# Sagnac effect in a rotating frame of reference. Relativistic Zeno paradox 

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After the publication of our review [1] dedicated to the critical consideration of different ways of explaining the Sagnac effect [2-4] (see also Refs [5, 6]), which showed the fallacy in nonrelativistic explanations of this effect, the PhysicsUspekhi editorial board received articles negating the existence of the Sagnac effect in the special theory of relativity (STR). As stated in these papers, the phase difference of counterpropagating waves in the reference system co-rotating with a ring interferometer, calculated in the context of the STR, is equal to zero. Regrettably, papers of this kind have been published before and continue to come out in different publications [7-13].

Negating the existence of the Sagnac effect in the STR, the authors of the above-mentioned papers draw quite different and absolutely wrong conclusions. In particular, Bashkov and Malakhaltsev $[9,11]$ arrive at the conclusion that the cause for the existence of the Sagnac effect lies with the Coriolis forces arising in the rotating frame of reference. Staroverova [7] and Bashkov and Sintsova [8, 10] believe that the magnitude of the effect should depend not on the interferometer area, but on the shape of its perimeter, and for a fiber ring interferometer (FRI) on the nature of winding of the optical fiber. Kupryaev $[12,13]$ casts doubt on the validity of the STR as a whole and, in particular, argued that the Sagnac effect should be treated in the context of 'light-carrying ether' theory.

We consider below the reason why the authors of Refs [7-13] were impelled to negate the feasibility of an adequate explanation of the Sagnac effect within the context of the STR. In a laboratory (inertial) frame of reference $K$, the magnitude of the Sagnac effect can be calculated in the context of the STR by proceeding from the invariance of an interval $x^{2}+y^{2}+z^{2}-c^{2} t^{2}$ [14] (where $x, y$, and $z$ are the wave front coordinates, $t$ is the time, and $c$ is the velocity of light in vacuum) or, which is simpler and physically clearer, by proceeding from the relativistic law of velocity composition - the phase wave velocity $v_{\phi}$ in a locally co-moving inertial reference system $K^{\prime}$ and the linear velocity $R \Omega$ of this system (the case in point is a ring interferometer of radius $R$, which rotates with an angular velocity $\Omega$ ):

$$
\begin{equation*}
v_{\phi}^{ \pm}=\frac{v_{\phi} \pm R \Omega}{1 \pm v_{\phi} R \Omega / c^{2}} \tag{1}
\end{equation*}
$$

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(We note that from here on we are dealing with the propagation time of phase fronts - the points of fixed phase of each of the waves - since the Sagnac effect is in reality recorded from the result of interference of the counterpropagating waves.) Hence, considering that the paths of counterpropagating waves in the system are different and equal to $l^{ \pm}=2 \pi R \pm R \Omega t^{ \pm}$, it is possible to obtain expressions for the times $t^{ \pm}$the counterpropagating waves take to travel through the ring and find the difference of these times. Correct to principal order, the magnitude of this difference is [1, 15]

$$
\begin{equation*}
\Delta t=t^{+}-t^{-}=\frac{4 \pi R^{2} \Omega}{c^{2}} \tag{2}
\end{equation*}
$$

Then, using the Lorentz transformations one can find the magnitude of $\Delta t^{\prime}$ in the inertial reference system $K^{\prime}$ instantaneously co-moving with the rotating ring interferometer, this magnitude coinciding with $\Delta t$ to small relativistic corrections on an order of $R^{2} \Omega^{2} / c^{2}$. However, if the propagation time difference $\Delta t^{\prime}$ for counterrunning waves is calculated in the context of the STR directly in the $K^{\prime}$ reference system, then at first glance some misunderstanding may arise. The phase velocities of the counterrunning waves in the $K^{\prime}$ reference system are in fact equal in magnitude and may be denoted by $v_{\phi}$, the paths of the phase fronts are seemingly also equal because the beam-splitting mirror is immobile in the $K^{\prime}$ frame, and the propagation time difference $\Delta t^{\prime}$ for counterrunning waves should seemingly be equal to zero as well, which signifies the absence of the Sagnac effect in the $K^{\prime}$ frame.

The situation resembles the Zeno paradox of an arrow (Zeno's aporia "The Arrow" [16]) in a sense. The arrow is at rest at each point in time, but it nevertheless moves. In our case, the velocities and paths of the counterrunning waves in a ring interferometer are equal in the $K^{\prime}$ frame, but nevertheless, as shown by measurements, their propagation times are different. As a consequence, the contradiction under discussion may, in our view, be termed the relativistic Zeno paradox by analogy with the so-called quantum Zeno paradox.

In reality, of course, an observer residing in the rotating frame of reference, which is noninertial, will find that the velocities of counterpropagating waves, defined as the ratios between the path lengths of the phase fronts and the times taken to travel these paths, are different [14, 17, 18]. The error of the authors of Refs [7-13] consists in that the $K^{\prime}$ frame is an inertial frame of reference, which instantaneously comoves with the noninertial rotating frame of reference $K_{\text {nonin }}^{\prime}$. At each single point in time, the velocity of $K^{\prime}$ relative to the $K$ frame is the same in magnitude ( $R \Omega$ ), but the
direction of this velocity changes in time. In fact, one should consider the set of inertial frames $K_{1}^{\prime}, K_{2}^{\prime}, K_{3}^{\prime}, \ldots$, which comove with the rotating ring interferometer at different points in time. If the time intervals between the neighboring frames $K_{i}^{\prime}, K_{i+1}^{\prime}$ are made to tend to zero and a continuous passage from one frame to another is effected, then it is possible to correctly calculate the propagation time difference between counterrunning waves in a ring interferometer. However, this is a rather intricate procedure. It is far easier to perform the corresponding calculations in the rotating (noninertial) reference system $K_{\text {nonin }}^{\prime}$ with the aid of a metric tensor in the absence of external gravitational field [14, 17]. As shown by Landau and Lifshitz [17], the physical cause of the existence of the Sagnac effect in the rotating reference system $K_{\text {nonin }}^{\prime}$ lies with a difference between the velocities of counterpropagating waves (making no difference in this case what kind of velocity we are dealing with: the phase velocities of wave fronts of continuous radiation or the group velocities of short light pulses), which to a first approximation are equal to $v^{ \pm}=v \pm R \Omega v^{2} / c^{2}$.

There exists an even simpler and physically somewhat formal derivation of the expression for the magnitude of the Sagnac effect in the $K_{\text {nonin }}^{\prime}$ frame, which is reliant on the equivalence principle and the effect of time dilation in a gravitational field [1]. To do this, one should consider some rotating reference system $K_{\mathrm{in}}^{\prime}$ which coincides with the rotating reference system $K_{\text {nonin }}^{\prime}$ and is equivalent to it, but is supplemented with a nonrelativistic (Newtonian) scalar potential of gravitational field, describing the Coriolis acceleration. Indeed, in a rotating reference system there occurs the Coriolis acceleration, apart from the centrifugal acceleration for moving bodies (for a watch, in particular) that follow the points of a fixed phase of counterpropagating waves. The magnitude of Coriolis acceleration is $2 \Omega v_{\phi}$, and its direction depends on whether the sense of motion coincides with the sense of rotation (in this case, its direction coincides with that of centrifugal acceleration) or is opposite to the sense of rotation (in this case, its direction is opposed to the centrifugal acceleration). If the limits of integration in the expression for the scalar potential $U$ are selected in such a way as to satisfy the condition $U=0[1,18]$ at the center of rotation, where the centrifugal acceleration is nonexistent, then the potentials for counterpropagating waves will be equal at the point corresponding to the center of rotation. The transformations in the $K_{\mathrm{in}}^{\prime}$ frame are formally performed as if it were inertial. The wave-front propagation times for the counterpropagating waves in the $K_{\text {in }}^{\prime}$ frame are [1]

$$
\begin{equation*}
\left(t^{ \pm}\right)^{K_{\mathrm{in}}^{\prime}}=t \sqrt{1+\frac{\left(U_{\mathrm{in}}^{\prime}\right)^{ \pm}}{c^{2}}}=t \sqrt{1-\frac{\Omega^{2} R^{2}}{2 c^{2}} \mp \frac{2 \Omega v_{\phi} R}{c^{2}}}, \tag{3}
\end{equation*}
$$

where $t=4 \pi R /\left(v_{\phi}^{+}+v_{\phi}^{-}\right)$, and the quantities $v_{\phi}^{ \pm}$are defined by expression (1). The last radicand in expression (3) is next expanded in terms of a small parameter $\Omega v_{\phi} R / c^{2} \ll 1$, and the effect of gravitational potential, which corresponds to the centrifugal acceleration, is neglected. There results an approximate expression for the difference in travelling times for fixed-phase points of counterpropagating waves, which coincides, to small relativistic corrections, with expression (1) obtained in the inertial frame $K$ (see also Ref. [1]).

So, in an instantaneously co-moving reference system $K^{\prime}$, the absolute velocity values of counterpropagating waves are equal, but in the rotating reference system $K_{\text {in }}^{\prime}$ (as well as in the $K_{\text {nonin }}^{\prime}$ frame) they differ. Therefore, the analogy with the

Zeno paradox is close enough: at each point in time, the phase velocities of counterpropagating waves are equal, but they diverge over an arbitrarily short, though finite, time interval. The Zeno paradox, which astounded people who lived 25 centuries ago, is known to have received an extremely trivial explanation after the advent of differential calculus: the velocity of a body (in the present context, an arrow) is defined not by its instantaneous position in space, but by the time derivative of its spatial coordinate. The explanation for the relativistic Zeno paradox is to an extent similar to the explanation for its antique analogue: the elements of the metric tensor in the inertial reference system $K^{\prime}$ instantaneously co-moving with the rotating ring interferometer coincide with the corresponding elements of the metric tensor in the rotating reference system $K_{\text {nonin }}^{\prime}$, but their derivatives with respect to the four space - time coordinates $x, y, z$, and $t$ (the Christoffel symbols) do not coincide in the two reference systems under consideration [17]. Therefore, the reference systems $K^{\prime}$ and $K_{\text {nonin }}^{\prime}$ are not completely equivalent, and the systems $K_{\text {nonin }}^{\prime}$ and $K_{\text {in }}^{\prime}$ are equivalent only formally.

As regards the explanation for the Sagnac effect with recourse to a consideration of the direct action of the Coriolis forces on counterrunning waves [9, 11], this explanation was shown to be fallacious by A Lunn even 80 years ago [19]. The fallacy in the explanation for this effect in the context of the 'light-carrying ether' theory [12, 13] was revealed by S I Vavilov 75 years ago [5] (see also Ref. [1]). As for the effect of the nature of optical fiber winding on the magnitude of the Sagnac effect in an FRI [7, 8, 10], it is pertinent to note that recent years have seen the publication of several papers (see, for instance, Refs [20, 21]) containing this statement, which was shown to be fallacious by Andronova and Malykin [22]. The rightfulness of considering the Sagnac effect in the framework of the STR was recently noted by Ginzburg [23].

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