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A joint scientific session of the Physical Sciences Division of the Russian Academy of Sciences (RAS) and the Joint Physical Society of the Russian Federation was held on 23 April 2003 at the P N Lebedev Physics Institute, RAS. The following reports were presented at the session:

(1) **Zabrodskiĭ A G** (A F Ioffe Physico-Technical Institute, RAS, St.-Petersburg) "Metal – insulator phase transition in disordered semiconductors";

(2) **Ipatov I P** (A F Ioffe Physico-Technical Institute, RAS, St.-Petersburg) "Instability of solid semiconductor solutions";

(3) **Yakovlev D G** (A F Ioffe Physico-Technical Institute, RAS, St.-Petersburg) "Pycnonuclear reactions and the structure of neutron stars";

(4) **Churazov E M** (Institute of Cosmic Research, RAS, Moscow) "High-temperature plasma spectroscopy and relativistic gas in galactic clusters";

(5) **Beskin V S** (P N Lebedev Physics Institute, RAS, Moscow) "Axisymmetric steady flows in astrophysics";

(6) **Ochkin V N, Tskhaĭ S N** (P N Lebedev Physics Institute, RAS, Moscow) "Coherent light scattering stimulated by a quasi-static electric field";

(7) **Belushkin A V** (Joint Institute for Nuclear Research, Dubna, Moscow region) "Novel approaches to the analysis of crystal structure. A nonstandard method for the study of atomic and molecular diffusion mechanisms";

(8) **Znamenskiĭ N V, Manykin E A, Samartsev V V** (Russian Research Centre 'Kurchatov Institute', Moscow) "Photon echo in impurity crystals".

An abridge version of reports 5, 6, and 7 is given below.

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Axisymmetric steady flows in astrophysics

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1. Introduction

Observations show that many astrophysical sources can be treated with good accuracy as axisymmetric and stationary. These include both accreting neutron stars and black holes, axisymmetric stellar (solar) wind, jet outflows from young stellar objects, and ejection of particles from magnetospheres of rotating neutron stars. Such flows also apparently form in magnetospheres of supermassive black holes which are thought to be the 'central engine' in active galactic nuclei and quasars [1, 2]. So, it is not surprising that ideal magnetohydrodynamics, which allows a sufficiently simple formalization of the problem, is actively applied when describing these flows.

The point is that due to axial symmetry and stationarity (as well as magnetic field line freezing-in), in the general case five 'integrals of motion' show themselves worth, being conserved at axisymmetric magnetic surfaces. This remarkable fact allows us to separate the problem of finding the poloidal field structure (the poloidal flow structure in hydrodynamics) from the problem of calculating the particle acceleration and structure of electric currents. The solution to the latter task in a given poloidal field can be obtained in terms of quite simple algebraic relations. It is important that such an approach can be straightforwardly generalized to flows in the vicinity of rotating black holes, as the Kerr metric is also axially symmetric and stationary.

On the other hand, it is much more difficult to find a two-dimensional poloidal magnetic field structure (the hydrodynamic flow structure). First of all, this is due to the complex structure of the equation describing axisymmetric steady flows. In the general case, it is a nonlinear equation of the mixed type, which changes from elliptical to hyperbolical at singular surfaces and in addition contains integrals of motion in the form of free functions. Generally speaking, similar equations, which stem from the classical Tricomi equation, have been discussed starting from the beginning of the last century in connection with transonic hydrodynamic flows [3]. Later on, axially symmetric stationary equilibrium equations were called Grad – Shafranov equations after the authors who formulated in the late 1950s an equation of such a type in connection with controlled thermonuclear fusion [4]. This equation, however, related to equilibrium static configurations only and required strong revision when generalizing to transonic flows. The full version of such an equation including all five integrals of motion was formulated by L S Solov'ev in 1963 in the third volume of *Problems of Plasma Theory* [5] and was well-known to physicists. However, as often occurs, the full version of the Grad – Shafranov equation was little known in the astrophysical literature, so it was 'rediscovered' several times [6].

As it turned out the difficulty lay in the fact that the very setting of the direct problem in the framework of the Grad – Shafranov equation method proved to be nontrivial. For example, in the hydrodynamic limit, when there are only three integrals of motion, the problem requires four boundary conditions for the transonic flow regime. This implies that,

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for instance, two thermodynamic functions and two velocity components either should be specified at some surface. However, to determine the Bernoulli integral, which naturally should be known in order to solve the equilibrium equation, all three components of the velocity must be specified, which is impossible since the third velocity component itself is to be obtained from the solution. Such inconsistency, as a matter of fact, is one of the main difficulties of the approach under consideration.

Nevertheless, there exist approaches allowing analytical solution of direct problems within the framework of the Grad–Shafranov equation method. For example, there is such a possibility if the exact solution of this equation is known and we explore flows weakly diverging from the known one. Spherically symmetric accretion (ejection) of matter could be such an exact solution. As a result, the known structure of the flow in the zeroth approximation enables us to determine with the required accuracy both the location of singular surfaces and all integrals of motion directly from boundary conditions, thus making it possible to solve the equilibrium equation within the direct formulation of the problem.

2. Grad – Shafranov equation

Let us consider axisymmetric steady plasma flow in the vicinity of a rotating black hole, i.e., in the most general axially symmetric stationary metric [2]:

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt)\,, \tag{1}$$

where

$$\alpha = \frac{\rho}{\Sigma}\sqrt{\Delta}\,, \quad \beta^r = \beta^\theta = 0\,, \quad \beta^\varphi = -\omega = -\frac{2a\mathcal{M}r}{\Sigma^2}\,,$$

$$g_{rr} = \frac{\rho^2}{\Delta}\,, \quad g_{\theta\theta} = \rho^2\,, \quad g_{\varphi\varphi} = \varpi^2\,. \tag{2}$$

Here, α is the gravitational redshift vanishing at the horizon $r_g = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}$, ω is the angular velocity of local nonrotating observers (the so-called Lense–Thirring angular velocity), and

$$\Delta = r^2 + a^2 - 2\mathcal{M}r\,, \quad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta\,, \quad \varpi = \frac{\Sigma}{\rho}\sin\theta\,. \tag{3}$$

As usual, $\mathcal{M}$ and a are the black hole mass and angular momentum per unit mass ($a = J/\mathcal{M}$), respectively. Here indices without hats denote components of vectors with respect to the coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\varphi$, and indices with hats stand for physical components of the vectors. Finally, we shall use below the system of units with $c = G = 1$, except as noted.

In what follows we shall utilize the $3+1$ split technique [2]. In this approach, the physical quantities are expressed through three-dimensional vectors which would be measured by local observers moving around the rotating black hole with angular velocity ω. The convenience of the $3+1$ split lies in the fact that it allows the representation of many expressions in the same form as in flat space. Here, all thermodynamic quantities are determined in the co-moving frame of reference.

Now, we shall demonstrate how the five 'integrals of motion', which are constant at the magnetic surfaces, arise in the general case of axisymmetric steady flows. It is convenient to introduce the scalar function $\Psi(r,\theta)$, which has a magnetic flux meaning. As a consequence, the magnetic field is defined in the following way:

$$\mathbf{B} = \frac{\nabla\Psi \times \mathbf{e}_{\hat\varphi}}{2\pi\varpi} - \frac{2I}{\alpha\varpi}\,\mathbf{e}_{\hat\varphi}\,, \tag{4}$$

where $I(r,\theta)$ is the total electric current inside the region $\Psi < \Psi(r,\theta)$.

As usual, we assume that the magnetosphere contains sufficient amount of plasma to provide the condition of magnetic field line freezing-in, which using the $3+1$ split is written, as in flat space, in the form of $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. On the other hand, the stationarity (as well as the condition for zero longitudinal electric field) implies that the field $\mathbf{E}$ can be written as

$$\mathbf{E} = -\frac{\Omega_F - \omega}{2\pi\alpha}\,\nabla\Psi\,. \tag{5}$$

Substituting relation (5) into the Maxwell equations it is easy to verify that the condition $\mathbf{B}\cdot\nabla\Omega_F = 0$ is satisfied, i.e., Ω_F must be constant at the magnetic surfaces (Ferraro's isorotation law): $\Omega_F = \Omega_F(\Psi)$.

Next, the Maxwell equation $\nabla\cdot\mathbf{B} = 0$, the continuity equation, and the magnetic field freezing-in condition allow us to write down the 4-velocity of matter in the form

$$\mathbf{u} = \frac{\eta}{\alpha n}\,\mathbf{B} + \gamma(\Omega_F - \omega)\,\frac{\varpi}{\alpha}\,\mathbf{e}_{\hat\varphi}\,, \tag{6}$$

where $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor of matter, and the quantity η has the meaning of the ratio between the particle flux and the magnetic field flux. Due to the relationship $\nabla\cdot(\eta\mathbf{B}_p) = 0$, it must also be constant at the magnetic surfaces $\Psi(r,\theta) = \text{const}$, i.e., $\eta = \eta(\Psi)$.

The next two integrals of motion are tied up with the axial symmetry and stationarity of the considered flows, with a consequent conservation of the flux of energy E and the z-component L_z of angular momentum:

$$E = E(\Psi) = \frac{\Omega_F I}{2\pi} + \mu\eta(\alpha\gamma + \omega u_\varphi)\,, \tag{7}$$

$$L = L(\Psi) = \frac{I}{2\pi} + \mu\eta\varpi u_{\hat\varphi}\,, \tag{8}$$

where $\mu = (\rho_m + P)/n$ is the relativistic enthalpy (ρ_m is the internal energy density, P is the pressure). Finally, in the axially symmetric case the isentropy condition yields $s = s(\Psi)$, so that the entropy per particle, $s(\Psi)$, is in fact the fifth integral of motion.

The five integrals of motion $\Omega_F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$, and $L(\Psi)$, as well as the poloidal magnetic field $\mathbf{B}_p$, allow us to recover the toroidal magnetic field $B_{\hat\varphi}$ and all other plasma parameters:

$$\frac{I}{2\pi} = \frac{\alpha^2 L - (\Omega_F - \omega)\varpi^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{9}$$

$$\gamma = \frac{1}{\alpha\mu\eta}\,\frac{\alpha^2(E - \Omega_F L) - M^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{10}$$

$$u_{\hat\varphi} = \frac{1}{\varpi\mu\eta}\,\frac{(E - \Omega_F L)(\Omega_F - \omega)\varpi^2 - LM^2}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{11}$$

where

$$M^2 = \frac{4\pi\eta^2\mu}{n} \,. \tag{12}$$

It is easy to see that the quantity M^2 represents to within the factor α^2 the Mach number squared of the poloidal velocity $u_{\rm p}$ with respect to the Alfvén velocity $u_{\rm A} = B_{\rm p}/\sqrt{4\pi n\mu}$, i.e., $M^2 = \alpha^2 u_{\rm p}^2/u_{\rm A}^2$.

Since $\mu = \mu(n, s)$, definition (12) allows us to express the concentration n (and hence the specific enthalpy μ) as a function of η, s, and M^2. This means that along with the five integrals of motion, the expressions for I, γ, and $u_{\hat\varphi}$ only depend on one additional quantity, namely the Mach number M. To determine the Mach number, we should use the obvious relation $\gamma^2 - \mathbf{u}^2 = 1$, which, owing to equations (10) and (11), can be rewritten in the form

$$\frac{K}{\varpi^2 A^2} = \frac{1}{64\pi^4}\,\frac{M^4(\nabla\Psi)^2}{\varpi^2} + \alpha^2\eta^2\mu^2 \,, \tag{13}$$

where

$$A = \alpha^2 - (\Omega_{\rm F} - \omega)^2\varpi^2 - M^2 \tag{14}$$

and

$$K = \alpha^2\varpi^2(E - \Omega_{\rm F}L)^2\left[\alpha^2 - (\Omega_{\rm F} - \omega)^2\varpi^2 - 2M^2\right] + M^4\left[\varpi^2(E - \omega L)^2 - \alpha^2L^2\right] . \tag{15}$$

As for the Grad–Shafranov equation itself, viz. the equilibrium equation for magnetic field lines, it can be written in the form [7]

$$\frac{1}{\alpha}\nabla_k\left\{\frac{1}{\alpha\varpi^2}\left[\alpha^2 - (\Omega_{\rm F} - \omega)^2\varpi^2 - M^2\right]\nabla^k\Psi\right\} + \frac{\Omega_{\rm F} - \omega}{\alpha^2}\,(\nabla\Psi)^2\frac{{\rm d}\Omega_{\rm F}}{{\rm d}\Psi} + \frac{64\pi^4}{\alpha^2\varpi^2}\,\frac{1}{2M^2}\,\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right) - 16\pi^3\mu n\,\frac{1}{\eta}\,\frac{{\rm d}\eta}{{\rm d}\Psi} - 16\pi^3 nT\,\frac{{\rm d}s}{{\rm d}\Psi} = 0 \,, \tag{16}$$

where

$$G = \alpha^2\varpi^2(E - \Omega_{\rm F}L)^2 + \alpha^2M^2L^2 - M^2\varpi^2(E - \omega L)^2 \,, \tag{17}$$

and the derivative $\partial/\partial\Psi$ only acts on the integrals of motion. We emphasize that, together with relation (13), equation (16) contains only the flux function Ψ and the five integrals of motion.

Equilibrium equation (16) is a second-order equation linear with respect to the highest derivatives. It changes from elliptical to hyperbolical at singular surfaces where the poloidal velocity of matter is compared with either fast or slow magnetosonic velocity (when $D = 0$), or with the cusp velocity (when $D = -1$). At the Alfvén surface $A = 0$, the type of equation does not change. Nonetheless, the Alfvén surface does represent a singular surface of the equilibrium equation, as the regularity condition must be satisfied there.

3. Examples

Bondi–Hoyle accretion. As the first example, we consider hydrodynamic accretion onto a moving black hole (the Bondi–Hoyle accretion), which is one of the classical problems of modern astrophysics [1]. First of all, let us formulate the hydrodynamic limit of the Grad–Shafranov equation, where we can neglect the electromagnetic field contribution. In this case, it is convenient to introduce a new potential $\Phi(\Psi)$ satisfying the condition $\eta(\Psi) = {\rm d}\Phi/{\rm d}\Psi$. Using definition (6) we obtain

$$\alpha n\mathbf{u}_{\rm p} = \frac{1}{2\pi\varpi}(\nabla\Phi \times \mathbf{e}_{\hat\varphi}) \,. \tag{18}$$

Surfaces $\Phi(r, \theta) = {\rm const}$ define the streamlines of matter.

In the hydrodynamic limit, there are only three integrals of motion. These are the energy flux and the z-component of the angular momentum:

$$E(\Phi) = \mu(\alpha\gamma + \varpi\omega u_{\hat\varphi}) \,, \tag{19}$$

$$L(\Phi) = \mu\varpi u_{\hat\varphi} \,, \tag{20}$$

as well as the entropy $s = s(\Phi)$. Now the algebraic Bernoulli equation (13) takes the form

$$(E - \omega L)^2 = \alpha^2\mu^2 + \frac{\alpha^2}{\varpi^2}\,L^2 + \frac{\hat M^4}{64\pi^4\varpi^2}\,(\nabla\Phi)^2 \,, \tag{21}$$

where the 'Mach number' squared $\hat M^2$ is defined as $\hat M^2 = 4\pi\mu/n$. Then Grad–Shafranov equation (16) is rewritten in the form [7]

$$-\frac{1}{\alpha}\nabla_k\left(\frac{\hat M^2}{\alpha\varpi^2}\,\nabla^k\Phi\right) + \frac{64\pi^4}{\alpha^2\varpi^2\hat M^2}\left[\varpi^2(E - \omega L) \times \left(\frac{{\rm d}E}{{\rm d}\Phi} - \omega\frac{{\rm d}L}{{\rm d}\Phi}\right) - \alpha^2L\frac{{\rm d}L}{{\rm d}\Phi}\right] - 16\pi^3 nT\frac{{\rm d}s}{{\rm d}\Phi} = 0 \,, \tag{22}$$

where now

$$D = -1 + \frac{1}{u_{\rm p}^2}\,\frac{c_{\rm s}^2}{1 - c_{\rm s}^2} \,. \tag{23}$$

As we see, equation (22) contains only one singular surface, viz. the sonic surface determined from the condition $D = 0$.

To construct the solution corresponding to the Bondi–Hoyle accretion, it turned out possible to search for the solution of the equation for the flux function $\Phi(r, \theta)$ in the form of a small correction to the spherically symmetric solution. As a result, if there is a small parameter $\varepsilon_1 = v_\infty/c_\infty$, which defines the ratio of the black hole velocity to the velocity of sound at infinity, then the variables are separated, so that the total solution can be represented in the form

$$\Phi(r, \theta) = \Phi_0\left[1 - \cos\theta + \varepsilon_1 g_1(r)\sin^2\theta\right] . \tag{24}$$

Here, the radial function $g_1(r)$ is the solution of an ordinary differential equation (see Refs [8, 9] for more details).

At the present level of PC development, this means that we managed to construct the analytical solution to the problem in hand allowing the full description of the flow structure. For example, the sonic surface now has the nonspherical form

$$r_*(\theta) = r_*\left[1 + \varepsilon_1\left(\frac{\Gamma + 1}{5 - 3\Gamma}\right)k_2\cos\theta\right] , \tag{25}$$

where the numerical coefficient $k_2 = r_* g_1'(r_*)$ is expressed through the derivative of the radial function $g_1(r)$ at the sonic point. As shown in Fig. 1, the analytical solution fully agrees with numerical calculations [10] in spite of the parameter $\varepsilon_1 = 0.6$ here being quite large.

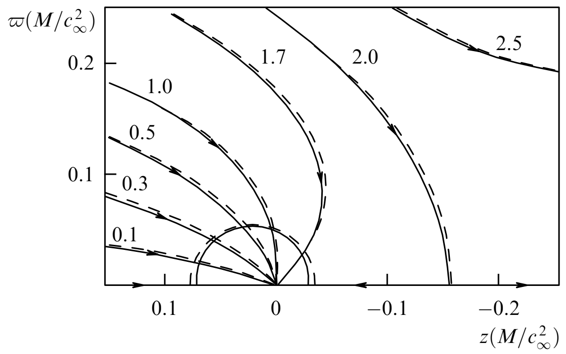

Figure 1. The flow structure and the sonic surface form for $\Gamma = 4/3$, $\varepsilon_1 = 0.6$ [8]. Numbers alongside the curves denote values of Φ/Φ_0, and the dashed lines show the streamlines and the sonic surface form obtained numerically in Ref. [10].

In connection with the above-obtained solution, we should make one note. As can be easily seen, outside the capture radius our main assumption, i.e., the smallness of the deviation from the spherically symmetric flow, is invalid. Nevertheless, the solution found remains valid. This remarkable property is due to the Grad – Shafranov equation becoming linear for constant concentration n. But as we learn from the spherically symmetric Bondi accretion, at large distances $r \gg r_*$ from the sonic surface the density of accreting matter is virtually constant. Accordingly, the density is constant for a homogeneous flow as well. As a result, under the condition that the capture radius $R_c \gg r_*$, which holds true for $\varepsilon_1 \ll 1$, near and beyond the capture radius (where the perturbation $\sim \varepsilon_1 g_1(r)$ becomes comparable to the value ~ 1 in the zeroth approximation) Eqn (22) becomes linear. So that the sum of two solutions, homogeneous and spherically symmetric, is also a solution.

Thin transonic disk. As the next example, we consider the internal two-dimensional structure of a thin accretion disk. Here, we consider for simplicity the case of a nonrotating (Schwarzschild) black hole [11]. We recall that according to the standard model [12] the accreting matter forms an equilibrium disk rotating around the gravitating center with the Keplerian velocity $v_K(r) = (G\mathcal{M}/r)^{1/2}$. The disk will be thin provided that its temperature is sufficiently small ($c_s \ll v_K$), since the vertical balance of the gravity force and the pressure gradient implies that $H \approx r c_s/v_K$. The general relativity effects lead to two important properties: the absence of stable circular orbits at $r < r_0 = 3r_g$, and the transonic regime of accretion. Here, it is important that the rapid gas fall inside the last stable orbit occurs with no viscosity, as well. So, we can assume that the ideal hydrodynamics approximation is relevant to describe flows in the innermost parts of the accretion disk.

So far, the vertical averaging procedure has been applied in most papers on thin accretion disks, with the vertical component $u_{\hat\theta}$ of the velocity being assumed zero [13]. Because of this, the vertical component $nu^b \nabla_b(\mu u_\theta)$ of the dynamic force in the Euler equation has also been assumed to be small down to the black hole horizon. For this reason, the disk thickness has been concluded to be determined by the pressure gradient in the supersonic area, too. However, as was shown in Ref. [11], the assumption $u_{\hat\theta} = 0$ near the sonic surface is irrelevant. As in the case of the Bondi accretion, the dynamic force becomes significant near the sonic surface.

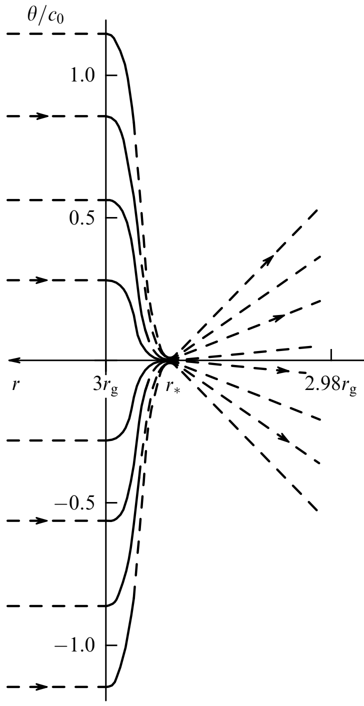

Figure 2. The structure of a thin accretion disk (at the real scale) following the passage of the last stable orbit $r = 3r_g$, numerically obtained by solving Eqn (22) for $c_0 = 10^{-2}$, $u_0 = 10^{-5}$. The solid lines correspond to the range of parameter values $u_p^2/c_0^2 < 0.2$. The dashed lines show the extrapolation of the solution towards the sonic surface. At the sonic surface, the flow takes the form of an ordinary nozzle.

Figure 2 shows the accretion disk structure near the sonic surface in the presence of a small parameter $\varepsilon_2 = u_0/c_0$, as inferred from the solution of Eqn (22). We emphasize that the existence of the small parameter $\varepsilon_2 \ll 1$, which is the ratio of the gas radial velocity to the velocity of sound in the last stable orbit, comes from the relation $v_r/v_K \approx \alpha_{SS} c_s^2/v_K^2$ for the radial velocity of gas flow in the accretion disk. In the vicinity of the last stable orbit this estimation is apparently inapplicable. Nevertheless, below we shall consider the parameter ε_2 to be small, as the presence of a small parameter allows us to investigate analytically many features of the flow. In addition, the small parameter makes the effect under discussion more pronounced.

As we see, the flow structure near the sonic surface is far from being radial. The appearance of a narrow waist has a simple physical meaning. Indeed, if there is a small parameter $\varepsilon_2 = u_0/c_0 \ll 1$, the density is nearly constant in the subsonic region, while the radial velocity changes from u_0 to $c_* \sim c_0$, i.e., by several orders of magnitude. And so, in consequence of the continuity equation the disk thickness H must change in the same proportion (see Fig. 2):

$$H(r_*) \approx \frac{u_0}{c_0} H(3r_g) . \tag{26}$$

Here, it is extremely important that both components of the dynamic force become comparable with the pressure gradient near the sonic surface:

$$\frac{u_{\hat\theta}}{r} \frac{\partial u_{\hat\theta}}{\partial \theta} \approx u_{\hat r} \frac{\partial u_{\hat\theta}}{\partial r} \approx \frac{\nabla_{\hat\theta} P}{\mu} \approx \frac{c_0^2}{u_0^2} \frac{\theta}{r} . \tag{27}$$

Here, the angle θ is reckoned from the equatorial plane.

In other words, if there appears a nonzero vertical velocity component, the dynamical term $(\mathbf{v}\nabla)\mathbf{v}$ cannot be neglected in striking the vertical force balance near the sonic surface [11]. It is clear that this property remains valid for arbitrary radial velocity of the flow, i.e., even when the transverse contraction of the disk is not so pronounced. Taking into account dynamic forces causes two additional degrees of freedom to appear, which relate to the higher derivatives in the Grad–Shafranov equation. This also leads to one more general conclusion independent of the value of the parameter ε_2. In a thin accretion disk, the critical condition at the sonic surface does not fix the accretion rate any more, but determines bending of the streamlines near the sonic surface. Finally, the inclusion of the vertical velocity inevitably leads to the appearance of a small longitudinal scale $\delta r_\parallel \approx H_*$ in the vicinity of a sonic surface, which for a thin disk proves to be much smaller than the distance to the black hole at any value of the parameter u_0/c_0. In the standard one-dimensional approach, this small scale does not emerge. As for the supersonic region (and, in particular, the region near the black hole horizon), the disk thickness here will be determined not by the pressure gradient, but by the form of ballistic trajectories, as is the case of supersonic flows.

The Blandford–Znajek process. In conclusion, we discuss the energy losses of a black hole embedded in the external magnetic field — the so-called Blandford–Znajek process [14], which is considered to be the most preferential mechanism of energy release in active galactic nuclei. Its main idea is based on the analogy with energy transfer in the internal regions of magnetospheres of radio pulsars. Indeed, let us suppose that there is a regular external magnetic field in the vicinity of a rotating black hole. The electric current I flows along this field. Then, the electric field $\mathbf{E}$ induced by plasma rotating with an angular velocity Ω_{F} and the toroidal magnetic field $\mathbf{B}_\varphi$ due to the longitudinal current I generate the electromagnetic energy flux (the Poynting vector flux) carrying away energy along the magnetic force lines.

Of course, by definition, general relativity effects are important near the black hole. Consequently, it is not obvious that the pulsar analogy can be useful in all cases. Indeed, in pulsar magnetospheres, the ponderomotive action of surface currents shorting electric currents in the magnetosphere results in the neutron star deceleration [15]. In the case of black holes such currents cannot lead to deceleration, though surface currents themselves can be formally introduced in the framework of the so-called membrane approach [2]. The point is that the horizon is not a physically preferred surface, not to mention the fact that by definition the horizon is not casually connected with the outer space (see, for example, paper [16] concluding that there is no energy flux along magnetic force lines passing through the black hole horizon).

However, a recent more accurate analysis [17] (in which, in fact, the first solution of the Grad–Shafranov equation for nonzero mass particles in the Kerr metric was obtained) indicated that in fact the retarding torque operates in the plasma generation region above the black hole horizon. Such a retarding torque appears due to the action of long-range gravitomagnetic forces which penetrate into regions causally connected with the outer magnetosphere. The horizon is indeed located inside the hyperbolic region of the total Grad–Shafranov equation and naturally cannot influence the flow far away from the black hole (we recall that the analysis in Ref. [14] was made in the force-free approximation when the Grad–Shafranov equation remains elliptical down to the black hole horizon). And so a rotating black hole, as well as a rotating neutron star, can work as a unipolar inductor and effectively transfer energy to long distances. As a result, the power loss $W_{\mathrm{tot}} \approx W_{\mathrm{BZ}}$, where

$$W_{\mathrm{BZ}} = \frac{\Omega_{\mathrm{F}}(\Omega_{\mathrm{H}} - \Omega_{\mathrm{F}})}{\Omega_{\mathrm{H}}^2} \left(\frac{a}{\mathcal{M}}\right)^2 B_0^2 r_{\mathrm{g}}^2 c \approx 10^{45} \left(\frac{a}{\mathcal{M}}\right)^2 \left(\frac{B_0}{10^4\,\mathrm{G}}\right)^2 \left(\frac{\mathcal{M}}{10^9 \mathcal{M}_\odot}\right)^2 \mathrm{erg\,s^{-1}}. \quad (28)$$

It is easy to check that for the ultimately fast rotating black hole and $B = B_{\mathrm{Edd}} \approx 10^4 (\mathcal{M}/10^9\mathcal{M}_\odot)^{-1/2}$ G, power loss W_{BZ} (28) coincides with the Eddington luminosity.

It should be emphasized, however, that as follows from equation (28), the rate of energy release needed to explain the characteristic luminosity of active galactic nuclei can be achieved only for limiting black hole masses $\sim 10^9 \mathcal{M}_\odot$, limiting magnetic fields $B \sim B_{\mathrm{Edd}}$ near the black hole, and limiting black hole rotation velocities $a \sim \mathcal{M}$. Therefore, papers have recently appeared in which the efficiency of the Blandford–Znajek process in real astrophysical conditions is in doubt [18]. In particular, it was pointed out that for rapid rotation the Wald solution for a vacuum magnetosphere leads to expulsion of the magnetic field into the ergosphere [2], which could cause the appearance of an additional factor $1 - a^2/\mathcal{M}^2$ in expression (28).

In fact, as shown in Fig. 3a, in the black hole magnetosphere filled with plasma, all magnetic force lines crossing the internal light surface $\alpha^2 = (\Omega_{\mathrm{F}} - \omega)^2 \varpi^2 + M^2$ ultimately cross the black hole horizon, that is why to an order of magnitude the energy release for the ultimately fast rotating black hole coincides with Eqn (28). Here, the situation is fully analogous to pulsar magnetospheres where force lines issuing out of the light cylinder do not intersect the equatorial plane (Fig. 3b).

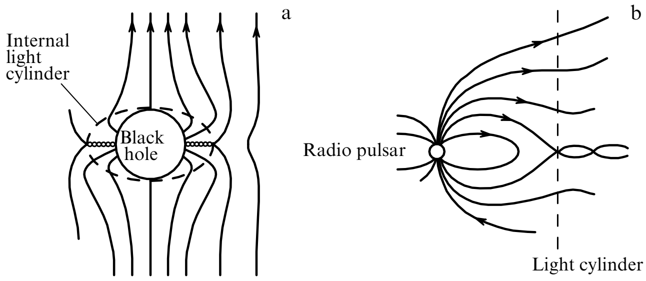

Figure 3. (a) The structure of the black hole magnetosphere completely filled with plasma. The longitudinal currents that flow along the magnetic field lines passing through the internal 'light cylinder' direct the field lines towards the black hole. (b) The structure of the radio pulsar magnetosphere. The force lines issuing out of the 'light cylinder' do not intersect the equatorial plane and go to infinity.

4. Conclusion

Thus, in some simple cases the Grad–Shafranov equation allows us to construct an exact analytical solution to the problem. In particular, this approach is very useful in studying analytical properties of transonic flows and in determining the required number of boundary conditions. On the other hand, in the general case no consistent procedure exists regarding the solution construction within the Grad – Shafranov equation method. The point is that the location of singular surfaces, at which critical conditions should be formulated, is not known beforehand and itself must be found from the solution to the problem. Moreover, it is impossible to generalize this approach to the case of nonideal, nonaxially symmetric and nonsteady flows. So it is not surprising that most investigators, who are primarily interested in astrophysical applications, have recently focused on a totally different class of equations, namely, on time relaxation problems, which can only be solved numerically [19]. Here, we would only like to hope that the key physical results obtained using the Grad – Shafranov equation, which, naturally, are independent of the computing method, are not forgotten.

References

1. Shapiro S, Teukolsky S A *Black Holes, White Dwarfs, and Neutron Stars* (New York: Wiley, 1983) [Translated into Russian (Moscow: Mir, 1985)]
2. Thorne K S, Price R H, MacDonald D A (Eds) *Black Holes: The Membrane Paradigm* (New Haven: Yale Univ. Press, 1986) [Translated into Russian (Moscow: Mir, 1988)]
3. Guderley K G *Theorie Schallnaher Stömungen* (Berlin: Springer, 1957) [Translated into English: *The Theory of Transonic Flow* (Oxford: Pergamon Press, 1962); Translated into Russian (Moscow: IL, 1960)]; von Mises R *Mathematical Theory of Compressible Fluid Flow* (New York: Academic Press, 1958) [Translated into Russian (Moscow: IL, 1961)]; Frankl' F I *Izbrannye Trudy po Gazovoĭ Dinamike* (Selected Works on Gas Dynamics) (Moscow: Nauka, 1973)
4. Shafranov V D *Zh. Eksp. Teor. Fiz.* **33** 710 (1957) [*Sov. Phys. JETP* **6** 545 (1958)]; Grad H *Rev. Mod. Phys.* **32** 830 (1960)
5. Solov'ev L S, in *Voprosy Teorii Plazmy* (Problems of Plasma Theory) Vol. 3 (Ed. M A Leontovich) (Moscow: Atomizdat, 1963) p. 245
6. Heinemann M, Olbert S *J. Geophys. Res.* **83** 2457 (1978); Okamoto I *Mon. Not. R. Astron. Soc.* **173** 357 (1975); Nitta S, Takahashi M, Tomimatsu A *Phys. Rev. D* **44** 2295 (1991)
7. Beskin V S, Par'ev V I *Usp. Fiz. Nauk* **163** (6) 95 (1993) [*Phys. Usp.* **36** 529 (1993)]
8. Beskin V S, Pidoprygora Yu N *Zh. Eksp. Teor. Fiz.* **107** 1025 (1995) [*JETP* **80** 575 (1995)]
9. Beskin V S *Usp. Fiz. Nauk* **167** 689 (1997) [*Phys. Usp.* **40** 659 (1997)]
10. Hunt R *Mon. Not. R. Astron. Soc.* **188** 83 (1979)
11. Beskin V S, Kompaneets R Yu, Tchekhovskoy A D *Pis'ma Astron. Zh.* **28** 616 (2002) [*Astron. Rep.* **28** 543 (2002)]
12. Shakura N I *Astron. Zh.* **49** 921 (1972) [*Sov. Astron.* **16** 756 (1973)]; Shakura N I, Sunyaev R A *Astron. Astrophys.* **24** 337 (1973); Novikov I D, Thorne K S, in *Black Holes* (Eds C DeWitt, B S DeWitt) (New York: Gordon and Breach, 1973) p. 343
13. Paczynski B, Bisnovatyi-Kogan G *Acta Astron.* **31** 283 (1981)
14. Blandford R D, Znajek R L *Mon. Not. R. Astron. Soc.* **179** 433 (1977)
15. Beskin V S *Usp. Fiz. Nauk* **169** 1169 (1999) [*Phys. Usp.* **42** 1071 (1999)]
16. Punsly B, Coroniti F V *Astrophys. J.* **350** 518 (1990)
17. Beskin V S, Kuznetsova I V *Nuovo Cimento B* **115** 795 (2000); astro-ph/0004021
18. Livio M, Ogilvie G I, Pringle J E *Astrophys. J.* **512** 100 (1999); Ghosh P, Abramowicz M A *Mon. Not. R. Astron. Soc.* **292** 887 (1997)
19. Hawley J F, Wilson J R, Smarr L L *Astrophys. J.* **277** 296 (1984); Petrich L I et al. *Astrophys. J.* **336** 313 (1989); Ustyugova G V et al. *Astrophys. J.* **439** L39 (1995); Koide S et al. *Astrophys. J.* **536** 668 (2000)

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Coherent light scattering stimulated by a quasi-static electric field

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1. Introduction

This work concerns radiation arising from the dipole-forbidden molecular transition in the case of nondegenerate four-wave interaction. One wave being unrelated to light emission, its frequency is zero or close to zero. Parallel monitoring of scattering intensity and anti-Stokes signal intensities during a degenerate CARS process provides for the absolute measurement of the static field strength, while polarization measurements permit us to determine its direction. Due to the coherent and multiphoton nature of the scattering process, the application of this approach, unlike other known methods of Stark spectroscopy, makes it possible to measure field characteristics in gases and plasma under elevated pressure. In what follows, we give examples of measurement in gas discharges.

In the presence of an electric field in an isotropic medium for centrally symmetric particles, the selection rules for dipole transitions can vary. E Condon [1] was the first to demonstrate as early as 1932 that the description of such transitions may be analogous to the description of Raman scattering of light at a scattered wave frequency tending to zero (constant field). Later on, spectroscopic constants were calculated more precisely based on the measurements of electric field-induced absorption of infrared (IR) radiation by homonuclear molecules H_2, D_2, and N_2 [2 – 5].

Progress in laser technology and nonlinear optics gave rise to many works on the generation of coherent harmonic and mixed-frequency radiation, including generation in an external electric field that permits us to ease the alternative forbidding. Early studies [6 – 8] have helped to elucidate the generation of radiation at the difference frequency during stimulated Raman scattering (SRS) of light from ruby and neodymium lasers in H_2 in an electric field. SRS is known to have a high transformation threshold; in experiments [6, 7], generation of radiation occurred at hydrogen pressure $p > 5.5$ atm. This and the fact that the limitation on the spectrum is only imposed by strongest transitions make it very difficult to use SRS for quantitative measurements.

This paper reports studies on electric field-induced IR radiation in hydrogen using biharmonic pumping of vibrational transition and the application of this radiation to measuring field parameters, in particular, for the diagnostics of gas-discharge plasma.

2. Method. Experimental technique

We shall consider electric field-induced transitions as proposed by Condon [1], i.e., by analogy with the well-developed scheme of coherent anti-Stokes Raman scattering (CARS) (see, for instance, monograph [9]). Figure 1 represents schematically degenerate (a) and nondegenerate (b) CARS transitions along with coherent IR transition (c) induced by field $E\,(\omega = 0)$. In CARS spectroscopy, biharmonic pumping by two waves of laser light ω_1 and ω_2 (such that the frequency difference $\omega_1 - \omega_2 = \Omega$ corresponds to the frequency of