

Dissipation and decoherence in quantum systems

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Abstract. The theory of dissipative quantum systems and its relation to the quantum theory of continuous measurements are reviewed. Constructing a correct theory of a dissipative quantum system requires that the system’s interaction with its environment (reservoir) be taken into account. Since information about the system is ‘recorded’ in the state of the reservoir, the quantum theory of continuous measurements can be used to account for the influence of the reservoir. If based on the use of restricted path integrals, this theory does not require an explicit reservoir model and is therefore much simpler technically.

1. Introduction

This review of the theory of dissipative quantum systems aims to show that a fundamental quantum theory of dissipation should be universal (i.e., should not depend on the model) and that such a model-independent theory is in fact identical with the quantum theory of continuous measurements.

The first attempts at the quantum theory of dissipative systems were to find a method of quantization for classical dissipative equations with a velocity-dependent friction force. These attempts invariably encountered difficulties and were in fact doomed as contradicting the uncertainty principle: in

quantum physics, position and velocity cannot be determined simultaneously to an arbitrary accuracy, so that a velocity-dependent force cannot exist.

Next, people learned to describe the dissipation of a quantum system by explicitly considering the system’s environment (reservoir), interaction with which is in fact the physical cause of the dissipation. To do this was extremely complicated technically due to the huge number of degrees of freedom in the reservoir. However, the equations for the dissipative system itself, which were obtained after the reservoir was removed from consideration (via a reduction procedure), turned out to be very simple and depended little on the particular reservoir model used (see, e.g., Refs [1, 2]).

The universality of quantum dissipation could in fact have been predicted because it is a consequence of the same uncertainty principle preventing the direct quantization of dissipative equations. To see this, note that the friction force exerted on the quantum system (by the reservoir) cannot be proportional to velocity but must become so in the classical limit. This means that the friction force must contain fluctuations which are of no significance in the classical limit. However, the entire experience of quantum mechanics shows that *quantum fluctuations are always due to the uncertainty principle* and do not depend on exactly which physical objects are behind them. Hence, dissipation effects should also be universal and independent of the specific reservoir model¹.

The approach which from the very start treats dissipation in a universal, model-independent fashion is the *quantum theory of continuous measurements* [3–5]. The efficiency of this approach as applied to dissipation is based on the fact

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that the interaction of a quantum system with its (macroscopic or mesoscopic) environment is always a measurement of the system — in the sense that the environment (reservoir) retains information about the system. In the quantum theory of measurement it is proved that the back action of the environment on the system (i.e., the disturbance of the system) consists of two parts: (1) a minimal disturbance, which depends in a standard way only on the result of the measurement (i.e., on what information about the system remains in its environment), and (2) a non-minimal disturbance, whose dependence on the result of the measurement is easy to describe — especially in the linear approximation (which is valid if the interaction with the environment is weak). With this result, the dynamics of a quantum system becomes simple to describe.

It is convenient to formulate such an approach in terms of *restricted path integrals*, because path integrals have a huge heuristic power in quantum mechanics. For specific calculations, path integrals are not necessary, however. Instead, an effective Schrödinger equation (with a modified complex Hamiltonian) for the state vector, or an equation for the density matrix with additional terms added to the conventional density matrix-Hamiltonian commutator, can be used.

Representing the evolution of a dissipative system by means of restricted path integrals or a complex Hamiltonian is a selective approach, i.e., one taking into account the result of the measurement (the final state of the reservoir). This aspect manifests itself in that the system is described by the state vector $|\psi_\alpha\rangle$ dependent on the result of the measurement, α . A non-selective description is obtained by presenting evolution, instead of the state vector, by a ‘partial’ density matrix $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$, and then passing to the total density matrix by summing over all possible measurement results, $\rho = \sum_\alpha \rho_\alpha$. The non-selective description is less detailed, but in many (although not all) cases there is actually no need for a higher level of detail.

Using restricted path integrals (or complex Hamiltonians) does not require a reservoir model and therefore provides a technically relatively simple and, most important, model-independent theory of dissipation for quantum systems. As in the classical case, such a theory enables one to describe a dissipative quantum system in system-specific terms without explicitly considering — but correctly introducing the effect of — the degrees of freedom of its environment.

In accord with what was said above, the present review is structured as follows. In the remainder of the introduction we get more specific about the ideas discussed above and we also give basic literature references. In Section 2 we will consider reservoir models (mainly the most popular of them, Caldeira – Leggett’s model) and discuss the equations they yield for describing dissipative systems. In doing this we will not go into detail and will give only a general outline of how these models are constructed, with a brief description of the assumptions made. Section 3 presents a dissipation theory which is based on the use of restricted path integrals and is independent of reservoir models. Finally, in Section 4 it is shown that the theory using restricted path integrals is general, i.e. is applicable to any open system, provided only that the environment of the system (the reservoir) is macroscopic, or at least mesoscopic.

Although Section 3, which presents the universal (model-independent) theory, is of central importance in this review, its content will be easier to understand if the reader first reads Section 2, which introduces some key terms (such as

decoherence) and discusses the physical causes of the key phenomena of interest. Section 4 (the most complex one technically) may be skipped in the first reading without even slightly compromising understanding of the main text of the paper.

1.1 ‘Quantization’ of a dissipative system

A simple and at the same time the most important example of a dissipative system is a particle moving in a viscous medium. The classical description of such a particle in the one-dimensional case is given by the equation

$$m\ddot{q} + 2m\gamma\dot{q} + V'(q) = F(t), \quad (1)$$

where m is the mass of the particle, q is the coordinate of the particle, γ is the friction coefficient, and $V(q)$ is the potential acting on the particle. $F(t)$ denotes the random force exerted on the particle by the molecules of the medium. The statistic characteristics of the random function $F(t)$ depend on the friction coefficient and the temperature of the medium,

$$\langle F(t) \rangle = 0, \quad \langle F(t)F(t') \rangle = 4m\gamma kT \delta(t - t'). \quad (2)$$

Equation (1) with the additional conditions (2) is called the Langevin equation and is used in the theory of Brownian motion.

At low temperatures, quantum effects have a significant influence on the behavior of a particle. To describe these effects, a quantum theory of a particle moving in a medium is needed. The development of such a theory — and indeed of the quantum theory of dissipative systems — has long remained an unsolved problem. To solve it, a variety of approaches have been proposed, and it is only in recent decades that they have led to success.

The first approaches to the quantum theory of dissipation involved attempts to extend to dissipative systems various methods of ‘quantizing’ classical dynamical systems. However, in the usual approach to quantizing a classical system one employs the Hamiltonian of the system. A dissipative system is non-Hamiltonian, making the usual quantization method inapplicable in this case. True, equations of the type (1) can be obtained from certain time-dependent Hamiltonians or from those containing additional variables, and this fact was indeed used in some works to construct a quantum theory of dissipative systems [6–8]. Also, approaches based on the nonlinear Schrödinger equation [9, 10] and on the use of complex observables [11] were proposed. These approaches, however, are formal and lead to a number of difficulties with, for example, the uncertainty principle [12]. One more direction in the search for the ‘quantum Langevin equation’ consisted in attempting to generalize the conditions (2) (see, for example, Refs [13, 14]).

The difficulties with the direct quantization of classical systems subject to dissipation appear to be of a fundamental nature. The main reason why such quantization attempts have not led (and probably cannot lead) to success is perhaps the fact that, whereas quantum theory implies the uncertainty principle, dissipative theories using classical approaches are at odds with this principle (see footnote 1). For example, in Eqn (1) the second term describes the friction force, which is proportional to the particle velocity. The dynamics of a particle described by this equation suggests that the particle’s coordinate and velocity must both have certain values at any

time simultaneously. In quantum mechanics they cannot be both known with an arbitrary accuracy, leading to difficulties in quantizing Eqn (1). While this argument does not prove this impossibility, it certainly identifies the source of fundamental difficulties that arise in the field (in this connection, see Ref. [12]).

Physically, dissipation is due to the system in question being affected by its environment. For example, Eqn (1) for a particle experiencing friction is a consequence of the fact that the particle is acted upon by the particles of the medium it moves through. Clearly, a reliable way of constructing a quantum dissipation theory is to use correct quantum mechanical methods to account for the influence of the environment (reservoir) on the system.

1.2 Models of the system-reservoir interaction

The most convincing in terms of physics — although complicated and model-dependent — are those approaches to dissipation theory that directly use the important fact that dissipation is due to the interaction of the system with its environment² (reservoir). So the system of interest here is an open one. To calculate its behavior we should consider a closed system which also includes the reservoir and then carry out a reduction procedure, i.e., sum over the reservoir's degrees of freedom. The result will be a description of our (sub)system in terms of a reduced density matrix.

Let us elaborate on this. The closed system $\mathcal{S} + \mathcal{E}$, consisting of our system \mathcal{S} and its environment \mathcal{E} , can be represented by a state vector. Let this be the state vector $|\Psi\rangle$. Instead of the state vector, the same state of the total system can be represented by the density matrix $\hat{\mathcal{R}} = |\Psi\rangle\langle\Psi|$. To change now to the description of our subsystem \mathcal{S} alone, we must sum the density matrix $\hat{\mathcal{R}}$ over the degrees of freedom of the system \mathcal{E} , i.e., we must take the partial trace of the total density matrix³,

$$\hat{\rho} = \text{Tr}_{\mathcal{E}} \hat{\mathcal{R}}. \quad (3)$$

The resulting operator $\hat{\rho}$ is a reduced density matrix of the system \mathcal{S} . The mean value of any observable \hat{A} of this system in the state $|\Psi\rangle$ is expressed in terms of the reduced density matrix as

$$\langle\Psi|\hat{A}|\Psi\rangle = \langle\hat{A}\rangle_{\hat{\rho}} = \text{Tr}(\hat{A}\hat{\rho}). \quad (4)$$

There have been many attempts to develop a theory of dissipative systems by first considering the system together with its environment (reservoir) and then carrying out a reduction — a procedure which removes the reservoir from the description (but takes into account its influence). We mention Refs [15–19] as examples of the realization of this program. Reviews of research on this theme can be found in Refs [20–22]. One of the most successful attempts was the 1983 quantum diffusion model of Caldeira and Leggett [1, 2] (see also Ref. [23], where this model is discussed from the point of view of Gell-Mann and Hartle's [24] theory of

'quantum histories'). We will discuss the Caldeira–Leggett model in Section 2.

In past decades models of this kind were also used to describe the physical nature of quantum measurements (see, for example, Refs [25–31]; reviews can be found in Refs [32, 33]). This has to do with the fact that the measurement of a quantum system is carried out through the interaction of the system with its environment, the latter playing the role of a measurement device⁴. True, in the theory of measurement, not only the behavior of the system undergoing measurement but also that of the environment is essential, because it is the state of this latter which represents the result of the measurement. However, in order to understand the physics of what is going on, often no regard has been given to what exactly the state is in which the environment of the measured system finds itself. Therefore, after first applying conventional quantum mechanical methods to describe the evolution of the composite system (i.e., the measured system and its environment), one would carry out a reduction by summing over all possible states of the environment. Although this treatment is aimed at the description of the nature of measurement, it is perfectly clear that there should also be a by-product of it — but a very important one — namely, the description of the quantum system as a dissipative system⁵.

It should be noted that the most general differential equation for the density matrix — one extending the von Neumann equation to the case of a dissipative quantum system — was found by Lindblad [34] in 1976 without using any model at all. In his purely mathematical analysis Lindblad assumed that the process is described by a semi-group of evolution operators, and imposed the condition that the density matrix maintain its positiveness. These requirements, Lindblad showed, are fulfilled in (and only in) the case where the equation for the density matrix has the form

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \sum_i (\hat{L}_i^\dagger \hat{L}_i \hat{\rho} - 2\hat{L}_i \hat{\rho} \hat{L}_i^\dagger + \hat{\rho} \hat{L}_i^\dagger \hat{L}_i), \quad (5)$$

where \hat{H} is the Hamiltonian of the system and the operators \hat{L}_i account for the system's dissipation. To specify the form of these operators means to specify the dissipative process, i.e., to indirectly characterize the system's environment and how the system and its environment interact.

Those density matrix equations derived for specific reservoir models — in particular, for the Caldeira–Leggett model — often do not belong to this class because of the approximations accepted in developing the model. Such equations describe well a wide range of phenomena in, for example, quantum optics and provide insight into the physical nature of dissipation in quantum systems. For some specific situations, however, they may prove inapplicable: they are not Lindblad-type equations and therefore may fail to maintain the density matrix's positiveness for some special initial conditions. A discussion of the difficulties that arise in this context can be found in Refs [35–38]. Note that this

² In some cases different degrees of freedom in one and the same material system can play the role of a dissipating system and its dissipation-producing environment.

³ If the vectors $|\psi_i\rangle$ constitute the orthonormal basis of the state space of the system \mathcal{S} and the vectors $|\Phi_a\rangle$ form a similar basis for \mathcal{E} , then the total trace $\text{Tr} \hat{\mathcal{R}} = \sum_{ai} \mathcal{R}_{ai, ai}$ of the density matrix $\hat{\mathcal{R}}$ is unity and the partial trace is defined as $(\text{Tr}_{\mathcal{E}} \hat{\mathcal{R}})_{ij} = \sum_a \mathcal{R}_{ai, aj}$.

⁴ In a more detailed description the environment is treated as a complex of a quantum system, which plays the role of a sensitive element, and the reservoir, which contains a large number of degrees of freedom — thus leading to classical features in the system.

⁵ True, the main aspect of the quantum theory of measurement is the appearance of classical features in the system — decoherence, to use a current term — whereas dissipation per se appears during a measurement as a concomitant — but, strictly speaking, not obligatory — phenomenon. We will discuss this in more detail below (see, for example, Section 2.2).

defect is absent in equations derived within the framework of the model-independent theory of continuous measurements, to which we now proceed.

1.3 Model-independent theory of measurement

It turns out that the effect of the reservoir (environment) on a quantum system — and thereby on its dissipation — can even be taken into account without specifying the reservoir model and the way the system interacts with the reservoir. This is done in the theory of continuous measurements. A review of different approaches to this theory can be found in Refs [3, 4]. The simplest approach is to approximately represent a continuous measurement as a series of instantaneous (short) measurements, each of which is described by a von Neumann reduction (see, for example, Refs [30, 39]). The reverse is also true, though: many (but not all) actual examples of continuous measurement are in fact series of interactions (say, collisions with gas molecules), each of which can be regarded as a measurement. If the duration of such an ‘elementary’ measurement is neglected, the measurement can be represented as instantaneous; otherwise, each short measurement turns out to be continuous as well. In either case, the entire series of interactions can be interpreted as a continuous measurement (see Ref. [3], Ch. 8).

Thus, the description of a continuous measurement as a series of instantaneous measurements is possible. However, in this description the evolution of the system under measurement is represented by the product

$$|\psi_t\rangle = U(t, t_N) P_{i_N} U(t_N, t_{N-1}) \dots U(t_3, t_2) P_{i_2} U(t_2, t_1) P_{i_1} U(t_1, 0) |\psi_0\rangle \quad (6)$$

of a large (in the limit, infinite) number of operators, where evolution operators of the free system acting at short time lengths alternate with von Neumann projectors corresponding to instantaneous measurements. Such a description is often cumbersome and inefficient.

As a realization of Feynman’s [40] idea, a quantum theory of continuous measurements based on the use of restricted path integrals was developed by the author [41, 42, 3, 4]. This theory and its applications to the description of dissipative systems are the subjects of Sections 3 and 4.

In this approach the propagator of a system undergoing continuous measurement (i.e., an open system, one interacting with its environment) is represented by a path integral of the type of Feynman’s integral [40, 43], but with the integration now restricted to those paths compatible with the measurement results. The restricted path integrals that emerge in this approach provide a description of a wide range of open quantum systems and, in particular, are efficient in representing the dissipation of a quantum system. Importantly, the description of such a system is only carried out in terms of the system itself. An explicit model of the environment (reservoir) is not needed. Instead, the influence of the environment on the evolution of the system is characterized by what information remains in the system’s environment (as encoded in the state of the environment).

The important point to realize here is that such an approach describes not only situations where the environment is specially designed to measure the system but also those with the measurement-type process occurring as a result of an uncontrollable interaction of the system with the environment (reservoir) existing naturally around it. For

dissipative systems this is generally the case. In Section 4 it is shown that for a very wide class of open systems evolution can be correctly considered as resulting from the continuous measurement of the system. Therefore, the restricted path integral approach allows a general theory of dissipation to be constructed.

In the very important special case where continuous measurement is the monitoring of one or several observables of the system, the restricted path integral description is equivalent to a Schrödinger equation with a complex Hamiltonian. For example, if the system is subject to the continuous monitoring of observable \hat{A} then, denoting the result of the monitoring by $a(t)$, we find that the system evolves in accord with the equation

$$|\dot{\psi}\rangle = -\frac{i}{\hbar} \hat{H}_{[a]} |\psi\rangle, \quad (7)$$

where

$$\hat{H}_{[a]} = \hat{H} + \lambda a(t) \hat{B} - i\hbar\alpha (\hat{A} - a(t))^2 \quad (8)$$

is the effective Hamiltonian containing terms depending on the measurement result $a(t)$ (an anti-Hermitian term among them). Equations of this type can account for dissipation (see Section 3).

The approach based on the use of restricted path integrals or complex Hamiltonians has a major advantage in that it has no complications related to considering (and subsequently removing from consideration, i.e., applying reduction to) the huge number of the reservoir’s degrees of freedom. The influence of the reservoir is included implicitly instead. All the reservoir characteristics relevant to this are contained in the description of what observables (and in what regime) are subject to measurement, i.e., what information about the system is ‘recorded’ in the reservoir (or is encoded in the reservoir’s state which emerges in the process of measurement). In the above example, for instance, the information recorded in the reservoir is that the observable \hat{A} varies with time as $a(t)$. The accuracy to which this is true depends on the constant α .

The restricted path integral approach is also distinctive in that the density matrix equation it produces always proves to be of the type of the Lindblad equation (5), thus ensuring the positiveness of the density matrix. The approach applies even if the process is not expected to be Markovian [3, 44], although a change from restricted path integrals to ‘time local’ equations is not possible in this case.

One more advantage of treating dissipation in the framework of the theory of continuous measurements is that the process is represented in a selective way, although a transition to the non-selective description can be made if desired. This means that such a theory allows one not only to handle a statistical mixture of all possible evolutions of a dissipative system (as represented by the density matrix) but also to consider each individual evolution separately (each evolution being represented by a wave function which solves the Schrödinger equation with a complex Hamiltonian).

Thus, in the case of the monitoring of observable \hat{A} (in the regime discussed above), the process is described in a selective way using Eqn (7). Such a description depends on the measurement result $[a] = \{a(t)\}$. A non-selective description of the same process is obtained by making a transition from the state vector $|\psi\rangle = |\psi_{[a]}\rangle$ to the ‘partial’ density matrix

$\rho_{[a]} = |\psi_{[a]}\rangle\langle\psi_{[a]}|$ and then to the total density matrix, the latter being obtained by summing over all possible measurement results $\rho = \int d[a] \rho_{[a]}$. The total density matrix turns out to satisfy the equation

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} = & -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\alpha}{2} [\hat{A}, [\hat{A}, \hat{\rho}]] - \frac{\lambda^2}{8\alpha\hbar^2} [\hat{B}, [\hat{B}, \hat{\rho}]] \\ & - \frac{i\lambda}{2\hbar} [\hat{B}, [\hat{A}, \hat{\rho}]_+], \end{aligned} \quad (9)$$

where $[\ ,]_+$ denotes an anticommutator. This equation automatically belongs to the class of Lindblad equations, Eqn (5), and describes a dissipative system, with the parameter λ proportional to the friction coefficient.

To describe continuous measurements and dissipation, selective approaches other than those based on restricted path integrals have been proposed. The most important two of these approaches employ stochastic equations and the quantum jump concept, respectively (their review can be found in Ref. [45]). In each of these approaches a source of noise is introduced into the Schrödinger equation of the system — white noise in the former case, and random quantum transitions in the system in the latter — to mimic the influence of the measurement system (reservoir). The statistics of the noise is chosen in such a way that the transition to the non-selective description yields a correct density matrix equation — the Lindblad equation (5), for example. These methods are applied successfully to the numerical calculation of real systems. There is, however, a common drawback to them, a lack of the physical interpretation of the noise and, as a consequence, arbitrariness in constructing the theory.

Thus, to different forms of stochastic equations for the state vector there correspond one and the same equation for the density matrix, and proposed criteria for choosing one of the forms are often purely mathematical and not grounded well enough. In the restricted path integral approach, by contrast, one of all possible alternative evolutions of the measured system is given by a function which has a clear physical meaning: it characterizes the result of the measurement [as the function $a(t)$ in the example above]. It should be noted in this connection that one of the forms of stochastic equations is equivalent [3, 46, 47] to the complex-Hamiltonian Schrödinger equation that arises in the theory of restricted path integrals. Because the restricted path integral method is thoroughly grounded physically (see Sections 3 and 4), its equivalent stochastic equation provides a correct selective description of the evolution of the measured system. With non-equivalent stochastic equations, evolution (the time dependence of the state vector) may be described incorrectly even if they lead — after the alternative evolutions are summed over — to a correct density matrix equation.

Now a question may arise: is any dissipation a continuous measurement of the quantum system by its environment? Because dissipation is always caused by the influence on the system of its environment, this question can be reformulated as follows: can any open quantum system be viewed as being measured by its environment? In Section 4 it is shown that this is indeed the case if the environment (reservoir) contains a sufficiently large number of degrees of freedom. Because this condition is practically always fulfilled, the continuous-measurement theory approach is a general one.

2. Reservoir models

In this section we consider briefly how models for the behavior of a dissipative system are built and discuss the conclusions these models suggest. Although some other models yield similar results, the well-known Caldeira–Leggett model is used as a typical example.

2.1 The Caldeira–Leggett model

The Caldeira–Leggett model [1, 2] was developed to describe the diffusion of a quantum particle through a substance, i.e., to construct a quantum analogue of the process described by the classical equations (1) and (2). The model is one-dimensional. It assumes that the test (Brownian) particle moves in a medium consisting of a large number of point ‘atoms’ interacting with one another. It is the ensemble of the atoms of this medium which constitutes the reservoir leading to the dissipation of the probe particle. It is assumed that the potential describing this set of atoms (reservoir) has an absolute minimum, corresponding to the equilibrium position of all the atoms; that the atoms exhibit small oscillations about their equilibrium positions; and that the interaction of the atoms with the test particle is weak. It can be said that the test particle moves through a one-dimensional crystal.

Under the assumption that displacements from equilibrium positions are small, the potential describing the field of the atoms can be considered quadratic in atomic coordinates, i.e., the ensemble of the atoms can be thought of as a set of harmonic oscillators interacting with each other. By redefining the coordinates of the set of oscillators (i.e., by passing to normal modes), we can replace them by a set of non-interacting oscillators. This procedure is typical for the description of a crystal and reduces its dynamics to that of isolated harmonic oscillators. The total Hamiltonian of the test particle in the reservoir is ultimately represented in the form

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{q}) + \hat{q} \sum_n c_n \hat{u}_n + \sum_n \left(\frac{\hat{v}_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 \hat{u}_n^2 \right), \quad (10)$$

where \hat{q} and \hat{p} are the coordinate and momentum of a particle of mass m experiencing the potential $V(\hat{q})$, and \hat{u}_n and \hat{v}_n are the same for the n th oscillator in the reservoir (for the n th mode of the system of atoms) of mass m_n and frequency ω_n .

We will not trace in detail further stages in the construction of the model but will only outline the general scheme of this process.

The effect of the reservoir on the test particle (provided the resulting state of the reservoir itself is of no interest to us) is described by the so-called Feynman–Vernon influence functional [43, 48]. The influence functional is obtained by writing the density matrix of the system (test particle + reservoir) in terms of the (double) Feynman path integral and subsequently integrating over the final states of the reservoir, thus passing to the reduced density matrix of the particle. As a result, the density matrix of the test particle is expressed as a double path integral for the particle itself only, but with a certain functional in the integrand. This functional describes the influence of the reservoir and is called the influence functional.

We will deal with and give a more accurate definition of the influence functional in Section 4. For now, just note that

the influence functional of an ensemble of non-interacting oscillators can be calculated exactly because the integrals it involves are of the Gaussian type. This calculation was done in the works by Feynman and Vernon [43, 48]. The assumptions adopted in the Caldeira–Leggett model allow Feynman and Vernon’s results to be employed.

It is essential to note that the family of oscillators that constitute the reservoir is taken to be infinite and even continuous. Formally, this is achieved by increasing the number of oscillators in a finite family to infinity. This limiting passage, however, is a very serious step in that it gives rise to a new property, that of irreversibility. This can be expressed simplest by using the language of measurement. As a result of the interaction of a particle with the reservoir, the state of the latter changes, and information about the state of the particle is encoded in this change in a certain way. To put it more precisely, entanglement between the states of the particles and the states of the reservoir takes place (see Refs [3, 4, 33]).

If the reservoir (the system of oscillators in our case) is finite, the reservoir can return to its initial state over time. Then the information which was in the reservoir about the particle will leave the reservoir and return to the particle (the particle-reservoir entanglement will disappear). As a result, the entire system can in principle return to its initial state, and the whole process turns out to be reversible. Of course, for a very large number of degrees of freedom of the reservoir, this needs a huge amount of time to occur, i.e., from a practical point of view the process is irreversible. A formal passage to the limit of the infinite number of degrees of freedom makes the time of return infinite, i.e., allows one to describe the process as irreversible in the absolute sense of the word.

In Section 4.2.2 we shall encounter this important assumption in a different (but essentially similar) context and in a different formalism. We shall see that the alternative evolution paths of an open system ‘decohere’ (i.e., no interference arises between them) if the environment affecting the system contains a large (macroscopic or at least mesoscopic) number of degrees of freedom.

After the density matrix of the particle is expressed in terms of the influence functional — which is in turn taken from Feynman and Vernon’s work — the law of evolution of the particle in the Caldeira–Leggett model is in principle known. There remains, however, the major problem of interpreting the obtained result in terms which have a clear physical meaning. For this purpose, first the transition to the classical limit is made and then this limit is compared with the existing classical theory, i.e., with Eqns (1) and (2). The classical limit implies, in particular, a high enough temperature ($kT \gg \hbar\omega_k$ for all characteristic reservoir frequencies ω_k) and a time exceeding the reservoir relaxation time.

Analysis of expressions obtained in the classical limit allows one to identify them with their classical theory counterparts. In particular, a correlation function identifiable with Eqn (2) is derived. This enables the parameters of a quantum model ultimately to be expressed in terms of those involved in the classical theory of Brownian motion: temperature T and friction coefficient γ . Then the equations for the density matrix of a quantum Brownian particle take the form [1, 2]

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}_R, \hat{\rho}] - \frac{D}{\hbar^2} [\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{i\gamma}{\hbar} [\hat{q}, [\hat{p}, \hat{\rho}]_+], \quad (11)$$

where $[\ ,]_+$ denotes an anticommutator, \hat{H}_R is the Hamiltonian of the particle ‘renormalized’ due to the interaction with the reservoir (see Sections 3.3 and 3.4), γ is the friction coefficient, and $D = 2m\gamma kT$ is the diffusion coefficient. Equation (11) is derived under the assumption of not too low temperatures (the thermal energy kT is larger than the quantum energies $\hbar\omega$ corresponding to the characteristic particle’s frequencies). A similar equation was earlier obtained in Ref. [11] by an entirely different method. Analogous equations for various reservoir models were also derived in Refs [27–29].

As applications, Caldeira and Leggett consider cases in which a diffusing particle moves in a zero [$V(q) = 0$] or a quadratic [$V(q) = m\omega^2 q^2/2$] potential, i.e., is either a free particle or a harmonic oscillator. This removes the limitation on the temperature, so that it can be arbitrarily low. Formulas for the time variation of the wave-packet width and center position are derived. It is shown, in particular, that for low friction ($\gamma \rightarrow 0$) the coordinate dispersion and the diffusion coefficient of a damped oscillator are asymptotically (for $t \rightarrow \infty$) equal to

$$\sigma^2 = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad D = m\gamma\hbar\omega \coth \frac{\hbar\omega}{2kT}. \quad (12)$$

For $T \rightarrow \infty$, the diffusion coefficient assumes the classical value $D = 2m\gamma kT$, and for $T \rightarrow 0$ it is equal $D = 2m\gamma\hbar\omega/2$.

2.2 Dissipation and decoherence

Let us take a closer look (see Refs [50, 33]) at the density matrix equation (11) obtained from the Caldeira–Leggett model. The goal of this analysis is to show that the process described by this equation not only leads to dissipation but also produces another important physical phenomenon, decoherence⁶. This phenomenon is the very essence of a measurement process of a quantum system [4, 3, 33], so that the dissipation of a quantum system is inseparably linked to its measurement.

We write Eqn (11) in the coordinate representation and retain on the right only its D term, which is the dominant one in the ‘classical’ regime, when the effective action becomes much larger than \hbar ,

$$\frac{\partial}{\partial t} \rho(q, q') = \dots - \frac{D}{\hbar^2} (q - q')^2 \rho(q, q'). \quad (13)$$

Hence, for large (but not too large, see below) times the solution to Eqn (11) will contain the factor

$$\exp \left[-\frac{D}{\hbar^2} (q - q')^2 t \right]. \quad (14)$$

From this it is seen that the nondiagonal (in coordinates) elements of the density matrix die out exponentially with time. This is namely the decoherence phenomenon. As decoherence in coordinate space appears, different values of a coordinate are only characterized by positive probabilities, but not by complex probability amplitudes (see, for example, Refs [4, 3, 33]).

Probabilities, unlike probability amplitudes, are characteristic of the classical theory, and the appearance of decoherence in coordinate space means that the coordinate

⁶ This is also true for equations obtained for other models used in the theory of dissipation.

acquires classical properties. An observable coordinate can then be assigned one or another specific value in accordance with the probability distribution over different values. This means that the coordinate has been measured. From Eqn (14) one can see that, over time, decoherence with respect to the coordinate appears on increasingly smaller scales, i.e., the coordinate is measured with increasingly higher accuracy.

The argument above suggests that quantum diffusion described by Eqn (11) is at the same time a coordinate measurement process: information about the coordinate is recorded in the reservoir. This is one of the illustrations of the fact that dissipation and the continuous measurement of a quantum system are closely related to one another. We will discuss this in more detail in Section 4.

It can be shown [33, Ch. 3] that the exponential decoherence regime described by Eqn (14) ultimately comes to an end, and the decoherence length is stabilized at a certain constant level ℓ of the order of the thermal de Broglie wavelength λ_{th} ,

$$\ell = \hbar \sqrt{\frac{\gamma}{2D}} = \frac{\lambda_{\text{th}}}{\sqrt{8\pi}}, \quad \lambda_{\text{th}} = \hbar \sqrt{\frac{2\pi}{mkT}}. \quad (15)$$

The time it takes for the decoherence length to reach a certain preassigned value Δa — i.e., the time which is enough for a measurement to distinguish between particle positions differing by Δa (decoherence time) — follows from the form of the exponential (14),

$$\tau_{\text{decoh}}(\Delta a) = \frac{\hbar^2}{D(\Delta a)^2}. \quad (16)$$

Relaxation effects related to friction — i.e., to dissipation — are characterized by a time constant proportional to the friction coefficient:

$$\tau_{\text{frict}} = \frac{1}{\gamma}. \quad (17)$$

Hence the rates of decoherence and friction-related relaxation have the ratio

$$\frac{\text{decoherence rate}}{\text{relaxation rate}} = \frac{\tau_{\text{frict}}}{\tau_{\text{decoh}}(\Delta a)} = \frac{mkT(\Delta a)^2}{\hbar^2} \sim \left(\frac{\Delta a}{\lambda_{\text{th}}}\right)^2. \quad (18)$$

The ratio (18) becomes equal to unity (the decoherence time becomes equal to the relaxation time) if we require that decoherence be a maximum, i.e., if we set $\Delta a = \lambda_{\text{th}}$. For decoherence on a macroscopic scale, however, this ratio is very large, i.e., decoherence occurs much faster than relaxation manifests itself. For a typical macroscopic situation ($m = 1$ g, $T = 300$ K, $\Delta a = 1$ cm) the ratio (18) is of the order of 10^{40} . In this case decoherence (transition to the classical regime) occurs much earlier than friction shows up, and the latter can be accounted for correctly in the framework of the classical theory [2, 51, 33].

Under certain conditions a situation can develop in which the ratio (18) is not large, decoherence is relatively slow, and the quantum properties of the process should be fully accounted for. This usually occurs in mesoscopic systems.

Thus, by using the methods of quantum optics it is possible to prepare [28, 52–55] several photons in a state

which is a superposition of two sharply distinct field configurations. After preparation, such a superposition is subject to a relatively slow decoherence process, gradually turning into a mixture of the same two configurations. The ratio of the type⁷ (18) in this case is of the order of the number of photons squared, and both components of this process — dissipation and decoherence — are of importance. States of this kind illustrate the specific features of quantum measurements and are therefore often called ‘Schrödinger cats’. They are, at the same time, fully realisable states which can be studied experimentally.

2.3 Decoherence by the internal structure of atoms

In the Caldeira–Leggett model the decoherence of a Brownian particle is due to the influence of the atoms interacting with it, which are assumed to be point-like (i.e., they have no inner structure but interact with one another). The particle excites the motion of the atoms, and the back action of them causes the particle’s decoherence (it partly loses its quantum properties) and dissipation. If the inner structure of the atoms is not neglected, then the interaction with the particle will excite their inner motions as well. This effect can be dominant under certain conditions.

Reference [56] (see also Ref. [3], Section 4.3) analyzes a model which resembles Caldeira–Leggett’s but differs in the treatment of the atoms of the medium (reservoir) through which the test particle moves: it is assumed that they are immobile (i.e., are sufficiently heavy for their displacements to be negligibly small) but still possess an inner structure. Under such assumptions the disturbance of the medium caused by the particle manifests itself not in the motion of the atoms (relative to their equilibrium positions) as in the Caldeira–Leggett model, but in the excitation of their internal degrees of freedom. From this model the equation for the density matrix of the particle was found to be

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\varkappa}{2} [\hat{\mathbf{r}}[\hat{\mathbf{r}}, \hat{\rho}]], \quad (19)$$

where $\hat{\mathbf{r}}$ is the operator of the (three-dimensional) position of the test particle, and the coefficient $\varkappa = 2/l^2\tau$ is expressed in terms of the radius l and the time constant τ for the interaction between the particle and the atoms of the reservoir. This equation was derived by two alternative methods in Ref. [56], first in the framework of the reservoir model, and second, using the theory of continuous measurements (restricted path integrals). We will discuss the second approach in more detail in Section 3.

Rewriting Eqn (19) in the coordinate representation and keeping only the ‘classical’ term we obtain

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, \mathbf{r}') = \dots - \frac{\varkappa}{2} (\mathbf{r} - \mathbf{r}')^2 \rho(\mathbf{r}, \mathbf{r}'). \quad (20)$$

This means that at large times the solution of Eqn (19) will contain the factor

$$\exp \left[-\frac{\varkappa}{2} (\mathbf{r} - \mathbf{r}')^2 t \right]. \quad (21)$$

⁷ In this case, instead of Δa , a parameter characterizing the difference between the two configurations (for example, the phase difference of two electromagnetic waves) must appear, and instead of λ_{th} , the limiting value this new parameter assumes when dissipation is taken into account.

This long-time asymptotic form shows that Eqn (19) describes the evolution of a system (quantum Brownian particle) which is subject to the measurement of the observable \hat{r} , i.e., of particle coordinates in three-dimensional space.

It was precisely the aim of Ref. [56] to build a model of the continuous measurement of a coordinate. The measurement model adopted for this purpose is the simplest one (describing the ‘minimally disturbing’ measurement of particle coordinates). This is why the model leads to decoherence (gradual loss of quantum properties in relation to the quantity under measurement) but not to dissipation. Recalling what we learned in Section 2.2 we can conclude that such a model is applicable if decoherence appears slowly and only at the early stage of the decoherence process, when dissipation does not yet manifest itself. In Section 3 we will consider, in the framework of a model-independent approach, a more complex regime of ‘non-minimally disturbing’ continuous measurement, in which both these phenomena, decoherence and dissipation, appear.

From the analysis of dissipation models we now proceed to the quantum theory of continuous measurements, which leads to a quantum system dissipation theory not relying on a specific model and, indeed, totally independent of models.

3. Dissipation — from the theory of continuous measurements

The effect of the reservoir on the quantum system with which it interacts can be accounted for in the framework of the quantum theory of continuous measurements. We will consider a theory of measurements which is based on the use of restricted path integrals. The derivation of equations for a dissipative quantum system from the theory of measurements is performed in accordance with Ref. [5]. In Section 4 it is shown that this approach to the theory of dissipation is a general one.

3.1 Measurement and restricted path integrals

Let us consider the restricted path integral method the present author proposed, starting from Feynman’s [40] idea, to describe continuous quantum measurements [57, 58, 41, 42, 3–5] (see also Refs [59–61, 47]). In this approach an open system subject to continuous measurement is described (as opposed to a closed system) not by a single unitary evolution operator but rather by a whole family of partial evolution operators (propagators) — in accordance with the set of alternative measurement results⁸. Each measurement result defines one quantum evolution channel by means of the corresponding partial propagator. Only the entire set of these channels (i.e., all the propagators taken together) provide a complete description of an open system subject to continuous measurement.

A starting point for developing the restricted path integral method is the Feynman version of quantum mechanics, with its path integral formalism. In the Feynman formulation of quantum mechanics the transition amplitude for a system going from one point to another in configuration space (the propagator of the system) is represented as a path integral in

the form

$$U_t(q'', q') = \int_{q'}^{q''} d[q] \exp \left[\frac{i}{\hbar} S[q] \right] \\ = \int_{q'}^{q''} d[p] d[q] \exp \left[\frac{i}{\hbar} \int_0^t dt (p\dot{q} - H(p, q, t)) \right]. \quad (22)$$

Here, $S[q]$ is the classical action of the system, which can be expressed as an integral of the Lagrangian along a given path,

$$S[q] = \int_0^t dt L(q, \dot{q}, t), \quad (23)$$

and H is its Hamiltonian. The first and the second integrals in Eqn (22) are path integrals in the configuration space and in the corresponding phase space, respectively. Both are taken along the path from point q' at time $t = 0$ to point q'' at time t . The coordinates and the corresponding momenta can be multi-dimensional.

The operator U_t with the kernel (22), i.e., the operator with the matrix elements

$$\langle q'' | U_t | q' \rangle = U_t(q'', q'), \quad (24)$$

is an evolution operator, i.e., it describes the evolution of the system in terms of its state vector or its density matrix by the equation

$$|\psi_t\rangle = U_t |\psi_0\rangle, \quad \rho' = U_t \rho_0 U_t^\dagger. \quad (25)$$

As is well-known, instead of evaluating the path integral (22), the vector $|\psi_t\rangle$ can be found by solving the Schrödinger equation

$$i\hbar \dot{|\psi_t\rangle} = -\frac{i}{\hbar} \hat{H} |\psi_t\rangle, \quad (26)$$

with the initial condition $|\psi_0\rangle$. The classical Hamiltonian of the system H is real, and the corresponding quantum operator \hat{H} is Hermitian. Therefore the evolution operator U_t is unitary,

$$U_t^\dagger U_t = 1, \quad (27)$$

and the vector $|\psi_t\rangle$ has unity norm (provided the initial vector $|\psi_0\rangle$ is normalised to unity).

We must now pass from the Feynman path integral (22) to a restricted path integral. To do this, we should recall the argument that led Feynman to the idea of a path integral. His argument was that the exponential appearing under the sign of the path integral and depending on a certain specific path is interpreted as the probability amplitude for the system to propagate precisely along this path. According to the standard rules of quantum mechanics, the total probability amplitude for the propagation of a system is represented as the sum (more precisely, the integral) of the amplitudes corresponding to all possible paths leading from point q' to point q'' . This is exactly what the Feynman integral (22) yields.

In this argument, however, the integration over all paths with specified boundary points, as in Eqn (22), is only justified for a closed system, when it is in principle impossible to know along which path the system propagates. The situation is different if the system is open, i.e., somehow interacts with its

⁸ Different measurement results actually correspond to different states into which the environment of the system (reservoir) can move as a result of the interaction with it. In the restricted path integral approach, however, the reservoir is not considered explicitly, and the results of measurement are described in terms characteristic of the system itself.

environment. In this case, the state of the environment changes due to the interaction, and it does this in a way that depends on the state of the system. This means that a continuous measurement of the system is taking place. The measurement provides partial information on what path the system has taken. In this case the path integral should be restricted to those paths compatible with the information obtained.

Thus a restricted path integral emerges. The idea of how to apply this integral to continuous measurements was briefly formulated by Feynman in his original work [40]. This approach was developed, conceptually and technically, in the work of the present author [57, 58, 41, 42, 4, 3] (see also [59–62]). Let us summarize some of its features briefly.

Let us denote the readout (result) of a continuous measurement by α . Suppose the system is subject to a continuous measurement and this is exactly the readout the measurement produced. Then, instead of Eqn (22), the evolution of the system is described by the propagator

$$U_t^\alpha(q'', q') = \int_{q'}^{q''} d[p] d[q] w_\alpha[p, q] \times \exp \left[\frac{i}{\hbar} \int_0^t dt (p\dot{q} - H(p, q, t)) \right], \quad (28)$$

where $w_\alpha[p, q]$ is a functional which depends on the path $w[p, q]$ in phase space and describes the information on the paths which is given by the measurement readout α . The latter point means that the functional is close in absolute value to unity for those paths which correspond to the measurement readout α and to zero for those which do not.

To clarify things, let us consider as an example a measurement which can be called the monitoring of the coordinate q . Suppose that the measurement is performed with a resolution of Δa and that the readout of the measurement is described by the curve $[a] = \{a(t)\}$ (which plays the role of α). Then to this readout there correspond paths $[q]$ lying in a certain corridor in configuration space. At the center of the corridor lies the curve $a(t)$ and the corridor width is $2\Delta a$. The phase space paths $[p, q] = [p][q]$ corresponding to a given measurement readout are characterized in exactly the same way: the path $[p]$ is arbitrary, and the path $[q]$ lies in the indicated corridor. The functional $w_{[q]}[p, q]$ representing such measurement readout is equal to unity for paths within the corridor and to zero for all the remaining paths. Below, a more general case of the monitoring of an arbitrary observable will be considered, and for this case the more realistic Gaussian approximation will be used.

In the general case the functional $w_\alpha[p, q]$ may assume complex values, and its absolute value lies between zero and unity. It is the functional $w_\alpha[p, q]$ by which the information given by the measurement readout α is characterized best. More precisely, the absolute value of the complex number $w_\alpha[p, q]$ characterizes the information obtained in the measurement while its argument corresponds to the back action of the measurement on the system being measured. If this argument is equal to zero, i.e., if the functional is real positive, then the measurement disturbs the state of the measured system in a minimal way (one cannot obtain information without introducing some disturbance into the system). If the argument is nonzero, i.e., the functional is complex, the measurement is a *non-minimally disturbing* one (see Section 3.3 and book [3] for more details).

The evolution of the system under measurement [as represented by the propagator (28)] thus depends on what readout α the measurement produced, i.e., on what state the system turned out to be in. Because measurement readouts may be different, the behavior of the measured system can be completely characterized only by the set of propagators (28) for all possible α . Therefore, each of these propagators is called partial.

If the evolution operator U_α is constructed as an integral operator with kernel (28), then the evolution of the system subject to measurement is described by the equations

$$|\psi_\alpha\rangle = U_\alpha |\psi_{\text{in}}\rangle, \quad \rho_\alpha = U_\alpha \rho_{\text{in}} U_\alpha^\dagger \quad (29)$$

and thus depends on the measurement readout α . If the measurement readout is not known, the evolution should be described by the total density matrix, which is obtained by integrating over all possible measurement readouts (non-selective description):

$$\rho = \int d\alpha \rho_\alpha = \int d\alpha U_\alpha \rho_{\text{in}} U_\alpha^\dagger. \quad (30)$$

It should be noted that the description of an open system by a state vector is an idealization: if various alternatives form a continuous set, a single alternative α forms a subset of measure zero and is never realized in practice. For example, in the measurement, the readings of any real measurement device enable one to separate only a certain interval \mathcal{A} of alternatives α , more or less narrow depending on the quality of the device. If the information about the measurement readout is limited to the fact that the interval \mathcal{A} is specified, then the evolution of the system, with this information taken into account, is described by the density matrix

$$\rho_{\mathcal{A}} = \int_{\mathcal{A}} d\alpha \rho_\alpha = \int_{\mathcal{A}} d\alpha U_\alpha \rho_{\text{in}} U_\alpha^\dagger. \quad (31)$$

The important thing to understand is that the width of the interval \mathcal{A} characterizes the purely classical uncertainty of the device, which has nothing to do with its quantum resolution as described by the width of the corridor along which the restricted path integral is taken (i.e., nothing to do with how the functionals w_α are structured). Let us clarify this difference as follows. The device, whose sensor in principle allows a measurement with an extremely high resolution, may have a poor detection system, whose output reading gives for the measured quantity a much worse estimate than the level of resolution would suggest. In this case the partially non-selective description (31) should be used. Furthermore, the detection system may be completely absent. Then the evolution of the system under measurement should be described by the total density matrix (30) — giving rise to a non-selective description of the measurement. This is precisely what happens if the measurement of a quantum system occurs in a spontaneous way, as a result of its interaction with the reservoir — as opposed to an organized way, with the aim of obtaining information about the system.

Strictly speaking, any real measurement should be described by density matrices of the type (31), i.e., the description should be partly non-selective. Under certain conditions, however, a selective description can be a good approximation (see Ref. [3] for details).

3.2 Monitoring of an observable

The expression (28) is the most general form of a restricted path integral, describing a continuous measurement of the most general kind. Consider now a special case in which measurement consists of continuously measuring the observable $\hat{A} = A(\hat{p}, \hat{q})$ (monitoring an observable). Suppose the monitoring yields a readout which is expressed by the function $a(t)$, i.e., as a readout of the measurement the value of the observable \hat{A} at time t is estimated as $a(t)$. Then the restricted path integral should be taken along the corridor of paths $[p(t), q(t)]$ such that for a path lying at the center of this corridor $A(t) = A(p(t), q(t)) = a(t)$. The width of the corridor depends on the resolution with which the measurement is being performed.

The exact definition of a restricted path integral consists of choosing a weight functional $w_{[a]}[p, q]$ dependent on $[a] = \{a(t') | 0 \leq t' \leq t\}$ as a measurement readout. The mathematically simplest choice, justified also from the physical point of view (see footnote 9), is the Gaussian functional

$$w_{[a]}[p, q] = \exp \left\{ \int_0^t dt [-\varkappa (A(p, q) - a(t))^2] \right\}. \quad (32)$$

Such a functional becomes smaller the farther (in the mean square sense) the curve $A(t) = A(p(t), q(t))$ is from the curve $a(t)$. The functional is positively defined and hence describes a *minimally disturbing* continuous measurement, the analogue of von Neumann's instantaneous measurement represented by a projector. Such a functional only suppresses in the path integral those paths that do not correspond to the information given by the measurement. The coefficient \varkappa specifies the resolution with which the measurement is being performed: the larger the coefficient, the more accurate the measurement.

Substitution of the functional (32) into the path integral in the partial propagator expression (28) reduces this integral to a form which is actually identical to the usual (non-restricted) Feynman integral (22) in which, however, instead of the original Hamiltonian $H(p, q)$ there is involved the effective Hamiltonian

$$H_{[a]}(p, q) = H(p, q) - i\hbar\varkappa(A(p, q) - a(t))^2, \quad (33)$$

with the imaginary term depending on the measurement readout $[a]$. Because the state vector arising under the action of the usual (Feynman) propagator satisfies the Schrödinger equation, in our case the same situation occurs — but with an effective Hamiltonian. Namely, the state vector

$$|\psi_t^{[a]}\rangle = U_t^{[a]}|\psi_0\rangle \quad (34)$$

satisfies the Schrödinger equation

$$i\dot{|\psi\rangle} = -\frac{i}{\hbar} \hat{H}_{[a]}|\psi\rangle = \left[-\frac{i}{\hbar} \hat{H} - \varkappa(\hat{A} - a(t))^2 \right] |\psi\rangle \quad (35)$$

with the effective Hamiltonian

$$\hat{H}_{[a]} = \hat{H} - i\hbar\varkappa(\hat{A} - a(t))^2. \quad (36)$$

The description of the evolution of the measured system which emerges in this way is selective, i.e., depends on the measurement readout $[a]$. In order to pass to a non-selective

description, it is necessary first to pass from the state vector to the equivalent density matrix,

$$\rho_t^{[a]} = |\psi_t^{[a]}\rangle\langle\psi_t^{[a]}|. \quad (37)$$

The description of the system by such a density matrix is, as before, selective. Now, however, we can integrate over all possible measurement readouts to obtain the total density matrix

$$\rho_t = \int d[a] |\psi_t^{[a]}\rangle\langle\psi_t^{[a]}|, \quad (38)$$

which no longer depends on the measurement readout (i.e., accounts for all possible alternative readouts). We have thereby a non-selective description of the evolution of the system being measured.

One can show [3, 63] that the total density matrix obtained in this way satisfies the equation

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\varkappa}{2} [\hat{A}[\hat{A}, \hat{\rho}]]. \quad (39)$$

This is an equation of the Lindblad type, Eqn (5). This equation has already been discussed in Section 2.3 and it has been shown that it leads to the decoherence phenomenon. In the present case decoherence manifests itself in that different values of the observable \hat{A} are characterized by probabilities instead of probability amplitudes (i.e., interference between them disappears and they acquire classical features). No wonder that this occurs in the present case because it was the aim of our entire effort to describe the measurement of the observable \hat{A} . In the following sections we will consider a more general measurement regime, which leads not only to decoherence but to dissipation as well.

There is an important point to make in conclusion. It has been shown above how, using restricted path integrals or complex Hamiltonians, a description of the continuous measurements of a quantum system can be derived from Feynman's form of quantum mechanics. This method of derivation has the advantage of being general and model-independent. It may, however, seem to be unreliable because it relies on a particular version of quantum mechanics — one proposed by Feynman — and in particular, on the theory of probability amplitudes which was extended significantly in Feynman's work [40, 43] (see also Ref. [64]).

Now the question arises: can the same conclusions be obtained by relying solely on the standard formalism of quantum mechanics? It has turned out they can. In Ref. [3, Ch. 8], it is shown by applying standard methods to a simple continuous measurement model (a two-level system subject to a series of short weak interactions) that the evolution of the model system is correctly described by means of restricted path integrals with a Gaussian weight functional⁹.

In Ref. [65] the same question is treated in its general formulation by looking for conditions under which the interaction of an open system with the environment

⁹ It seems the Gaussian functional appears each time when the continuous measurement consists of a large number of 'weak' measurements (i.e., those which provide very little information but, on the other hand, affect very weakly the system being measured). If this trend is indeed a general one (which is very plausible), then it represents the quantum analogue of the Central Limiting Theorem of the probability theory.

(reservoir) can be viewed as the measurement of the system and can be described in terms of restricted path integrals. It turns out that this is possible under very general assumptions, in particular, each time when the environment (reservoir) contains a large number of degrees of freedom. This shows that the description of open (in particular, dissipative) systems within the framework of the theory of continuous measurement is, in fact, general. We will address these arguments in Section 4. For now, we proceed to a more general model of monitoring, one which is capable of demonstrating both decoherence and dissipation.

3.3 Non-minimally disturbing measurement

If we add a pure imaginary term to the argument of the exponential in the functional (32), the same paths as before will be suppressed, meaning that such a functional describes the monitoring of the same observable and with the same resolution. However, in the presence of the imaginary term the disturbance experienced by the measured system is stronger than is needed to obtain information¹⁰. In this case we obtain the description of a *non-minimally disturbing* continuous measurement [3, 5].

Suppose that the imaginary term depends linearly on the measurement readout,

$$w_{[a]}[p, q] = \exp \left\{ \int_0^t dt \left[-\kappa(A(p, q) - a(t))^2 - \frac{i}{\hbar}(\lambda a(t)B(p, q) + C(p, q)) \right] \right\}. \quad (40)$$

This expression for the weight functional means that the variable \hat{A} is subject to monitoring and that the operators \hat{B} and \hat{C} determine the non-minimal part of the disturbance caused by the measurement. As we shall see later on, the operator \hat{C} leads to a change in the Hamiltonian of the system, resulting from the system being subject to measurement. The comment following Eqn (49) will present a more detailed interpretation of the operator \hat{B} for the special case of a harmonic oscillator.

If the weight functional of a restricted path integral is taken to be of the form (40), then the partial propagator describing the evolution of the measured system [provided the measurement yields the readout $a(t)$] is written in terms of the restricted path integral as

$$U_{[a]}(q, q', t) = \int d[p] \int_q^{q'} d[q] \exp \left\{ \int_0^t dt \left[\frac{i}{\hbar}(p\dot{q} - H(p, q)) - \kappa(A(p, q) - a(t))^2 - \frac{i}{\hbar}(\lambda a(t)B(p, q) + C(p, q)) \right] \right\}. \quad (41)$$

The corresponding evolution operator $U_{[a]}(t)$ allows one to express the state vector or the density matrix of the system at an arbitrary time instant provided the measurement readout is $a(t)$,

$$|\psi_{[a]}(t)\rangle = U_{[a]}(t) |\psi(0)\rangle, \quad \hat{\rho}_{[a]}(t) = U_{[a]}(t) \hat{\rho}(0) U_{[a]}^\dagger(t). \quad (42)$$

¹⁰ In the case of an instantaneous measurement the minimally disturbing measurement of a coordinate with a resolution Δq leads to an additional momentum uncertainty, $\Delta p = \hbar/2\Delta q$, whereas a non-minimally disturbing measurement can, in addition to that, transfer to the system a certain momentum whose value depends on the readout of the measurement.

The total density matrix is obtained by summing over all possible measurement readouts $[a]$, i.e.,

$$\hat{\rho}(t) = \int d[a] \hat{\rho}_{[a]}(t).$$

The integral over $[a]$ involved in this expression turns out to be Gaussian and therefore is evaluated exactly¹¹. This yields

$$\begin{aligned} \hat{\rho}(t) = & \int d[p', q'] \int d[p'', q''] \rho_0(q', q'') \\ & \times \exp \left[\int_0^t dt \left(\frac{i}{\hbar}(p'\dot{q}' - H' - C') \right. \right. \\ & - \frac{i}{\hbar}(p''\dot{q}'' - H'' - C'') - \frac{\kappa}{2}(A' - A'')^2 - \frac{\lambda^2}{8\kappa\hbar^2}(B' - B'')^2 \\ & \left. \left. + \frac{i\lambda}{2\hbar}(A' + A'')(B'' - B') \right) \right]. \quad (43) \end{aligned}$$

Here we use the notation $A' = A(p', q')$, $B'' = B(p'', q'')$, etc.

Differentiating the density matrix expression (43) with respect to time, one can derive a differential equation for the density matrix. In doing this it should be kept in mind that all singly (doubly) primed operators should be on the left (right) of the density matrix $\hat{\rho}$.

Another concern is that the operators \hat{A} and \hat{B} should be ordered correctly when they refer to the same time instant. The physical interpretation of the problem suggests a way in which to resolve this indeterminacy. Because the operator \hat{B} describes the back action (non-minimal disturbance) caused by the measurement of the operator \hat{A} , the operator \hat{A} should operate earlier than \hat{B} . Thus we obtain the equation for the density matrix of the measured system in the form

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} = & -\frac{i}{\hbar}[\hat{H} + \hat{C}, \hat{\rho}] - \frac{\kappa}{2}[\hat{A}, [\hat{A}, \hat{\rho}]] \\ & - \frac{\lambda^2}{8\kappa\hbar^2}[\hat{B}, [\hat{B}, \hat{\rho}]] - \frac{i\lambda}{2\hbar}[\hat{B}, [\hat{A}, \hat{\rho}]_+], \quad (44) \end{aligned}$$

where $[\ ,]_+$ denotes an anticommutator.

When rewritten in terms of the operator

$$\hat{L} = \hat{A} - \frac{i\lambda}{2\kappa\hbar} \hat{B} \quad (45)$$

and its conjugate, Eqn (44) becomes

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} = & -\frac{i}{\hbar} \left[\hat{H} + \hat{C} - \frac{i\kappa\hbar}{4}(\hat{L}^{\dagger 2} - \hat{L}^2), \hat{\rho} \right] \\ & - \frac{\kappa}{2}(\hat{L}^\dagger \hat{L} \hat{\rho} - 2\hat{L} \hat{\rho} \hat{L}^\dagger + \hat{\rho} \hat{L}^\dagger \hat{L}), \quad (46) \end{aligned}$$

showing that this is an equation of the type (5) — or of the Lindblad type — i.e., the evolution it represents maintains the positiveness of the density matrix. The initial Hamiltonian of the measured system is renormalized due to the measurement procedure.

¹¹ For the calculation of Gaussian functional integrals see, for example, Ref. [41].

3.4 A harmonic oscillator with friction

Let us now consider the special case of a harmonic oscillator,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2. \quad (47)$$

Suppose that the subject of the monitoring is the *momentum* operator and that the non-minimal disturbance is represented by the *coordinate* operator,

$$\hat{A} = \hat{p}, \quad \hat{B} = \omega \hat{q}. \quad (48)$$

Then the weight functional (40) takes the form [for $C(p, q) \equiv 0$]

$$w_{[a]}[p, q] = \exp \left\{ \int_0^t dt \left[-\varkappa (p - a(t))^2 - \frac{i}{\hbar} (\lambda \omega a(t) q) \right] \right\}. \quad (49)$$

The coefficient λ turns out to be dimensionless in this case.

The exponential of the observed coordinate \hat{q} (times a pure imaginary factor) is the displacement operator of the observed momentum \hat{p} . More precisely, the operator

$$\exp \left(-\frac{i}{\hbar} \delta p \hat{q} \right) \quad (50)$$

with an arbitrary factor δp displaces the momentum by δp . Hence, the non-minimal disturbance represented in Eqn (49) by a term proportional to \hat{q} manifests itself in that the momentum is displaced by an amount depending on its measured value.

In other words, in this case the momentum of the oscillator is being continuously measured and, as the measurement process goes on, is continuously subject to displacement, the amount of this displacement depending on the current readout of the measurement (i.e., the readout obtained at a given time instant). More precisely, the momentum is displaced by an amount proportional to its value (as estimated by the measurement). If $\lambda > 0$, a positive momentum acquires a negative increment, and a negative momentum gets a positive increment. Thus, the absolute value of the momentum always decreases. It is intuitively clear that such a process must cause the oscillator to attenuate — i.e., must lead to dissipation.

This type of process is involved in the laser cooling of an ion in a magnetic trap. In this case, an ion absorbs a laser radiation photon and then emits a photon in such a way that the momentum decreases in absolute value. At the same time, during the photon rescattering process information about the momentum of the ion remains in the environment — i.e., we are in fact dealing with the measurement of the momentum. If the photons interact frequently enough with the ions, the process can be described as a continuous, non-minimally disturbing measurement of the ion momentum which leads to dissipation. It is this dissipation which causes the decrease in ion energy (i.e., cools the ion).

Upon substitution of Eqn (48) the density matrix equation (46) takes the form

$$\begin{aligned} \frac{\partial \hat{\rho}}{\partial t} = & -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\varkappa}{2} [\hat{p}, [\hat{p}, \hat{\rho}]] \\ & - \frac{\lambda^2 \omega^2}{8\varkappa \hbar^2} [\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{i\lambda\omega}{2\hbar} [\hat{q}, [\hat{p}, \hat{\rho}]_+]. \end{aligned} \quad (51)$$

This equation is known as representing the Brownian motion of an oscillator [38]. In the present case we have obtained it as an equation describing the evolution of an oscillator with continuously measured momentum, the measurement process involving a momentum transfer which acts to decrease the absolute value of the momentum.

In actual fact, we have thus far never used a specific form of the oscillator's Hamiltonian, so that the equation obtained describes the non-minimally disturbing measurement of momentum in a one-dimensional system with an arbitrary potential. Comparing this equation with Eqn (11) derived from the Caldeira–Leggett model, we see that Eqn (51) differs by the presence of a term accounting for the decoherence of momentum¹² (the term with the double commutator of the density matrix and momentum; for a discussion of decoherence see Section 2.2). The appearance of this term is no surprise because we demanded from the very beginning that the system's momentum be measured, and introduced the parameter \varkappa precisely as a constant to determine the momentum resolution. We see, however, that in the density matrix equation there has also appeared a double commutator with a coordinate, implying that the evolution of the system also involves the decoherence (continuous measurement) of the system's coordinate with a resolution determined by the constant $\lambda^2 \omega^2 / 4\varkappa \hbar^2$.

This comes as no surprise, though, because the *continuous* measurement of momentum provides information on the coordinate, and vice versa. Intuitively, the measurement of the coordinate is easier to understand. Suppose we are continuously measuring the coordinate of a particle. The readout of the measurement looks like a corridor of paths in coordinate space — meaning that the particle is moving within this corridor. But this also gives some information on the particle's velocity at each time instant, with the implication that its velocity — and hence momentum — is also simultaneously being measured.

Now, the question may arise, which of the two equations, Eqn (11) or Eqn (51), describes more accurately what is going on? Besides the purely intuitive argument just above, there is more definitive evidence in favor of Eqn (51). The point is that [as Eqn (46) demonstrates] Eqn (51) belongs to the class of Lindblad equations, i.e., maintains the positiveness of the density matrix, whereas Eqn (11) does not possess this property.

It is interesting to find the energy shift due to the measurement procedure. Taking the shift from Eqn (46) for $\hat{C} = 0$ and using the definition (48), we obtain for the oscillator energy shift due to the influence of the measurement

$$\Delta \hat{H} = -\frac{i\varkappa \hbar}{4} (\hat{L}^{\dagger 2} - \hat{L}^2) = \frac{\lambda \omega}{4} (\hat{p} \hat{q} + \hat{q} \hat{p}), \quad (52)$$

which, when combined with the Hamiltonian (47), yields the effective (measurement-renormalized) Hamiltonian in the form

$$\begin{aligned} \hat{H}_{\text{eff}} = & \frac{1}{2m} (\hat{p}^2 + m^2 \omega^2 \hat{q}^2) + \frac{\lambda \omega}{4} (\hat{p} \hat{q} + \hat{q} \hat{p}) \\ = & \frac{1}{2m} \left(\hat{p} + \frac{\lambda \omega m}{2} \hat{q} \right)^2 + \frac{m}{2} \left(\omega^2 - \frac{\lambda^2 \omega^2}{4} \right) \hat{q}^2. \end{aligned} \quad (53)$$

¹² The dissipative term in these equations is the same [see the definition (54)].

If we set

$$\gamma = \frac{\lambda\omega}{2}, \quad (54)$$

then the new oscillator frequency involved in Eqn (53) is identical to the familiar expression

$$\Omega = \sqrt{\omega^2 - \gamma^2} \quad (55)$$

for the effective frequency of a classical oscillator with friction described by the equation

$$m\ddot{q} + 2m\gamma\dot{q} + m\omega^2q = F(t). \quad (56)$$

The expression (54) for the friction coefficient (relaxation coefficient) is also consistent with the form of Eqn (58) below. Besides, it shows that the dissipative term in Eqns (51) and (11) is the same.

Equations for the first moments (i.e., for the mean values)¹³ of the observables \hat{p} and \hat{q} follow from Eqn (51) as

$$\left\langle \frac{\partial \hat{p}}{\partial t} \right\rangle = -m\omega^2 \langle \hat{q} \rangle - \lambda\omega \langle \hat{p} \rangle, \quad (57)$$

$$\left\langle \frac{\partial \hat{q}}{\partial t} \right\rangle = \frac{1}{m} \langle \hat{p} \rangle.$$

The second moments (averages of the quadratic combinations of these observables) satisfy the equations

$$\begin{aligned} \frac{\partial}{\partial t} \langle \hat{p}^2 \rangle &= -m\omega^2 \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle - 2\lambda\omega \langle \hat{p}^2 \rangle + \frac{\lambda^2\omega^2}{4\kappa}, \\ \frac{\partial}{\partial t} \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle &= -\lambda\omega \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle + 2 \left(\frac{1}{m} \langle \hat{p}^2 \rangle - m\omega^2 \langle \hat{q}^2 \rangle \right), \\ \frac{\partial}{\partial t} \langle \hat{q}^2 \rangle &= \frac{1}{m} \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle + \kappa\hbar^2. \end{aligned} \quad (58)$$

The last term in the third of Eqns (58) is proportional to \hbar^2 and hence is purely quantum. This term is necessary for the system's evolution to have a Lindblad character, but it is not easy to interpret [38, 66]. In the framework of the quantum theory of measurements, however, this term is interpreted in a natural way as a consequence of the uncertainty principle (see Ref. [56] and Ref. [3], Section 4.3.4, where the dual situation with continuous coordinate measurement is considered). To see this, note that over time, as a result of its being continuously measured, the momentum becomes known with less and less uncertainty: $\Delta p_t^2 = 1/\kappa t$ (see Ref. [3], Section 5.1.3). At the same time, the uncertainty of the coordinate increases due to the uncertainty relation. For small enough times, the increase in the coordinate uncertainty is described quantitatively correctly by the quantum term in Eqn (58).

The system of equations (58) has the stationary solution

$$\begin{aligned} \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle &= -m\kappa\hbar^2, \\ \frac{1}{2m} \langle \hat{p}^2 \rangle &= \frac{\lambda\omega}{16\kappa m} + \frac{m\omega\kappa\hbar^2}{4\lambda}, \\ \frac{1}{2} m\omega^2 \langle \hat{q}^2 \rangle &= \frac{\lambda\omega}{16\kappa m} + \frac{m\omega\kappa\hbar^2}{4\lambda} + \frac{1}{4} m\lambda\omega\kappa\hbar^2, \end{aligned} \quad (59)$$

which shows where the measured system stabilizes itself as it evolves. One can see that the solution (59) is consistent with the classical virial theorem (the average kinetic energy should be equal to the average potential energy) to within $m\lambda\omega\kappa\hbar^2/4$, a quantity whose value disappears in the classical limit.

From equations (59) we obtain for the mean value of the free Hamiltonian

$$\langle \hat{H} \rangle = \frac{1}{2} \left\langle \frac{1}{m} \hat{p}^2 + m\omega^2 \hat{q}^2 \right\rangle = \frac{\lambda\omega}{8m\kappa} + \left(\frac{1}{4} + \frac{1}{2\lambda^2} \right) m\lambda\omega\kappa\hbar^2. \quad (60)$$

This gives for the mean value of the renormalized Hamiltonian the expression

$$\langle \hat{H}_{\text{eff}} \rangle = \frac{\hbar\omega}{2} \left(\frac{m\kappa\hbar}{\lambda} + \frac{\lambda}{4m\kappa\hbar} \right), \quad (61)$$

which depends on the ratio of two dimensionless parameters: the constant $m\kappa\hbar$ characterizing the strength of the measurement (its resolution), and the constant λ determining the value of the non-minimal disturbance of the measured system that leads to the system's dissipation.

It is easy to see that the quantity (61) reaches its minimum value $\hbar\omega/2$ at

$$\lambda = 2m\kappa\hbar \quad (62)$$

and increases indefinitely as the ratio $\lambda/m\kappa\hbar$ tends to zero or infinity. Because the finite energy of the oscillator is a minimum under the condition (62), the relation (62) may be called the condition for the most efficient cooling by measurement.

3.5 An oscillator at thermal equilibrium

Suppose that the reservoir performing a measurement on an oscillator is a thermal bath at a given temperature T and that the oscillator is in equilibrium with it (which is possible if the oscillator is in a stationary state). Then the density matrix of the oscillator corresponds to the Gibbs distribution over its energies. Applying this distribution to the calculation of the average energy or, which is the same, of the mean value of the quantum number $\hat{n} = a^\dagger a$, we obtain

$$\langle \hat{n} \rangle = \frac{1}{\exp(\hbar\omega/kT) - 1}, \quad (63)$$

$$\langle \hat{H} \rangle = \hbar\omega \left\langle \hat{n} + \frac{1}{2} \right\rangle = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2kT}.$$

Using Eqn (60), for the coefficient λ this gives the equation

$$\lambda \left(\frac{1}{4m\kappa\hbar} + \frac{m\kappa\hbar}{2} \right) + \frac{m\kappa\hbar}{\lambda} = \coth \frac{\hbar\omega}{2kT}. \quad (64)$$

From this equation the coefficient λ is expressed in terms of the reservoir temperature and the strength of the measurement κ . This means that the degree of the non-minimal disturbance the oscillator experiences when being measured depends on the temperature of the reservoir. Note that the formula (64) is derived under the assumption that the oscillator is in thermal equilibrium with the reservoir.

Equation (64) has a solution only under the condition that

$$\sqrt{1 + 2(m\kappa\hbar)^2} < \coth \frac{\hbar\omega}{2kT}. \quad (65)$$

¹³ The mean value of operator \hat{A} in a state $\hat{\rho}$ is defined as $\langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A})$.

At high temperatures ($kT \gg \hbar\omega$) the solution under the assumption $m\kappa\hbar \ll 1$ has the form

$$\lambda = 4m\kappa\hbar \frac{2kT}{\hbar\omega} = \frac{8m\kappa kT}{\omega}. \quad (66)$$

Planck's constant does not enter this relation, as indeed should be the case in the classical regime. At low temperatures ($kT \ll \hbar\omega$), the solution exists for

$$2(m\kappa\hbar)^2 < \coth \frac{\hbar\omega}{2kT} - 1 \quad (67)$$

and has the form

$$\lambda = 2m\kappa\hbar, \quad (68)$$

which is identical to the condition for the most efficient cooling at low dissipation (see the end of Section 3.4).

Recall that the above conclusions are made under the assumption of thermal equilibrium between the oscillator and the reservoir. There are likely to be other relations between the parameters λ and κ — but ones for which equilibrium is not possible.

Thus we have obtained a correct description of the Brownian motion of a quantum oscillator in terms of continuous measurements. All conclusions were reached in the framework of a specific model of continuous measurement. In this model, momentum is subject to continuous monitoring, being disturbed in a non-minimal way in the process [see the comment following Eqn (49)]. Also, we have shown — albeit in less detail — that the non-minimally disturbing monitoring of an observable leads to dissipation when described by restricted path integrals with a weight functional of the form (40).

In the general case, the influence of the reservoir is described by more complex restricted path integrals (see Section 4). However, an important class of influences can be represented as the monitoring of one or several observables by introducing additional terms into the argument of the exponential in the functional (40). The equations for the state vector and density matrix of a system subject to such an influence is readily obtained similarly to the way it was done in Section 3.3.

An interesting question is how to realize in practice an influence described formally by this or that weight functional in restricted path integral formulation. One of the ways to solve this question is demonstrated in Ref. [3, Ch. 8] using a two-level system as an example. The 'Gaussian' monitoring of an arbitrary observable of such a system is represented by a long series of short interactions with an auxiliary system (a meter). One can also expect that in the case of an arbitrary system a long series of weak short interactions (with statistically independent outcomes) will always lead to an evolution described by restricted path integrals with a Gaussian functional. This statement, if true under sufficiently general assumptions, is the quantum analogue of the Central Limiting Theorem of probability theory.

4. Open system viewed as being measured

In Section 3 we showed that the continuous measurement of a quantum system leads to its dissipation. Now we will show (following Ref. [65]) that the reverse is actually also true: any dissipative system can be considered as a system subject to continuous measurement by its environment (reservoir). This

follows from the fact that open quantum systems can, under very general assumptions, be correctly described as being continuously measured by their environment. Their evolution in this case is represented by restricted path integrals. This means that the restricted path integral approach of Section 3 to the theory of dissipative systems is a general one.

Usually, the evolution of an open quantum system is described non-selectively, i.e., without considering in what state the system's environment finds itself. The state of the open system in this case is represented by its density matrix. The variation of the density matrix with time is generally described, to good approximation, by an equation containing the time derivative of the density matrix. In the literature, this equation is called the master equation. The most general form of the master equation (5) that ensures the positiveness of the density matrix was derived by Lindblad [34] from the requirement that the density matrix maintain its positiveness.

The same process can be represented in a selective way. The state of an open system is then described by a state vector (wave function), which depends on what state the environment of the system under study finds itself in as a result of interaction with the system. If an open system is interpreted as a measured one, and its interaction with the environment as the measurement of the system, then the selective description of the measured system depends on the readout of the measurement, i.e., on what information about the system is being recorded (encoded) in the state of its environment. A selective description is more detailed, and it is possible to make a transition from it to a non-selective description by summing over all possible states of the environment (over all possible readouts of the measurement).

In the quantum theory of continuous measurements the selective description of an open (continuously measured) system is given by restricted path integrals, in terms of which the *partial propagators* of the measured system are expressed. In the Markovian approximation the process can be represented by the Schrödinger equation with a complex Hamiltonian (see Sections 3.1–3.3). Let us now find out under what conditions an open quantum system can be considered as being continuously measured and can be described in terms of restricted path integrals.

We will start with the most general description of an open system in terms of the Feynman–Vernon influence functional [48] and then introduce the expansion of the influence functional in *partial influence functionals*. Thereby, the evolution of the system is expanded in 'partial evolutions'. We will then figure out under what conditions various partial evolutions decohere (i.e., interference effects between them become insignificant). The decoherence condition generalizes the familiar Gell-Mann–Hartle condition for the consistency of 'quantum histories' [24].

Analysis shows that a set of decohering evolutions exists if the environment of the system (reservoir) contains a large number of degrees of freedom, i.e., is macro- or mesoscopic. Finally, we will see that under the decoherence condition, the restricted path integral description of the process is correct, i.e., an open system is correctly described as being continuously measured. Partial evolutions then correspond to different measurement readouts and are characterized by probabilities rather than probability amplitudes.

4.1 Evolution of an open system

Let us consider an open system \mathcal{S} which interacts with its environment \mathcal{E} . We will consider first the evolution of a

composite system $\mathcal{S} + \mathcal{E}$ and interpret this evolution in terms of a set of ‘partial evolutions’. The set of partial evolutions can be chosen with a large degree of arbitrariness. Our aim will be to formulate such conditions for the choice of the set which would allow us to interpret the evolution as the description of the continuous measurement of the system \mathcal{S} by its environment \mathcal{E} , and the partial evolutions, as corresponding to alternative measurement readouts.

Note that the environment \mathcal{E} may be a measurement device specially designed to perform the continuous measurement of the system \mathcal{S} , but it may as well be a natural surrounding interacting with the system \mathcal{S} in an uncontrolled way. However, even in the latter case the process can be interpreted as measurement because, in the way the environment \mathcal{E} evolves, certain information about the evolution of the system \mathcal{S} is ‘recorded’ (i.e., encoded in some way). In the case where the environment of the system is a natural surrounding medium and is not controlled, it is often called a reservoir. However, we will use both terms, environment and reservoir, as equivalent in either situation because, from the point of view of the influence of the system \mathcal{E} on our system \mathcal{S} , there is no significant difference between these situations.

4.1.1 Evolution of a system and of its environment. The composite system consisting of a system \mathcal{S} and its environment \mathcal{E} is assumed to be closed (isolated). Let the Hamiltonian of this system be given by

$$H_{\text{tot}}(p, q, P, Q) = H(p, q) + H_{\mathcal{E}}(P, Q) + H_I(p, q, P, Q), \quad (69)$$

where the Hamiltonians H , $H_{\mathcal{E}}$, and H_I describe the system of interest \mathcal{S} , its environment, and the interaction between them, respectively. (All observables featured in Eqn (69) may be multicomponent, and the Hamiltonians, time-dependent.)

The evolution of the composite system $\mathcal{S} + \mathcal{E}$ is given by the formula

$$\mathcal{R} = U^{\text{tot}} \mathcal{R}_{\text{in}} (U^{\text{tot}})^{\dagger}, \quad (70)$$

where \mathcal{R} is the density matrix of the composite system at an arbitrary instant of time, \mathcal{R}_{in} is the same at the initial instant, and U^{tot} is the evolution operator of the composite system over the corresponding time interval.

The evolution operator can be expressed in the form of the Feynman path integral as¹⁴

$$U^{\text{tot}} = \int d[p, q] \int d[P, Q] \times \exp \left\{ \frac{i}{\hbar} \left[A[p, q] + A_{\mathcal{E}}[P, Q] - \int_0^t dt H_I(p, q, P, Q) \right] \right\}, \quad (71)$$

where we introduced the notation

$$A[p, q] = \int_0^t dt [p\dot{q} - H(p, q)], \quad (72)$$

$$A_{\mathcal{E}}[P, Q] = \int_0^t dt [P\dot{Q} - H_{\mathcal{E}}(P, Q)].$$

The evolution of the subsystem \mathcal{S} can be represented by a reduced density matrix ρ obtained from \mathcal{R} by taking the trace

over the degrees of freedom of the subsystem \mathcal{E} ,

$$\rho = \text{Tr}_{\mathcal{E}} \mathcal{R} = \text{Tr}_{\mathcal{E}} [U^{\text{tot}} \mathcal{R}_{\text{in}} (U^{\text{tot}})^{\dagger}]. \quad (73)$$

4.1.2 Expansion of the evolution operator. To proceed further we need the expansion of the evolution operator U^{tot} in a sum (more precisely, an integral) of *partial evolution operators* U_{α}^{tot} :

$$U^{\text{tot}} = \int d\alpha U_{\alpha}^{\text{tot}} \quad (74)$$

(later on, the alternatives α will, under certain conditions, be interpreted as the alternative readouts of continuous measurement). The partial evolution operators entering the expansion (74) can be defined as

$$U_{\alpha}^{\text{tot}} = \int d[p, q] \int d[P, Q] W_{\alpha}[P, Q] \times \exp \left\{ \frac{i}{\hbar} \left[A[p, q] + A_{\mathcal{E}}[P, Q] - \int_0^t dt H_I(p, q, P, Q) \right] \right\}, \quad (75)$$

where the *weight functionals*¹⁵ $W_{\alpha}[P, Q]$ form the expansion of unity, i.e., the relation

$$\int d\alpha W_{\alpha}[P, Q] = 1 \quad (76)$$

holds, with the integration in a certain measure $d\alpha$ over the set of all possible alternatives α .

Using the expansion (74), one can write the density matrix \mathcal{R} of the composite system $\mathcal{S} + \mathcal{E}$ as the integral of *partial density matrices* representing ‘partial evolutions’:

$$\mathcal{R} = \int d\alpha d\beta \mathcal{R}_{\alpha\beta}, \quad \mathcal{R}_{\alpha\beta} = U_{\alpha}^{\text{tot}} \mathcal{R}_{\text{in}} (U_{\beta}^{\text{tot}})^{\dagger}. \quad (77)$$

If the density matrix of the composite system is expanded as in Eqn (77), this determines the corresponding expansion of the reduced density matrix for the subsystem \mathcal{S} ,

$$\rho = \int d\alpha d\beta \rho_{\alpha\beta}, \quad \rho_{\alpha\beta} = \text{Tr}_{\mathcal{E}} \mathcal{R}_{\alpha\beta}. \quad (78)$$

4.2 Decoherence

Let us formulate the condition under which the alternatives denoted by α are characterized not by probability amplitudes but by probabilities themselves, so that no interference effects appear between the alternatives. This is necessary for the alternatives α to be classical (or, in other terminology, incompatible). Only under this condition can such alternatives be interpreted as the readouts of a measurement because alternative measurement readouts should, by definition, be classical (incompatible) — i.e., cannot interfere.

4.2.1. Decoherence of a system and of its environment. The total trace of partial density matrices (total meaning taken over all degrees of freedom) yields the set of *generalized decoherence functionals*

$$P_{\alpha\beta} = \text{Tr} \mathcal{R}_{\alpha\beta}, \quad \int d\alpha d\beta P_{\alpha\beta} = 1. \quad (79)$$

¹⁴ We use the integral over paths $[p, q, P, Q]$ in the phase space of the composite system; accordingly, the action is expressed in terms of the Hamiltonian of this system.

¹⁵ Note that weight functionals [for example, in Eqns (79) and (80)] can, generally speaking, be chosen in a more general form $W_{\alpha}[p, q, P, Q]$, but here we only need the special case $W_{\alpha}[P, Q]$.

These functionals are analogues of the decoherence functionals introduced by Gell-Mann and Hartle [24]. The generalized decoherence functionals correspond to the alternatives α , which in their turn are represented by weight functionals W_α ('path corridors'). Their analogues, Gell-Mann and Hartle's decoherence functionals, correspond instead to 'quantum histories', i.e., to sequences of projectors¹⁶.

By analogy with Gell-Mann and Hartle's definition of the consistency condition, we will say that alternatives α decohere if the *generalized consistency condition*

$$P_{\alpha\beta} = \text{Tr} \mathcal{R}_{\alpha\beta} = \delta(\alpha, \beta) P_\alpha, \quad \int d\alpha P_\alpha = 1 \quad (80)$$

holds. Here, $\delta(\alpha, \beta)$ is the delta-function defined with respect to the measure $d\alpha$ used in the integral (76)¹⁷. Equation (80) implies that *alternatives α decohere*, i.e., that they are characterized by *probabilities* P_α (more precisely, by probability densities with respect to the measure $d\alpha$), rather than by probability amplitudes. This is necessary for the set $\{\alpha\}$ to be treated as a set of *classical alternatives*. The condition (80) is usually only approximately satisfied.

4.2.2 Environment-induced decoherence. In the general case decoherence in the composite system $\mathcal{S} + \mathcal{E}$ is due equally to the properties of its component parts, \mathcal{S} and \mathcal{E} . Sometimes, however, in particular in the case of measurement, it is a consequence of the special properties of the environment \mathcal{E} only, but not of the system \mathcal{S} . It is precisely this situation which interests us. It is intuitively clear that this should occur if the environment contains a large number of degrees of freedom, i.e., is macro- or at least mesoscopic. However, decoherence can be caused by the special properties of the system \mathcal{E} (the orthogonality of propagation amplitudes of the subsystem \mathcal{E} corresponding to different alternatives) instead of by a large number of degrees of freedom.

A situation in which decoherence is caused only by the properties of the environment can be characterized mathematically as follows: the delta function $\delta(\alpha, \beta)$ [whose presence in Eqn (80) means decoherence] arises not only after taking the full trace, as in Eqn (80), but already after taking the trace over the degrees of freedom of the environment. This means that the relation

$$\rho_{\alpha\beta} = \text{Tr}_{\mathcal{E}} \mathcal{R}_{\alpha\beta} = \delta(\alpha, \beta) \rho_\alpha \quad (81)$$

should hold with certain operators ρ_α . These operators can be called decoherent partially reduced density matrices.

If the condition (81) is satisfied, then, instead of Eqn (78), the total reduced density matrix is expanded in terms of the partial density matrices as follows:

$$\rho = \int d\alpha \rho_\alpha. \quad (82)$$

This expansion is easily obtained by using the condition (81) and delta function properties (see footnote 17) in Eqn (78).

¹⁶ 'Quantum histories', as defined in Ref. [24], can be viewed as special cases of path corridors defined by weight functionals in path integrals, but the weight functionals in this case will be singular ones concentrated in a discrete set of time instants.

¹⁷ This means that $\delta(\alpha, \beta) = \delta(\beta, \alpha)$ and $\int \delta(\alpha, \beta) f(\beta) d\beta = f(\alpha)$.

The condition (81) is stronger than the (generalized) consistency condition (80) and can be called a *condition for decoherence by environment*. It can be expected that the condition (81) is approximately satisfied if the environment contains a large number of degrees of freedom, but not in this case only. We will show later that a somewhat stronger condition allows us to represent the evolution of the system \mathcal{S} as resulting from the system being continuously measured by its environment \mathcal{E} . The alternatives α can then be interpreted as the *readouts of measurement* which arise in the course of the continuous measurement.

4.3 Partial influence functionals and restricted path integrals

We will now proceed by introducing the Feynman–Vernon influence functional, expanding it into the integral of partial influence functionals, and then expressing in terms of the partial influence functionals the (earlier introduced) condition for decoherence by environment as well as a certain stronger decoherence condition. We will then be in a position to describe the evolution of an open system \mathcal{S} in terms of restricted path integrals. After that the process can be interpreted as a continuous measurement.

4.3.1 Partial influence functionals. By means of Eqn (71), the reduced density matrix (73) can be expressed in the form of a multiple path integral. Assuming that the initial density matrix of the composite system is factorized,

$$\mathcal{R}_{\text{in}}(q, Q|\bar{q}, \bar{Q}) = \mathcal{R}_{\text{in}}^{\mathcal{E}}(Q, \bar{Q}) \rho_{\text{in}}(q, \bar{q}), \quad (83)$$

and carrying out the functional integration in two stages, one can write the reduced density matrix as

$$\begin{aligned} \rho(q', \bar{q}') &= \int dq \int d\bar{q} \int_q^{q'} d[p, q] \int_{\bar{q}}^{\bar{q}'} d[\bar{p}, \bar{q}] \\ &\times \exp \left\{ \frac{i}{\hbar} (A[p, q] - A[\bar{p}, \bar{q}]) \right\} F[p, q|\bar{p}, \bar{q}] \rho_{\text{in}}(q, \bar{q}), \quad (84) \end{aligned}$$

where

$$\begin{aligned} F[p, q|\bar{p}, \bar{q}] &= \int dQ' \int dQ \int d\bar{Q} \int_Q^{Q'} d[P, Q] \\ &\times \int_{\bar{Q}}^{\bar{Q}'} d[\bar{P}, \bar{Q}] \mathcal{R}_{\text{in}}^{\mathcal{E}}(Q, \bar{Q}) \\ &\times \exp \left\{ \frac{i}{\hbar} \left[A_{\mathcal{E}}[P, Q] - \int_0^t dt H_I(p, q, P, Q) \right] \right\} \\ &\times \exp \left\{ -\frac{i}{\hbar} \left[A_{\mathcal{E}}[\bar{P}, \bar{Q}] - \int_0^t dt H_I(\bar{p}, \bar{q}, \bar{P}, \bar{Q}) \right] \right\} \quad (85) \end{aligned}$$

is the Feynman–Vernon *influence functional* [48] in the phase space path integral representation.

The partial reduced density matrices (78) are now expressed as

$$\begin{aligned} \rho_{\alpha\beta}(q', \bar{q}') &= \int dq \int d\bar{q} \int_q^{q'} d[p, q] \int_{\bar{q}}^{\bar{q}'} d[\bar{p}, \bar{q}] \\ &\times \exp \left\{ \frac{i}{\hbar} (A[p, q] - A[\bar{p}, \bar{q}]) \right\} F_{\alpha\beta}[p, q|\bar{p}, \bar{q}] \rho_{\text{in}}(q, \bar{q}) \quad (86) \end{aligned}$$

in terms of the *partial influence functionals*

$$\begin{aligned}
F_{\alpha\beta}[p, q|\bar{p}, \bar{q}] &= \int dQ' \int dQ \int d\bar{Q} \int_Q^{Q'} d[P, Q] \int_{\bar{Q}}^{Q'} d[\bar{P}, \bar{Q}] \\
&\times W_\alpha[P, Q] \mathcal{R}_{\text{in}}^\mathcal{E}(Q, \bar{Q}) W_\beta^*[\bar{P}, \bar{Q}] \\
&\times \exp \left\{ \frac{i}{\hbar} \left[A_\mathcal{E}[P, Q] - \int_0^t dt H_I(p, q, P, Q) \right] \right\} \\
&\times \exp \left\{ -\frac{i}{\hbar} \left[A_\mathcal{E}[\bar{P}, \bar{Q}] - \int_0^t dt H_I(\bar{p}, \bar{q}, \bar{P}, \bar{Q}) \right] \right\}. \quad (87)
\end{aligned}$$

The total influence functional F is equal to the integral of the partial influence functionals $F_{\alpha\beta}$ by analogy with Eqns (77) and (78).

The decoherence-by-environment condition (81) means that the partial influence functional $F_{\alpha\beta}$ is diagonal, i.e., contains the delta function $\delta(\alpha, \beta)$ as a factor. We will present arguments in favor of the assertion that under sufficiently general assumptions the stronger condition,

$$F_{\alpha\beta}[p, q|\bar{p}, \bar{q}] = \delta(\alpha, \beta) w_\alpha[p, q] w_\beta^*[\bar{p}, \bar{q}], \quad (88)$$

is fulfilled. The condition (88) can be called the *condition for decoherence of partial influence functionals*. In Section 4.3.3 it is shown that this condition leads to the description of the system \mathcal{S} in terms of restricted path integrals, so that the influence of the environment on the system can be treated as a continuous measurement.

4.3.2 Analysis of the alternative-decoherence condition. The condition (88) is (approximately) satisfied if the alternatives α defined by the weight functionals W_α [see Eqn (75)] are chosen correctly, so that the sets (path corridors) $[P, Q]$ determined by these functionals are sufficiently — but not too — wide¹⁸.

We denote those corridors of paths $[P, Q]$, which are determined by functionals W_α , as $\alpha_\mathcal{E}$. The corridors should be chosen in such a way that (1) if the corridor does not contain any classical trajectory (of the system \mathcal{E}), then the integral along it is negligibly small; (2) if the corridor does contain a certain classical trajectory, then the integral along the corridor is well approximated by the exponential of the action along this trajectory (stationary phase approximation).

Such requirements mean that the system \mathcal{E} can, to a good approximation, be treated as classical, and each of its path corridors $\alpha_\mathcal{E}$ as a quantum image corresponding to the classical notion of motion along a classical trajectory. If the system \mathcal{E} contains a sufficiently large number of degrees of freedom — i.e., is macro- or at least mesoscopic — then it is possible to find corridors $\alpha_\mathcal{E}$ satisfying these conditions.

If the conditions (1) and (2) are satisfied, there is only one case in which the quantity $F_{\alpha\beta}$ is not negligibly small: if both corridors $\alpha_\mathcal{E}$ and $\beta_\mathcal{E}$ contain classical trajectories and all these classical trajectories are close to one another. Hence, the quantity $F_{\alpha\beta}$ is negligibly small each time when one of the corridors $\alpha_\mathcal{E}$, $\beta_\mathcal{E}$ contains no classical trajectories at all or when both do contain classical trajectories, but

those of them belonging to $\alpha_\mathcal{E}$ differ strongly from those belonging to $\beta_\mathcal{E}$ ¹⁹.

These features are approximately represented by the formula (88). The functional $w_\alpha[p, q]$ in it turns out to be negligibly small if the corresponding set $\alpha_\mathcal{E}$ of paths of the environment — the set determined by the weight functional $W_\alpha[P, Q]$ — contains no classical trajectories.

Let us be more specific about what we have said, and introduce specific formulas for those partial influence functionals which are not negligibly small — i.e., for such $F_{\alpha\beta}$ for which corridors $\alpha_\mathcal{E}$ and $\beta_\mathcal{E}$ (determined by the functionals $W_\alpha[P, Q]$ and $W_\beta[P, Q]$, respectively) both contain classical trajectories. We have assumed that the corridors are wide enough. For this reason, the path integrals in the expression (87) can be approximated by the exponentials of the classical action as calculated along these classical trajectories:

$$\begin{aligned}
F_{\alpha\beta}[p, q|\bar{p}, \bar{q}] &\approx \int_{I_{\text{in}}(\alpha) \cup I_{\text{in}}(\beta)} dQ' \int_{I_{\text{in}}(\alpha)} dQ \int_{I_{\text{in}}(\beta)} d\bar{Q} \mathcal{R}_{\text{in}}^\mathcal{E}(Q, \bar{Q}) \\
&\times \exp \left\{ \frac{i}{\hbar} [S_{\text{cl}}(Q, Q', [p, q]) - S_{\text{cl}}(\bar{Q}, Q', [\bar{p}, \bar{q}])] \right\}. \quad (89)
\end{aligned}$$

Here $I_{\text{in}}(\alpha)$ [$I_{\text{in}}(\beta)$] is the set of initial (end) points for the classical trajectories belonging to the corridor $\alpha_\mathcal{E}$. The classical action $S_{\text{cl}}(Q, Q', [p, q])$ depends on the initial and end points Q, Q' of the corresponding classical trajectory. The dependence on $[p, q]$ arises because according to Eqn (87) this path determines the force acting on the system \mathcal{E} , so that the shape of the classical trajectory of the system depends on $[p, q]$. The same is true for $S_{\text{cl}}(\bar{Q}, Q', [\bar{p}, \bar{q}])$.

The structure of the double path integral (87) allows one to consider the integral as a single path integral — but along trajectories that are ‘closed in time’ (i.e., go from the initial instant of time to time instant t and then back to the initial instant again). In the part of the integral corresponding to the reversed direction of time, the signs of the momentum P and velocity \dot{Q} should be changed.

For this reason, the classical trajectory which, during the time interval $[0, t]$, goes from point Q to point Q' , must then come back close to the initial point Q during the time period $[t, 0]$. Hence, points Q and \bar{Q} in the integral (87) must be close to one another — otherwise the value of the integral will be negligibly small²⁰. It may be taken approximately that $\bar{Q} = Q$. Then Eqn (89) takes the form

$$\begin{aligned}
F_{\alpha\beta}[p, q|\bar{p}, \bar{q}] &\propto \int_{I_{\text{in}}(\alpha) \cup I_{\text{in}}(\beta)} dQ' \int_{I_{\text{in}}(\alpha) \cup I_{\text{in}}(\beta)} dQ \mathcal{R}_{\text{in}}^\mathcal{E}(Q, Q) \\
&\times \exp \left\{ \frac{i}{\hbar} [S_{\text{cl}}(Q, Q', [p, q]) - S_{\text{cl}}(Q, Q', [\bar{p}, \bar{q}])] \right\}. \quad (90)
\end{aligned}$$

With Eqn (90), we can justify the approximate expression (88) for partial influence functionals and improve the relation between the weight functionals $W_\alpha[P, Q]$ (or the corresponding path corridors $\alpha_\mathcal{E}$) for the environment \mathcal{E} and the functionals $w_\alpha[p, q]$ (and the corresponding corridors, to be denoted by α) for the system \mathcal{S} . Let us carry out this improvement now.

¹⁸ The sets in question are sets of paths $[P, Q]$ in the phase space of the environment \mathcal{E} . The sets (corridors) of paths $[p, q]$ in the phase space of the system \mathcal{S} , which are described by weight functionals $w_\alpha[p, q]$, may prove to be arbitrarily narrow.

¹⁹ Due to the assumptions made concerning the properties of the corridors, classical trajectories in one and the same corridor cannot differ much from one another.

²⁰ Points Q and \bar{Q} coincide exactly if paths $[p, q]$ and $[\bar{p}, \bar{q}]$ coincide.

Note first the regions of integration in Eqn (90), $I_{\text{in}}(\alpha) \cup I_{\text{in}}(\beta)$ and $I_{\text{fin}}(\alpha) \cup I_{\text{fin}}(\beta)$; it is clear that the integral is nonzero only if the alternatives α and β are very close to one another. This is approximately expressed by the delta function $\delta(\alpha, \beta)$.

We next consider the dependence of the classical trajectories, along which the classical actions in Eqn (90) are calculated, on the paths $[p, q]$ and $[\bar{p}, \bar{q}]$, respectively. In doing so, we note that the path $[p, q]$ ($[\bar{p}, \bar{q}]$) of the system \mathcal{S} determines the force acting on the system \mathcal{E} and thereby determines this latter's classical trajectory lying in the corridor $\alpha_{\mathcal{E}}$ ($\beta_{\mathcal{E}}$); see the comment following Eqn (89). The expression (90) is written under the assumption that the paths $[p, q]$ and $[\bar{p}, \bar{q}]$ are such that both classical trajectories are close to the middle of the corresponding corridors $\alpha_{\mathcal{E}}$, $\beta_{\mathcal{E}}$.

Let us denote by $[p_{\alpha}, q_{\alpha}]$ that path of the system \mathcal{S} in the choice of which the classical trajectory of the system \mathcal{E} lies in the middle of the corridor $\alpha_{\mathcal{E}}$. If the alternatives α and β coincide with one another and if the paths $[p, q]$ and $[\bar{p}, \bar{q}]$ coincide with $[p_{\alpha}, q_{\alpha}]$, the expression (90) is real and has the maximum absolute value. If $[p, q]$ and $[\bar{p}, \bar{q}]$ deviate from $[p_{\alpha}, q_{\alpha}]$, the expression (90) is not true; we should return to Eqn (87) instead, with the integral there becoming negligibly small. The factors $w_{\alpha}[p, q]$ and $w_{\beta}[\bar{p}, \bar{q}]$ in Eqn (88) become smaller and smaller in absolute value as the path deviation increases, and they can become complex. The factor $w_{\alpha}[p, q]$ ($w_{\beta}[\bar{p}, \bar{q}]$) becomes negligibly small when the path $[p, q]$ ($[\bar{p}, \bar{q}]$) is such that the corridor $\alpha_{\mathcal{E}}$ ($\beta_{\mathcal{E}}$) contains no classical trajectories.

4.3.3 Derivation of restricted path integrals. Suppose that the alternatives α are chosen correctly (in accordance with the requirements formulated in Section 4.3.2), so that the condition (88) is fulfilled. Then the alternatives α are decoherent, i.e., can be considered classical. It is easy to show that in this case the system \mathcal{S} evolves in exact correspondence with the way the evolution of a continuously measured system is described in the restricted path integral approach (see Sections 3.1 and 3.3). The alternatives α then play the role of different *measurement readouts*.

To see this, note that if the condition (88) is fulfilled then, in accord with Eqn (86), the partial density matrices of the system \mathcal{S} take the form

$$\rho_{\alpha\beta} = \delta(\alpha, \beta) U_{\alpha} \rho_{\text{in}} U_{\alpha}^{\dagger}, \quad (91)$$

involving the *decoherent* partial evolution operators²¹ for the open system \mathcal{S} :

$$U_{\alpha} = \int d[p, q] w_{\alpha}[p, q] \exp \left\{ \frac{i}{\hbar} \int_0^t [p\dot{q} - H(p, q)] \right\}. \quad (92)$$

The law of evolution for the open system \mathcal{S} takes the form

$$\rho_{\alpha} = U_{\alpha} \rho_{\text{in}} U_{\alpha}^{\dagger}, \quad \rho = \int d\alpha \rho_{\alpha} = \int d\alpha U_{\alpha} \rho_{\text{in}} U_{\alpha}^{\dagger}. \quad (93)$$

The first of Eqns (93) is a selective description of the evolution of the system \mathcal{S} (i.e., of its evolution under the condition of a specific alternative α), whereas the second equation gives a nonselective description of the same process (when all possible alternatives are included).

The selective form of the law of evolution can be equivalently expressed in terms of state vectors (instead of the density matrix),

$$|\psi_{\alpha}\rangle = U_{\alpha} |\psi_{\text{in}}\rangle. \quad (94)$$

A nonselective description is only possible in density matrix terms.

Equations (92)–(94) are characteristic of the restricted path integral description of continuous quantum measurements (see Section 3.1). A point to note here is that the alternatives α are interpreted as the alternative readouts of continuous measurements. Hence, we have here developed a restricted path integral derivation of a continuous measurement by starting from the standard description of an open system and choosing the set of alternatives $\{\alpha\}$ in such a way as to provide the fulfillment of the decoherence-by-environment condition (81) and a similar condition (88) on the partial influence functionals.

Thereby, we have shown that under sufficiently general assumptions (in particular, if the environment contains a large enough number of degrees of freedom — i.e., is macro- or mesoscopic), an open system can be treated as a system subject to continuous measurement by its environment. Because dissipation arises due to the system being affected by its environment (reservoir) — i.e., as a result of the openness of the system — we conclude that it may be considered as resulting from the continuous measurement of the system and can be described using restricted path integrals, as was done in Section 3.

4.4 Decoherence of alternatives and measurement

Let us summarize what has been accomplished in Sections 4.1–4.3. We have considered an open system and have expanded its evolution into a sum of partial evolutions. We have shown that the evolution of an open system can be correctly described in terms of the restricted path integral approach, provided the set of alternative partial evolutions is chosen such that the alternatives are decoherent due to the specific features of the system's environment.

In mathematical terms, the condition for the system to evolve as a continuously measured one is formulated as the *condition for the decoherence of partial influence functionals*, Eqn (88), or as a weaker (but — for an environment with sufficient degrees of freedom — practically equivalent) *condition for decoherence by the environment*, Eqn (81).

Each alternative α is represented by a corridor of paths $[p, q, P, Q]$ of the composite system $\mathcal{S} + \mathcal{E}$, but for the purpose of analysis the paths $[p, q]$ (of the system \mathcal{S}) and $[P, Q]$ (of the system \mathcal{E}) contained in this corridor should be considered. The latter are denoted by $\alpha_{\mathcal{E}}$. For the decoherence condition to be fulfilled, it is essential that the corridors $\alpha_{\mathcal{E}}$ possess classical properties, i.e., that they be quantum images of the classical notion of the 'motion along a classical trajectory'. This proves possible if the system \mathcal{E} has many degrees of freedom (is macroscopic or mesoscopic).

There is a correlation between the paths $[p, q]$ contained in α , on the one hand, and the paths $[P, Q]$ contained in $\alpha_{\mathcal{E}}$, on the other. The correlation results from the fact that in the framework of the influence functional formalism, each path $[p, q]$ of the system \mathcal{S} determines the force acting on the system \mathcal{E} , thus determining the latter's classical trajectory. The correlation arises because, whatever the choice of the path $[p, q]$ in α , the classical trajectory of the system \mathcal{E} lies

²¹ Not to be confused with the partial evolution operators U_{α}^{tot} for the closed composite system $\mathcal{S} + \mathcal{E}$.

close to the middle of the corridor $\alpha_{\mathcal{E}}$ (otherwise the influence functional is close to zero). It is due to this correlation that the classical trajectories (or more precisely, the corridors of paths $\alpha_{\mathcal{E}}$) of the system \mathcal{E} contain information on the paths of the system \mathcal{S} . This means, in fact, that the system \mathcal{E} performs the measurement of the system \mathcal{S} , and that the readout of the measurement is recorded in the state of the system \mathcal{E} , by determining, namely, the classical trajectory (more precisely, the corridor $\alpha_{\mathcal{E}}$) along which the system \mathcal{E} moves.

The readouts of the present analysis provide the most general justification for the restricted path integral approach. These readouts confirm that Feynman's theory of amplitudes and his path integral techniques are valid not only for closed systems but for open systems as well. This is important because the Feynman formalism provides a physically transparent representation of quantum mechanics and proves to be both computationally and heuristically efficient.

Finally, our analysis shows that dissipative quantum systems can be treated as systems that are being continuously measured by their environment. The description of dissipation in terms of restricted path integrals or (in the Markovian approximation) in terms of complex Hamiltonians (see Section 3) proves applicable under very general assumptions and hence can be used to develop a dissipation theory which does not rely on specific reservoir models.

5. Conclusion

We have considered a variety of approaches to the theory of quantum systems with dissipation, such as a particle diffusing through a medium. It has been shown that attempts at 'quantizing' classical dissipative equations encounter difficulties since they come into conflict with the uncertainty principle. A physically justified derivation of dissipation is one in which the system's environment (reservoir) is explicitly considered. The principal device for overcoming concomitant technical challenges was to model the reservoir as a system of a large number of oscillators and then to pass to the normal modes of the system. This technically complex method is not necessary, however, because the same uncertainty principle suggests that a universal, model-independent description of the dissipation of a quantum system is possible.

Such a universal, model-independent description of dissipation can be obtained within the framework of the quantum theory of continuous measurements. What makes this possible is the fact that any open (in particular, dissipative) system is being continuously measured by its environment — provided this latter is large enough (i.e., if it is macro- or at least mesoscopic). The most general approach to the theory of continuously measured systems (including dissipative ones) is provided by the technique of restricted path integrals. This means that integration in the Feynman path integral representing the evolution of the system is restricted to those paths compatible with the information obtained as a result of the measurement. The transparent physical meaning of this procedure makes this approach heuristically valuable.

In the very important case where the continuous measurement is a monitoring, i.e., proceeds step-by-step, locally in time, a restricted path integral reduces to the Schrödinger equation with a complex Hamiltonian or to an equation for the density matrix. The latter equation automatically turns out to be a Lindblad-type equation, i.e., maintains the

positiveness of the density matrix. This permits the improvement of some of the equations obtained earlier in the framework of model approaches.

Compared to considering reservoir models, the approach based on the use of restricted path integrals and complex Hamiltonians turns out to be preferable, first, because it is model-independent and hence allows general conclusions to be made, and, second, due to its mathematical simplicity: in this approach there is no need to first introduce and then exclude from consideration (by a reduction operation) a huge number of the reservoir's degrees of freedom.

An important point to add is that the conclusions of the theory of dissipative quantum systems can now be verified experimentally. With the quantum-optical devices developed over the past few decades, experimental research into quantum measurement processes and hence into processes of quantum dissipation has become possible [31, 54, 55].

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