was performed on the r_{b} scale, i.e., the distribution functions $\hat{f}_2^0(+\infty)$, $\hat{f}_2^0(-\infty)$ are calculated for the velocities $\hat{\mathbf{v}}_1'$, $\hat{\mathbf{v}}_{j\in N_\delta}'$ and $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_{j\in N_\delta}$ in the situation where the particles are found outside of their region of interaction — in other words, before or after the collision (with primed velocities in the latter case). If before the collision the conditions of molecular chaos are fulfilled on the λ-scale, then the two-particle DFs can be expressed as a product of one-particle DFs. In this case $\hat{J}^{\,\mathrm{st},0}$ is the Boltzmann collision ('stoß') integral:

$$\hat{J}^{\,\mathrm{st},0} = \sum_{\delta=1}^{\mu}\frac{N_\delta}{N}\int\big[\hat{f}_1'^{\,0}\hat{f}_{j\in N_\delta}'^{\,0} - \hat{f}_1^0\hat{f}_{j\in N_\delta}^0\big]\,\hat{g}_{j\in N_\delta,1}\hat{b}\,\mathrm{d}\hat{b}\,\mathrm{d}\varphi\,\mathrm{d}\hat{\mathbf{v}}_{j\in N_\delta}\,. \tag{2.23}$$

Lenard [30] and Balescu [31] solved the equation for the correlation function W_2 under the assumptions of a weakened initial correlation, no time delay, and spatially uniform DF f_1. The corresponding collision integral (the Balescu – Lenard collision integral) incorporates the polarization of the plasma and allows elimination of the logarithmic divergence of the Boltzmann collision integral for a Coulomb plasma [30 – 33]. If, however, the Boltzmann collision integral is still used in plasma description, a cut-off procedure involving Debye screening must be employed.

Using expressions (2.20), the kinetic equation (2.5) is written down in the form

$$\begin{aligned}
&\frac{\partial \hat{f}_1^1}{\partial \hat{t}_{\mathrm{b}}} + \hat{\mathbf{v}}_{1\mathrm{b}}\cdot\frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + (\alpha\hat{\mathbf{F}}_1 + \varepsilon\hat{\mathbf{F}}_1^{\mathrm{a}})\cdot\frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \varepsilon_2\frac{\partial \hat{f}_1^0}{\partial \hat{t}_\lambda} + \hat{\mathbf{v}}_{1\mathrm{b}}\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1\lambda}}\\
&\quad + \varepsilon_1\varepsilon_2\varepsilon_3\frac{\partial \hat{f}_1^0}{\partial \hat{t}_L} + \varepsilon_1\hat{\mathbf{v}}_{1\mathrm{b}}\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1L}} + \hat{\mathbf{F}}_1^{\mathrm{a}}\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \varepsilon_2\hat{\mathbf{F}}_1\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\lambda}}\\
&\quad + \frac{\varepsilon_2}{\varepsilon_3}\,\hat{\mathbf{F}}_1\cdot\frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1L}} = \hat{J}^{\,\mathrm{st},0}\,,
\end{aligned} \tag{2.24}$$

where

$$\hat{\mathbf{F}}_1^{\mathrm{a}} = \sum_{\delta=1}^{\mu} \frac{N_\delta}{N} \hat{\mathbf{F}}_{1\delta}^{\mathrm{a}} .$$

It should be emphasized that in its dimensional form the factor

$$\varepsilon \hat{\mathbf{F}}_{1\delta}^{\mathrm{a}} = \frac{1}{v_{0\mathrm{b}}^2 / r_\mathrm{b}} \int f_{j\in N_\delta} \mathbf{F}_{1,\, j\in N_\delta} \,\mathrm{d}\mathbf{v}_{j\in N_\delta} \,\mathrm{d}\mathbf{r}_{j\in N_\delta} , \qquad (2.25)$$

if it is remembered that

$$\varepsilon = n r_\mathrm{b}^3 , \quad \hat{f} = f v_{0\mathrm{b}}^3 n^{-1} , \quad \hat{v} = \frac{v}{v_{0\mathrm{b}}} ,$$
$$\hat{r} = \frac{r}{r_\mathrm{b}} , \quad \hat{F}_{1j} = \frac{F_{1j}}{v_{0\mathrm{b}}^2 / r_\mathrm{b}} .$$

The scale of the internal force F_{1j} corresponds to choosing the Landau length l as r_b. Let us now write Eqn (2.24) in the form (cf. Eqn (2.10) of Ref. [5])

$$\frac{\mathrm{D}_1 \hat{f}_1^1}{\mathrm{D}\hat{t}_\mathrm{b}} + \frac{\mathrm{d}_1 \hat{f}_1^0}{\mathrm{d}\hat{t}_{\mathrm{b},\lambda,L}} = \hat{J}^{\mathrm{st},0} , \qquad (2.26)$$

where we have introduced the notation

$$\frac{\mathrm{D}_1 \hat{f}_1^1}{\mathrm{D}\hat{t}_\mathrm{b}} = \frac{\partial \hat{f}_1^1}{\partial \hat{t}_\mathrm{b}} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} + (\alpha \hat{\mathbf{F}}_1 + \varepsilon \hat{\mathbf{F}}_1^{\mathrm{a}}) \cdot \frac{\partial \hat{f}_1^1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} , \qquad (2.27)$$

$$\frac{\mathrm{d}_1 \hat{f}_1^0}{\mathrm{d}\hat{t}_{\mathrm{b},\lambda,L}} = \varepsilon_2 \frac{\partial \hat{f}_1^0}{\partial \hat{t}_\lambda} + \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1\lambda}} + \varepsilon_1 \varepsilon_2 \varepsilon_3 \frac{\partial \hat{f}_1^0}{\partial \hat{t}_L} + \varepsilon_1 \hat{\mathbf{v}}_{1\mathrm{b}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{r}}_{1L}}$$
$$+ \hat{\mathbf{F}}_1^{\mathrm{a}} \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} + \varepsilon_2 \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1\lambda}} + \frac{\varepsilon_2}{\varepsilon_3} \hat{\mathbf{F}}_1 \cdot \frac{\partial \hat{f}_1^0}{\partial \hat{\mathbf{v}}_{1L}} . \qquad (2.28)$$

We now wish to use Eqn (2.26) for describing the evolution of the distribution function $\hat{f}_1^0$ — but the trouble is, this equation involves a single-order term $\mathrm{D}_1 \hat{f}_1^1 / \mathrm{D}_1 \hat{t}_\mathrm{b}$ linked with respect to upper index. So we are faced with the problem of how to approximate this term — a problem which is similar in a sense to that of approximating the two-particle distribution function in the collision integral. Using the series (2.3) allows an exact representation for the term of interest:

$$\frac{\mathrm{D}_1 \hat{f}_1^1}{\mathrm{D}\hat{t}_\mathrm{b}} = \frac{\mathrm{D}_1}{\mathrm{D}\hat{t}_\mathrm{b}} \left[\frac{\partial \hat{f}_1}{\partial \varepsilon} \right]_{\varepsilon=0} . \qquad (2.29)$$

The term $\mathrm{D}_1 \hat{f}_1^1 / \mathrm{D}_1 \hat{t}_\mathrm{b}$ describes how the distribution function varies over times of the order of the collision time, or equivalently on the r_b-scale. If this term is left out of account then, from the viewpoint of the derivation of the hierarchy of kinetic equations, this means that

(1) the distribution function does not vary on the r_b-scale [provided we also neglect the average internal force that gives rise to the second and third terms on the right in Eqn (2.21)];

(2) the particles are point-like and structureless;

(3) changes in DF due to collisions take place instantaneously and are described by the source term $\hat{J}^{\mathrm{st},0}$.

In the field description, however, the DF f_1 on the interaction scale (r_b-scale) depends on ε through the dynamic variables $\mathbf{r}$, $\mathbf{v}$, t related by the laws of classical mechanics, thus allowing the approximation [3–5]

$$\frac{\mathrm{D}_1}{\mathrm{D}\hat{t}_\mathrm{b}} \left[\left(\frac{\partial \hat{f}_1}{\partial \varepsilon} \right)_{\varepsilon=0} \right] \cong \frac{\mathrm{D}_1}{\mathrm{D}(-\hat{t}_\mathrm{b})} \left[\frac{\partial \hat{f}_1}{\partial(-\hat{t}_\mathrm{b})} \left(\frac{\partial(-\hat{t}_\mathrm{b})}{\partial \varepsilon} \right)_{\varepsilon=0} \right.$$
$$+ \frac{\partial \hat{f}_1}{\partial \hat{\mathbf{r}}_{1\mathrm{b}}} \cdot \frac{\partial \hat{\mathbf{r}}_\mathrm{b}}{\partial(-\hat{t}_\mathrm{b})} \left(\frac{\partial(-\hat{t}_\mathrm{b})}{\partial \varepsilon} \right)_{\varepsilon=0}$$
$$\left. + \frac{\partial \hat{f}_1}{\partial \hat{\mathbf{v}}_{1\mathrm{b}}} \cdot \frac{\partial \hat{\mathbf{v}}_{1\mathrm{b}}}{\partial(-\hat{t}_\mathrm{b})} \left(\frac{\partial(-\hat{t}_\mathrm{b})}{\partial \varepsilon} \right)_{\varepsilon=0} \right]$$
$$= -\frac{\mathrm{D}_1}{\mathrm{D}\hat{t}_\mathrm{b}} \left[\left(\frac{\partial \hat{t}_\mathrm{b}}{\partial \varepsilon} \right)_{\varepsilon=0} \frac{\mathrm{D}_1 \hat{f}_1}{\mathrm{D}\hat{t}_\mathrm{b}} \right] \cong -\frac{\mathrm{D}_1}{\mathrm{D}\hat{t}_\mathrm{b}} \left[\left(\frac{\partial \hat{t}_\mathrm{b}}{\partial \varepsilon} \right)_{\varepsilon=0} \frac{\mathrm{D}_1 \hat{f}_1^0}{\mathrm{D}\hat{t}_\mathrm{b}} \right] . \qquad (2.30)$$

In expression (2.30) we have introduced an approximation proceeded against the flying direction of an arrow of time, which corresponds to the condition of there being no correlations for $t_0 \to -\infty$, with t_0 being a certain instant of time on the r_b-scale at which the particles start to interact. In this way, Markov processes are separated out from all stochastic processes possible in the system.

For the particles of the chemical sort α ($\alpha = 1, \ldots, \mu$) we employ the following normalized DF:

$$f_\alpha = \frac{f_1 N_\alpha}{N} , \quad \int f_\alpha \,\mathrm{d}\mathbf{v}_\alpha = n_\alpha , \quad \int n_\alpha \,\mathrm{d}\mathbf{r} = N_\alpha . \qquad (2.31)$$

In Eqn (2.31), f_1 is a one-particle DF. Returning to the expression (2.26) written in the dimensional form, we convolute the multiscale substantial derivatives to obtain

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} - \frac{\mathrm{D}}{\mathrm{D}t} \left[\tau_\alpha \frac{\mathrm{D}f_\alpha}{\mathrm{D}t} \right] = \sum_{\beta=1}^{\mu} \int \left[f'_\alpha f'_\beta - f_\alpha f_\beta \right] g_{\beta\alpha} b \,\mathrm{d}b \,\mathrm{d}\varphi \,\mathrm{d}\mathbf{v}_\beta , \qquad (2.32)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F}_\alpha^{\mathrm{sc}} \cdot \frac{\partial}{\partial \mathbf{v}_\alpha} , \quad \mathbf{F}_\alpha^{\mathrm{sc}} = \mathbf{F}_\alpha + \mathbf{F}_\alpha^{\mathrm{a}} . \qquad (2.33)$$

Let us comment on equation (2.32).

(1) We consider that the particle numbered 1 in the multicomponent mixture belongs to a component α, which is exactly what the subscript α on the symbol of the DF indicates. Note also that we dropped the superscript 0 from this symbol: carrying it no longer makes sense because all the equations hereinafter already contain only functions of zero order (in the sense of the series expansion in terms of the density parameter ε).

(2) The parameter τ_α is written in the form

$$\tau_\alpha = \frac{\varepsilon}{\left[\partial \varepsilon / \partial t \right]_{\varepsilon=0}} , \qquad (2.34)$$

where ε is the number of particles of all kinds that find themselves in the interaction volume of an α particle by the instant of time t; introducing $\varepsilon^{\mathrm{eq}}$ (the 'equilibrium' particle density in the close interaction volume), Eqn (2.34) is written in a typical relaxation form

$$\frac{\partial \varepsilon}{\partial t} = -\frac{\varepsilon(t) - \varepsilon^{\mathrm{eq}}}{\tau_\alpha} . \qquad (2.35)$$

The denominator in Eqn (2.34) is interpreted as the number of particles that find themselves within the interaction volume of a certain particle belonging to the αth component per unit time; the derivative is calculated under the additional condition $\varepsilon = 0$. Clearly, this number is equal to the number of collisions occurring in the interaction volume per unit time. Hence, τ_α is the mean time between collisions of a particle of the αth sort with particles of all other sorts. The procedure includes the action, during the collision of the particles, of the self-consistent force $\mathbf{F}^{\mathrm{sc}}$ being the sum of the external force and the force $\mathbf{F}^{\mathrm{a}}$ of internal origin.

As the derivation of formula (2.34) suggests, τ_α is determined by close collisions occurring in the plasma. By analogy with expression (2.2) we have

$$\tau_\alpha^n = \Lambda^{-1}\tau_\alpha\,, \tag{2.36}$$

where τ_α^n is the mean time between collisions.

In the hydrodynamic approximation, the time τ_α can be expressed in terms of the viscosity η_α of the component α [27, 34]; for example, for ions one has

$$\tau_\alpha = \Lambda \Pi \eta_\alpha p_\alpha^{-1}\,. \tag{2.37}$$

Equation (2.37) involves the coefficient Π which is determined by the interaction model (for ions $\Pi = 1.04$ [27, 35], and the static pressure

$$p_\alpha = n_\alpha k T_\alpha\,. \tag{2.38}$$

The generalized Boltzmann equation is invariant under the Galileo transformation and has a correct free-molecular and Maxwellian asymptotic behavior. Alternative approaches to the derivation of the KE$_{f_1}$ are discussed elsewhere [5].

We shall now write down the system of generalized hydrodynamic equations. These equations have been obtained previously [3 – 5] for gaseous systems in an external field of forces. The distinguishing feature of the generalized Enskog equations we display below is the inclusion of the self-consistent forces $\mathbf{F}^{\mathrm{sc}}$ [see formulas (2.33)]. The continuity equation for the component α is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho_\alpha - \tau_\alpha\left[\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot(\rho_\alpha \overline{\mathbf{v}}_\alpha)\right]\right\} \\
&\quad + \frac{\partial}{\partial \mathbf{r}}\cdot\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}\right.\right. \\
&\quad \left.\left. - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}^{\mathrm{sc}}\right]\right\} = R_\alpha\,,
\end{aligned} \tag{2.39}$$

the equation of motion is written as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}}\right.\right. \\
&\quad \left.\left. - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}^{\mathrm{sc}}\right]\right\} - \mathbf{F}_\alpha^{(1)\mathrm{sc}}\left[\rho_\alpha - \tau_\alpha\left(\frac{\partial\rho_\alpha}{\partial t} + \frac{\partial}{\partial\mathbf{r}}\cdot(\rho_\alpha\overline{\mathbf{v}}_\alpha)\right)\right] \\
&\quad - \frac{q_\alpha}{m_\alpha}\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}\right.\right. \\
&\quad \left.\left. - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}^{\mathrm{sc}}\right]\right\}\times\mathbf{B}^{\mathrm{sc}} \\
&\quad + \frac{\partial}{\partial\mathbf{r}}\cdot\left\{\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}) + \frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\overline{(\mathbf{v}_\alpha\mathbf{v}_\alpha)\mathbf{v}_\alpha}\right.\right. \\
&\quad - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}}\,\overline{\mathbf{v}}_\alpha - \rho_\alpha\overline{\mathbf{v}}_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{(\mathbf{v}_\alpha\times\mathbf{B}^{\mathrm{sc}})\mathbf{v}_\alpha} \\
&\quad \left.\left. - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}_\alpha(\mathbf{v}_\alpha\times\mathbf{B}^{\mathrm{sc}})}\right]\right\} = \overline{\mathbf{I}}_{\alpha,\,\mathrm{mot}}\,,
\end{aligned} \tag{2.40}$$

and the equation of energy has the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\frac{\rho_\alpha\overline{v_\alpha^2}}{2} + \varepsilon_\alpha n_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{\rho_\alpha\overline{v_\alpha^2}}{2} + \varepsilon_\alpha n_\alpha\right)\right.\right. \\
&\quad \left.\left. + \frac{\partial}{\partial\mathbf{r}}\cdot\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha\right) - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}}\cdot\overline{\mathbf{v}}_\alpha\right]\right\} \\
&\quad + \frac{\partial}{\partial\mathbf{r}}\cdot\left\{\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha\right)\right.\right. \\
&\quad + \frac{\partial}{\partial\mathbf{r}}\cdot\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}\right) - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}}\cdot\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} \\
&\quad \left.\left. - \frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{F}_\alpha^{\mathrm{sc}}} - \varepsilon_\alpha n_\alpha\overline{\mathbf{F}_\alpha^{\mathrm{sc}}}\right]\right\} \\
&\quad - \left\{\rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}}\cdot\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\mathbf{F}_\alpha^{(1)\mathrm{sc}}\cdot\left(\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha)\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{\partial}{\partial\mathbf{r}}\cdot\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)\mathrm{sc}} - q_\alpha n_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}^{\mathrm{sc}}\right)\right]\right\} = \overline{\mathbf{I}}_{\alpha,\,\mathrm{en}} \\
&\quad (\alpha = 1, \ldots, \mu)\,,
\end{aligned} \tag{2.41}$$

where $\mathbf{F}_\alpha^{\mathrm{sc}}$ is the total self-consistent force acting on the unit mass of species of the αth kind, $\mathbf{F}_\alpha^{(1)\mathrm{sc}}$ is the component of the self-consistent force independent of the velocity of the charged particle, $\mathbf{B}^{\mathrm{sc}}$ is the magnetic induction, q_α the charge of the particle α, ε_α its internal energy, and ρ_α the density of component α; the bar indicates an average over the velocities.

Thus, the generalized Enskog hydrodynamic equations involve self-consistent forces due to the collective nature of plasma particle interactions. In the following sections we discuss the applicability of the above theory to plasma physics problems.

3. Generalized Boltzmann equation in the physics of a weakly ionized gas. Hydrodynamic aspects of the theory

The traditional area of application of the Boltzmann kinetic theory (BKT) is the physics of a weakly ionized gas. It is interesting to see what the GBE yields in this case and how its results differ from those of the classical theory. To answer this fundamental question, let us consider the classical Lorentz formulation of the problem. We consider a spatially homogeneous, weakly ionized gas, for which it is assumed that collisions between charged particles may be ignored:

$$\nu_e \ll \delta\nu_{ea}\,,$$

where ν_e is the collision rate between charged particles; ν_{ea} is the collision rate between charged and neutral particles, and δ is the relative amount of energy which a charged particle loses in one collision with a neutral particle. We assume that the magnetic field is either absent or has a static component B_z, and that the electric field is along the x-axis; all inelastic interactions are neglected. The classical BE in this case takes the form

$$\frac{\mathrm{D}f_e}{\mathrm{D}t} \equiv \frac{\partial f_e}{\partial t} + \mathbf{F}_e\cdot\frac{\partial f_e}{\partial \mathbf{v}_e} = J_{ea}\,, \tag{3.1}$$

where $\mathbf{F}_e = q_e\mathbf{E}/m_e$ is a force acting on a unit mass of the charged particle, and q_e is the particle charge. The GBE is written in the following way:

$$\frac{\mathrm{D}f_e}{\mathrm{D}t} - \frac{\mathrm{D}}{\mathrm{D}t}\left(\tau_{ea}\,\frac{\mathrm{D}f_e}{\mathrm{D}t}\right) = J_{ea}\,. \tag{3.2}$$

In Eqn (3.2), τ_{ea} is the mean time between the collisions of neutral and charged particles. As the GBE theory suggests [3–5], the collision integral can be taken in the Boltzmann form. Multiplying Eqns (3.1) and (3.2) by the collision invariants m_e, $m_e\mathbf{v}_e$, $m_e v_e^2/2$ and then integrating them over the velocities $\mathbf{v}_e$, we arrive at the classical hydrodynamic equations (HEs) and the generalized hydrodynamic equations (GHEs), which assume a closed form provided we know how to evaluate the moments of the collision integrals involved. Note that in this case the following relation holds:

$$\int J_{ea} m_e\,\mathrm{d}\mathbf{v}_e = 0 \tag{3.3}$$

owing to the law of conservation of mass in elastic nonrelativistic collisions. But the integrals

$$\int J_{ea} m_e \mathbf{v}_e\,\mathrm{d}\mathbf{v}_e \quad \text{and} \quad \int J_{ea}\,\frac{m_e v_e^2}{2}\,\mathrm{d}\mathbf{v}_e$$

can be taken explicitly only for special models of particle interaction. Let us adopt the Maxwell model, in which the force F_{ea} of the intermolecular interaction depends on the inverse fifth power of the interparticle spacing:

$$F_{ea} = \frac{\chi_{ea}}{r^5}\,. \tag{3.4}$$

For this model, the integrals mentioned above are well known [36, 37], and the quantity τ_{ea} (hereinafter the subscript *ea* is dropped) was calculated to be

$$\tau = \left[0.422\;\;2\pi\left(\frac{\chi(m_e+m_a)}{m_e m_a}\right)^{1/2} n_a\right]^{-1}. \tag{3.5}$$

Introducing the quantities

$$A = \frac{8\sqrt{\pi}}{3}\,\Gamma\!\left(\frac{5}{2}\right) 0.422(m_e+m_a)^{-1}\left(\frac{\chi}{m_a M_e}\right)^{1/2} \tag{3.6}$$

and

$$M_a = \frac{m_a}{m_a+m_e}\,, \qquad M_e = \frac{m_e}{m_a+m_e}\,, \tag{3.7}$$

the rate ν of collisions between charged and neutral particles can be written in the form

$$\nu = n_a(m_a+m_e)A\,, \tag{3.8}$$

where $\tau = \nu^{-1}$, and n_a is the number density of neutral particles.

The continuity equations obtained from Eqns (3.1) and (3.2) yield the condition $n_e = \text{const}$, and the GBE in this case becomes

$$\frac{\partial f_e}{\partial t} + \mathbf{F}_e\cdot\frac{\partial f_e}{\partial \mathbf{v}_e} - \tau\left(\frac{\partial^2 f_e}{\partial t^2} + 2\mathbf{F}_e\cdot\frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial t} + \frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial \mathbf{v}_e} : \mathbf{F}_e\mathbf{F}_e\right) = J_{ea}\,. \tag{3.9}$$

From here on, a colon denotes the double scalar product of tensors.

We now introduce the drift velocity v_{ex} defined by the expression

$$\overline{v}_{ex} = \frac{1}{n_e}\int f_e v_{ex}\,\mathrm{d}\mathbf{v}_e\,. \tag{3.10}$$

Then the equation of motion entering the system of GHEs takes the form

$$\tau\,\frac{\mathrm{d}^2\overline{v}_{ex}}{\mathrm{d}t^2} - \frac{\mathrm{d}\overline{v}_{ex}}{\mathrm{d}t} - A m_a n_a \overline{v}_{ex} + q_e E m_e^{-1} = 0\,. \tag{3.11}$$

In writing Eqn (3.11) we have used the result [36, 37]

$$\int m_e v_{ex} J_{ea}\,\mathrm{d}\mathbf{v}_e = -A m_a n_a \overline{v}_{ex} m_e n_e\,.$$

The solution of Eqn (3.11) takes the form

$$\overline{v}_{ex}^{\,\mathrm{GBE}} = \left(\overline{v}_{ex}^{\,0} - \frac{q_e E}{m_e m_a n_a A}\right)\exp\left[-\frac{t}{2\tau}\left(\sqrt{4M_a+1}-1\right)\right] + \frac{q_e E}{m_e m_a n_a A}\,. \tag{3.12}$$

The superscript 0 here refers to the initial instant of time. The problem of the time relaxation of Maxwell particles in an electric field is known to be amenable to a BE solution [37] giving for the drift velocity the result

$$\overline{v}_{ex}^{\,\mathrm{BE}} = \left(\overline{v}_{ex}^{\,0} - \frac{q_e E}{m_e m_a n_a A}\right)\exp\left(-tAn_a m_a\right) + \frac{q_e E}{m_e m_a n_a A}\,. \tag{3.13}$$

For example, let us assume that $m_e \ll m_a$. Then, from Eqns (3.12) and (3.13), it follows that

$$\overline{v}_{ex}^{\,\mathrm{GBE}} = (\overline{v}_{ex}^{\,0} - F_{ex}\tau)\exp\left(-\frac{0.618t}{\tau}\right) + F_{ex}\tau\,, \tag{3.14}$$

$$\overline{v}_{ex}^{\,\mathrm{BE}} = (\overline{v}_{ex}^{\,0} - F_{ex}\tau)\exp\left(-\frac{t}{\tau}\right) + F_{ex}\tau\,. \tag{3.15}$$

Thus, all other things being equal, the relaxation of the drift velocity $\overline{v}_{ex}$ in the framework of BKT proceeds faster than in the generalized BKT, whereas the steady-state drift velocities are found to be the same.

We now turn our attention to the equation of energy and introduce the energy temperatures $\hat{T}_e$ and $\tilde{T}_e$ in accord with the definitions

$$\begin{aligned}\hat{T}_e &= \frac{m_e}{3n_e}\int f_e v_e^2\,\mathrm{d}\mathbf{v}_e\,,\\ \tilde{T}_e &= \frac{m_e}{3n_e}\int f_e(\mathbf{v}_e - \overline{\mathbf{v}}_e)^2\,\mathrm{d}\mathbf{v}_e\,.\end{aligned} \tag{3.16}$$

Clearly, which of these temperatures is used is a matter of convenience, and in our case one obtains

$$\hat{T}_e = \tilde{T}_e + \frac{1}{3}\,m_e\overline{v}_{ex}^{\,2}\,. \tag{3.17}$$

We next evaluate the moments on the left-hand side of the kinetic equations. For example, the following relations hold true:

$$\frac{\partial}{\partial t}\int \frac{m_e v_e^2}{2}\, f_e\, \mathrm{d}\mathbf{v}_e = \frac{3}{2}\, n_e\, \frac{\partial \hat{T}_e}{\partial t}\,, \tag{3.18}$$

$$\int \mathbf{F}_e \cdot \frac{\partial^2 f_e}{\partial \mathbf{v}_e\, \partial t}\, \frac{m_e v_e^2}{2}\, \mathrm{d}\mathbf{v}_e = -F_{ex} m_e n_e\, \frac{\partial \overline{v}_{ex}}{\partial t}\,, \tag{3.19}$$

$$\int \frac{m_e v_e^2}{2}\, \frac{\partial^2 f_e}{\partial \mathbf{v}_e\, \partial \mathbf{v}_e} : \mathbf{F}_e \mathbf{F}_e\, \mathrm{d}\mathbf{v}_e = F_{ex}^2 m_e n_e\,. \tag{3.20}$$

The corresponding integral on the right-hand side was calculated in Ref. [37] and is found to be

$$\int J_{ea}\, \frac{m_e v_e^2}{2}\, \mathrm{d}\mathbf{v}_e = -\frac{3(\hat{T}_e - \hat{T}_a)}{m_e + m_a}\, A m_e m_a n_e n_a\,. \tag{3.21}$$

We have then the following inhomogeneous linear second-order differential equation

$$\begin{aligned}
&\frac{\mathrm{d}^2 \tilde{T}_e}{\mathrm{d}t^2} - \frac{1}{\tau}\, \frac{\mathrm{d}\tilde{T}_e}{\mathrm{d}t} - 2\, \frac{\hat{T}_e - \hat{T}_a}{m_e + m_a}\, \frac{A}{\tau}\, m_a m_e n_a \\
&\quad = \frac{1}{3}\, \frac{m_e}{\tau}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \overline{v}_{ex}^{\,2} - \frac{2}{3}\, \frac{m_e}{\tau}\, F_{ex} \overline{v}_{ex} - \frac{1}{3}\, m_e\, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\, \overline{v}_{ex}^{\,2} \\
&\quad + \frac{4}{3}\, F_{ex} m_e\, \frac{\mathrm{d}\overline{v}_{ex}}{\mathrm{d}t} - \frac{2}{3}\, m_e F_{ex}^2\,.
\end{aligned} \tag{3.22}$$

Omitting the straightforward but tedious algebra we arrive at the following results. For example, by setting $m_e \ll m_a$, the GBE yields

$$\begin{aligned}
\tilde{T}_e^{\mathrm{GBE}} &= \tilde{T}_a + C_2 \exp\left(-2\, \frac{m_e}{m_a}\, \frac{t}{\tau}\right) \\
&- 2.175 m_e \tau F_{ex} (\overline{v}_{ex}^{\,0} - F_{ex}\tau) \exp\left(-0.618\, \frac{t}{\tau}\right) \\
&- \frac{1}{3}\, m_e (\overline{v}_{ex}^{\,0} - F_{ex}\tau)^2 \exp\left(-1.236\, \frac{t}{\tau}\right) + \frac{2}{3}\, F_{ex}^2 \tau^2 m_a\,,
\end{aligned} \tag{3.23}$$

where the following notation was used:

$$\begin{aligned}
C_2 &= \tilde{T}_e^0 - \tilde{T}_a + 2.157 m_e \tau F_{ex} (\overline{v}_{ex}^{\,0} - F_{ex}\tau) \\
&\quad + \frac{1}{3}\, m_e (\overline{v}_{ex}^{\,0} - F_{ex}\tau)^2 - \frac{2}{3}\, F_{ex}^2 \tau^2 m_a\,.
\end{aligned}$$

Similar BE results are as follows

$$\begin{aligned}
\tilde{T}_e^{\mathrm{BE}} &= \tilde{T}_a + C_2 \exp\left(-2\, \frac{m_e}{m_a}\, \frac{t}{\tau}\right) \\
&- \frac{4}{3}\, m_e \tau F_{ex} (\overline{v}_{ex}^{\,0} - F_{ex}\tau) \exp\left(-\frac{t}{\tau}\right) \\
&- \frac{1}{3}\, m_e (\overline{v}_{ex}^{\,0} - F_{ex}\tau)^2 \exp\left(-2\, \frac{t}{\tau}\right) + \frac{1}{3}\, F_{ex}^2 \tau^2 m_a\,.
\end{aligned} \tag{3.24}$$

Here, the notation was used:

$$\begin{aligned}
C_2 &= \tilde{T}_e^0 - \tilde{T}_a + \frac{4}{3}\, m_e \tau F_{ex} (\overline{v}_{ex}^{\,0} - F_{ex}\tau) \\
&\quad + \frac{1}{3}\, m_e (\overline{v}_{ex}^{\,0} - F_{ex}\tau)^2 - \frac{1}{3}\, F_{ex}^2 \tau^2 m_a\,.
\end{aligned}$$

In the steady-state regime, the above solutions are related by the expression

$$(\tilde{T}_e - \tilde{T}_a)_{\mathrm{st}}^{\mathrm{GBE}} = 2(\tilde{T}_e - \tilde{T}_a)_{\mathrm{st}}^{\mathrm{BE}}\,. \tag{3.25}$$

As before, the superscripts on the energy temperature differences in Eqn (3.25) refer to the type of the solution.

We are now in a position to write down the solutions for the energy temperatures $\hat{T}_e$; in the GBE scheme we have

$$\begin{aligned}
\hat{T}_e = \hat{T}_a &+ \Bigg[\hat{T}_{ea}^0 - \frac{2}{3}\, F_{ex}\tau\, \frac{m_e}{M_a(2M_e - 1)} \\
&\times \sqrt{4M_a + 1}\left(\overline{v}_{ex}^{\,0} - \frac{F_{ex}\tau}{M_a}\right) \\
&- \frac{\tau^2}{3}\, \frac{F_{ex}^2}{M_a^2}\, (1 + M_a)(m_e + m_a)\Bigg] \\
&\times \exp\left[-\frac{t}{2\tau}\left(\sqrt{8M_a M_e + 1} - 1\right)\right] \\
&+ \frac{2}{3}\, F_{ex}\tau\, \frac{m_e}{M_a(2M_e - 1)}\, \sqrt{4M_a + 1}\left(\overline{v}_{ex}^{\,0} - \frac{F_{ex}\tau}{M_a}\right) \\
&\times \exp\left[-\frac{t}{2\tau}\left(\sqrt{4M_a + 1} - 1\right)\right] \\
&+ \frac{\tau^2}{3}\, \frac{F_{ex}^2}{M_a^2}\, (m_e + m_a)(1 + M_a)\,,
\end{aligned} \tag{3.26}$$

with $\hat{T}_{ea}^0 = \hat{T}_e^0 - \hat{T}_a^0$.

In the framework of the classical Boltzmann equation we find [29]

$$\begin{aligned}
\hat{T}_e = \hat{T}_a &+ \Bigg[\hat{T}_{ea}^0 - \frac{2}{3}\, F_{ex}\tau\, \frac{m_e}{M_a(2M_e - 1)}\left(\overline{v}_{ex}^{\,0} - \frac{F_{ex}\tau}{M_a}\right) \\
&- \frac{\tau^2}{3}\, \frac{F_{ex}^2}{M_a^2}\, (m_e + m_a)\Bigg] \exp\left(-2M_a M_e\, \frac{t}{\tau}\right) \\
&+ \frac{2}{3}\, F_{ex}\tau\, \frac{m_e}{M_a(2M_e - 1)}\left(\overline{v}_{ex}^{\,0} - \frac{F_{ex}\tau}{M_a}\right) \exp\left(-M_a\, \frac{t}{\tau}\right) \\
&+ \frac{\tau^2}{3}\, \frac{F_{ex}^2}{M_a^2}\, (m_e + m_a)\,.
\end{aligned} \tag{3.27}$$

Notice that the vanishing of the term $2M_e - 1$ in the denominators in Eqns (3.26) and (3.27) does not actually lead to singularities at $M_e = 0.5$, because the corresponding terms cancel due to the exponential factors being equal. From Eqns (3.26), (3.27) it follows that

$$\hat{T}_{ea,\,\mathrm{st}}^{\mathrm{GBE}} = (1 + M_a)\, \hat{T}_{ea,\,\mathrm{st}}^{\mathrm{BE}}\,. \tag{3.28}$$

Thus, unlike drift velocity calculations, not only the GBE changes the trend of the relaxation curves but it also leads to different steady-state values of the energy temperatures. For a weakly ionized Lorentz gas, the effect of the self-consistent forces of electromagnetic origin can be neglected. Then, multiplying the GBE [see also Eqns (2.39)–(2.41)] by the collision invariants

$$m_\alpha,\ m_\alpha \mathbf{v}_\alpha,\ \frac{m_\alpha v_\alpha^2}{2} + \varepsilon_\alpha$$

(with ε_α being the internal energy of the particles of the component) and integrating with respect to the velocities we arrive at Enskog's system of generalized hydrodynamic equations (GHEs), in which only the effect of external forces is included:

the continuity equation for the component α

$$\frac{\partial}{\partial t}\left\{\rho_\alpha - \tau_\alpha\left[\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot(\rho_\alpha \overline{\mathbf{v}}_\alpha)\right]\right\} + \frac{\partial}{\partial \mathbf{r}}\cdot\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}\right]\right\} = R_\alpha\,, \tag{3.29}$$

the equation of motion

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}\right]\right\}\\ &- \left\{\mathbf{F}_\alpha^{(1)}\left[\rho_\alpha - \tau_\alpha\left(\frac{\partial\rho_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot(\rho_\alpha\overline{\mathbf{v}}_\alpha)\right)\right]\right\}\\ &- \frac{q_\alpha}{m_\alpha}\left\{\rho_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}\right]\right\}\times\mathbf{B}\\ &+ \frac{\partial}{\partial \mathbf{r}}\cdot\left\{\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \tau_\alpha\left[\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}) + \frac{\partial}{\partial \mathbf{r}}\cdot\overline{(\mathbf{v}_\alpha\mathbf{v}_\alpha)\mathbf{v}_\alpha}\right.\right.\\ &\left.\left. - \mathbf{F}_\alpha^{(1)}\rho_\alpha\overline{\mathbf{v}}_\alpha - \rho_\alpha\overline{\mathbf{v}}_\alpha\mathbf{F}_\alpha^{(1)} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{[\mathbf{v}_\alpha\times\mathbf{B}]\mathbf{v}_\alpha} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}_\alpha[\mathbf{v}_\alpha\times\mathbf{B}]}\right]\right\} = \bar{\mathbf{J}}_{\alpha,\,\mathrm{mot}}\,,\end{aligned} \tag{3.30}$$

and the equation of energy

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{\rho_\alpha\overline{v_\alpha^2}}{2} + \varepsilon_\alpha n_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{\rho_\alpha\overline{v_\alpha^2}}{2} + \varepsilon_\alpha n_\alpha\right) + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha\right) - \mathbf{F}_\alpha^{(1)}\cdot\rho_\alpha\overline{\mathbf{v}}_\alpha\right]\right\}\\ &+ \frac{\partial}{\partial \mathbf{r}}\cdot\left\{\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha - \tau_\alpha\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}}_\alpha\right) + \frac{\partial}{\partial \mathbf{r}}\cdot\left(\frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{v}_\alpha\mathbf{v}_\alpha} + \varepsilon_\alpha n_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}\right) - \rho_\alpha\mathbf{F}_\alpha^{(1)}\cdot\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha}\right.\right.\\ &\left.\left. - \frac{1}{2}\,\rho_\alpha\overline{v_\alpha^2\mathbf{F}_\alpha} - \varepsilon_\alpha n_\alpha\overline{\mathbf{F}}_\alpha\right]\right\}\\ &- \left\{\overline{\mathbf{v}}_\alpha\cdot\rho_\alpha\mathbf{F}_\alpha^{(1)} - \tau_\alpha\left[\overline{\mathbf{F}}_\alpha^{(1)}\cdot\left(\frac{\partial}{\partial t}(\rho_\alpha\overline{\mathbf{v}}_\alpha) + \frac{\partial}{\partial \mathbf{r}}\cdot\rho_\alpha\overline{\mathbf{v}_\alpha\mathbf{v}_\alpha} - \rho_\alpha\mathbf{F}_\alpha^{(1)} - \frac{q_\alpha}{m_\alpha}\,\rho_\alpha\overline{\mathbf{v}}_\alpha\times\mathbf{B}\right)\right]\right\} = \bar{J}_{\alpha,\,\mathrm{en}}\,,\end{aligned} \tag{3.31}$$

where v_0 is the hydrodynamical velocity, B is the magnetic induction, $\mathbf{F}_\alpha^{(1)}$ are external nonmagnetic forces per unit mass of the particle α, and q_α is the charge. The right-hand sides of Eqns (3.30), (3.31) involve the integral relaxation terms $\bar{\mathbf{J}}_{\alpha,\,\mathrm{mot}}$ and $\bar{J}_{\alpha,\,\mathrm{en}}$ which, due to momentum and energy conservation laws, satisfy the relations

$$\sum_{\alpha=1}^{\mu}\bar{\mathbf{J}}_{\alpha,\,\mathrm{mot}} = 0\,, \qquad \sum_{\alpha=1}^{\mu}\bar{J}_{\alpha,\,\mathrm{en}} = 0\,. \tag{3.32}$$

However, for the systems being far from equilibrium one has to introduce approximations for $\bar{\mathbf{J}}_{\alpha,\,\mathrm{mot}}$, $\bar{J}_{\alpha,\,\mathrm{en}}$. This can be done in a number of ways, including the Bhatnagar–Gross–Krook (BGK) method or its extensions [38].

The generalized Boltzmann equation and the system of GHEs can be used to study plasma in an electric field, in particular to understand the electron energy runaway effect [39, 40]. We now proceed to apply the generalized Boltzmann kinetic theory to the classical problems of plasma physics.

4. Charged particle distribution function for a Lorentz gas

Calculating the distribution function for charged particles added as an impurity to a neutral gas in an external electric field is a classical problem in gas discharge physics, whose long history dates back to Pidduck's 1913 attempt to calculate the ion drift velocity in gases [41]. Mention should also be made of Compton's work concerned with computing the charge particle distribution function and its moments [42, 43]. Later on, Druyvesteyn [44, 45] and Davydov [46] obtained analytical expressions for the distribution function and transport coefficients for the special case of elastic collisions. More recent work (note, in particular, the monograph [47]) has been aimed principally at investigating the effect of inelastic collisions on the DF and transport processes within the BKT framework. It is important to note that the calculation of the DF depends heavily on what model of particle interaction is adopted — and hence ultimately on the collision cross sections involved. For example, the Davydov–Druyvesteyn distribution obtained under the assumption of a constant mean free path l for elastically colliding, charged gas particles significantly underpredicts the number of 'hot' particles on the tail of the DF and leads ultimately to unacceptable results when the theory is extended to calculating the kinetics of inelastic processes [47].

We apply the generalized Boltzmann equation

$$\mathbf{F}_e\cdot\frac{\partial f_e}{\partial \mathbf{v}_e} - \tau\,\frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial \mathbf{v}_e} : \mathbf{F}_e\mathbf{F}_e = J_{ea} \tag{4.1}$$

to consider charged particles in a steady state in a Lorentz gas subject to a stationary external electric field, where $\mathbf{F}_e = e\mathbf{E}/m_e$. The Boltzmann kinetic equation is usually solved by expanding the DF in a power series of zero-order solid spherical harmonics, i.e., of Legendre polynomials. The corresponding system of linked equations was obtained elsewhere [48, 49]. The solution to the GBE (4.1) is conveniently sought as an expansion in terms of solid spherical harmonics:

$$f(\mathbf{v}_e) = f_0(v_e) + \mathbf{F}_e\cdot\mathbf{v}_e f_1(v_e) + \mathbf{F}_e\mathbf{F}_e : \mathbf{v}_e^0\mathbf{v}_e f_2(v_e) + \ldots \tag{4.2}$$

Here $\mathbf{v}_e^0\mathbf{v}_e$ is the zero-trace tensor. For our further calculations in this section we assume that the force $\mathbf{F}_e$ is along the positive direction of a certain chosen coordinate axis.

We now substitute expansion (4.2) into Eqn (4.1) and transform the corresponding terms; we have, for example, the following relations:

$$\mathbf{F}_e \cdot \frac{\partial}{\partial \mathbf{v}_e} (f_2 \mathbf{F}_e \mathbf{F}_e : \mathbf{v}_e \mathbf{v}_e) = \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^3}{v_e} \frac{\partial f_2}{\partial v_e} - \frac{1}{3} F_e^2 v_e (\mathbf{F}_e \cdot \mathbf{v}_e) \frac{\partial f_2}{\partial v_e} + \frac{4}{3} F_e^2 f_2 (\mathbf{v}_e \cdot \mathbf{F}_e) , \tag{4.3}$$

$$\frac{\partial^2}{\partial \mathbf{v}_e \partial \mathbf{v}_e} : \left\{ f_2 (\mathbf{F}_e \mathbf{F}_e : \mathbf{v}_e^0 \mathbf{v}_e) \mathbf{F}_e \mathbf{F}_e \right\} = 5 F_e^2 \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^2}{v_e} \frac{\partial f_2}{\partial v_e} + 2 f_2 F_e^4 + \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^4}{v_e^2} \frac{\partial^2 f_2}{\partial v_e^2} - \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^4}{v_e^3} \frac{\partial f_2}{\partial v_e} - \frac{1}{3} \sum_{i=1}^{3} F_{ei}^2 F_e^2 \frac{\partial^2}{\partial v_{ei}^2} (f_2 v_e^2) . \tag{4.4}$$

The left-hand side Λ of the generalized Boltzmann equation then takes the form

$$\begin{aligned} \Lambda = {} & (\mathbf{F}_e \cdot \mathbf{v}_e) \frac{1}{v_e} \frac{\partial f_0}{\partial v_e} + F_e^2 f_1 + \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^2}{v_e} \frac{\partial f_1}{\partial v_e} + \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^3}{v_e} \frac{\partial f_2}{\partial v_e} \\ & - \frac{1}{3} F_e^2 v_e (\mathbf{F}_e \cdot \mathbf{v}_e) \frac{\partial f_2}{\partial v_e} + \frac{4}{3} F_e^2 f_2 (\mathbf{F}_e \cdot \mathbf{v}_e) \\ & - \tau \left[\frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^2}{v_e} \frac{\partial}{\partial v_e} \left(\frac{1}{v_e} \frac{\partial f_0}{\partial v_e} \right) + \frac{F_e^2}{v_e} \frac{\partial f_0}{\partial v_e} + 3 F_e^2 \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)}{v_e} \frac{\partial f_1}{\partial v_e} \right. \\ & - \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^3}{v_e^3} \frac{\partial f_1}{\partial v_e} + \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^3}{v_e^2} \frac{\partial^2 f_1}{\partial v_e^2} + 4 F_e^2 \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^2}{v_e} \frac{\partial f_2}{\partial v_e} \\ & + \frac{4}{3} f_2 F_e^4 + \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^4}{v_e^2} \frac{\partial^2 f_2}{\partial v_e^2} - \frac{1}{3} F_e^2 (\mathbf{F}_e \cdot \mathbf{v}_e)^2 \frac{\partial^2 f_2}{\partial v_e^2} \\ & \left. - \frac{(\mathbf{F}_e \cdot \mathbf{v}_e)^4}{v_e^3} \frac{\partial f_2}{\partial v_e} - \frac{1}{3} F_e^4 v_e \frac{\partial f_2}{\partial v_e} \right] . \end{aligned} \tag{4.5}$$

Denoting the angle between the vectors $\mathbf{F}_e$ and $\mathbf{v}_e$ by ϑ ($0 \leqslant \vartheta \leqslant \pi$), multiplying the GBE by $\mathrm{d}\cos\vartheta$, and integrating over the entire range of angles, we arrive at

$$f_1 + \frac{1}{3} v_e \frac{\partial f_1}{\partial v_e} - \frac{\tau}{3} \left(\frac{2}{v_e} \frac{\partial f_0}{\partial v_e} + \frac{\partial^2 f_0}{\partial v_e^2} + \frac{12}{5} v_e \frac{\partial f_2}{\partial v_e} F_e^2 + 4 f_2 F_e^2 + \frac{4}{15} F_e^2 v_e^2 \frac{\partial^2 f_2}{\partial v_e^2} \right) = \frac{1}{F_e^2} J_{ea} . \tag{4.6}$$

Multiplying the GBE by $\cos\vartheta \, \mathrm{d}\cos\vartheta$, a similar procedure gives

$$\frac{\partial f_0}{\partial v_e} + \frac{4}{15} F_e^2 v_e^2 \frac{\partial f_2}{\partial v_e} + \frac{4}{3} F_e^2 v_e f_2 - \frac{3}{5} \tau F_e^2 \left(4 \frac{\partial f_1}{\partial v_e} + v_e \frac{\partial^2 f_1}{\partial v_e^2} \right) = - \frac{3}{2} \frac{1}{F_e} J_1 . \tag{4.7}$$

As has been indicated, the collision terms J_{ea} and J_1 in the generalized Boltzmann kinetic theory (GBKT) can be taken in the form in which they are usually written in the Boltzmann equation. In the case we consider below, assuming that the change in the electron energy due to an elastic collision [approximately equal to $(m_e/m_a)^{1/2} \varepsilon_e$] is much less that the electron energy prior to the collision, in the Fokker – Planck approximation (see, for example, Ref. [50]) we have

$$J_{ea} = \frac{m_e}{m_a} \frac{\hat{T}_a}{v_e^2} \frac{\partial}{\partial v_e} \left[v_e^3 \nu \left(\frac{f_0}{\hat{T}} + \frac{1}{m_e v_e} \frac{\partial f_0}{\partial v_e} \right) \right] , \tag{4.8}$$

$$J_1 = \frac{2}{3} F_e \frac{v_e^2}{l} f_1 , \tag{4.9}$$

where $\hat{T}_a$ is the energy temperature of the neutral gas ($\hat{T}_a = k T_a$), ν is the collision rate which generally depends on the velocity, and l is the mean free path for collisions of neutral and charged particles. It is relations (4.6)–(4.9) which provide the required basis for determining the DF and its moments. Traditionally, two limiting situations are considered in detail: (1) a constant frequency rate, $\nu = \mathrm{const}$, $\nu = \tau^{-1} = v_e l^{-1}$, and (2) a constant mean free path between the collisions of charged and neutral particles, $l = \mathrm{const}$.

We take up the former case first. Multiplying Eqn (4.6) through by $3 v_e^2$ and using Eqn (4.8), we find, after some algebra, that

$$F_e^2 \frac{\mathrm{d}}{\mathrm{d}v_e} \left(v_e^3 f_1 - \tau v_e^2 \frac{\mathrm{d}f_0}{\mathrm{d}v_e} \right) = \frac{3 \hat{T}_a m_e}{\tau m_a} \frac{\mathrm{d}}{\mathrm{d}v_e} \left[v_e^3 \left(\frac{f_0}{\hat{T}} + \frac{1}{m_e v_e} \frac{\mathrm{d}f_0}{\mathrm{d}v_e} \right) \right] , \tag{4.10}$$

or upon integration over v_e:

$$f_1 v_e = \left(\tau + \frac{3 \hat{T}_a}{F_e^2 m_a \tau} \right) \frac{\mathrm{d}f_0}{\mathrm{d}v_e} + \frac{3}{F_e^2} \frac{m_e}{m_a} \frac{v_e}{\tau} f_0 , \tag{4.11}$$

because the constant of integration is zero due to the fact that both the left-hand and right-hand sides of Eqn (4.11) vanish for $v_e = 0$. Equation (4.11) was obtained under the condition (which will also be used in the following analysis) that small terms proportional to f_2 may be dropped. Substituting now Eqn (4.11) into Eqn (4.9) and making use of the result produced to eliminate f_1 from Eqn (4.7), we arrive at the following equation in f_0:

$$\begin{aligned} & v_e^2 \left(\tau + \frac{3 \hat{T}_a}{F_e^2 m_a \tau} \right) \frac{\mathrm{d}^3 f_0}{\mathrm{d}v_e^3} + v_e \left(2\tau + \frac{3 m_e}{F_e^2 m_a \tau} v_e^2 + \frac{6 \hat{T}_a}{F_e^2 m_a \tau} \right) \frac{\mathrm{d}^2 f_0}{\mathrm{d}v_e^2} \\ & + \left(-2\tau - \frac{10}{3 \tau F_e^2} v_e^2 - \frac{5 \hat{T}_a}{\tau^3 F_e^4 m_a} v_e^2 + \frac{12 m_e}{F_e^2 m_a \tau} v_e^2 - \frac{6 \hat{T}_a}{F_e^2 m_a \tau} \right) \frac{\mathrm{d}f_0}{\mathrm{d}v_e} - \frac{5 m_e}{F_e^4 \tau^3 m_a} v_e^3 f_0 = 0 . \end{aligned} \tag{4.12}$$

To solve Eqn (4.12), three boundary conditions are needed. These are in fact quite obvious. Indeed, for $v_e = 0$, we can specify a certain value of f_0, determined only by the normalization of the function. From Eqn (4.12) it is also seen that $f_0' = 0$ for $v_e = 0$. Finally, dividing the above equation by v_e^3 we find that $f_0 \to 0$ for $v_e \to \infty$. Thus, Eqn (4.12) is easily solved by, for example, the sweep method. To do this, it is convenient to first bring the equation to the dimensionless form by introducing the following dimensionless quantities labelled with arcs over the symbols:

$$\breve{v}_e = \frac{v_e}{F_e \tau} , \quad \breve{\varepsilon} = \frac{m_e F_e^2 \tau^2}{\hat{T}_a} , \quad \breve{f}_0 = \frac{f_0}{f_0 (v_e = 0)} . \tag{4.13}$$

The procedure is realized to yield

$$\breve{v}_e^2\left(1+\frac{3m_e}{m_a\breve{\varepsilon}}\right)\frac{\mathrm{d}^3\breve{f}_0}{\mathrm{d}\breve{v}_e^3}+\breve{v}_e\left(2+\frac{6m_e}{m_a\breve{\varepsilon}}+\frac{3m_e\breve{v}_e^2}{m_a}\right)\frac{\mathrm{d}^2\breve{f}_0}{\mathrm{d}\breve{v}_e^2}$$
$$-\left[2+\frac{6m_e}{m_a\breve{\varepsilon}}+\breve{v}_e^2\left(\frac{10}{3}+\frac{5m_e}{m_a\breve{\varepsilon}}-12\,\frac{m_e}{m_a}\right)\right]\frac{\mathrm{d}\breve{f}_0}{\mathrm{d}\breve{v}_e}$$
$$-5\,\frac{m_e}{m_a}\,\breve{v}_e^3\breve{f}_0=0\,. \tag{4.14}$$

Let us define the energy temperature of charged particles as follows

$$\hat{T}_e=\frac{1}{3n_e}\int f_e\,m_e v_e^2\,\mathrm{d}\mathbf{v}_e\cong\frac{1}{3n_e}\int f_0\,m_e v_e^2\,\mathrm{d}\mathbf{v}_e\,. \tag{4.15}$$

This means, for example, that, in terms of definitions (4.13), the Maxwellian function $\breve{f}_{\mathrm{M}}$ has the form

$$\breve{f}_{\mathrm{M}}=\exp\left(-\frac{\hat{T}_a\breve{\varepsilon}}{2\hat{T}_e}\,\breve{v}_e^2\right). \tag{4.16}$$

Let us examine the asymptotics of the function f_0 at large velocities v_e. From Eqn (4.14) it follows that for $v_e\to\infty$ the equation

$$\frac{\mathrm{d}^2f_0}{\mathrm{d}v_e^2}-\frac{5}{3F_e^2\tau^2}\,f_0=0 \tag{4.17}$$

holds, which has the solution

$$f_0\approx\exp\left(-\sqrt{\frac{5}{3}}\,\frac{v_e}{F_e\tau}\right). \tag{4.18}$$

Note that, in the limiting case we are considering, the classical solution of the Boltzmann equation [50] leads to a Maxwellian distribution function with a temperature $\hat{T}_e$ different from the neutral gas temperature $\hat{T}_a$. Thus, the solution of the GBE results in a large number of 'hot' charged particles on the tail of the distribution function.

Of course, the moments of the distribution function — the temperature $\hat{T}_e$ and the drift velocity $\overline{v}_{ex}$ — can be found by properly integrating the DF after the solution of Eqn (4.14) has been found. There is no need to do this, however. Indeed, multiplying Eqn (4.14) by v_e and integrating term by term we obtain

$$\left(\frac{3\hat{T}_a}{m_a\tau^2F_e^2}+2\right)\int_0^\infty f_0v_e^2\,\mathrm{d}v_e=\frac{m_e}{m_a\tau^2F_e^2}\int_0^\infty f_0v_e^4\,\mathrm{d}v_e\,. \tag{4.19}$$

Assuming that

$$\int f\,\mathrm{d}\mathbf{v}_e\cong\int f_0\,\mathrm{d}\mathbf{v}_e=4\pi\int_0^\infty f_0v_e^2\,\mathrm{d}v_e=n_e\,, \tag{4.20}$$

as was done in Eqn (4.15), it is found that

$$\hat{T}_e=\hat{T}_a+\frac{2}{3}\,m_a\tau^2F_e^2\,. \tag{4.21}$$

In a similar way, without explicitly solving Eqns (4.11) and (4.12), we can determine the drift velocity. To accomplish this, we multiply Eqn (4.11) termwise by v_e^3 and integrate the resulting expression to yield

$$\int_0^\infty f_1v_e^4\,\mathrm{d}v_e=\left(\tau+\frac{3\hat{T}_a}{F_e^2m_a\tau}\right)\int_0^\infty v_e^3\,\frac{\mathrm{d}f_0}{\mathrm{d}v_e}\,\mathrm{d}v_e$$
$$+\frac{3}{F_e^2}\,\frac{m_e}{m_a\tau}\int_0^\infty f_0v_e^4\,\mathrm{d}v_e\,, \tag{4.22}$$

leading to

$$\overline{v}_{ex}=\frac{3(\hat{T}_e-\hat{T}_a)}{m_a\tau F_e}-\tau F_e\,, \tag{4.23}$$

because, by definition, the following relations hold true:

$$\overline{v}_{ex}=\frac{1}{n_e}\int fv_{ex}\,\mathrm{d}\mathbf{v}_e=\frac{4\pi F_e}{3n_e}\int_0^\infty f_1v_e^4\,\mathrm{d}v_e\,. \tag{4.24}$$

Using expressions (4.21) and (4.23), we achieve the result sought:

$$\overline{v}_{ex}=\tau F_e\,. \tag{4.25}$$

Comparing relations (4.21) and (4.25) with known classical results [Ref. 50, p. 108] suggests that in the limiting case $v=\mathrm{const}$ the drift velocity remains unchanged and that $\hat{T}_e$ increases [the classical analogue of Eqn (4.21) contains the numerical coefficient $1/2$ instead of $2/3$]. In concluding the discussion of this limiting case, we present the corresponding form of Eqn (4.14) ($m_e\ll m_a$) for $\breve{\varepsilon}\gtrsim 1$:

$$\breve{v}_e^2\,\frac{\mathrm{d}^3\breve{f}_0}{\mathrm{d}\breve{v}_e^3}+\left(2+3\,\frac{m_e}{m_a}\,\breve{v}_e^2\right)\breve{v}_e\,\frac{\mathrm{d}^2\breve{f}_0}{\mathrm{d}\breve{v}_e^2}-\left(2+\frac{10}{3}\,\breve{v}_e^2\right)\frac{\mathrm{d}\breve{f}_0}{\mathrm{d}\breve{v}_e}$$
$$-5\,\frac{m_e}{m_a}\,\breve{v}_e^3\breve{f}_0=0\,. \tag{4.26}$$

As a check on the correctness of the above results, note that if $F_e\equiv 0$, then Eqn (4.11) leads, as it should, to the Maxwellian distribution function $f_{0\mathrm{M}}$:

$$\frac{\mathrm{d}f_0}{\mathrm{d}v_e}=-\frac{m_ev_e}{\hat{T}}\,f_0\,, \tag{4.27}$$

$$f_{0\mathrm{M}}=C\exp\left(-\frac{m_ev_e^2}{2\hat{T}}\right). \tag{4.28}$$

We proceed now to the second limiting case, $l=\mathrm{const}$ [51]. In this case, the analogue of Eqn (4.11) is as follows

$$f_1v_e=\left(\tau+\frac{3\hat{T}v_e}{F_e^2m_al}\right)\frac{\mathrm{d}f_0}{\mathrm{d}v_e}+\frac{3}{F_e^2}\,\frac{m_e}{m_a}\,\frac{v_e^2}{l}\,f_0\,. \tag{4.29}$$

By the same procedure used in the limiting case $v=\mathrm{const}$, we arrive at the following equation in f_0:

$$v_e^2\left(\tau+\frac{3\hat{T}_av_e}{F_e^2m_al}\right)\frac{\mathrm{d}^3f_0}{\mathrm{d}v_e^3}+\left(2\tau+\frac{12\hat{T}_a}{m_alF_e^2}\,v_e+\frac{3m_ev_e^3}{m_alF_e^2}\right)v_e\,\frac{\mathrm{d}^2f_0}{\mathrm{d}v_e^2}$$
$$+\left(18\,\frac{m_e}{m_alF_e^2}\,v_e^3-2\tau-\frac{5v_e^2}{3\tau F_e^2}\right.$$
$$\left.-\frac{5}{3F_e^2l}\,v_e^3-\frac{5\hat{T}_a}{\tau F_e^4m_al^2}\,v_e^4\right)\frac{\mathrm{d}f_0}{\mathrm{d}v_e}$$
$$+v_e^2\left(12\,\frac{m_e}{F_e^2m_al}-5\,\frac{m_e}{\tau F_e^4m_al^2}\,v_e^3\right)f_0=0\,. \tag{4.30}$$

Again, it is easily seen by multiplying Eqn (4.30) termwise by F_e^4 that the vanishing of the external force F_e leads to

Eqn (4.27) and then, upon integration, to the Maxwellian distribution function (4.28). The boundary conditions for Eqn (4.30) are as follows: f_0 is specified for $v_e = 0$ in accord with the chosen normalization; for $v_e = 0$, as Eqn (4.30) suggests, $f_0' = 0$, and, finally, $f_0 \to 0$ when $v_e \to \infty$. The last result becomes evident if one first divides Eqn (4.30) through by v_e^5.

In order to numerically integrate Eqn (4.30), it is convenient to bring this equation to the dimensionless form by using the dimensionless quantities

$$\breve{v}_e = \frac{v_e}{l/\tau}\,, \qquad \breve{\varepsilon} = \frac{m_e F_e^2 \tau^2}{\hat{T}_a}\,, \qquad \breve{A} = \frac{F_e^2 \tau^4}{l^2} \tag{4.31}$$

to give the ordinary differential equation

$$\begin{aligned} &\breve{v}_e^2 \breve{A}\left(1 + 3\,\frac{m_e}{m_\alpha}\,\frac{\breve{v}_e}{\breve{\varepsilon}}\right)\frac{\mathrm{d}^3 \breve{f}_0}{\mathrm{d}\breve{v}_e^3} \\ &\quad + \left[\breve{A}\left(2 + 12\,\frac{m_e}{m_a}\,\frac{\breve{v}_e}{\breve{\varepsilon}}\right) + 3\,\frac{m_e}{m_a}\,\breve{v}_e^3\right]\breve{v}_e\,\frac{\mathrm{d}^2 \breve{f}_0}{\mathrm{d}\breve{v}_e^2} \\ &\quad + \left[-2\breve{A} - \frac{5}{3}\,\breve{v}_e^2 - \left(\frac{5}{3} - 18\,\frac{m_e}{m_a}\right)\breve{v}_e^3 - 5\,\frac{m_e}{m_a}\,\frac{\breve{v}_e^4}{\breve{\varepsilon}}\right]\frac{\mathrm{d}\breve{f}_0}{\mathrm{d}\breve{v}_e} \\ &\quad + \breve{v}_e^2\,\frac{m_e}{m_a}\left(12 - 5\,\frac{\breve{v}_e^3}{\breve{A}}\right)\breve{f}_0 = 0 \end{aligned} \tag{4.32}$$

with the boundary conditions

$$\breve{f}_0(0) = 1\,, \qquad \breve{f}_0'(0) = 0\,, \qquad \breve{f}_0(\infty) = 0\,. \tag{4.33}$$

Here the term-by-term integration no longer leads to elegant results like Eqns (4.21) and (4.25). We can, however, give a useful formula for computing the drift velocity $\overline{v}_{ex}$, which is obtained from Eqn (4.32) by multiplying it by v_e^3 and then integrating, to yield

$$\begin{aligned} \overline{v}_{ex} = \frac{4\pi F_e}{3 n_e}\bigg(&\frac{18}{5}\,\frac{\hat{T}_a \tau}{m_a}\int_0^\infty f_0\,\mathrm{d}v_e + C\,\frac{6}{5}\,\tau^2 l F_e^2 \\ &+ 2l\int_0^\infty f_0 v_e\,\mathrm{d}v_e - \frac{9}{10}\,\frac{m_e}{\pi m_a}\,\tau n_e\bigg)\,, \end{aligned} \tag{4.34}$$

with $C = f_0$ $(v_0 = 0)$. Although Eqn (4.34) can of course be used only after numerically integrating Eqn (4.33), it is of interest to note that, unlike Eqn (4.25), the drift velocity is a nonlinear function of F_e in this limiting case.

Let us consider here some numerical results for the distribution function of charged particles in an external electric field, produced when employing the generalized Boltzmann equation [51].

In Fig. 2, the dimensionless distribution function $\breve{f}_0$ is plotted versus the dimensionless velocity $\breve{v}_e$ for $\breve{\varepsilon} = 10^{-3}$ and $\tau = \text{const}$. The line *1* corresponds to the Maxwellian distribution function, and the curve *2*, to the distribution function obtained using the GBE. As $\breve{\varepsilon}$ is decreased, the two distributions approach each other. Note that the function $\breve{f}_0^{\,\mathrm{GBE}}$ lies above the Maxwellian function. Figures 3 and 4 present $\breve{f}_0$ calculated in the case of $l = \text{const}$ under the conditions $\breve{\varepsilon} = 10^{-2}$, $\breve{A} = 1$, and $\breve{\varepsilon} = 10^{-2}$, $\breve{A} = 10^{-1}$, respectively. The curves *1*, *2*, and *3* in Figs 3 and 4 correspond to the Maxwellian distribution function, the generalized Boltzmann equation, and the Druyvesteyn distribution function, respectively. It is interesting to note that the distribution function $\breve{f}_0^{\,\mathrm{GBE}}$ may lie both between the Maxwell and Druyvesteyn

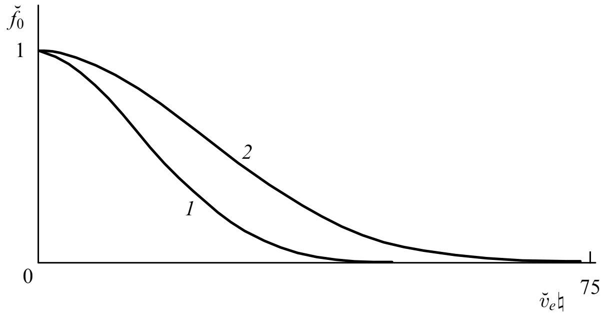

Figure 2. Dependence of $\breve{f}_0$ on $\breve{v}_e$ for $\tau = \text{const}$: *1*, Maxwellian distribution function; *2*, $\breve{f}_0^{\,\mathrm{GBE}}$.

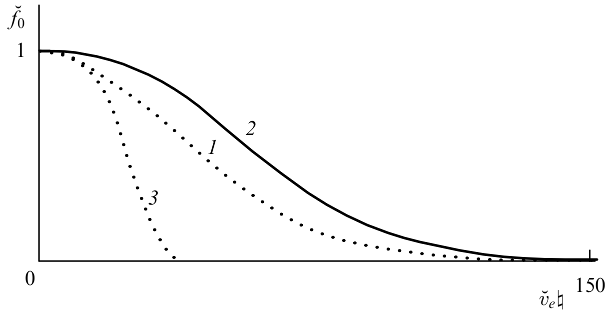

Figure 3. Dependence of $\breve{f}_0$ on $\breve{v}_e$ for $l = \text{const}$, $\breve{\varepsilon} = 10^{-2}$, $\breve{A} = 1$: *1*, Maxwellian distribution function; *2*, $\breve{f}_0^{\,\mathrm{GBE}}$; *3*, Druyvesteyn distribution function.

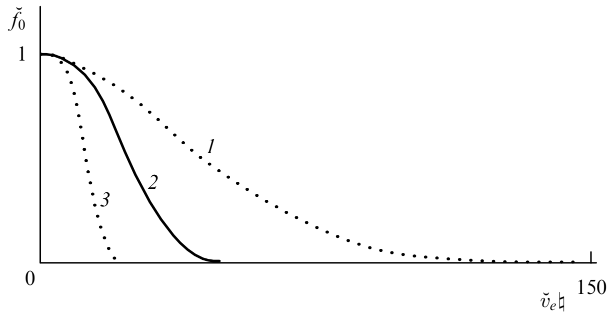

Figure 4. Dependence of $\breve{f}_0$ on $\breve{v}_e$ for $l = \text{const}$, $\breve{\varepsilon} = 10^{-2}$, $\breve{A} = 10^{-1}$: *1*, Maxwellian distribution function; *2*, $\breve{f}_0^{\,\mathrm{GBE}}$; *3*, Druyvesteyn distribution function.

functions and above the two. In practical computations, to reemphasize, the distribution functions can be normalized to the number density of the charged particles involved.

5. Charged particles in an alternating electric field

As another example of the application of the GBE, let us consider the time evolution of the DF of charged particles moving in an alternating electric field. In this problem, only elastic collisions will be considered; the GBE takes the form

$$\begin{aligned} &\left(\frac{\partial f_e}{\partial t} + \mathbf{F}_e \cdot \frac{\partial f_e}{\partial \mathbf{v}_e}\right)\left(1 - \frac{\partial \tau}{\partial t}\right) - \tau\bigg(\frac{\partial^2 f_e}{\partial t^2} + 2\mathbf{F}_e \cdot \frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial t} \\ &\qquad + \frac{\partial \mathbf{F}_e}{\partial t} \cdot \frac{\partial f_e}{\partial \mathbf{v}_e} + \frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial \mathbf{v}_e} : \mathbf{F}_e \mathbf{F}_e\bigg) = J_{ea}\,. \end{aligned} \tag{5.1}$$

If we make use of the expansion

$$f(\mathbf{v}_e, t) = f_0(v_e, t) + \mathbf{F}_e \cdot \mathbf{v}_e f_1(v_e, t) + \mathbf{F}_e\mathbf{F}_e : \mathbf{v}_e^0\mathbf{v}_e f_2(v_e, t)\,, \tag{5.2}$$

then utilizing a procedure analogous to that described above we arrive at the following equations in the functions f_0 and f_1:

$$\left(1 - \frac{\partial\tau}{\partial t}\right)\left(\frac{\partial f_0}{\partial v_e} + F_e^2 f_1 + \frac{1}{3}F_e^2 v_e \frac{\partial f_1}{\partial v_e}\right) - \tau\left(\frac{\partial^2 f_0}{\partial t^2} + \frac{1}{3}F_e^2\frac{\partial^2 f_0}{\partial v_e^2} + \frac{2}{3}F_e^2\frac{1}{v_e}\frac{\partial f_0}{\partial v_e} + \frac{3}{2}f_1\frac{\partial F_e^2}{\partial t} + 2F_e^2\frac{\partial f_1}{\partial t} + \frac{1}{2}\frac{\partial F_e^2}{\partial t}v_e\frac{\partial f_1}{\partial v_e} + \frac{2}{3}F_e^2 v_e\frac{\partial^2 f_1}{\partial v_e\,\partial t}\right) = J_{ea}\,, \tag{5.3}$$

$$\left(1 - \frac{\partial\tau}{\partial t}\right)\left(F_e\frac{\partial f_0}{\partial v_e} + v_e\frac{\partial F_e}{\partial t}f_1 + F_e v_e\frac{\partial f_1}{\partial t}\right) - \tau\left(2F_e\frac{\partial^2 f_0}{\partial v_e\,\partial t} + \frac{\partial F_e}{\partial t}\frac{\partial f_0}{\partial v_e} + 2v_e\frac{\partial F_e}{\partial t}\frac{\partial f_1}{\partial t} + v_e\frac{\partial^2 F_e}{\partial t^2}f_1 + F_e v_e\frac{\partial^2 f_1}{\partial t^2} + \frac{12}{5}F_e^3\frac{\partial f_1}{\partial v_e} + \frac{3}{5}F_e^3 v_e\frac{\partial^2 f_1}{\partial v_e^2}\right) = -\frac{2}{3}J_1\,, \tag{5.4}$$

where, for example, $J_1 = (2/3)F_e f_1 v_e^2/l$.

Now consider a case in which the mean time between collisions $\tau = 1/v_e$ is independent of the velocity. It proves possible to determine the distribution function moments $\overline{v}_{ex}$ and $\hat{T}_e$ without directly solving Eqns (5.3) and (5.4). Multiply Eqn (5.4) termwise by v_e^3 and integrate the resulting expression over all absolute velocities. Using the additional conditions

$$\int_0^\infty f_1 v_e^4\,\mathrm{d}v_e = \frac{3n_e}{4\pi F_e}\overline{v}_{ex}\,, \qquad \int_0^\infty f_0 v_e^2\,\mathrm{d}v_e = \frac{n_e}{4\pi}\,,$$

$$\int_0^\infty f_0 v_e^4\,\mathrm{d}v_e = \frac{3}{4\pi}n_e\frac{\hat{T}_e}{m_e}\,,$$

we derive the following equation

$$\tau\frac{\mathrm{d}^2\overline{v}_{ex}}{\mathrm{d}t^2} - \frac{\mathrm{d}\overline{v}_{ex}}{\mathrm{d}t} - \frac{1}{\tau}\overline{v}_{ex} + F_e - \tau\frac{\mathrm{d}F_e}{\mathrm{d}t} = 0\,. \tag{5.5}$$

Suppose that the time dependence of the external force can be represented as

$$F_e = \frac{eE_0}{m_0}\cos\omega t\,,$$

where the frequency ω is related to the external electric field strength. The solution of the inhomogeneous differential equation (5.5) is written as

$$\overline{v}_{ex}^{\,\mathrm{GBE}} = C_1\exp\left(-\frac{at}{\tau}\right) + \frac{b\tau}{(\omega\tau)^4 + 3(\omega\tau)^2 + 1}\left[\cos\omega t + \omega t(2 + \omega^2\tau^2)\sin\omega t\right], \tag{5.6}$$

where $b = eE_0/m_e$, and C_1 is the constant of integration, which is determined by the initial conditions of the problem.

The classical result which can be obtained from the BE for the quasi-stationary case is given by

$$\overline{v}_{ex}^{\,\mathrm{BE}} = \frac{b\tau}{(\omega\tau)^2 + 1}(\cos\omega t + \omega t\sin\omega t)\,. \tag{5.7}$$

The introduction of the mobility K usually defined through the expression

$$\overline{v}_{ex} = K\frac{m_e}{e}F_e$$

would serve no purpose due to singularities that can appear for $F_e = 0$.

We now turn our attention to Eqn (5.3). We multiply this equation termwise by v_e^4 and integrate the resulting expression over all v_e to obtain

$$\frac{\mathrm{d}^2\hat{T}_e}{\mathrm{d}t^2} - \frac{1}{\tau}\frac{\mathrm{d}T_e}{\mathrm{d}t} - \frac{2m_e}{m_a\tau^2}(\hat{T}_e - \hat{T}_a) = -\frac{2}{3}m_e\frac{F_e}{\tau}\overline{v}_{ex} - \frac{2}{3}m_e F_e^2 + \frac{2}{3}m_e\overline{v}_{ex}\frac{\mathrm{d}F_e}{\mathrm{d}t} + \frac{4}{3}m_e F_e\frac{\mathrm{d}\overline{v}_{ex}}{\mathrm{d}t}\,, \tag{5.8}$$

where $F_e = b\sin\omega t$, and consequently

$$\overline{v}_{ex} = C_1\exp\left(-a\frac{t}{\tau}\right) + \frac{b\tau}{(\omega\tau)^4 + 3(\omega\tau)^2 + 1}\times\left[\sin\omega t - \left(2 + (\omega t)^2\right)\omega t\cos\omega t\right]. \tag{5.9}$$

The differential equation (5.8) integrates in the finite form to the following expression

$$\begin{aligned}
\hat{T}_{ea} = {} & C_1\exp\left(-d\frac{t}{\tau}\right) + \exp\left(-a\frac{t}{\tau}\right) \\
& \times\frac{\tau^2 Z\omega\tau}{\left[(d+a+1)^2 + \omega^2\tau^2\right]\left[(d-a)^2 + \omega^2\tau^2\right]} \\
& \times\Big\{\sin\omega t\big[\omega^2\tau^2\big(2\sqrt{5} - 2(d+a+1) \\
& + \omega^2\tau^2(\sqrt{5} - 1 - d - a)\big) + (d-a) \\
& \times\big(2\omega^2\tau^2 + \omega^4\tau^4 + \omega^2\tau^2\sqrt{5}\,(d+a+1) \\
& + 2\sqrt{5}\,(d+a+1)\big)\big] \\
& + \cos\omega t\big[(d-a)\,\omega\tau\big(2\sqrt{5} + \omega^2\tau^2(\sqrt{5} - 1 - d - a) \\
& - 2(d+a+1)\big) - \omega\tau\big(2\omega^2\tau^2 + \omega^4\tau^4 \\
& + \omega^2\tau^2\sqrt{5}\,(d+a+1) + 2\sqrt{5}\,(d+a+1)\big)\big]\Big\} \\
& + Z\tau^2\bigg\{\frac{\sin^2\omega t}{2(d^2 + 4\omega^2\tau^2)\left[(d+1)^2 + 4\omega^2\tau^2\right]} \\
& \times\big[4d(d+1) - 2d(d+1)\omega^2\tau^2 - 4\omega^2\tau^2 \\
& - 2d(d+1)\omega^4\tau^4 - 28\omega^4\tau^4 - 16\omega^6\tau^6\big] \\
& + \cos^2\omega t\,\frac{\omega^2\tau^2(2 + \omega^2\tau^2)\,d(d+1)}{\left[(d+1)^2 + 4\omega^2\tau^2\right]\left[d^2 + 4\omega^2\tau^2\right]} \\
& - \omega\tau\cos\omega t\sin\omega t\,\frac{\omega^2\tau^2(d^2 + d + 14) + 4d + 4}{(d^2 + 4\omega^2\tau^2)\left[(d+1)^2 + 4\omega^2\tau^2\right]} \\
& + \omega^2\tau^2\big[9d^2 + 9d + 4 + \omega^2\tau^2(12d^2 + 10d + 18) \\
& + 4\omega^4\tau^4(d^2 + d + 2)\big] \\
& \times\left[d(d+1)\big((d+1)^2 + 4\omega^2\tau^2\big)(d^2 + 4\omega^2\tau^2)\right]^{-1}\bigg\},
\end{aligned} \tag{5.10}$$

where the notation was used:

$$a = \frac{\sqrt{5}-1}{2}\,, \qquad d = \frac{1}{2}\left(\sqrt{1+8\,\frac{m_e}{m_a}}-1\right) \cong 2\,\frac{m_e}{m_a} \ll 1\,,$$

$$Z = \frac{2}{3}\,\frac{m_e b^2}{\omega^4\tau^4 + 3\omega^2\tau^2 + 1}\,.$$

In the quasi-stationary limiting case, under the condition $\omega\tau \gg 1$, one finds

$$\hat{T}_{ea} = -\tau^2 Z\,\frac{\omega^2\tau^2}{2}\,\sin^2\omega t + \tau^2 Z\,\frac{\omega^2\tau^2}{2d} \tag{5.11}$$

or, taking into account that the time average

$$\overline{\sin^2\omega\tau} = \frac{1}{2} \ll d^{-1}\,,$$

we arrive at

$$\hat{T}_e = \frac{m_a}{6}\left(\frac{eE_0}{m_e\omega}\right)^2. \tag{5.12}$$

Thus, in the limiting case $l = \mathrm{const}$, the following equality holds true:

$$\hat{T}_e^{\,\mathrm{BE}} = \hat{T}_e^{\,\mathrm{GBE}}\,. \tag{5.13}$$

In the opposite limit of $\omega\tau \ll d \ll 1$, one has

$$\hat{T}_{ea} = \tau^2 Z\,\frac{2}{d}\,\sin^2\omega t$$

or, computing the average over the time, we obtain

$$\hat{T}_{ea} = \frac{m_a\tau^2}{3}\left(\frac{eE_0}{m_e}\right)^2, \tag{5.14}$$

i.e., a result which corresponds to the solution of the classical Boltzmann equation [50].

6. Conductivity of a weakly ionized gas in crossed electric and magnetic fields

In this section, the conductivity of a weakly ionized gas subject to crossed magnetic and (alternating) electric field will be examined using the GBE with the BGK approximation for the elastic collision integral. The BGK-approximated kinetic equation takes the form

$$\frac{\partial f_e}{\partial t} + \mathbf{F}_e\cdot\frac{\partial f_e}{\partial \mathbf{v}_e} - \tau\left(\frac{\partial^2 f_e}{\partial t^2} + 2\mathbf{F}_e\cdot\frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial t} + \frac{\partial \mathbf{F}_e}{\partial t}\cdot\frac{\partial f_e}{\partial \mathbf{v}_e} + \frac{\partial^2 f_e}{\partial \mathbf{v}_e\,\partial \mathbf{v}_e} : \mathbf{F}_e\mathbf{F}_e + \frac{\partial f_e}{\partial \mathbf{v}_e}\,\mathbf{F}_e : \frac{\partial}{\partial \mathbf{v}_e}\,\mathbf{F}_e\right) = -\frac{f_e - f_e^0}{\tau}\,, \tag{6.1}$$

where $\mathbf{F}_e = \mathbf{F}_e^{(1)} + \mathbf{F}_B$ is the Lorentz force which, in our case, includes the effect of the alternating electric field

$$F_e^{(1)} = \frac{eE^0}{m_e}\,\exp(\mathrm{i}\omega t)$$

directed along the x-axis, and of a static magnetic field, whose induction is along the z-axis. The equation of motion (3.30) reduces to the form

$$\frac{\partial}{\partial t}\left[\overline{\mathbf{v}}_e - \tau\left(\frac{\partial \overline{\mathbf{v}}_e}{\partial t} - \frac{e\mathbf{E}^0}{m_e}\exp(\mathrm{i}\omega t) - \frac{e}{m_e}\,\overline{\mathbf{v}}_e\times\mathbf{B}\right)\right] - \frac{e\mathbf{E}^0}{m_e}\exp(\mathrm{i}\omega t) - \frac{e}{m_e}\left[\overline{\mathbf{v}}_e - \tau\left(\frac{\partial \overline{\mathbf{v}}_e}{\partial t} - \frac{e\mathbf{E}^0}{m_e}\exp(\mathrm{i}\omega t) - \frac{e}{m_e}\,\overline{\mathbf{v}}_e\times\mathbf{B}\right)\right]\times\mathbf{B} = -\frac{\overline{\mathbf{v}}_e}{\tau}\,. \tag{6.2}$$

The components of the drift velocity $\overline{\mathbf{v}}_e$ along the axes x and y are determined by the following set of equations ($\overline{v}_{ez} = 0$):

$$\frac{\partial}{\partial t}\left[\overline{v}_{ex} - \tau\left(\frac{\partial \overline{v}_{ex}}{\partial t} - \frac{eE^0}{m_e}\exp(\mathrm{i}\omega t) - \frac{e}{m_e}\,\overline{v}_{ey}B\right)\right] - \frac{eE^0}{m_e}\exp(\mathrm{i}\omega t) - \frac{e}{m_e}\left(B\,\overline{v}_{ey} - \tau B\,\frac{\partial \overline{v}_{ey}}{\partial t}\right) = -\frac{\overline{v}_{ex}}{\tau} - \frac{e^2\tau}{m_e^2}\,B^2\,\overline{v}_{ex}\,, \tag{6.3}$$

$$\frac{\partial}{\partial t}\left[\overline{v}_{ey} - \tau\left(\frac{\partial \overline{v}_{ey}}{\partial t} + \frac{e}{m_e}\,\overline{v}_{ex}B\right)\right] - \frac{e}{m_e}\left(-\overline{v}_{ex}B + \tau B\,\frac{\partial \overline{v}_{ex}}{\partial t} - \frac{eE^0}{m_e}\exp(\mathrm{i}\omega t)\,\tau B\right) = -\frac{\overline{v}_{ey}}{\tau} - \frac{e^2\tau}{m_e^2}\,B^2\,\overline{v}_{ey}\,. \tag{6.4}$$

The solution to Eqn (6.2) is naturally sought in the form $\overline{v} = \overline{v}^{\,0}\exp(\mathrm{i}\omega t)$, thus leading to the following system of algebraic equations

$$\overline{v}_{ex}^{\,0}\left[\mathrm{i}\omega + \tau(\omega^2+\omega_B^2) + \frac{1}{\tau}\right] = \frac{eE^0}{m_e} + \overline{v}_{ey}^{\,0}\omega_B(1-2\mathrm{i}\omega\tau) - \mathrm{i}\omega\tau\,\frac{eE^0}{m_e}\,, \tag{6.5}$$

$$\overline{v}_{ey}^{\,0}\left[\mathrm{i}\omega + \tau(\omega^2+\omega_B^2) + \frac{1}{\tau}\right] = \overline{v}_{ex}^{\,0}\omega_B(2\mathrm{i}\omega\tau - 1) - \omega_B\tau\,\frac{eE^0}{m_e}\,, \tag{6.6}$$

where $\omega_B = eB/m_e$. From these equations it is not difficult to find the components $\overline{v}_{ex}$ and $\overline{v}_{ey}$ of the drift velocity, and hence to determine the components of the electrical conductivity tensor. In our case, the complex conductivity σ_x assumes the form

$$\sigma_x = \sigma_0\left[1 + 2\omega^2\tau^2 + \mathrm{i}(\omega_B^2\tau^2 - \omega^2\tau^2)\,\omega\tau\right] \times \left[1 + \omega^2\tau^2 + \omega^4\tau^4 + 3\omega_B^2\tau^2 + \omega_B^4\tau^4 - 2\omega^2\tau^2\omega_B^2\tau^2 + 2\mathrm{i}\omega\tau(1+\omega^2\tau^2-\omega_B^2\tau^2)\right]^{-1}\,, \tag{6.7}$$

where we have used the notation:

$$\sigma_x = \frac{en_e\overline{v}_{ex}^{\,0}}{E^0}\,, \qquad \sigma_0 = \frac{n_e e^2\tau}{m_e}\,.$$

Separating the real part of σ_x now yields

$$\mathrm{Re}\,\sigma_x = \sigma_0\left[1 + 3\omega^2\tau^2 + \omega^4\tau^4 + \omega_B^2\tau^2(3 + 6\omega_B^2\tau^2 + \omega_B^2\tau^2)\right] \times \left\{\left[1 + \omega^2\tau^2 + \omega^4\tau^4 + \omega_B^2\tau^2(3 + \omega_B^2\tau^2 - 2\omega_B^2\tau^2)\right]^2 + 4\omega^2\tau^2(1+\omega^2\tau^2-\omega_B^2\tau^2)^2\right\}^{-1}. \tag{6.8}$$

The Boltzmann theory, as is known, leads to the following results

$$\operatorname{Re}\sigma_x = \frac{\sigma_0(1+\omega^2\tau^2+\omega_B^2\tau^2)}{1+2\omega^2\tau^2+\omega^4\tau^4+\omega_B^2\tau^2(2-2\omega^2\tau^2+\omega_B^2\tau^2)}\,, \tag{6.9}$$

$$\sigma_x = \sigma_0\,\frac{1+\mathrm{i}\omega\tau}{1+(\omega_B^2-\omega^2)\tau^2+2\mathrm{i}\omega\tau}\,. \tag{6.10}$$

For the traditionally considered limiting cases, Eqns (6.7)–(6.10) give the following results:

(a) $\omega = 0$, a constant electric field:

$$\operatorname{Re}\sigma_x^{\mathrm{GBE}} = \sigma_0\,\frac{1}{1+3\omega_B^2\tau^2+\omega_B^4\tau^4}\,,$$

$$\operatorname{Re}\sigma_x^{\mathrm{BE}} = \sigma_0\,\frac{1}{1+\omega_B^2\tau^2}\,;$$

(b) $\omega_B = 0$, no magnetic field:

$$\operatorname{Re}\sigma_x^{\mathrm{GBE}} = \frac{\sigma_0(1+3\omega^2\tau^2+\omega^4\tau^4)}{(1+\omega^2\tau^2+\omega^4\tau^4)^2+4\omega^2\tau^2(1+\omega^2\tau^2)^2}\,,$$

$$\operatorname{Re}\sigma_x^{\mathrm{BE}} = \sigma_0\,\frac{1}{1+\omega^2\tau^2}\,;$$

(c) $\omega = \omega_B$:

$$\operatorname{Re}\sigma_x^{\mathrm{GBE}} = \sigma_0\,\frac{1+6\omega^2\tau^2+8\omega^4\tau^4}{1+12\omega^2\tau^2+16\omega^4\tau^4}\,,$$

$$\operatorname{Re}\sigma_x^{\mathrm{BE}} = \sigma_0\,\frac{1+2\omega^2\tau^2}{1+4\omega^2\tau^2}\,;$$

(d) $\omega = \omega_B$, $\omega\tau \gg 1$, the cyclotron resonance condition:

$$\operatorname{Re}\sigma_x^{\mathrm{GBE}} = \operatorname{Re}\sigma_x^{\mathrm{BE}} = \frac{1}{2}\,\sigma_0\,.$$

Finally, from the system of equations (6.3), (6.4) the drift velocity along the y-axis is found as

$$\begin{aligned}\overline{v}_{ey}^{0\,\mathrm{GBE}} &= \frac{eE^0}{m_e}\,\omega_B\tau^2\big[(\omega^4-\omega_B^4)\,\tau^4-3(\omega^2+\omega_B^2)\,\tau^2\\ &\qquad -2+\mathrm{i}\omega\tau(3\omega^2\tau^2+\omega_B^2\tau^2)\big]\,D^{-1}\,,\end{aligned}$$

$$\begin{aligned}D &= 1+(\omega^6+\omega_B^6)\,\tau^6+4\omega_B^2\tau^2+4\omega^2\tau^2\omega_B^2\tau^2\\ &\quad -\omega^4\tau^4\omega_B^2\tau^2+4\omega_B^4\tau^4-\omega^2\tau^2\omega_B^4\tau^4\\ &\quad +\mathrm{i}\omega\tau(3\omega^4\tau^4-5\omega^2\tau^2+3+3\omega_B^2\tau^2\\ &\quad -2\omega^2\tau^2\omega_B^2\tau^2-\omega_B^4\tau^4)\,.\end{aligned}$$

Notice that the BE implies that

$$\overline{v}_{ey}^{0\,\mathrm{BE}} = -\frac{eE^0}{m_e}\,\omega_B\tau^2\,\frac{1}{1+(\omega_B^2-\omega^2)\,\tau^2+2\mathrm{i}\omega\tau}\,.$$

In particular, it follows that

(a) if $\omega = 0$,

$$\operatorname{Re}\overline{v}_{ey}^{0\,\mathrm{GBE}} = -\frac{eE^0}{m_e}\,\omega_B\tau^2\,\frac{\omega_B^4\tau^4+3\omega_B^2\tau^2+2}{\omega_B^6\tau^6+4\omega_B^4\tau^4+4\omega_B^2\tau^2+1}\,,$$

$$\operatorname{Re}\overline{v}_{ey}^{0\,\mathrm{BE}} = -\frac{eE^0}{m_e}\,\omega_B\tau^2\,\frac{1}{\omega_B^2\tau^2+1}\,;$$

(b) if $\omega = \omega_B$,

$$\begin{aligned}\operatorname{Re}\overline{v}_{ey}^{0\,\mathrm{GBE}} &= -\frac{eE^0}{m_e}\,\omega\tau^2\\ &\times\frac{2+14\omega^2\tau^2+28\omega^4\tau^4+56\omega^6\tau^6}{(1+4\omega^2\tau^2+8\omega^4\tau^4)^2+\omega^2\tau^2(3-2\omega^2\tau^2)^2}\,,\end{aligned}$$

$$\operatorname{Re}\overline{v}_{ey}^{0\,\mathrm{BE}} = -\frac{eE^0}{m_e}\,\omega\tau^2\,\frac{1}{1+4\omega^2\tau^2}\,.$$

The calculations show that while the BE and GBE results may happen to be identical, they may also be significantly different, both qualitatively and quantitatively. The question of exactly how significantly can only be answered through the solution of concrete problems. In particular, the generalized Boltzmann equation has been applied successfully to transport processes in a partially ionized gas of inelastically colliding particles [52].

7. Plasma dispersion relations for the GBE model with a collision term

The generalized Boltzmann equation describes how the one-particle distribution function f_α $(\alpha = 1, \ldots, \mu)$ in a μ-component gas mixture changes over times of the order of the time between collisions, of the order of the hydrodynamic flow time, and, unlike the conventional Boltzmann equation, over a time of the order of the collision time. The GBE for a plasma medium has the form

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} - \frac{\mathrm{D}}{\mathrm{D}t}\left(\tau_\alpha\,\frac{\mathrm{D}f_\alpha}{\mathrm{D}t}\right) = J_\alpha\,, \tag{7.1}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}_\alpha\cdot\frac{\partial}{\partial\mathbf{r}} + \mathbf{F}_\alpha\cdot\frac{\partial}{\partial\mathbf{v}_\alpha} \tag{7.2}$$

is the substantial (particle) derivative containing the self-consistent force $\mathbf{F}_\alpha$, J_α is the classical (Boltzmann) collision integral, and τ_α is the mean time between the close particle collisions. In the hydrodynamic regime τ_α can be expressed in terms of the Coulomb logarithm Λ, viscosity η_α, static pressure p_α, and the coefficient Π dependent on the particle interaction model [see Eqn (2.37)].

The generalized Boltzmann equation in general and that for plasma in particular have a fundamentally important feature that the additional GBE terms prove to be of the order of the Knudsen number. This does not mean that in the hydrodynamic (small Kn) limit these terms may be neglected: the Knudsen number in this case appears as a small parameter of the higher derivative in the GBE. Consequently, the additional GBE terms (as compared to the BE) are significant for any Kn, and the order of magnitude of the difference between the BE and GBE solutions is impossible to tell beforehand (see Ref. [3]).

In this connection, it is of interest to apply the GBE model to obtain the dispersion relation for a plasma in the absence of a magnetic field. In doing so, we will make the same assumptions that were used in the BE-based derivation, namely: (a) the integral collision term is neglected; (b) the evolution of electrons and ions in a self-consistent electric field corresponds to a nonstationary one-dimensional model; (c) the distribution functions for ions, f_{i}, and for electrons, f_{e}, deviate little from their equilibrium counterparts $f_{0\mathrm{i}}$ and $f_{0\mathrm{e}}$; (d) a wave number k and a complex frequency ω are appropriate to the wave mode considered; (e) the quadratic GBE terms determining the deviation from the equilibrium DF are neglected, and (f) the self-consistent forces F_{i} and F_{e} are small.

Results of the calculations done under these assumptions are given in Appendix A.2. Proceeding now to the dispersion relation, we lift one of these assumptions, the first, by introducing the Bhatnagar – Gross – Krook (BGK) collision term

$$J_\alpha = -\frac{f_\alpha - f_{0\alpha}}{v_\alpha^{-1}} \tag{7.3}$$

into the right-hand side of the GBE. Here, $f_{0\alpha}$ and $v_\alpha^{-1} = \tau_{\mathrm{p}\alpha}$ are respectively a certain equilibrium distribution function and the relaxation time for species of the αth kind. Using Eqns (A2.9) and (7.1), we arrive at the dispersion relation

$$1 = -\frac{e^2}{\varepsilon_0 k}\left\{\frac{1}{m_{\mathrm{e}}}\int_{-\infty}^{+\infty}\frac{(\partial f_{0\mathrm{e}}/\partial u)\left[\mathrm{i} - \tau_{\mathrm{e}}(\omega - ku)\right]}{\mathrm{i}(\omega - ku) - \tau_{\mathrm{e}}(\omega - ku)^2 - v_{\mathrm{e}}}\,\mathrm{d}u + \frac{1}{m_{\mathrm{i}}}\int_{-\infty}^{+\infty}\frac{(\partial f_{0\mathrm{i}}/\partial u)\left[\mathrm{i} - \tau_{\mathrm{i}}(\omega - ku)\right]}{\mathrm{i}(\omega - ku) - \tau_{\mathrm{i}}(\omega - ku)^2 - v_{\mathrm{i}}}\,\mathrm{d}u\right\}. \tag{7.4}$$

In the Boltzmann kinetic theory, the analogue of Eqn (7.4) is the equation [53]

$$1 = -\frac{e^2}{\varepsilon_0 k}\left\{\frac{1}{m_{\mathrm{e}}}\int_{-\infty}^{+\infty}\frac{\partial f_{0\mathrm{e}}/\partial u}{\omega - ku + \mathrm{i}v_{\mathrm{e}}}\,\mathrm{d}u + \frac{1}{m_{\mathrm{i}}}\int_{-\infty}^{+\infty}\frac{\partial f_{0\mathrm{i}}/\partial u}{\omega - ku + \mathrm{i}v_{\mathrm{i}}}\,\mathrm{d}u\right\}. \tag{7.5}$$

To solve Eqn (7.4), we take advantage of the additional conjectures. Let us assume that the ions are at rest and that both the temperature and average velocity of the electrons are zero. Then the electron distribution function can be expressed in terms of the delta function:

$$f_{0\,\mathrm{e}}(u) = n_{\mathrm{e}}\,\delta(u)\,. \tag{7.6}$$

Upon integration by parts in Eqn (7.5), we arrive at the equation (the subscript 'e' on v_{e} and τ_{e} is dropped for brevity)

$$1 + \frac{e^2 n_{\mathrm{e}}}{\varepsilon_0 m_{\mathrm{e}}}\int_{-\infty}^{+\infty}\frac{\delta(u)\left\{\left[1 + \mathrm{i}\tau(\omega - ku)\right]^2 + v\tau\right\}}{\left[\mathrm{i}(\omega - ku) - \tau(\omega - ku)^2 - v\right]^2}\,\mathrm{d}u = 0\,. \tag{7.7}$$

In the special case of Boltzmann collisionless plasma, Eqn (7.7) leads to the classical formula

$$1 - \frac{e^2 n_{\mathrm{e}}}{\varepsilon_0 m_{\mathrm{e}}}\int_{-\infty}^{+\infty}\frac{\delta(u)}{(\omega - ku)^2}\,\mathrm{d}u = 0\,. \tag{7.8}$$

Using the properties of the delta function and performing the integration in Eqn (7.7), it is found that

$$\omega_{\mathrm{e}}^2 = -\frac{(v\tau + \omega^2\tau^2 - \mathrm{i}\omega\tau)^2}{\tau^2(1 + v\tau - \omega^2\tau^2 + 2\mathrm{i}\omega\tau)}\,, \tag{7.9}$$

with $\omega_{\mathrm{e}} = \sqrt{e^2 n_{\mathrm{e}}/\varepsilon_0 m_{\mathrm{e}}}$ being the plasma frequency.

Let us consider the limiting cases inherent in Eqn (7.9). If $|\omega|\tau \ll 1$, then separating the real and imaginary parts of relation (7.9) leads to the result

$$\omega_{\mathrm{e}}^2 = -\frac{v}{1 + v\tau}\,\frac{(1 + v\tau)v + 2\omega''}{1 + v\tau - 4\omega''\tau}\,, \tag{7.10}$$

$$\omega'\,\frac{1 + 2\tau v}{(1 + v\tau)(1 + v\tau - 4\omega''\tau)} = 0\,, \tag{7.11}$$

where $\omega = \omega' + \mathrm{i}\omega''$. From the last equation it follows that

$$\omega' = 0\,, \tag{7.12}$$

and from Eqn (7.10) one obtains

$$\omega'' = -\frac{1}{2}\,\frac{(1 + v\tau)\left[(1 + v\tau)\omega_{\mathrm{e}}^2 + v^2\right]}{v - 2\omega_{\mathrm{e}}^2\tau(1 + v\tau)}\,. \tag{7.13}$$

If $\omega_{\mathrm{e}}^2\tau^2 \ll 1$, then

$$\omega'' = -\frac{1}{2}\,v(1 + v\tau)\,. \tag{7.14}$$

Thus, the condition $|\omega|\tau \ll 1$ leads to the fast decay of the perturbation (A2.2) in accordance with the solution (A2.4).

In the opposite limit of $|\omega|\tau \gg 1$ ($\omega'\tau \gg 1$, $\omega''\tau \gg 1$), from the relation

$$\omega_{\mathrm{e}}^2 = -\frac{(\omega^2\tau^2 - \mathrm{i}\omega\tau)^2}{\tau^2(-\omega^2\tau^2 + 2\mathrm{i}\omega\tau)}$$

we find that

$$\omega^2 = \omega_{\mathrm{e}}^2\,. \tag{7.15}$$

Thus, for the election distribution function of the form (7.6), the asymptotic solutions of the dispersion relation (7.9) have a transparent physical meaning.

Now let the ions be at rest, and the electron component have a Maxwellian velocity distribution:

$$f_{0\,\mathrm{e}} = n_{\mathrm{e}}\left(\frac{m_{\mathrm{e}}}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{m_{\mathrm{e}}V^2}{2k_{\mathrm{B}}T}\right),$$

where k_{B} is the Boltzmann constant. Then equation (7.4) becomes

$$1 + \frac{e^2 n_{\mathrm{e}}}{\varepsilon_0 k m_{\mathrm{e}}}\left(\frac{m_{\mathrm{e}}}{2\pi k_{\mathrm{B}}T}\right)^{1/2}\int_{-\infty}^{+\infty}\left[\mathrm{i} - \tau(\omega - ku)\right] \times \frac{\partial}{\partial u}\exp\left(-\frac{m_{\mathrm{e}}u^2}{2k_{\mathrm{B}}T}\right)\left[\mathrm{i}(\omega - ku) - \tau(\omega - ku)^2 - v\right]^{-1}\mathrm{d}u = 0\,, \tag{7.16}$$

where we have reintroduced the notation ($u \equiv V_x$) for the velocity of the one-dimensional, unsteady wave motion. From the above equation one derives the expression

$$1 + \frac{1}{r_{\mathrm{D}}^2 k^2}\left[1 - \sqrt{\frac{m_{\mathrm{e}}}{2\pi k_{\mathrm{B}}T}} \times \int_{-\infty}^{+\infty}\frac{\left\{\left[\mathrm{i} - \tau(\omega - ku)\right]\omega - v\right\}\exp\left(-m_{\mathrm{e}}u^2/2k_{\mathrm{B}}T\right)}{\mathrm{i}(\omega - ku) - \tau(\omega - ku)^2 - v}\,\mathrm{d}u\right] = 0\,, \tag{7.17}$$

where $r_{\mathrm{D}} = \sqrt{\varepsilon_0 k_{\mathrm{B}}T/n_{\mathrm{e}}e^2}$ is the Debye – Hückel radius.

Introducing now the dimensionless variables

$$\hat{u} = u\sqrt{\frac{m_e}{2k_B T}}\,, \qquad \hat{\omega} = \omega\,\frac{1}{k}\sqrt{\frac{m_e}{2k_B T}}\,,$$
$$\hat{v} = v\,\frac{1}{k}\sqrt{\frac{m_e}{2k_B T}}\,, \qquad \hat{\tau} = \tau k\sqrt{\frac{2k_B T}{m_e}}\,, \tag{7.18}$$

we can rewrite Eqn (7.17) in the form

$$1+\frac{1}{r_D^2 k^2}\times\left[1-\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\left\{\left[\mathrm{i}-\hat{\tau}(\hat{\omega}-\hat{u})\right]\hat{\omega}-\hat{v}\right\}\exp(-\hat{u}^2)}{\mathrm{i}(\hat{\omega}-\hat{u})-\hat{\tau}(\hat{\omega}-\hat{u})^2-\hat{v}}\,\mathrm{d}\hat{u}\right]=0\,. \tag{7.19}$$

Now consider a situation in which the denominator of the complex integrand in Eqn (7.19) becomes zero. The quadratic equation

$$\hat{\tau}y^2-\mathrm{i}y+\hat{v}=0\,, \qquad y=\hat{\omega}-\hat{u}\,, \tag{7.20}$$

has the roots

$$y_1=\frac{\mathrm{i}}{2\hat{\tau}}\left(1+\sqrt{1+4\hat{\tau}\hat{v}}\,\right), \qquad y_2=\frac{\mathrm{i}}{2\hat{\tau}}\left(1-\sqrt{1+4\hat{\tau}\hat{v}}\,\right). \tag{7.21}$$

Hence, Eqn (7.19) can be rewritten as

$$1+\frac{1}{r_D^2 k^2}\times\left[1+\frac{1}{\hat{\tau}\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\left\{\left[\mathrm{i}+\hat{\tau}(\hat{u}-\hat{\omega})\right]\hat{\omega}-\hat{v}\right\}\exp(-\hat{u}^2)}{(\hat{u}-\hat{u}_1)(\hat{u}-\hat{u}_2)}\,\mathrm{d}u\right]=0\,, \tag{7.22}$$

where

$$\hat{u}_1=\hat{\omega}-y_1\,, \qquad \hat{u}_2=\hat{\omega}-y_2\,. \tag{7.23}$$

Let us transform equation (7.22) to the following one:

$$1+\frac{1}{r_D^2 k^2}\left\{1+\frac{1}{\sqrt{\pi}}\left[\left(\frac{\mathrm{i}\hat{v}+0.5\hat{\omega}}{\sqrt{1+4\hat{\tau}\hat{v}}}-0.5\hat{\omega}\right)\int_{-\infty}^{+\infty}\frac{\exp(-\hat{u}^2)}{\hat{u}_1-\hat{u}}\,\mathrm{d}\hat{u} -\left(\frac{\mathrm{i}\hat{v}+0.5\hat{\omega}}{\sqrt{1+4\hat{\tau}\hat{v}}}+0.5\hat{\omega}\right)\int_{-\infty}^{+\infty}\frac{\exp(-\hat{u}^2)}{\hat{u}_2-\hat{u}}\,\mathrm{d}\hat{u}\right]\right\}=0\,. \tag{7.24}$$

The last equation contains improper Cauchy type integrals which can be evaluated using the theory of residues. Let us first analyze the conditions under which plasma waves can be damped. This requires, first, that [see Eqn (A2.4)] the imaginary part of the complex frequency fulfil the condition

$$\omega''<0 \tag{7.25}$$

and, second, that the poles involved in the calculation of the integrals in Eqn (7.24) lie in the upper half-plane (see Fig. 5, in which the integration contour is shown). Since

$$\hat{u}_1=\hat{\omega}'+\mathrm{i}\left(\hat{\omega}''-\frac{1+\sqrt{1+4\hat{\tau}\hat{v}}}{2\hat{\tau}}\right), \tag{7.26}$$

$$\hat{u}_2=\hat{\omega}'+\mathrm{i}\left(\hat{\omega}''-\frac{1-\sqrt{1+4\hat{\tau}\hat{v}}}{2\hat{\tau}}\right), \tag{7.27}$$

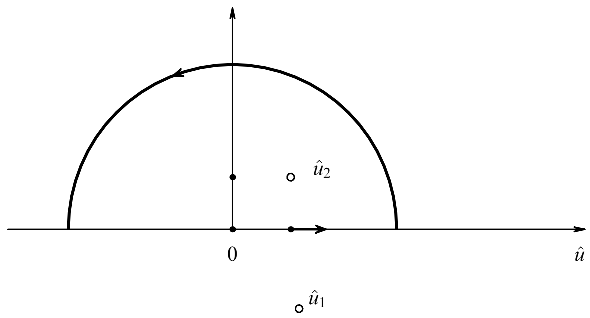

Figure 5. Integration contour for evaluating complex integrals in Eqn (7.24). The two open circles are the possible positions of the poles $\hat{u}_1$ and $\hat{u}_2$ [see Eqns (7.26) and (7.27)] involved in the solution of the dispersion equation in the regime of damped plasma oscillations.

the second condition is fulfilled for the integral

$$I_2=\int_{-\infty}^{+\infty}\frac{\exp(-\hat{u}^2)}{\hat{u}-\hat{u}_2}\,\mathrm{d}\hat{u}$$

if the inequality

$$\hat{\omega}''+\frac{\sqrt{1+4\hat{\tau}\hat{v}}-1}{2\hat{\tau}}>0 \tag{7.28}$$

is satisfied.

A similar condition for the integral

$$I_1=\int_{-\infty}^{+\infty}\frac{\exp(-\hat{u}^2)}{\hat{u}-\hat{u}_1}\,\mathrm{d}\hat{u} \tag{7.29}$$

cannot be satisfied. For this integral, the poles lie in the lower half-plane, and hence

$$I_2=2\pi\mathrm{i}\,\mathrm{res}\,(\hat{u}=\hat{u}_2)\,, \tag{7.30}$$

$$I_1=0\,. \tag{7.31}$$

Then Eqn (7.24) produces the dispersion relation which admits a damped plasma wave solution:

$$\exp(\hat{u}_2^2)\,\frac{1+r_D^2k^2}{2\sqrt{\pi}}=\frac{\hat{v}}{\sqrt{1+4\hat{\tau}\hat{v}}}-\frac{\mathrm{i}\hat{\omega}}{2}\left(1+\frac{1}{\sqrt{1+4\hat{\tau}\hat{v}}}\right), \tag{7.32}$$

where

$$\hat{u}_2^2=\left[\hat{\omega}'+\mathrm{i}\left(\hat{\omega}''+\frac{\sqrt{1+4\hat{\tau}\hat{v}}-1}{2\hat{\tau}}\right)\right]^2 =\hat{\omega}'^2-\hat{\omega}''^2-\hat{\omega}''\,\frac{\sqrt{1+4\hat{\tau}\hat{v}}-1}{\hat{\tau}}-\frac{1+2\hat{\tau}\hat{v}-\sqrt{1+4\hat{\tau}\hat{v}}}{2\hat{\tau}^2} +\mathrm{i}\left(2\hat{\omega}''+\frac{\sqrt{1+4\hat{\tau}\hat{v}}-1}{\hat{\tau}}\right)\hat{\omega}'\,. \tag{7.33}$$

The relaxation time τ_{rel} can be estimated in terms of the mean time τ between close collisions and the Coulomb logarithm [35]:

$$\tau_{rel}=\tau\Lambda^{-1}\,. \tag{7.34}$$

We can then write

$$\tau v=\Lambda\,, \qquad \hat{\tau}\hat{v}=\Lambda\,. \tag{7.35}$$

Now, in Eqn (7.32) we write down the complex part of the exponential in the trigonometrical form:

$$\exp\left[-\mathrm{i}\hat\omega'\left(2\hat\omega''+\frac{\sqrt{1+4\hat\tau\hat v}-1}{\hat\tau}\right)\right]$$
$$=\cos\left[\hat\omega'\left(2\hat\omega''+\frac{\sqrt{1+4\hat\tau\hat v}-1}{\hat\tau}\right)\right]$$
$$-\,\mathrm{i}\sin\left[\hat\omega'\left(2\hat\omega''+\frac{\sqrt{1+4\hat\tau\hat v}-1}{\hat\tau}\right)\right]$$

and then separate the real and imaginary parts. For the real part we have

$$\frac{1+r_{\mathrm D}^2k^2}{2\sqrt{\pi}}\exp\left(\hat\omega'^2-\hat\omega''^2-\hat\omega''\hat v\,\frac{\sqrt{1+4\Lambda}-1}{\Lambda}\right.$$
$$\left.-\,\hat v^2\,\frac{1+2\Lambda-\sqrt{1+4\Lambda}}{2\Lambda^2}\right)$$
$$=\left(\frac{\hat v}{\sqrt{1+4\Lambda}}+0.5\hat\omega''+\frac{0.5\hat\omega''}{\sqrt{1+4\Lambda}}\right)$$
$$\times\cos\left[\hat\omega'\left(2\hat\omega''+\hat v\,\frac{\sqrt{1+4\Lambda}-1}{\Lambda}\right)\right]$$
$$-\,0.5\hat\omega'\left(1+\frac{1}{\sqrt{1+4\Lambda}}\right)\sin\left[\hat\omega'\left(2\hat\omega''+\hat v\,\frac{\sqrt{1+4\Lambda}-1}{\Lambda}\right)\right]. \tag{7.36}$$

Similarly, for the imaginary part we find

$$0.5\hat\omega'\left(1+\frac{1}{\sqrt{1+4\Lambda}}\right)\cos\left[\hat\omega'\left(2\hat\omega''+\hat v\,\frac{\sqrt{1+4\Lambda}-1}{\Lambda}\right)\right]$$
$$+\left(\frac{\hat v}{\sqrt{1+4\Lambda}}+0.5\hat\omega''+\frac{0.5\hat\omega''}{\sqrt{1+4\Lambda}}\right)$$
$$\times\sin\left[\hat\omega'\left(2\hat\omega''+\hat v\,\frac{\sqrt{1+4\Lambda}-1}{\Lambda}\right)\right]=0\,. \tag{7.37}$$

The system of complicated transcendent equations (7.36), (7.37) can generally be solved only on a computer. If, however, the Coulomb logarithm Λ is large enough for terms of order $\Lambda^{-1/2}$ to be negligible, then a common calculator will do. The system of equations (7.36), (7.37) in this case simplifies to

$$\frac{1+r_{\mathrm D}^2k^2}{\sqrt{\pi}}\exp\left(\hat\omega'^2-\hat\omega''^2\right)$$
$$=\hat\omega''\cos\left(2\hat\omega'\hat\omega''\right)-\hat\omega'\sin\left(2\hat\omega'\hat\omega''\right), \tag{7.38}$$
$$\hat\omega'\cos\left(2\hat\omega'\hat\omega''\right)+\hat\omega''\sin\left(2\hat\omega'\hat\omega''\right)=0\,. \tag{7.39}$$

Let us introduce the notation

$$\alpha=2\hat\omega'\hat\omega''\quad\text{and}\quad\beta=1+r_{\mathrm D}^2k^2 \tag{7.40}$$

and note that

$$\hat\omega'^2=-\frac12\,\alpha\tan\alpha\,,\qquad\hat\omega''^2=-\frac12\,\alpha\cot\alpha\,,$$
$$\hat\omega'^2-\hat\omega''^2=\alpha\cot2\alpha\,.$$

Then from Eqns (7.38), (7.39) one finds

$$-\sin2\alpha\exp\left(2\alpha\cot2\alpha\right)=\frac{\pi}{\beta^2}\,\alpha\,.$$

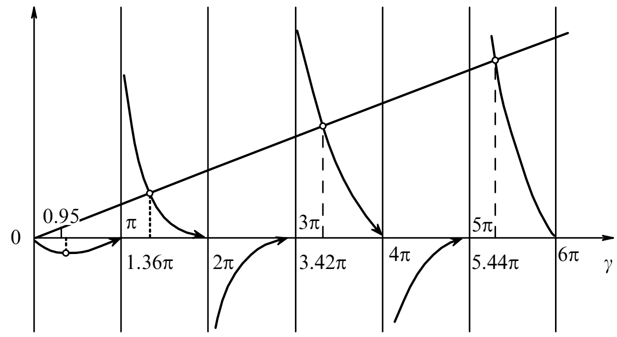

Figure 6. Graphical solution of Eqn (7.41), producing a discrete spectrum.

Now if one introduces the variable $\gamma=-2\alpha=-4\hat\omega'\hat\omega''$, the problem reduces to that of solving the transcendent equation

$$-\exp\left(\gamma\cot\gamma\right)\sin\gamma=\frac{\pi}{2\beta^2}\,\gamma\,, \tag{7.41}$$

which can be solved either graphically or iteratively [54]. The graphical solution for $\beta^2=1$ (corresponding to $r_{\mathrm D}^2k^2\ll1$) is illustrated in Fig. 6, which shows that in this case a discrete plasma oscillation spectrum appears even for an unbounded medium. The first seven values of γ are as follows: $\gamma_1=1.361\pi$, $\gamma_2=3.418\pi$, $\gamma_3=5.439\pi$, $\gamma_4=7.449\pi$, $\gamma_5=9.460\pi$, $\gamma_6=11.465\pi$, and $\gamma_7=13.469\pi$.

In the asymptotic limit of large, odd positive integers n ($n\geqslant301$), we have

$$\gamma_n=\left(n+\frac12\right)\pi\,. \tag{7.42}$$

The dimensionless frequencies $\hat\omega_n'$, $\hat\omega_n''$ are calculated using the formulas

$$\hat\omega_n'=\frac12\sqrt{-\gamma_n\tan\frac{\gamma_n}{2}}\,, \tag{7.43}$$
$$\hat\omega_n''=-\frac12\sqrt{-\gamma_n\cot\frac{\gamma_n}{2}}\,. \tag{7.44}$$

Their asymptotic values are given by

$$\hat\omega_n'=\frac{\sqrt{\gamma_n}}{2}=\frac12\sqrt{\pi\left(n+\frac12\right)}\,, \tag{7.45}$$
$$\hat\omega_n''=-\frac{\sqrt{\gamma_n}}{2}=-\frac12\sqrt{\pi\left(n+\frac12\right)}\,. \tag{7.46}$$

For example, $\hat\omega_2'=1.866$, $\hat\omega_2''=-1.438$, and $\hat\omega_3'=2.221$, $\hat\omega_3''=-1.510$.

In the classical Boltzmann kinetic theory, the search for damped wave modes in collisionless plasma leads to necessity of taking complex integrals of the type (A2.12), whose integrands have no poles in the upper half-plane [the 'upper' being specified by the choice of the solution in the form (A2.4)]. This problem is overcome by an artifice, the Landau rule for making a detour from below around a pole located on the real axis. We will show how the dispersion relation (7.32) produces results corresponding to 'classical' damping in collisional and collisionless plasmas. To do this, let us examine the asymptotic behavior of the dispersion relations

$$\frac{1+r_{\mathrm D}^2 k^2}{2\sqrt{\pi}}\exp\left\{\hat\omega'^2-\hat\omega''^2-\hat\omega''\hat\nu\,\frac{\sqrt{1+4\hat\tau\hat\nu}-1}{\hat\tau\hat\nu}-\hat\nu^2\,\frac{1+2\hat\tau\hat\nu-\sqrt{1+4\hat\tau\hat\nu}}{2(\hat\tau\hat\nu)^2}\right\}$$
$$=\left[\frac{\hat\nu}{\sqrt{1+4\hat\tau\hat\nu}}+0.5\hat\omega''\left(1+\frac{1}{\sqrt{1+4\hat\tau\hat\nu}}\right)\right]\times\cos\left[\hat\omega'\left(2\hat\omega''+\hat\nu\,\frac{\sqrt{1+4\hat\tau\hat\nu}-1}{\hat\tau\hat\nu}\right)\right]$$
$$-0.5\hat\omega'\left(1+\frac{1}{\sqrt{1+4\hat\tau\hat\nu}}\right)\sin\left[\hat\omega'\left(2\hat\omega''+\hat\nu\,\frac{\sqrt{1+4\hat\tau\hat\nu}-1}{\hat\tau\hat\nu}\right)\right], \qquad (7.47)$$

$$0.5\hat\omega'\left(1+\frac{1}{\sqrt{1+4\hat\tau\hat\nu}}\right)\cos\left[\hat\omega'\left(2\hat\omega''+\hat\nu\,\frac{\sqrt{1+4\hat\tau\hat\nu}-1}{\hat\tau\hat\nu}\right)\right]$$
$$+\left[\frac{\hat\nu}{\sqrt{1+4\hat\tau\hat\nu}}+0.5\hat\omega''\left(1+\frac{1}{\sqrt{1+4\hat\tau\hat\nu}}\right)\right]\times\sin\left[\hat\omega'\left(2\hat\omega''+\hat\nu\,\frac{\sqrt{1+4\hat\tau\hat\nu}-1}{\hat\tau\hat\nu}\right)\right]=0\,. \qquad (7.48)$$

The passage to the case of classical collisions is achieved by proceeding to the limit $\tau\to 0$ [see the generalized Boltzmann equations (7.1)–(7.3)] at a fixed frequency ν. The indeterminate forms involved in the calculation are evaluated by expanding the corresponding terms in a power series of a small parameter $\hat\tau\hat\nu$ and retaining the first two terms in the expansion in Eqns (7.47), (7.48) [in the last term in the curly brackets in Eqns (7.47), the quadratic term is also retained, though]. We follow this procedure to give

$$\frac{1+r_{\mathrm D}^2 k^2}{2\sqrt{\pi}}\exp\left[\hat\omega'^2-(\hat\omega''+\nu)^2\right]$$
$$=(\hat\omega''+\hat\nu)\cos\left[2\hat\omega'(\hat\omega''+\hat\nu)\right]-\hat\omega'\sin\left[2\hat\omega'(\hat\omega''+\hat\nu)\right], \qquad (7.49)$$
$$\hat\omega'\cos\left[2\hat\omega'(\hat\omega''+\hat\nu)\right]+(\hat\omega''+\hat\nu)\sin\left[2\hat\omega'(\hat\omega''+\hat\nu)\right]=0\,. \qquad (7.50)$$

Equations (7.49) and (7.50) can be brought to the same form as the system of equations (7.38), (7.39):

$$\frac{1+r_{\mathrm D}^2 k^2}{2\sqrt{\pi}}\exp\left(\hat\omega'^2-\hat\omega_1''^2\right)$$
$$=\hat\omega_1''\cos\left(2\hat\omega'\hat\omega_1''\right)-\hat\omega'\sin\left(2\hat\omega'\hat\omega_1''\right), \qquad (7.51)$$
$$\hat\omega'\cos\left(2\hat\omega'\hat\omega_1''\right)+\hat\omega_1''\sin\left(2\hat\omega'\hat\omega_1''\right)=0 \qquad (7.52)$$

by replacing ω'' with the variable $\omega_1''=\omega''+\nu$. It should also be noted that, in the asymptotics we are considering, an additional factor 0.5 appears on the left-hand side of Eqn (7.51). Equations (7.51) and (7.52) are then solved in exactly the same manner to give (for large n):

$$\hat\omega_n'=\frac{1}{2}\sqrt{\pi\left(n+\frac{1}{2}\right)}\,,\qquad \hat\omega_n''=-\frac{1}{2}\sqrt{\pi\left(n+\frac{1}{2}\right)}-\hat\nu\,. \qquad (7.53)$$

The relevant pole here lies in the upper half-plane (see Fig. 5) inside the integration contour and has the ordinate $\omega''+\nu$. We are now in a position to determine the frequency spectrum corresponding to the classical collisionless damping regime (Landau damping). For this we proceed to the limit $\nu\to 0$ in formulas (7.53). Using the notation introduced in Eqn (7.40), it then follows from Eqns (7.51) and (7.52) that

$$-\sin 2\alpha\,\exp\left(2\alpha\cot 2\alpha\right)=\frac{4\pi}{\beta^2}\,\alpha\,. \qquad (7.54)$$

If we introduce the variable $\gamma=-2\alpha=-4\hat\omega'\hat\omega''$, then the problem reduces to that of solving the transcendent equation

$$-\exp\left(\gamma\cot\gamma\right)\sin\gamma=\frac{2\pi}{\beta^2}\,\gamma\,, \qquad (7.55)$$

which can be solved either graphically or iteratively. The graphical solution for the $\beta^2=1$, corresponding to the long-wavelength limit $r_{\mathrm D}^2 k^2\ll 1$, is illustrated in Fig. 7. This solution shows that in the case of interest a discrete oscillation spectrum appears even for an unbounded medium. The first seven values of γ are as follows: $\gamma_1=1.271\pi$, $\gamma_2=3.379\pi$, $\gamma_3=5.410\pi$, $\gamma_4=7.432\pi$, $\gamma_5=9.444\pi$, $\gamma_6=11.452\pi$, and $\gamma_7=13.458\pi$. In the asymptotic limit of large, odd positive integers n ($n>301$) we have

$$\gamma_n=\left(n+\frac{1}{2}\right)\pi\,. \qquad (7.56)$$

The dimensionless frequencies $\hat\omega_n'$ and $\hat\omega_n''$ are calculated using formulas (7.43), (7.44), and their dimensional forms are represented as follows

$$\omega_n'=k\sqrt{-\frac{k_{\mathrm B}T}{2m_{\mathrm e}}\,\gamma_n\tan\frac{\gamma_n}{2}}\,,\qquad \omega_n''=-k\sqrt{-\frac{k_{\mathrm B}T}{2m_{\mathrm e}}\,\gamma_n\cot\frac{\gamma_n}{2}}\,. \qquad (7.57)$$

The asymptotic values of the frequencies are

$$\hat\omega_n'=\frac{\sqrt{\gamma_n}}{2}=\frac{1}{2}\sqrt{\pi\left(n+\frac{1}{2}\right)}\,,\qquad \hat\omega_n''=-\frac{\sqrt{\gamma_n}}{2}=-\frac{1}{2}\sqrt{\pi\left(n+\frac{1}{2}\right)}\,. \qquad (7.58)$$

It is also worthwhile to list the first seven pairs of dimensionless frequencies [55]:

$$\hat\omega_1'=1.484\,,\quad \hat\omega_1''=-0.673\,;\qquad \hat\omega_2'=1.979\,,\quad \hat\omega_2''=-1.341\,;$$
$$\hat\omega_3'=2.379\,,\quad \hat\omega_3''=-1.786\,;\qquad \hat\omega_4'=2.691\,,\quad \hat\omega_4''=-2.169\,;$$
$$\hat\omega_5'=2.975\,,\quad \hat\omega_5''=-2.493\,;\qquad \hat\omega_6'=3.235\,,\quad \hat\omega_6''=-2.780\,;$$
$$\hat\omega_7'=3.473\,,\quad \hat\omega_7''=-3.043\,. \qquad (7.59)$$

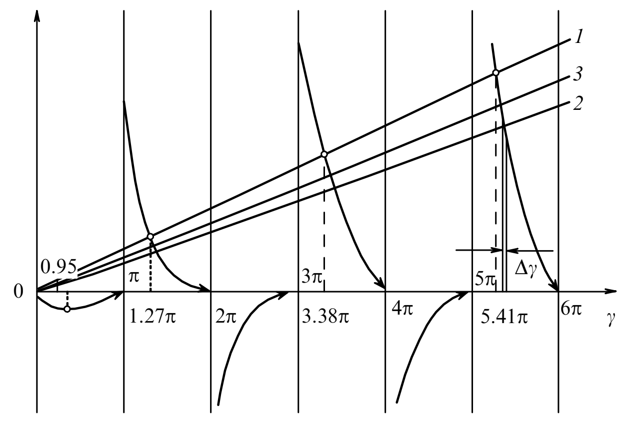

Figure 7. Graphical solution of the dispersion equation (7.55).

8. Generalized dispersion relations for plasma: theory and experiment

In this section theoretical results are discussed in the context of Looney and Brown's experiments [56] on the detection of plasma waves excited by an electron beam. The experimental setup of Looney and Brown (see Fig. 8) consists of a bulb about 10 cm in diameter, in which mercury plasma at a pressure of as low as 3×10^{-3} mm Hg was created between two cathodes C and an anode ring A using an electric discharge. An electron beam produced in a lateral tap was accelerated by a voltage of several hundred volts and then introduced into the plasma through a hole of diameter 1 mm in the bulb wall. In the region between the accelerating anode A and the anode disk D, with a separation of 1.5 cm, an ion cloud formed. The beam electrons excited oscillations in the region AD. The oscillations were registered by a movable probe attached to the detector. The results of the experiment are presented in Fig. 9 reproduced from Looney and Brown's paper. Because the density of the electron beam is proportional to the discharge current, Looney and Brown presented their results as the dependences of the oscillation frequency squared on the electron number density n_e. The inset on the left shows the electric field distribution over the region AD, and the dashed straight line corresponds to the dispersion relation

$$\omega = \omega_{\mathrm{pe}}\,, \tag{8.1}$$

where ω_{pe} is the Langmuir frequency of plasma electron oscillations. Equation (8.1) follows from the one-dimensional hydrodynamic equation of motion without considering convective terms and the pressure gradient. In a more general form including the electron pressure, this equation becomes

$$\omega^2 - \omega_{\mathrm{pe}}^2 = \frac{3}{2}\, k^2 v_T^2\,, \tag{8.2}$$

where $v_T^2 = 2k_{\mathrm{B}}T/m_{\mathrm{e}}$. The term on the right-hand side of Eqn (8.2) was dropped in constructing the dashed line in Fig. 9 in order to achieve a better agreement between the theoretical and experimental results [56, 57].

Looney and Brown noted the fundamental disagreement between the dispersion relations (8.1), (8.2), which lead to a continuous oscillation spectrum, and the experimental data displaying a discrete oscillation spectrum. Furthermore, as the electron density increases, one can see from Fig. 9 that the curve grows stepwise, with discontinuities and a slight increase in the slope of the steps within the confines of the plateau, i.e., the oscillation spectrum one observes is in fact of a band type. Band spectral structures were also seen in later experiments on the damping of electron waves in collisionless plasmas (see, for instance, Ref. [58]). Neither Eqns (8.1), (8.2) nor qualitative considerations based upon the theory of standing waves can explain these experiments. We proceed now to the interpretation of the experimental data based on the generalized dispersion relations (7.51), (7.52). We note from the start that a study on the level of dispersion relations is inadequate to give a complete picture of processes occurring in the system under study here. Therefore, the calculation of $\omega(\lambda)$ only reflects major qualitative and quantitative features of the system — provided we know the wavelengths of the observed waves and the hydrodynamic parameters, primarily the concentrations of the components and the ion and electron temperatures.

A. The dispersion relation (7.55) produces a discrete spectrum of solutions γ_n and, hence, of $\omega_n'(\lambda_n, T)$, $\omega_n''(\lambda_n, T)$. The discrete frequency spectrum is observed in experiment. To proceed to quantitative estimates, however, it is necessary first to estimate the beam temperature. From the requirement that the theoretical value of the square of the linear frequency

$$f_1'^2 = \hat{\omega}_1'^2\, \frac{2k_{\mathrm{B}}T}{\lambda_1^2 m_{\mathrm{e}}} \tag{8.3}$$

be equal to its experimental value ($f_1'^2 = 3 \times 10^{17}$ Hz^2), we have $T \approx 40$ eV. While the temperature was not measured directly in the experiment, there is indirect experimental evidence to support this value. Based on the parameters of the experiment, for the lower frequency level the wavelength $\lambda_1 \approx 1$ cm and $r_{\mathrm{D}}^2 k^2 \approx 0.2$. Consequently, this experiment fails to satisfy the conditions

$$r_{\mathrm{D}}^2 k^2 \ll 1\,, \qquad \lambda \gg r_{\mathrm{D}}\,,$$

which formally underlie the Landau-damping solution of the classical dispersion relation [59]. The solutions of equation

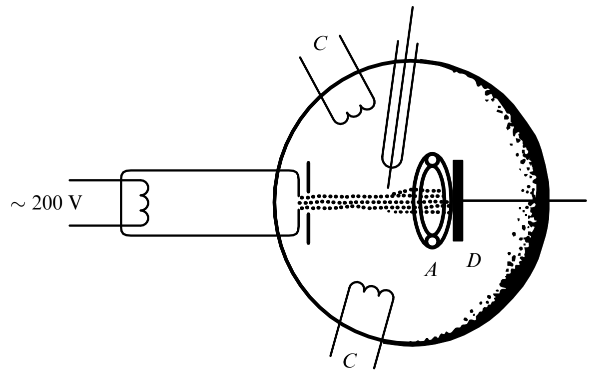

Figure 8. Schematic of Looney and Brown's experiment on the excitation of plasma oscillations.

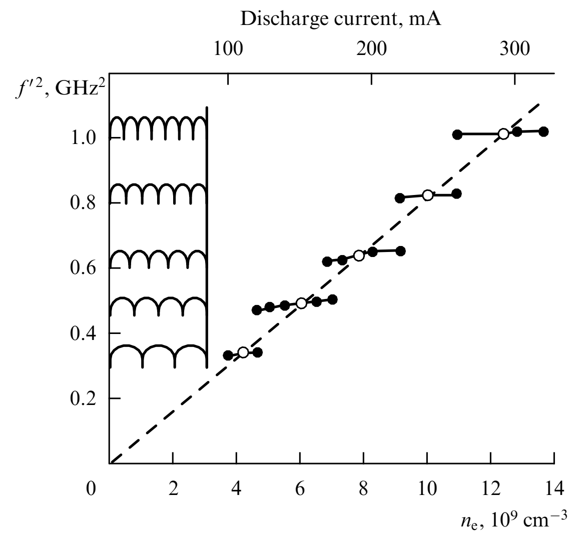

Figure 9. Square of the observed frequency versus plasma electron number density, as measured in the Looney – Brown experiment. The inset shows the oscillations observed in the anode gap AD.

(7.55) presented above has been found for the limiting case

$$\beta^2 = 1\,. \tag{8.4}$$

Note, however, that the region where the straight line $2\pi\gamma$ intersects the curve $\Phi(\gamma) = -\exp(\gamma\cot\gamma)\sin\gamma$ is that of the steep rise of the function $\Phi(\gamma)$ (see Fig. 7). Consequently, varying the slope of this line in the region of existence of solutions to Eqn (7.55) has a little effect on the solution γ_n. To estimate

$$f_n'^2 = \hat{\omega}_n'^2 \, \frac{2k_{\mathrm{B}}T}{\lambda_n^2 m_{\mathrm{e}}}\,, \tag{8.5}$$

one should take the calculated values of $\hat{\omega}_n'^2$ and the experimental values of λ_n and T. The electron temperatures in experiments yielding the frequencies f_n' can be estimated from the simplest form of the beam equilibrium condition:

$$p_{\mathrm{e}} = p_{\mathrm{pl}}\,, \tag{8.6}$$

where p_{e} is the pressure produced by the beam electrons, and p_{pl} is that of the mercury plasma. As a result, the quantitative agreement between the theory and experiment is quite reasonable (to within 20–50%) for the second through the fifth of the observed levels $f_n'^2$.

B. Under the condition (8.4), the straight line *1* in Fig. 7 has the maximum slope possible. The straight line *2* corresponds to a certain nonzero value of $r_{\mathrm{D}}^2 k^2$ and is drawn for illustrative purposes. Now suppose that the concentration n_{e} of the beam electrons starts to increase, whereas other plasma parameters remain, to a first approximation, unchanged. Increasing n_{e} reduces [see Eqn (A1.5)] the Debye–Hückel radius r_{D} and increases the slope of the straight line *2*, which now takes position *3*. The straight line approaches position *1*. Instead of a certain discrete set of γ_n's, we will have a set of possible intervals $\Delta\gamma_n$, and hence of intervals $\Delta\omega_n'$, $\Delta\omega_n''$ — giving rise to the plateau regions of the experimentally revealed values of $\omega_n'^2$.

C. It is easily verified by direct calculation that the function

$$F(\gamma_n) = \frac{\gamma_n}{4}\tan\left(-\frac{\gamma_n}{2}\right) \tag{8.7}$$

increases with decreasing γ_n. Hence, within the confines of a plateau the square of the frequency $\omega_n'^2$ will grow slightly with concentration of the beam electrons:

$$\omega_{n,\gamma-\Delta\gamma}'^2 = \omega_{n,\gamma}'^2 + o\big(|\Delta\gamma_n|\big)\,. \tag{8.8}$$

This effect is also observed in experiment.

D. The square of the oscillation frequency of plasma waves, $\omega_n'^2$, is proportional to the wave energy. Hence, the energy of plasma waves is quantized, and as n grows we have the asymptotic expression

$$\hat{\omega}_n'^2 = \frac{\pi}{4}\left(n + \frac{1}{2}\right), \tag{8.9}$$

and the squares of possible dimensionless frequencies become equally spaced:

$$\hat{\omega}_{n+1}'^2 - \hat{\omega}_n'^2 = \frac{\pi}{4}\,. \tag{8.10}$$

E. Let us see how the motion of ions affects the solution of the dispersion equation. The velocity distribution of electrons is taken to be Maxwellian; the thermal motion of ions is neglected. The ion distribution function is then written as

$$f_{\mathrm{i}} = n_{\mathrm{i}}\,\delta(\mathbf{v}_{\mathrm{i}})\,. \tag{8.11}$$

The generalized dispersion relation in the collisionless limit becomes

$$1 + \frac{1}{r_{\mathrm{D}}^2 k^2}\left[1 + 2\pi\mathrm{i}\hat{\omega}\exp\left(-\hat{\omega}'^2 + \hat{\omega}''^2 - 2\mathrm{i}\hat{\omega}'\hat{\omega}''\right)\right] = \frac{\omega_{\mathrm{pi}}^2}{\omega^2}\,, \tag{8.12}$$

where ω_{pi} is the Langmuir ion frequency.

Introducing the parameter

$$\varepsilon = \frac{m_{\mathrm{e}} n_{\mathrm{i}}}{2 m_{\mathrm{i}} n_{\mathrm{e}}}\,, \tag{8.13}$$

equation (8.12) is written in the following way

$$1 + r_{\mathrm{D}}^2 k^2 + 2\pi\mathrm{i}\hat{\omega}\exp\left(-\hat{\omega}'^2 + \hat{\omega}''^2 - 2\mathrm{i}\hat{\omega}'\hat{\omega}''\right) = \frac{\varepsilon}{\hat{\omega}^2}\,. \tag{8.14}$$

Under the conditions of the experiment being discussed, ε is a small quantity. In the general case, however — if $n_{\mathrm{i}} \gg n_{\mathrm{e}}$ and if the parameter ε is not too small — it may happen that the ion motion must be accounted for. Equation (8.14) can be solved perturbatively by expanding the frequencies in power series:

$$\hat{\omega}' = \sum_{k=0}^{\infty} \varepsilon^k \hat{\omega}'^{(k)}\,, \tag{8.15}$$

$$\hat{\omega}'' = \sum_{k=0}^{\infty} \varepsilon^k \hat{\omega}''^{(k)}\,. \tag{8.16}$$

Then for $k = 0$, in the first approximation, we arrive at a special case of Eqns (7.49), (7.50) as a result of separating the real and imaginary parts of Eqn (8.14).

In the second approximation ($k = 1$), we have

$$\hat{\omega}'^{(1)}\left(\cos\frac{\gamma_0}{2} - \frac{\gamma_0}{2}\cot\frac{\gamma_0}{2}\right) + \hat{\omega}''^{(1)}\left(\sin\frac{\gamma_0}{2} + \frac{\gamma_0}{2}\right) = \frac{\sin^2\gamma_0}{\gamma_0\sqrt{\pi}}\exp\left(\frac{\gamma_0}{2}\cot\gamma_0\right) - \hat{\omega}'^{(0)}\,, \tag{8.17}$$

$$\hat{\omega}'^{(1)}\left[\frac{\beta}{\sqrt{\pi}}\,\hat{\omega}'^{(0)}\exp\left(\frac{\gamma_0}{2}\cot\gamma_0\right) - \frac{\gamma_0}{2} - \sin\frac{\gamma_0}{2}\right] - \hat{\omega}''^{(1)}\left[\frac{\beta}{\sqrt{\pi}}\,\hat{\omega}''^{(0)}\exp\left(\frac{\gamma_0}{2}\cot\gamma_0\right) + \cos\frac{\gamma_0}{2} + \frac{\gamma_0}{2}\tan\frac{\gamma_0}{2}\right] = \frac{\sin 2\gamma_0}{2\gamma_0\sqrt{\pi}}\exp\left(\frac{\gamma_0}{2}\cot\gamma_0\right) + \hat{\omega}''^{(0)}\,, \tag{8.18}$$

where γ_0 is the first-order approximation to the solution of Eqn (8.14). We can estimate the magnitude of the effect by giving the second-order results for the coefficients in the expansions (8.15), (8.16):

$$\hat{\omega}'^{(1)} = 3.05\,, \qquad \hat{\omega}''^{(1)} = -0.87\,.$$

The complete solution of the boundary value problem concerning the spatial and temporal evolution of plasma in the Looney–Brown apparatus can only be obtained by solving the GBE. Still, the dispersion relation we have considered correctly reflects the essential features of the experimental results.

We now proceed to compare the theoretical results with those of the computer experiment. Extensive simulations of the attenuation of Langmuir waves in plasma have been performed at the SB RAS Institute of Nuclear Physics (Novosibirsk) in the 1970s and 1980s (see, for example, Refs [60 – 63]). Of interest to us here is the formulation of the problem close to the classical Landau formulation [64 – 66]. The problem involves a one-dimensional plasma system subject to periodic boundary conditions. The velocity distribution of plasma electrons is taken to be Maxwellian, and ions are assumed to be at rest ($m_i/m_e = 10^4$) and distributed uniformly over the length of the system. It is also assumed that at some initial point in time the system is subjected to small electron velocity and electron density perturbations of the form

$$\frac{\delta n}{n_0} = \frac{k_0 E_0}{4\pi e n_0} \sin(\omega_0 t - k_0 x)\,, \qquad \delta v = \frac{\omega_0 E_0}{4\pi e n_0} \sin(\omega_0 t - k_0 x)\,, \tag{8.19}$$

corresponding to the linear wave

$$E(x, t) = E_0 \sin(\omega_0 t - k_0 x)\,,$$

with $\omega_0^2 = \omega_{pe}^2 + (3/2)k_0^2 v_T^2$, and $k_0 = 2\pi/\lambda_0$. The quantities E_0, φ_0, λ_0, ω_0, and k_0 are the initial values of the field amplitude, potential, wavelength, frequency, and wave number, respectively. The numerical integration is performed using the 'particles-in-cells' method. The number of particles is not large (in Refs [60 – 63], the authors usually put $N = 10^4$, with about 10^2 particles per cell). To reduce the initial noise level, the 'easy start' method is used, in which neither the coordinate nor velocity distribution functions of the electrons change from one cell to another. In this case, it was noted in Ref. [61] that the noise level is determined by computation errors but increases for the computation scheme chosen; the noise level increases with increasing E_0 and with decreasing λ_0. The computation only makes sense until the noise level remains small compared to the harmonics of the effect under study that arise in the calculation.

The calculations in Refs [60 – 63] were performed over a wide range of initial wave parameters. The time dependence of the field strength is quite complicated, but the initial stage always corresponds to the wave damping regime in which an increase in the amplitude and the phase velocity v_ϕ in the range $e\varphi_0/(k_B T) > 1$ (and the corresponding decrease in the parameter $k_0 r_D$) dramatically increases the damping decrement compared to the Landau value [64]. Table 2 summarizes the results of the numerical simulation series I-1 to I-8 [62, 63].

Table 2. Comparison of analytical predictions with numerical simulation results.

N	I-1	I-2	I-3	I-4	I-5	I-6	I-7	I-8
$\frac{v_\phi}{\sqrt{k_B T/m}}$	2.46	2.95	4.2	6.9	9.4	11.2	16	22.4
$\frac{\lambda_0}{r_D}$	11	15	24	42	58	70	100	140
$k_0 r_D$	0.57	0.42	0.26	0.15	0.11	0.09	0.063	0.045
$(k_0 r_D)^2$	3.3×10^{-1}	1.7×10^{-1}	6.8×10^{-2}	2.2×10^{-2}	1.2×10^{-2}	8×10^{-3}	3.9×10^{-3}	2×10^{-3}
$\hat{E}_0$	1 – 60	11 – 60	26 – 70	70 – 170	119 – 250	170 – 250	240 – 450	333 – 591
$\frac{\sqrt{e\varphi_0/m}}{\sqrt{k_B T/m}}$	0.2 – 1.6	0.8 – 1.9	1.6 – 2.6	3.5 – 5.4	5.3 – 7.6	6.9 – 8.4	9.8 – 13.5	13.7 – 18.3
$\frac{\sqrt{e\varphi_0/m}}{v_\phi}$	8×10^{-2} – 0.8	0.28 – 0.65	0.38 – 0.62	0.5 – 0.78	0.56 – 0.81	0.61 – 0.75	0.61 – 0.84	0.61 – 0.82
$\frac{e\varphi_0}{k_B T}$	4×10^{-2} – 2.7	0.7 – 3.6	2.5 – 6.8	11.9 – 28.5	28 – 58.4	48 – 70.5	96.7 – 181	188 – 334
$\frac{\Delta N}{N}$, %	0 – 20	1 – 13	1 – 7	1 – 11	1 – 12	2 – 7.5	1 – 13	0.5 – 10
$\frac{\gamma_L}{\omega_{pe}}$	0.32	0.17	6×10^{-3}	10^{-8}	4×10^{-17}	2×10^{-25}	6×10^{-53}	2×10^{-105}
$\frac{\gamma}{\omega_{pe}}$	0.32 – 1	0.17 – 0.96	0.03 – 0.4	0.03 – 0.65	0.03 – 0.8	0.04 – 0.2	0.03 – 0.8	0.02 – 0.3
$\frac{\gamma_A}{\omega_{pe}}$	0.522	0.4	0.247	0.143	0.105	0.0857	0.06	0.0428
$\frac{\gamma}{\gamma_L}$	1.0 – 3.4	1.0 – 5.6	5 – 60	$\sim 10^6 – 10^8$	$\sim 10^{15} – 10^{16}$	$\sim 10^{23} – 10^{24}$	$\sim 10^{50} – 10^{52}$	$\sim 10^{102} – 10^{104}$

In the leftmost column of the table, the following definitions need some comment: $\hat{E}_0 = E_0 e \tau_{0p}^2 / m r_D$ is the dimensionless, normalized wave amplitude, τ_{0p} is the period of plasma oscillations, $\Delta N/N$ the fraction of trapped electrons (in %), γ is the damping decrement determined in the simulation experiment, $\gamma_A = -\omega_1''$ the damping decrement found as the asymptotics of the solution to the generalized Boltzmann equation and calculated from the first decrement involved in the discrete spectrum of solutions. Table 2 also presents the damping decrement $\gamma_L = -\omega_L''$ calculated from a modified Landau formula [67]. There is strong disagreement between the decrements γ, γ_A, and γ_L in the long wavelength limit $k_0 r_D \ll 1$. This disagreement is easy to explain from the computation point of view. Let us write down the Landau formula in its classic form

$$\frac{\gamma}{\omega_{pe}} = \sqrt{\frac{\pi}{8}} \frac{1}{k_0^3 r_D^3} \exp\left(-\frac{1}{2k_0^2 r_D^2}\right). \tag{8.20}$$

For $k_0 r_D \ll 1$, the damping decrement calculated by Eqn (8.20) becomes very small, whereas its simulation counterpart does not differ much from the plasma frequency. Applying the GBE asymptotics to the solution of classical problem of Landau damping makes it possible, even in Landau's linear formulation, to obtain a quite satisfactory agreement with both physical and mathematical experiments.

To conclude, the results presented here are only a small part of what the generalized kinetic theory has produced during the 15 years of its development — the years, one is safe to say, which have showed it to be a highly effective tool for solving many physical problems in areas where the classical theory runs into difficulties.

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9. Appendices

A.1 Characteristic scales in plasma physics

The fundamental feature of plasma physics is the existence of a multiparticle interaction in the system under study. Consequently, care must be exercised when choosing typical scales for describing the evolution of a plasma volume. The Landau length l over which the characteristic kinetic energy $k_B T$ of thermal motion equals the potential energy of interaction between charges e is determined by the relation

$$l = \frac{e^2}{4\pi\varepsilon_0 k_B T} = \frac{1.67 \times 10^{-5}}{T} \text{ m}, \tag{A1.1}$$

where ε_0 is the electric constant, and k_B is the Boltzmann constant.

Binary collisions for which impact parameters are less than or equal to the Landau length are said to be 'close'. It is useful to introduce the ratio of the Landau length to the average distance $n^{-1/3}$ between plasma particles:

$$\beta = l n^{1/3} = 1.67 \times 10^{-5} n^{1/3} T^{-1}, \tag{A1.2}$$

where n is the number density of particles, in m^{-3}.

While the interaction parameter β is usually small in laboratory plasma, the solar corona and the solar atmosphere, in the ionosphere and interstellar gas, but for free electrons in a metal it can reach a value of $\sim 10^2$. The cross section σ_b for close collisions is determined by the relation

$$\sigma_b = \pi l^2, \tag{A1.3}$$

and the mean free path of a probe particle between binary close collisions is

$$\lambda = \frac{1}{\pi n l^2} = \frac{1}{\pi n^{1/3} \beta^2} = 1.1 \times 10^9 \frac{T^2}{n} \text{ m}. \tag{A1.4}$$

The pair interaction between particles in a plasma effectively extends to the distance determined by the Debye – Hückel radius r_D defined as follows

$$r_D = \sqrt{\frac{\varepsilon_0 k T}{n e^2}} = \frac{1}{n^{1/3}\sqrt{4\pi\beta}} = 0.69 \times 10^2 \sqrt{\frac{T}{n}} \text{ m}. \tag{A1.5}$$

It can be argued that collective plasma properties disappear in systems less than r_D in size. The following relationship between the characteristic plasma lengths should be noted:

$$l : n^{-1/3} : r_D : \lambda = \beta : 1 : \frac{1}{2\sqrt{\pi\beta}} : \frac{1}{\pi\beta^2}. \tag{A1.6}$$

Equation (A1.6) should be complemented by the hydrodynamic scale L being the characteristic size of the system; L is usually much larger than λ.

The above list of characteristic plasma scales is not exhaustive, though. For processes in rapidly alternating fields — when the distance a particle travels over a period of oscillation of the field is less than the range of the forces involved — additional scales may appear in the problem.

A.2 Dispersion relations in the generalized Boltzmann kinetic theory neglecting the integral collision term

We are concerned with developing (within the GBE framework) the dispersion relation for plasma in the absence of a magnetic field. We make the same assumptions used in developing this relation within the BE model, namely: (a) the integral collision term is neglected; (b) the evolution of electrons and ions in a self-consistent electric field corresponds to a one-dimensional, unsteady model; (c) distribution functions for ions f_i and electrons f_e deviate only slightly from their respective equilibrium values f_{0i} and f_{0e}:

$$f_i = f_{0i}(u) + \delta f_i(x, u, t), \tag{A2.1}$$

$$f_e = f_{0e}(u) + \delta f_e(x, u, t); \tag{A2.2}$$

(d) we consider a wave mode corresponding to a certain wave number k and a complex frequency ω, so that the solution of the GBE can be written in the form

$$\delta f_i = \langle \delta f_i \rangle \exp\left[i(kx - \omega t)\right], \tag{A2.3}$$

$$\delta f_e = \langle \delta f_e \rangle \exp\left[i(kx - \omega t)\right]; \tag{A2.4}$$

(e) the quadratic terms in the GBE, determining the deviation from the equilibrium DFs, are neglected, and (f) the self-consistent forces F_i and F_e are small:

$$F_i = -\frac{e}{m_i}\frac{\partial\varphi}{\partial x}\,, \tag{A2.5}$$

$$F_e = -\frac{e}{m_e}\frac{\partial\varphi}{\partial x}\,, \tag{A2.6}$$

where e is the absolute electron charge, m_i are the ion masses, m_e the electron mass, and finally

$$\varphi = \langle\varphi\rangle \exp\left[\mathrm{i}(kx - \omega t)\right]. \tag{A2.7}$$

Under these assumptions, the GBE is written as follows (we seek the solution for the ion plasma component, to be specific):

$$\begin{aligned}&\frac{\partial f_i}{\partial t} + u\frac{\partial f_i}{\partial x} + F_i\frac{\partial f_i}{\partial u} - \tau_i\bigg(\frac{\partial^2 f_i}{\partial t^2} + 2u\frac{\partial^2 f_i}{\partial t\,\partial x} + u^2\frac{\partial^2 f_i}{\partial x^2}\\ &\quad + 2F_i\frac{\partial^2 f_i}{\partial t\,\partial u} + \frac{\partial F_i}{\partial t}\frac{\partial f_i}{\partial u} + F_i\frac{\partial f_i}{\partial x} + u\frac{\partial F_i}{\partial x}\frac{\partial f_i}{\partial u}\\ &\quad + F_i^2\frac{\partial^2 f_i}{\partial u^2} + 2uF_i\frac{\partial^2 f_i}{\partial u\,\partial x}\bigg) = 0\,.\end{aligned} \tag{A2.8}$$

Using the assumptions listed above, we find the relations

$$\begin{aligned}&\frac{\partial f_i}{\partial t} = -\mathrm{i}\omega\,\delta f_i\,, \qquad u\frac{\partial f_i}{\partial x} = \mathrm{i}ku\,\delta f_i\,,\\ &F_i\frac{\partial f_i}{\partial u} = -\frac{e}{m_i}\frac{\partial\varphi}{\partial x}\frac{\partial f_{0i}}{\partial u}\,, \qquad \frac{\partial^2 f_i}{\partial t^2} = -\omega^2\,\delta f_i\,,\\ &2u\frac{\partial^2 f_i}{\partial t\,\partial x} = 2\omega uk\,\delta f_i\,, \qquad u^2\frac{\partial^2 f_i}{\partial x^2} = -u^2k^2\,\delta f_i\,,\\ &2F_i\frac{\partial^2 f_i}{\partial u\,\partial t} = 0\,, \qquad \frac{\partial F_i}{\partial t}\frac{\partial f_i}{\partial u} = -\frac{e}{m_i}\,\omega k\varphi\,\frac{\partial f_{0i}}{\partial u}\,,\\ &F_i\frac{\partial f_i}{\partial x} = 0\,, \qquad u\frac{\partial f_i}{\partial u}\frac{\partial F_i}{\partial x} = \frac{e}{m_i}\,k^2u\varphi\,\frac{\partial f_{0i}}{\partial u}\,,\\ &F_i^2\frac{\partial^2 f_i}{\partial u^2} = 0\,, \qquad \frac{\partial^2 f_i}{\partial u\,\partial x}\,2uF_i = 0\,,\end{aligned} \tag{A2.9}$$

which when substituted into Eqn (A2.8) yield

$$\begin{aligned}&\mathrm{i}(ku-\omega)\langle\delta f_i\rangle - \mathrm{i}\,\frac{e}{m}\,k\langle\varphi\rangle\frac{\partial f_{0i}}{\partial u}\\ &\quad - (ku-\omega)\,\tau_i\left[-(ku-\omega)\langle\delta f_i\rangle + \langle\varphi\rangle\frac{ek}{m_i}\frac{\partial f_{0i}}{\partial u}\right] = 0\,,\end{aligned} \tag{A2.10}$$

giving the ion density fluctuation

$$\langle\delta n_i\rangle = -\frac{e}{m_i}\langle\varphi\rangle\,k\int\frac{\partial f_{0i}}{\partial u}\,\frac{1}{\omega - ku}\,\mathrm{d}u\,, \tag{A2.11}$$

and the electron density fluctuation

$$\langle\delta n_e\rangle = \frac{e}{m_e}\langle\varphi\rangle\,k\int\frac{\partial f_{0e}}{\partial u}\,\frac{1}{\omega - ku}\,\mathrm{d}u\,. \tag{A2.12}$$

Equations (A2.11) and (A2.12) are identical to their BE analogues. Substituting Eqns (A2.11) and (A2.12) into the Poisson equation

$$\varepsilon_0 k^2\varphi = e(\delta n_i - \delta n_e)\,, \tag{A2.13}$$

we arrive at the classical dispersion relation (see, for instance, Ref. [53])

$$\begin{aligned}1 = &-\frac{e^2}{\varepsilon_0 k}\left(\frac{1}{m_e}\int_{-\infty}^{+\infty}\frac{\partial f_{0e}}{\partial u}\,\frac{1}{\omega - ku}\,\mathrm{d}u\right.\\ &\left. + \frac{1}{m_i}\int_{-\infty}^{+\infty}\frac{\partial f_{0i}}{\partial u}\,\frac{1}{\omega - ku}\,\mathrm{d}u\right).\end{aligned} \tag{A.14}$$

Although Eqns (A2.11) and (A2.12) are a consequence of the general statement that in the absence of the integral collision term the relation

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} = 0 \tag{A2.15}$$

(the Vlasov equation) is the solution of the equation

$$\frac{\mathrm{D}f_\alpha}{\mathrm{D}t} - \frac{\mathrm{D}}{\mathrm{D}t}\left(\tau_\alpha\,\frac{\mathrm{D}f_\alpha}{\mathrm{D}t}\right) = 0\,, \tag{A2.16}$$

the above argument shows that the GBE can produce correct and expected results, when treated perturbatively.

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