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The phase of an oscillator in quantum theory. What is it 'in reality'?

Yu I Vorontsov

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Contents

Abstract. An analysis of the current theory of the quantum oscillator phase is presented. Predictions using existing approaches to the phase problem differ not only quantitatively but also qualitatively. The question in the title has not yet been given a generally accepted answer. However, it is logical to argue that all the theoretically predicted properties of the phase are physically meaningful if appropriate measurements are possible. Current phase measurement methods either involve the simultaneous (approximate) measurement of the amplitude and the phase or rely on the simultaneous measurement of quadrature amplitudes.

1. Introduction

Is there any concept on earth that is simpler than the 'phase of oscillations'? A schoolboy and a sophomore will tell you what it is; even professors of physics will know that, unless they have read what has been written about the phase in quantum theory. The most doubtful are probably those who have spent part of their lives studying the quantum phase. The main mind-breaker is the question whether the phase is a dynamic observable (like the coordinate or momentum), or a parameter (like, for example, the time).

Yu I Vorontsov Physics Department, M V Lomonosov Moscow State University, Vorob'evy Gory, 119892 Moscow, Russian Federation
Tel. (7-095) 939 39 03
E-mail: yury@hbar.phys.msu.su

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In addition, studies have revealed certain strange nonclassical properties of the phase difference between two oscillators. Various trains of thought led to the conclusion that the values of phase difference may be discrete. Different approaches, however, give different discrete values. At the same time it was demonstrated that the change of phase difference could be continuous. Several theories of quantum phase have emerged over the years of discussions. For relatively large energies of oscillations all theories give very similar results that agree well with experiment. Disagreements arise when the average energy of oscillations is a few quanta or less.

This paper is devoted to a critical review of the main approaches to the problem of phase in quantum theory.

The transition from classical to quantum description consists in matching the classical dynamic variables (coordinate, momentum, energy, etc.) to the corresponding linear Hermitian operators. Almost every dynamic observable was found to have a canonically conjugate counterpart. The problem arose with selecting the observable that is conjugate to the number of quanta. From the principle of correspondence between a commutator of operators and classical Poisson brackets, it is the phase that must be conjugate to the number of quanta. Accordingly, the classical phase ought to be put in correspondence with the linear Hermitian operator.

The first such attempt was made by Dirac in 1927 [1]. Representing the destruction operator in the form of the polar decomposition

$$\hat{a} = \sqrt{\hat{N}} \exp(\mathrm{i}\hat{\varphi}), \tag{1.1}$$

he argued that operator $\hat{\varphi}$ is Hermitian and is the sought phase operator.

Using the matrix representation of operators $\hat{U} = \exp(\mathrm{i}\hat{\varphi})$ and $\hat{U}^{\dagger} = \exp(-\mathrm{i}\hat{\varphi})$, London [2] in the same year proved that they are not unitary, and therefore the operator $\hat{\varphi}$ is not Hermitian. Dirac accepted this remark, and reproduced it in the first two editions of his textbook on quantum mechanics. At the same time, he noted the following relations:

$$[\hat{U}, \hat{N}] = \hat{U}\,, \qquad [\hat{N}, \hat{\varphi}] = \mathrm{i}\,. \tag{1.2}$$

Using these, Robertson derived the uncertainty relation [3]

$$\Delta n\, \Delta\varphi \geqslant \frac{1}{2}\,. \tag{1.3}$$

The incorrectness of relation (1.3) is obvious if we recall that the values of phase are confined to the interval 2π. As $\Delta n \to 0$ inequality (1.3) does not hold. However, this problem was publicly admitted only in 1936 [4].

The flow of publications on the problem of phase in quantum theory started only in 1963 [5–7], and kept on growing till 1995. As many as 50 papers appeared each year between 1991 and 1995. (A complete list of publications up to 1996 can be found in Ref. [9].) The studies of phase were stimulated by the interest in quantum fields generated by masers and lasers. Once again it was demonstrated that the operator $\hat{U}$ is not unitary, and that phase cannot be put into correspondence with a Hermitian operator in the conventional Hilbert space, because the spectrum of eigenvalues of the operator $\hat{N}$ is bounded [8]. (London's proof was not remembered at that time, probably because Dirac omitted the discussion about the phase operator in the third edition of his textbook.)

One of the proofs runs as follows. By definition,

$$\hat{a}|n\rangle = \sqrt{n}\,|n\rangle, \qquad \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,$$
$$\hat{a}|0\rangle = 0\,, \qquad \hat{a}^{\dagger}\hat{a}|0\rangle = 0\,.$$

If we represent the annihilation operator in the form

$$\hat{a} = \hat{U}\hat{N}^{1/2}\,,$$

then we should have

$$\hat{U}|n\rangle = |n-1\rangle\,, \qquad \hat{U}^{\dagger}|n\rangle = |n+1\rangle\,, \qquad \hat{U}|0\rangle = 0\,.$$

Accordingly,

$$\hat{U}\hat{U}^{\dagger}|n\rangle = |n\rangle\,, \qquad n = 0, 1, 2, \ldots\,,$$

but

$$\hat{U}^{\dagger}\hat{U}|n\rangle = \begin{cases} |n\rangle\,, & n = 1, 2, 3, \ldots\,, \\ 0\,, & n = 0\,. \end{cases} \tag{1.4}$$

Thus, $\hat{U}\hat{U}^{\dagger} = \hat{1}$, but $\hat{U}^{\dagger}\hat{U} \neq \hat{1}$.

Having run into problems with the introduction of the Hermitian phase operator via the relation $\hat{U} = \exp(\mathrm{i}\hat{\varphi})$ in the conventional Hilbert space, physicists started proposing most diverse solutions to the problem of phase in quantum electrodynamics. Eventually three main approaches emerged, which differ in the physical interpretation of the concept of phase.

1. The first approach is based on the concept of phase in the spirit of the traditional understanding of phase in quantum electrodynamics as a physical property of an oscillator or field mode. The phase operator is considered as conjugated with the operator of the number of quanta. The main difficulty in this approach consists in overcoming the problems related to the effects of the lower limit of the spectrum of eigenvalues $\hat{N}$. To resolve this problem, some proposed a fictitious extension of the n-space to $-\infty$ [10], while others introduced the phase operator in the finite-dimensional space of excitations $n \in [0, s]$ bounded from above, with a subsequent limiting procedure [11–13].

As the alternative to the phase operator, the Hermitian operators $\hat{S}$ and $\hat{C}$ are introduced, which correspond to the classical phase functions $\sin\varphi$ and $\cos\varphi$ [6, 8]. In spite of the absence of any formal problems, this approach has not become popular because of a number of physical implications that do not fit in with the traditional notions of the phase properties of oscillator. In particular, in this approach the very definition of phase becomes ambiguous. Since the operators $\hat{S}$ and $\hat{C}$ do not commute, the results of calculations of phase in terms of arcsine and arccosine functions are not the same. However, the traditional concepts are not always adequate to the quantum properties. Therefore, this approach to the description of phase properties should not be discarded, especially because in real experiments we deal for the most part with $\sin\varphi$ or $\cos\varphi$ rather than with the phase itself.

2. Even in the first approach, the definition of the phase operator is not an end in itself. The ultimate objective in all cases is the same: the development of a mathematical tool for calculating the results of the relevant experiments. Formulas are required that for the given state of the object allow calculation of the probability density of the phase, the moments of phase, and the correlation functions. These formulas can be derived aside from the phase operator and phase states, representing the phase as a parameter of state in the phase space [14, 15, 43].

For each mode, the complex phase space is introduced, each point of which is the complex amplitude α. The complex amplitude can be expressed in terms of the real intensity I and phase φ:

$$\alpha = \sqrt{I}\exp(-\mathrm{i}\varphi)\,.$$

The results of measurements, including the equal-time correlation functions, can be calculated with the help of $P(\alpha)$ and $Q(\alpha)$ quasi-probability functions and the Wigner function $W(\alpha)$. The probability density of phase is defined as the marginal distribution of the joint quasi-probability $w(I, \varphi)$. The drawback of this method is that for certain states of the field the probability calculated in this way can be negative, or it may be completely impossible to define the P-function. This disadvantage can be eliminated by employing the method of the double phase plane, where the intensity and the phase can be complex-valued rather than real [16].

3. In quantum theory it is tacitly assumed that every Hermitian operator corresponds to a certain physical observable. This, however, has not been proved (or, for that matter, refuted). Because of this, the operational approach that defines both the scheme of measurement and the corresponding Hermitian operator is interesting. Both approaches described above do not answer the question how to realize the measurement that would yield results corresponding to the results of the calculations. In the first approach the theory allows definition of the probability

operator measure characterizing the measurement [17–19], but not the method of its realization. Not every probability operator measure will correspond to a real measurement. In practice, primal is the scheme of measurement, the physical operation that serves for obtaining the required information.

In the operational approach, the physical variable is defined as the result of measurement in some particular scheme of measurement. In order to provide for the calculation of the probability density and other characteristics of the results of measurements in the given scheme of measurement, the operator is defined (invented) that acts in the space of the object under consideration, so as to achieve best fit with the experiment. With this approach, the information about the quantum phase is obtained from the experimental data in the same way as in the analysis of classical experiment. The results of measurement relate to the 'object-instrument' system rather than to the object alone. Because of this, different schemes of measurement may give different distributions of probability density of 'phase' for one and the same state of the object under investigation.

The description of phase properties of the state in all approaches described above uses the same term 'phase', which may not have the same meaning. The words 'change of phase', 'dispersion of phase', etc., may imply different quantitative characteristics in different approaches.

What then is the correct quantum description of the phase of harmonic oscillator (field mode)? One of the checkpoints is the agreement between quantum and classical calculations in the case of the quasi-classical state of oscillator, which is a coherent state with sufficiently large amplitude. Let us consider the characteristics of the classical phase.

2. Phase of classical oscillator

Properties of deterministic classical phase. Oscillations of coordinate and momentum of a harmonic oscillator can be represented in the form

$$x(t) = \left(\frac{2J}{m\omega}\right)^{1/2} \cos\varphi(t)\,, \qquad p(t) = (2m\omega J)^{1/2} \sin\varphi(t)\,, \tag{2.1}$$

where J is the action, and $\varphi(t)$ is the angle. In the new variables the Hamiltonian is $H = \omega J$.

Using the Poisson brackets, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\cos\varphi(t) &= \big\{\cos\varphi(t), H\big\} = \omega\sin\varphi(t)\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\sin\varphi(t) &= \big\{\sin\varphi(t), H\big\} = -\omega\cos\varphi(t)\,. \end{aligned} \tag{2.2}$$

Under condition (2.1), the time dependence of phase is

$$\varphi(t) = \varphi_0 - \omega t\,. \tag{2.3}$$

However, if in (2.1) we interchange sin and cos, we get

$$\varphi(t) = \varphi_0 + \omega t\,.$$

The sign of phase vs. time is not a matter of principle — it is a matter of convention. In quantum electrodynamics the operator of phase is introduced as the phase of the annihilation operator. The annihilation operator is related to that part of the spectrum of oscillations which in the complex form is written as $\exp(-\mathrm{i}\omega t)$. This corresponds to the time dependence of phase in the form (2.3).

In what interval does one have to calculate the values of phase? On the one hand, in the classical region one can observe the change of phase in an unlimited range of values (for example, such is the change of phase that is actually shown by the clock). On the other hand, the states of the oscillator that differ in phase by 2π are indistinguishable. Therefore, in quantum theory it would be logical to delimit the values of phase by the interval 2π, and in place of (2.3) use the following relation for the phase:

$$\phi(t) = \varphi(t) + 2\pi\sum_{p=1}^{\infty}\Theta\big(\varphi(t) - \theta_0 + 2p\pi\big)\,, \tag{2.4}$$

where $\Theta(z)$ is a step function that equals zero at $z > 0$, and equals one at $z < 0$, while θ_0 is the conventional limit of the interval of phase values $[\theta_0, \theta_0 + 2\pi]$. The change of phase with time is illustrated in Fig. 1.

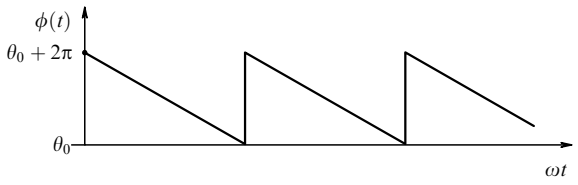

Figure 1. Phase vs. time.

The rate of phase variation, defined by relations (2.4) and (2.3), is

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -\omega\big[1 + 2\pi\delta_{2\pi}(\phi - \theta_0)\big]\,, \tag{2.5}$$

$$\delta_{2\pi}(\phi - \theta_0) = \sum_{1}^{\infty}\delta(\phi - \theta_0 + 2p\pi)\,.$$

Characteristics of random classical phase. The probability density $P(\phi)$ of classical phase ϕ has the following obvious properties:

$$\int_{\theta_0}^{\theta_0+2\pi} P(\phi)\,\mathrm{d}\phi = 1\,, \qquad P(\phi + 2\pi) = P(\phi)\,. \tag{2.6}$$

In polar coordinates, the function $P(\phi, t)$ can be visualized as an endless rotating tape whose width is proportional to $P(\phi)$.

The rate of variation of the mean value of phase is

$$\frac{\mathrm{d}\big\langle\phi(t)\big\rangle}{\mathrm{d}t} = -\omega\big[1 - 2\pi P(\theta_0, t)\big]\,. \tag{2.7}$$

This can be proved by calculating as $\mathrm{d}t \to 0$ the difference

$$\begin{aligned} \big\langle\phi(t + \mathrm{d}t)\big\rangle - \big\langle\phi(t)\big\rangle = &\int_{\theta_0}^{\theta_0+2\pi} (\phi + \delta\varphi)\,P(\phi + \delta\varphi)\,\mathrm{d}\phi \\ &- \int_{\theta_0}^{\theta_0+2\pi} \phi\,P(\phi)\,\mathrm{d}\phi\,, \end{aligned}$$

where $\delta\varphi = -\omega\,\mathrm{d}t$.

As follows from (2.5), the mean rate of phase variation equals the rate of change of the mean value of phase.

For illustration of the features of phase characteristics related to the jumps of phase values upon transition across the

boundary θ_0, let us consider the free evolution in time of the mean value of phase and the dispersion in the case of 'rectangular' distribution function $P(\phi)$ of width δ. As long as the distribution function does not reach the boundary θ_0, we have $P(\theta_0) = 0$, and, accordingly, $\mathrm{d}\langle\phi\rangle/\mathrm{d}t = -\omega$, while the dispersion of phase is constant and equal to $\delta^2/12$.

When the distribution function passes the boundary, the mean value of phase increases, and the dispersion of phase at first increases and then returns to the initial value. The form of the function $P(\phi)$ does not change. Accordingly, the probability of occurrence of phase values in a certain interval $\phi \in [\phi_1, \phi_2]$ does not depend on the selection of the reference phase θ_0:

$$\int_{\phi_2}^{\theta_0+2\pi} P(\phi)\,\mathrm{d}\phi + \int_{\theta_0}^{\phi_1} P(\theta_0)\,\mathrm{d}\phi = 1 - \int_{\phi_1}^{\phi_2} P(\phi)\,\mathrm{d}\phi\,.$$

This example demonstrates that dispersion as a characteristic of phase distribution is not informative. Its use in this capacity is only justified in the case of a narrow Gaussian distribution, when the value of $P(\theta_0)$ is negligibly small.

A random variation of phase

$$\delta\varphi = -\int \omega\,\mathrm{d}t\,,$$

associated, for example, with the uncertainty of frequency or the time of evolution, leads to the corresponding distribution of probability density $P_\varphi(\varphi(t))$ over the infinite interval of phase values φ. The probability density of phase ϕ defined over the interval $[\theta_0, \theta_0 + 2\pi]$ is related to $P_\varphi(\varphi)$ by formula

$$P_\phi(\phi, t) = \sum_{k=-\infty}^{\infty} P_\varphi(\varphi = \phi + 2k\pi)\,.$$

3. Quantum phase

3.1 Phase states. Pegg – Barnett distribution

The vector of phase state must satisfy the conditions (2.3), (2.6), (2.7). In addition, if the phase is considered as the observable conjugate to the number of quanta, it would be logical to assume complete uncertainty regarding the number of quanta in the phase state — that is, all values of n are equiprobable.

These conditions are satisfied by vectors known as the Susskind – Glogower phase states [6]

$$|\theta\rangle = \frac{1}{(2\pi)^{1/2}} \sum_{n=0}^{\infty} \exp(\mathrm{i}n\theta)\,|n\rangle\,, \tag{3.1}$$

which are the eigenfunctions of the exponential Susskind – Glogower phase operator

$$\widehat{\exp}(\mathrm{i}\theta) = \sum_{n=0}^{\infty} |n\rangle\langle n+1|\,. \tag{3.2}$$

Here we use a new designation for the phase θ to distinguish between the different approaches to the definition of phase.

The free evolution of vector $|\theta\rangle$ is described by the relation

$$|\theta(t)\rangle = \exp(-\mathrm{i}\hat{N}\omega t)|\theta(0)\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(\mathrm{i}n\theta)\exp(-\mathrm{i}n\omega t)\,|n\rangle = |\theta - \omega t\rangle\,. \tag{3.3}$$

The vectors $|\theta\rangle$ have another property that is characteristic of the phase states as conjugate to the n-states — equal probability (complete uncertainty) of all possible values of the number of quanta.

If the values of phase are limited to the interval $[\theta_0, \theta_0 + 2\pi]$, then in accordance with the relation (2.4) and Fig. 1 the quantity $\theta(t)$ must be replaced by

$$\theta - \omega t + 2\pi \sum_{p=1}^{\infty} \Theta(\theta - \omega t - \theta_0 + 2p\pi)\,.$$

Given that the values of θ are confined to the interval 2π, we may write the expansion of unity with respect to the states $|\theta\rangle$:

$$\int_0^{2\pi} |\theta\rangle\langle\theta|\,\mathrm{d}\theta = \sum_{m,n=0}^{\infty} |m\rangle\langle n| \int_0^{2\pi} \exp\left[\mathrm{i}\theta(n-m)\right] \frac{\mathrm{d}\theta}{2\pi} = \sum_{m,n} |m\rangle\langle n|\delta_{mn} = \hat{1}\,.$$

Accordingly, any state of the oscillator may be represented as an expansion in vectors $|\theta\rangle$. If in the n-representation a state is described by

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle\,,$$

then in the phase θ-representation it can be described by the relation

$$|\psi\rangle = \int_0^{2\pi} \sum_n \psi_n |\,\theta\rangle\langle\theta|n\rangle\,\mathrm{d}\theta = \int_0^{2\pi} \psi(\theta)\,|\theta\rangle\,\mathrm{d}\theta\,,$$

where

$$\psi(\theta) = \langle\theta|\psi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \psi_n \exp(-\mathrm{i}n\theta)\,. \tag{3.4}$$

The integral of the squared modulus of function $\psi(\theta)$ equals one:

$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m} \psi_n \psi_m^* \exp\left[\mathrm{i}\theta(m-n)\right]\mathrm{d}\theta = 1\,.$$

In this way, the function

$$P(\theta) = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} \psi_n \exp(-\mathrm{i}n\theta)\right|^2 \tag{3.5}$$

has the necessary properties of the probability density function of phase θ.

Formula (3.5) is known as the Pegg – Barnett distribution. It is derived, for example, in Ref. [9], where the function $\psi_n(\theta)$ is defined as the eigenfunction of operator $\hat{N}$ in the phase representation $\hat{N} = \mathrm{i}\,\mathrm{d}/\mathrm{d}\theta$. Although the result does not cause any doubts, such a definition of the eigenfunction of operator $\hat{N}$ in the phase representation was earlier criticised in Ref. [8], because it leads to continuous values of n from $-\infty$ to $+\infty$. Relation (3.5) is verified by the analysis based on the Pegg – Barnett phase operator (see Section 3.2).

According to the quantum theory of measurements [17 – 21], the probability density distribution of the results of measurement θ is determined by the state of the object $\hat{\rho}$ and

the probability operator measure $\hat{\Pi}(\theta)$:

$$P(\theta) = \mathrm{Tr}\left[\hat{\rho}\hat{\Pi}(\theta)\right]. \tag{3.6}$$

Function (3.5) corresponds to

$$\hat{\Pi}(\theta) = |\theta\rangle\langle\theta|. \tag{3.7}$$

Calculation of the phase distribution does not require knowing the phase operator. It suffices to know the probability operator measure. The physical quantities that are matched with a Hermitian operator in the case of exact measurement correspond to the orthogonal probability operator measures. For example, the exact measurement of the coordinate corresponds to the probability operator measure $|x\rangle\langle x|$, the exact measurement of the number of quanta corresponds to $|n\rangle\langle n|$.

The probability operator measure $\hat{\Pi}(\theta)$ added new difficulties to the analysis of quantum phase: it turned out to be non-orthogonal. This fact makes questionable the measurability of phase in the sense that is ascribed, for example, to the measurability of the coordinate. The non-orthogonality of the probability operator measure $\Pi(\theta)$ is caused by the non-orthogonality of vectors $|\theta\rangle$ [8]:

$$\langle\theta|\theta'\rangle = \sum_{n=0}^{\infty} \exp\left[-\mathrm{i}n(\theta-\theta')\right]$$

$$= \frac{1}{4\pi} + \frac{1}{2}\,\delta(\theta-\theta') - \frac{\mathrm{i}}{4\pi}\cot\frac{\theta-\theta'}{2}.$$

The cause of non-orthogonality is obvious — it is the fact that the summation is carried out over only the non-negative values of n.

As a result, on the one hand we have formula (3.5), which, in the opinion of the authors of Ref. [9], has everything that is necessary for calculating the phase properties of light. On the other hand, owing to the non-orthogonality of the eigenvectors, the phase cannot be matched to a Hermitian operator. Accordingly, one cannot consider the phase θ as a physical observable in the conventional sense of this notion.

The arguments developed above do not imply that the quantity θ does not have any physical meaning at all. In any case, this quantity can be regarded as a parameter of state responsible for the phase properties. If the direct measurement of θ is not possible, then it is possible to get an estimate by processing the results of measurements of phase-dependent observables. Because of this, physical meaning can only be ascribed to the probability density (3.5) and the mean values of phase θ and its functions:

$$\langle\theta\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta)\,\mathrm{d}\theta, \quad \left\langle F(\theta)\right\rangle = \int_{\theta_0}^{\theta+2\pi} F(\theta)\,P(\theta)\,\mathrm{d}\theta. \tag{3.8}$$

The mean values of phase and squared phase can be represented in terms of the mean values of self-conjugate operators [22]:

$$\hat{\Theta} = \int_{\theta_0}^{\theta_0+2\pi} \theta|\theta\rangle\langle\theta|\,\mathrm{d}\theta, \qquad \widehat{\Theta^2} = \int_{\theta_0}^{\theta_0+2\pi} \theta^2|\theta\rangle\langle\theta|\,\mathrm{d}\theta.$$

However, in the phase representation operator $\hat{\Theta}$ is not a multiplicative operator $\theta\times$, and the squared operator $\hat{\Theta}$ is not equal to operator $\widehat{\Theta^2}$:

$$\hat{\Theta}\cdot\hat{\Theta} \neq \widehat{\Theta^2}.$$

If vectors (3.1), which satisfy the condition of equidistribution of n, are not orthogonal, is it possible that there are vectors of phase that satisfy all other criteria except the condition of equidistribution of n? It has been proved (see, for example, Ref. [9]), that there are no such vectors in the conventional Hilbert space.

The number – phase uncertainty relation. The right-hand side of the generalized Robertson – Schrödinger uncertainty relation for any pair of Hermitian operators equals 1/2 of the modulus of the mean value of the commutator of these operators. In case of the number – phase uncertainty relation, we should have

$$\Delta\theta\,\Delta n \geqslant \frac{1}{2}\left|\left\langle[\hat{\Theta},\hat{N}]\right\rangle\right|.$$

At the same time, the mean value of the commutator $[\hat{\Theta},\hat{H}] = \hbar\omega[\hat{\Theta},\hat{N}]$ determines the mean rate of phase variation. Accordingly, if the phase operator satisfies the classical condition (2.7) for $\langle \mathrm{d}\phi/\mathrm{d}t\rangle$, we have

$$\left\langle[\hat{\Theta},\hat{N}]\right\rangle = -\mathrm{i}\left[1 - 2\pi P(\theta_0)\right].$$

Then we get

$$\Delta\theta\,\Delta n \geqslant \frac{1}{2}\left[1 - 2\pi P(\theta_0)\right]. \tag{3.9}$$

As demonstrated above, the quantities $\Delta\theta$ and $P(\theta_0)$ are periodic functions of time. The uncertainty of the initial phase $\theta(0)$ does not depend on time, but depends on the selection of the zero point θ_0. Relation (3.9) is free from the drawbacks of the traditional uncertainty relation (1.3). For example, since it is assumed that as $\Delta n \to 0$ the phase distribution tends to the equiprobable distribution with $P(\theta) = 1/(2\pi)$, both the left-hand and the right-hand sides of relation (3.9) tend to zero.

It should be kept in mind that the Robertson – Schrödinger uncertainty relation is justified for operators that satisfy the condition $\hat{C}\hat{C} = \hat{C}^2$ [23]. Operator $\hat{\Theta}$ does not answer this criterion. Therefore, there are reasons to question the validity of relation (3.9). Nevertheless, these relations are in good agreement with the results of the experiment [79].

Figure 2 shows theoretical and experimental curves for the dispersion of phase, the product $\Delta\theta\,\Delta n$, and the mean value of the commutator as a function of the mean number of photons in optical mode in the coherent state.

Quantum states with minimal uncertainty of phase. Usually the state with minimal uncertainty is understood as the state with the least product of dispersions of two non-commuting observables. Physically more natural, however, is a different statement of the problem: states with minimal uncertainty are the states with the least value of ΔA at certain preset parameters (for example, $\langle B\rangle$, ΔB, $\langle A\rangle$).

As demonstrated in Ref. [25], the search for the minimal value of ΔA at different values of ΔB yields solutions that minimize the product $\Delta A\Delta B$. It is possible also to find solutions that cannot be obtained by directly minimizing the product $\Delta A\Delta B$. In the context of such a statement of the problem, the authors of Ref. [26] were seeking the states that have the minimal phase uncertainty for the given values of the mean number of quanta and the dispersion of the number of quanta. Using the method of Lagrange coefficients, the

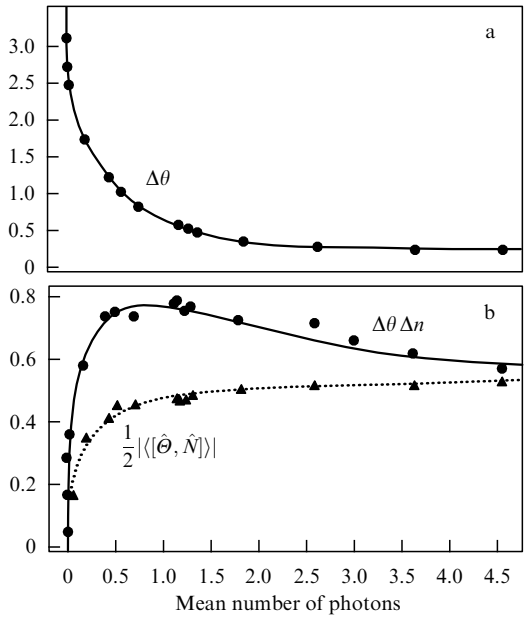

Figure 2. (a) Phase dispersion vs. the mean number of photons in coherent state: points represent experimental measurements, solid curve is calculated by the Pegg–Barnett formula. (b) Experimental and theoretical curves for the product of uncertainties and the mean value of the commutator [9, 79].

equation

$$[\hat{\Theta}^2 + \lambda_2\hat{N}^2 + \lambda_1\hat{N} + \lambda_0]\,|\Psi\rangle = 0 \tag{3.10}$$

was solved, where $\lambda_{1,2,0}$ are the Lagrange coefficients, and $|\Psi\rangle$ is the sought optimal state.

When the mean number of quanta is of the order of unity, the numerical solution of equation (3.10) reveals that the equality $\Delta n\,\Delta\theta = 1/2$ for the states with minimal uncertainty only holds in the range $0.5 < \Delta n < \langle n\rangle$. If $\Delta n \to 0$, the dispersion of phase tends to the asymptotic value of $\pi^2/3$. Then the following relation holds:

$$\Delta n\,\Delta\theta = \frac{\pi}{\sqrt{3}}\,\Delta n\,.$$

For each value of $\langle n\rangle$ there is a maximum value of Δn at which equation (3.10) still has a solution.

In the case when the mean number of quanta is much greater than one, it is possible to express the solution in the general form:

$$\psi(x) = \psi_0 H_v(x)\exp\left(-\frac{x^2}{2}\right), \tag{3.11}$$

where ψ_0 is the normalization coefficient, $H_v(x)$ is the Hermite function, and v is determined by certain additional conditions [26]. At $\Delta n/\langle n\rangle \ll 1$ the distribution of the number of quanta is close to Gaussian, and $\Delta n\,\Delta\theta = 1/2$ up to $\Delta n/\langle n\rangle \leqslant 0.3$. At $\Delta n/\langle n\rangle \sim 1$ the distribution is considerably distorted.

In the case when the uncertainty of phase is only minimized at a certain mean number of quanta much greater than one, the phase uncertainty is related to the mean number of quanta by the asymptotic relation

$$\Delta\theta \approx \frac{1.376\ldots}{\langle n\rangle}\,. \tag{3.12}$$

The right-hand side of (3.12) may be regarded as the refined *Heisenberg limit of phase uncertainty*. The uncertainty relation for this optimal state is

$$\Delta n\,\Delta\theta \approx 0.615\ldots$$

The dispersion of phase is not a universal characteristic of phase distribution. Theory knows four more *measures of phase uncertainty*: the inverse of the maximal value

$$\delta_m\phi = \left[P_{\max}(\phi)\right]^{-1};$$

the inverse of the averaged distribution (Süssman measure)

$$\delta\phi_s = \left[\int_{-\pi}^{\pi} P(\phi)\,P(\phi)\,\mathrm{d}\phi\right]^{-1};$$

the squared modulus of the mean value of phase constant

$$\delta_p\phi = \left|\int_{-\pi}^{\pi}\exp(\mathrm{i}\phi)\,P(\phi)\,\mathrm{d}\phi\right|^2;$$

and the entropy measure of uncertainty

$$\delta_E\phi = -\int_{-\pi}^{\pi} P(\phi)\ln P(\phi)\,\mathrm{d}\phi\,.$$

In Ref. [27] the states are defined that minimize these measures provided that the maximum number of quanta is limited. The uncertainty of phase in different states was also analyzed in Refs [28, 30, 31, 61].

3.2 Pegg – Barnett phase operator

First Popov and Yarunin (whose paper of 1973 went unnoticed) [12], Garrison and Wong [11], then Pegg and Barnett [13] proposed defining the phase operator in finite-dimensional with respect to n space Ψ_s, in which $0 \leqslant n \leqslant s$, with subsequent transition to the limit $s \to \infty$. The results of Pegg and Barnett [13] gained the broadest recognition.

The problems associated with the existence of the boundary values of n were eliminated by introducing the cyclic space Ψ_s. The operator $\hat{U}_s = \exp(\mathrm{i}\hat{\Phi}_s)$ introduced by Pegg and Barnett, and its Hermitian conjugate $\hat{U}_s^\dagger$ act upon $|n\rangle$ in such a way that

$$\hat{U}_s|n\rangle = \begin{cases} |n-1\rangle & \text{for } n > 0\,, \\ |s\rangle & \text{for } n = 0\,, \end{cases} \tag{3.13}$$

$$\hat{U}_s^\dagger|n\rangle = \begin{cases} |n+1\rangle & \text{for } n < s\,, \\ |0\rangle & \text{for } n = s\,. \end{cases} \tag{3.14}$$

If in the conventional space $\hat{U}|0\rangle = 0$, then in the space Ψ_s the operator $\hat{U}_s$ transforms the vacuum state $|n=0\rangle$ into state $|s\rangle$, and operator $\hat{U}_s^\dagger$ transforms the state $|s\rangle$ into state $|0\rangle$. This is accomplished by taking the operator $\hat{U}_s$ in the n-representation in the form [13]

$$\hat{U}_s = |0\rangle\langle 1| + |1\rangle\langle 2| + \cdots + |s-1\rangle\langle s| + \exp\left(\mathrm{i}(s+1)\theta_0\right)|s\rangle\langle 0|\,. \tag{3.15}$$

Operators $\hat{U}_s$ and $\hat{U}_s^\dagger$ are unitary, and as a consequence operator $\hat{\Phi}_s$ is Hermitian.

The eigenvalues of operator $\hat{U}_s$ are

$$\Theta_k = \theta_0 + \frac{2\pi k}{s+1}$$

$(k = 0, 1, 2, \ldots)$, where θ_0 is the arbitrary reference phase. The eigenvectors

$$|\Theta_k\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(\mathrm{i}\Theta_k n)|n\rangle$$

are orthonormal: $\langle\Theta_k|\Theta_m\rangle = \delta_{km}$. The corresponding self-conjugate phase operator is

$$\hat{\Phi}_s = \sum_{k=0}^{s} \Theta_k |\Theta_k\rangle\langle\Theta_k| . \tag{3.16}$$

The orthogonality of the phase states $|\Theta_k\rangle$ allows representing the operator functions expandable in a power series as

$$F(\hat{\Theta}_s) = \sum_{k=0}^{s} F(\Theta_k)\,|\Theta_k\rangle\langle\Theta_k| .$$

The commutator $[\hat{N}, \hat{\Phi}_s]$ is not the same as the Dirac commutator, and is

$$[\hat{N}, \hat{\Phi}_s] = -\mathrm{i}\big[\,\hat{1} - |\theta_0\rangle\langle\theta_0|\,\big] .$$

Here the unit operator is

$$\hat{1} = \sum_{k=0}^{s} |\Theta_k\rangle\langle\Theta_k| .$$

It is very important to note that the vectors $|\Theta_k\rangle$ are orthogonal only for integer k. The discreteness of values of Θ_k is associated with the cyclic nature of Ψ_s-space, and it does not vanish when $s \to \infty$. In the conventional space the values of phase are continual. Because of this, for any arbitrarily large value of s there will be such values of phase that do not comply with the criterion of orthogonality. Because of this, there are problems with the transition to the conventional infinite-dimensional space. There exists only a weak limit [33, 58].

The sequence of operators $\hat{A}_s$ converges to the weak limit $\hat{A}_w$ in H-space if

$$\lim_{s\to\infty} \langle g|\hat{A}_s|\,f\rangle = \langle g|\hat{A}_w|\,f\rangle$$

for all $|g\rangle$ and $|\,f\rangle$ from H. The weak limit does not conserve the operator algebra. For example, the square of weak limit $(\hat{\Phi}_w)^2$ of $\hat{\Phi}_s$ is not equal to the weak limit $(\hat{\Phi}^2)_w$ of $\hat{\Phi}_s^2$, and $\big[\exp(\mathrm{i}\Phi)\big]_w \neq \exp(\mathrm{i}\hat{\Phi}_w)$. As a result [9, 58], the two ways of going to the limit — either at the end of all computations, or going to the limit in the operators — give physically incomparable results.

The eigenstates of the weak limit of operator Φ_s do not comply with the criterion of equidistribution of n, as required by the complementarity condition. The results that agree with the traditional notion of phase are only yielded by the transition to the limit in the non-operator functions Θ_k. This is because in the Ψ_s space the functions of the discrete variable Θ_k are the exact representation of the function of continuous variable Θ.

For operator $\hat{\Phi}_s$ in any power m there exists a weak limit [26]

$$\lim_{s\to\infty} \langle f|\hat{\Phi}_s^m|g\rangle = \lim_{s\to\infty} \frac{1}{s+1} \sum_{k=0}^{s} \Theta_k^m f_s^*(\Theta_k)\, g_s(\Theta_k)$$

$$= \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+2\pi} \theta^m f^*(\theta)\, g(\theta)\, \mathrm{d}\theta = \langle f|\hat{\Phi}^m|g\rangle ,$$

where

$$f(\Theta_k) = \sqrt{s+1}\,\langle\Theta_k|\,f\rangle , \qquad g(\Theta_k) = \sqrt{s+1}\,\langle\Theta_k|g\rangle ,$$

$$f(\theta) = \sum_{0}^{\infty} f_n \exp(-\mathrm{i}n\theta) , \qquad g(\theta) = \sum_{0}^{\infty} g_n \exp(-\mathrm{i}n\theta) ,$$

$\hat{\Phi}^m$ is the weak limit of operator $\hat{\Phi}_s^m$. Since $f(\theta)$ and $g(\theta)$ are non-orthogonal, we have, for example,

$$\langle f|\hat{\Phi}^2|g\rangle \neq \langle f|\hat{\Phi}\hat{\Phi}|g\rangle .$$

Therefore, operators $\hat{\Phi}$ and $\hat{\Phi}^2$ are similar to operators $\hat{\Theta}$ and $\hat{\Theta}^2$ introduced above.

The limit relations for the moments of phase are

$$\langle\hat{\Phi}^m\rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta^m \langle\theta|\,\hat{\rho}\,|\theta\rangle\,\mathrm{d}\theta = \int_{\theta_0}^{\theta_0+2\pi} \theta^m P(\theta)\,\mathrm{d}\theta .$$

Here $|\theta\rangle$ are the same vectors as (3.1), and the probability density

$$P(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} \langle\theta_k|\,\hat{\rho}\,|\theta_k\rangle = \langle\theta|\,\hat{\rho}\,|\theta\rangle = \mathrm{Tr}\left[\hat{\rho}\,\hat{\Pi}(\Theta)\right] \tag{3.17}$$

coincides with function (3.6). From the limit of the mean value of commutator $[\hat{N}, \hat{\Phi}_s]$ it was found in Ref. [13] that the Pegg–Barnett uncertainty relation coincides with (3.9).

In this way, the transition to the limit in the non-operator functions Θ_k (that is, after the algebraic manipulations) confirms the distribution (3.6). However, the attempt to find the phase operator in H-space failed. The weak limit of operator $\hat{U}_s$ turned out to be the non-unitary Susskind–Glogower operator (3.2).

The fact that the weak limit of operator $\hat{\Phi}_s$ is not a satisfactory phase operator, adds weight to the conclusion that there is no such operator in the H-space. Therefore, phase is a quantity that cannot be measured exactly. This view is also supported by the analysis of the limiting transition from measurement in Ψ_s space to the measurement in H space [35–37].

The existing methods of phase measurement in optics are actually measuring not the phase but rather the phase-dependent variables, and reduce, as a rule, to a photon count. These measurements give information about phase distribution in the initial state, but there is no reduction — that is, the measurements do not prepare a state where the uncertainty of phase would be less than that in the initial state. If an indirect measurement of phase could be possible — similar, for example, to the measurement of coordinate, then the oscillator would have moved to a state with the lower uncertainty of phase. Such a measurement would necessarily be accompanied by a random disturbance of the distribution of quanta caused by the instrument.

The operator of shift of the number of quanta without the change of phase can be represented in the form $\exp(\mathrm{i}\hat{\nu}\hat{\Phi}_s)$,

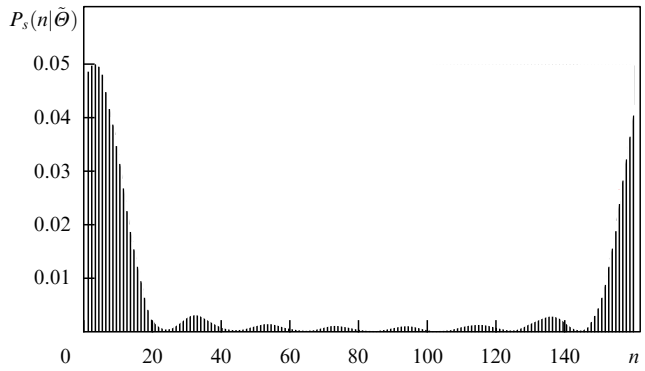

Figure 3. Distribution of n after the nonperturbing measurement of phase in the cyclic space Ψ_s ($s = 160$) [37].

similar to the form of the operator $\hat{U}_s$, where $\hat{y}$ is a certain operator of the instrument. As follows from conditions (3.13), even with the low values of uncertainty y (low measurement accuracy), the excitation of high levels n by the jump across $n = 0$ is possible.

In other words, after the approximate measurement of phase Φ_s the levels near $n \approx s$ may become excited (Fig. 3). The probability of such excitation does not change as $s \to \infty$. The transition to the limit in the function $P_s(n|\tilde{\Theta})$ of the posterior distribution of n does not eliminate this physically absurd excitation of the infinitely high energy levels in the case of an approximate measurement of phase. There is only one way of eliminating this effect — by banning the jump across the boundaries. This, however, will bring us back to the original undecidable problem of the Hermitian phase operator.

The arguments developed above indicate that the Pegg–Barnett phase operator actually cannot be matched to any real physical observable. This agrees with the conclusion regarding the absence of the Hermitian phase operator in the H-space.

4. Cosine and sine of phase operators

The discussions of physicists around the phase operator undeservedly pushed into the shadow the Hermitian operators $\hat{C}$ and $\hat{S}$, proposed by Susskind and Glogower as early as 1964 and corresponding to the cosine and sine of the phase [6, 8]. These operators were not popular with physicists probably because some of the implications contradict the deep-rooted notions that in the Fock state (including the vacuum state) the phase is completely undefined (that is, all of its values are equiprobable), and all moments of $\sin\varphi$ and $\cos\varphi$ entirely correspond to the phase distributions.

However, the quantum world 'in the person of the harmonic oscillator' keeps surprising our classical outlook. Today physicists take for granted the discreteness of energy levels of the oscillator, and the definite energy value in the vacuum state. The consequence of this definiteness (or, conversely, its cause) is the hundred percent anticorrelation between the squared coordinate x^2 and the squared momentum p^2, and the completely inexplicable from the classical standpoint negative mean value of the operator corresponding to the classical product x^2p^2 [38]. Therefore, it would be quite reasonable to assume that certain phase characteristics in vacuum and other states deviate from the traditional views.

The definitions and characteristics of operators $\hat{C}$ and $\hat{S}$ are described in detail in Ref. [8]. Let us briefly reproduce them here with some variations. For the starting point we take relations (2.2). Operators $\hat{C}$ and $\hat{S}$ are defined as the operators for which the replacement of Poisson brackets with the commutator yields the same algebraic structure as the brackets themselves. Hence, we should have

$$\dot{\hat{C}} = \frac{1}{\mathrm{i}\hbar}[\hat{C}, \hat{H}] = \omega\hat{S}, \qquad \dot{\hat{S}} = \frac{1}{\mathrm{i}\hbar}[\hat{S}, \hat{H}] = -\omega\hat{C}, \tag{4.1}$$

which is equivalent to

$$[\hat{C}, \hat{N}] = \mathrm{i}\hat{S}, \qquad [\hat{S}, \hat{N}] = -\mathrm{i}\hat{C}. \tag{4.2}$$

Operators $\hat{C}$ and $\hat{S}$ obey the standard equations for the normal modes of oscillations

$$\ddot{\hat{C}} + \omega^2\hat{C} = 0, \qquad \ddot{\hat{S}} + \omega^2\hat{S} = 0. \tag{4.3}$$

These equations do not have a unique solution. The solution used in Refs [6, 8] is

$$\hat{C} = \frac{1}{2}(\hat{E}_- + \hat{E}_+), \qquad \hat{S} = \frac{1}{2\mathrm{i}}(\hat{E}_- - \hat{E}_+), \tag{4.4}$$

where

$$\hat{E}_- = \left(\hat{N} + 1\right)^{-1/2}\hat{a} = \hat{a}\hat{N}^{-1/2}, \tag{4.5}$$

$$\hat{E}_+ = \hat{a}^\dagger\left(\hat{N} + 1\right)^{-1/2} = \hat{N}^{-1/2}\,\hat{a}^\dagger. \tag{4.6}$$

Operators $\hat{E}_\pm$ are identical to the phase operators of Susskind and Glogower (3.2): $\widehat{\exp}\,(\pm\mathrm{i}\phi)$ [6]. They are non-unitary (because the range of n is bounded):

$$\hat{E}_-\hat{E}_+ = 1, \qquad \hat{E}_+\hat{E}_- = 1 - \hat{P}_0, \qquad \hat{P}_0 = |0\rangle\langle 0|. \tag{4.7}$$

Nevertheless, operators $\hat{C}$ and $\hat{S}$ are definitely Hermitian: $\hat{C} = \hat{C}^\dagger$ and $\hat{S} = \hat{S}^\dagger$, because $\hat{E}_\pm = \hat{E}_\mp^\dagger$. Since the operators $\hat{E}_\pm$ are non-unitary, the operators $\hat{C}$ and $\hat{S}$ do not commute:

$$[\hat{C}, \hat{S}] = \frac{1}{2\mathrm{i}}\,\hat{P}_0. \tag{4.8}$$

The sum of their squares is not equal to the unit operator:

$$\hat{C}^2 + \hat{S}^2 = 1 - \frac{1}{2}\,\hat{P}_0. \tag{4.9}$$

For each operator $\hat{C}$ and $\hat{S}$ it is possible to define the independent non-commuting phase operators:

$$\hat{\varphi}_C \equiv \cos^{-1}\hat{C}, \qquad \hat{\varphi}_S \equiv \sin^{-1}\hat{S}.$$

In this way, the definition of phase operator in terms of operators $\hat{C}$ and $\hat{S}$ leads to two non-commuting Hermitian phase operators. This can be seen either as the cause of futility of all attempts to introduce a common phase operator [8], or as an indication that the phases defined in terms of operators $\hat{C}$ and $\hat{S}$ and the phase defined via the probability operator measure (3.7) are different physical quantities.

The implications of formulas (4.2), (4.8) and (4.9) are the uncertainty relations

$$\Delta n\,\Delta C \geqslant \frac{1}{2}\left|\langle S\rangle\right|, \tag{4.10}$$

$$\Delta n\,\Delta S \geqslant \frac{1}{2}\left|\langle C\rangle\right|, \tag{4.11}$$

$$\Delta S\,\Delta C \geqslant \frac{1}{2}\left|\langle \hat{P}_0\rangle\right|. \tag{4.12}$$

The eigenvectors of operator $\hat{C}$

$$|\cos\theta\rangle = \left(\frac{2}{\pi}\right)^{1/2}\sum_0^\infty \sin\left[(n+1)\theta\right]|n\rangle$$
$$= -\mathrm{i}\left[\exp(\mathrm{i}\theta)|\theta\rangle - \exp(-\mathrm{i}\theta)|-\theta\rangle\right], \tag{4.13}$$

are orthogonal and admit the resolution of unity. The spectrum of eigenvalues of $\hat{C}$ lies in the interval from -1 to 1. The independent eigenfunctions of $\hat{C}$ are contained in the interval $[0,\pi]$ of the values of variable θ.

The eigenvectors of operator $\hat{S}$

$$|\sin\theta\rangle = \left[\exp(\mathrm{i}\theta)|\theta\rangle + \exp(-\mathrm{i}\theta)|\pi-\theta\rangle\right], \tag{4.14}$$

exhibit the same properties. The independent eigenfunctions $\hat{S}$ are contained in the interval $[-\pi/2,\pi/2]$ of the values of variable θ. Vectors $|\theta\rangle$ are identical to the vectors (3.1).

The action of operators $\hat{C}$ and $\hat{S}$ on arbitrary vectors is illustrated by the relations

$$\langle f|\hat{S}|g\rangle = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi}\sin(\theta)\,f^*(\theta)\,g(\theta)\,\mathrm{d}\theta,$$
$$\langle f|\hat{C}|g\rangle = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi}\cos(\theta)\,f^*(\theta)\,g(\theta)\,\mathrm{d}\theta.$$

At the same time, vectors $|\cos\theta\rangle$ and $|\sin\theta\rangle$ are the eigenvectors of operators $\hat{\varphi}_C$ and $\hat{\varphi}_S$ [26]:

$$\hat{\varphi}_C|\cos\theta\rangle = \theta|\cos\theta\rangle, \qquad \hat{\varphi}_S|\sin\theta\rangle = \theta|\sin\theta\rangle.$$

As a consequence, operators $\hat{\varphi}_C$ and $\hat{\varphi}_S$ can be represented in the form

$$\hat{\varphi}_C = \int_0^\pi \theta|\cos\theta\rangle\langle\cos\theta|\,\mathrm{d}\theta,$$
$$\hat{\varphi}_S = \int_{-\pi/2}^{\pi/2} \theta|\sin\theta\rangle\langle\sin\theta|\,\mathrm{d}\theta.$$

The eigenstates of operators $\hat{C}$ and $\hat{S}$ are degenerate. Vector $|\cos\theta\rangle$ corresponds to two values of phase φ_C: θ and $-\theta$, and vector $|\sin\theta\rangle$ corresponds to two values of phase φ_S: θ and $\pi-\theta$. Therefore, if at $t=0$ the oscillator occurred in the state $|\cos\theta\rangle$, then the vector

$$|\cos\theta(t)\rangle \equiv \exp(-\mathrm{i}\hat{N}\omega t)\,|\cos\theta\rangle$$
$$= -\mathrm{i}\left[\exp(\mathrm{i}\theta)\,|\theta-\omega t\rangle - \exp(-\mathrm{i}\theta)\,|-\theta-\omega t\rangle\right] \tag{4.15}$$

will be the eigenvector of operator $\hat{C}$ only at the times $t=0$ and $k\pi/\omega$. Similarly, twice in every period the eigenvector of operator $\hat{S}$ is the vector

$$|\sin\theta(t)\rangle = \exp(\mathrm{i}\theta)\,|\theta-\omega t\rangle + \exp(-\mathrm{i}\theta)\,|\pi-\theta-\omega t\rangle.$$

In case of the initial state $|\cos\theta\rangle$, the distribution of phase φ_C changes with the time as follows. The initial δ-distribution splits into two similar packets of the same width that move with the same speed in opposite directions. After 'bouncing' off the boundaries of the interval $[0,\pi]$ these packets travel towards each other. After half a period they meet at the point that is symmetrical (with respect to $\pi/2$) to the initial point θ, and the distribution again becomes δ-shaped.

In a case of the coherent state $|\alpha\rangle$, where $\alpha = |\alpha|\exp(\mathrm{i}\varphi)$, at large values of $|\alpha|$ the moments of $\hat{C}$ and $\hat{S}$ are close to classical:

$$\langle C\rangle \sim \cos\varphi, \qquad \langle C^2\rangle \sim \cos^2\varphi, \qquad \langle C^2+S^2\rangle \sim 1.$$

Similar relations hold for operators $\hat{S}$. At small values of $|\alpha|$ — that is, in the case of a considerable overlap of the coherent state with the vacuum state,

$$\langle C\rangle \sim |\alpha|\cos\varphi, \quad \langle S\rangle \sim |\alpha|\sin\varphi, \quad \langle S^2\rangle \sim \frac{1}{4}, \quad \langle C^2\rangle \sim \frac{1}{4}.$$

In the case of the n-state we have

$$\langle n|C|n\rangle = \langle n|S|n\rangle = 0,$$

like all other odd moments of operators $\hat{C}$ and $\hat{S}$. This is in line with the idea that in the n-states the phase is 'completely undefined' — that is, its distribution is uniform. However, the even moments of operators $\hat{C}$ and $\hat{S}$ do not always correspond to such notion of phase distribution:

$$\langle n|C^2|n\rangle = \frac{1}{2}, \quad n \geqslant 1, \quad \text{but} \quad \langle n|C^2|n\rangle = \frac{1}{4}, \quad n=0,$$
$$\langle n|C^4|n\rangle = \frac{3}{8}, \quad n \geqslant 2, \quad \text{but} \quad \langle n|C^4|n\rangle = \frac{5}{16}, \quad n=1,$$
$$\langle n|C^4|n\rangle = \frac{1}{8}, \quad n=0.$$

The sixth moment coincides with the classical moment only at $n \geqslant 3$. Naturally, the equidistribution of phase in n-states does not correspond to the distribution of the eigenvalues of operator $\hat{C}$ itself, which, according to (4.13), can be written as a function of θ in the form

$$P_n(\cos\theta) = \left|\langle n|\cos\theta\rangle\right|^2 = \frac{2}{\pi}\sin^2\left[(n+1)\theta\right]. \tag{4.16}$$

Relation (4.16) defines also the distribution of phase φ_C in the interval $[0,\pi]$. The probability density related to the interval $[0,2\pi]$ is half of that. The dispersion of phase in the n-state depends on the value of n:

$$\Delta^2\varphi_C = \frac{\pi^2}{3} - \frac{1}{2(n+1)^2}.$$

In this connection, it would be logical to state that, contrary to expectations, there is no equidistribution of phase in the vacuum state and other energy states [8].

This statement is based on the fact that the phase of quantum oscillator exists irrespective of the measurement, and the values of the observables $\hat{C}$ and $\hat{S}$ are equal to the values of the corresponding trigonometric functions of phase. A different view is also possible: the observables $\hat{C}$ and $\hat{S}$ do not reduce to the trigonometric functions of phase, but rather are independent observables, in the same way as the observable $\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2$ does not reduce to the product of

squares of coordinate and momentum [38]. Phases φ_C and φ_S are not 'true' phases, but correspond instead to the inverse transform of $\hat{C}$ and $\hat{S}$.

Relation (4.16) can also be regarded as distribution of n (not normalised) in the state $|\cos\theta\rangle$. This agrees with the fact that such state is a superposition (4.13) of two 'phase' states: $|\theta\rangle$ and $|-\theta\rangle$. In each separate state $|\theta\rangle$ we have equidistribution of n, but (4.16) holds for the superposition. According to (4.16), the distribution of n tends to equidistribution as $\theta \to m\pi$ ($m = 0, 1, 2, \ldots$), that is, as $|\theta\rangle \to |-\theta\rangle$.

As noted above, the solution (4.4) of equations (4.1)–(4.3) is not unique. The properties of other solutions have not been analyzed.

Let us summarize our results. The distribution of phase defined as $\varphi_C = \cos^{-1} C$ or $\sin^{-1} S$ does not coincide with the distribution (3.5) of phase θ. Accordingly, the distribution of values of C and S cannot be calculated as the distribution of the trigonometric functions of phase θ.

At the same time, if we are interested not in the distribution of the trigonometric functions of phase as such, but only in its moments, then the values of $\langle C^m\rangle$ and $\langle S^m\rangle$ will be close to the values of $\langle\cos^m\theta\rangle$ and $\langle\sin^m\theta\rangle$, given that for the even values of m the state of the oscillator weakly overlaps with the levels $m/2 - 1$.

5. Phase and phase space

5.1 Wigner phase distribution

The ultimate objective of all approaches to the description of phase properties of the field is the calculation of the distribution of the results of phase measurements. The distribution of phase can be calculated skipping the stage of the definition of phase operator [39]. In this case the phase is regarded as a parameter — the coordinate in the phase plane.

It is known that squared modulus of scalar product of vectors up to the factor of 2π equals to the scalar product in the phase space of the corresponding Wigner functions. Hence, the distribution of phase in the state $|\psi\rangle$ can be represented as

$$P(\phi) = \left|\langle\psi|\phi\rangle\right|^2 = 2\pi \int W_\psi(x, p)\, W_\phi(x, p)\, \mathrm{d}x\, \mathrm{d}p\,. \quad (5.1)$$

Here W_ψ and W_ϕ are the Wigner functions of states $|\psi\rangle$ and $|\phi\rangle$ (where $|\phi\rangle$ is the phase state). In polar coordinates

$$x = r\cos\varphi\,, \qquad p = r\sin\varphi$$

from (5.1) we get

$$P(\phi) = 2\pi \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} W_\psi(r, \varphi)\, W_\phi(r, \varphi)\, \mathrm{d}\varphi\,. \quad (5.2)$$

The next step is the replacement of the Wigner function of the phase state by the $2\pi\delta(\varphi - \phi)$ function. If by the phase state $|\phi\rangle$ we mean the state $|\theta\rangle$ in the form (3.1), then such a replacement must be regarded as an approximation. The actual function W_θ has finite width, and is alternating [40]. For this reason the actual distribution of phase θ may differ from the approximation:

$$P_{\mathrm{W}}(\theta) = \int_0^\infty W_\psi(r, \theta)\, r\,\mathrm{d}r\,. \quad (5.3)$$

In this way, the marginal distribution (5.3) of the joint quasi-probability $W_\psi(r, \theta)$ is the approximate distribution of phase θ. It only tends to the exact distribution as $r \to \infty$ [40].

It should be borne in mind that, even though in many cases the relation (5.3) gives a reasonable result, it may give negative probability for some values of phase [41]. The possibility of getting the negative value of $P_{\mathrm{W}}(\phi)$ proves that there is no phase state $|\phi\rangle$ that would correspond to the Wigner function of the form $\delta(\varphi - \phi)$.

Aside from the Wigner function, the quasi-probabilities that depend on the phase of the complex amplitude $\alpha = r\exp(-\mathrm{i}\phi)$ are Glauber's $P(\alpha)$-function, and Husimi's $Q(\alpha)$-function. The Wigner function simplifies averaging of the correlation functions symmetrically ordered with respect to $\hat{a}$ and $\hat{a}^\dagger$ [42]. For the normally ordered correlation functions it is convenient to use the P-function [43], and the Q-function is more convenient in the case of anti-normally ordered correlation functions [44].

The relevant phase distributions, aside from (5.3), can be the distributions $P_P(\phi)$ and $P_Q(\phi)$ obtained from the $P(\alpha)$ and $Q(\alpha)$ functions by integration with respect to the radial coordinate (amplitude). Like in case of the Wigner function, nonphysical results can be obtained for certain states. For example, the P function does not exist as a normal mathematical function in such important cases as the idealized compressed state and the n-state.

Nothing can stop the imagination of theoreticians. Some proposed modifying the method of defining the phase as a parameter of the phase plane by doubling the number of variables in the phase space, so that the intensity and the phase become complex c-numbers [45].

5.2 Illustration of differences in the approaches

The above-described approaches to the definition of the phase of quantum oscillator [via the phase operators, operators $\hat{C}$ and $\hat{S}$, quasi-probabilities $W(\alpha)$, $P(\alpha)$, $Q(\alpha)$] give different results. For illustration, let us consider the mixed state of the mode [15]

$$\varrho = p|\psi_1\rangle\langle\psi_1| + (1 - p)|\psi_2\rangle\langle\psi_2|\,,$$

where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle + |1\rangle\right\}, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\left\{|1\rangle - |2\rangle\right\}.$$

Now let us compare at what values of parameter p the distribution function of phase describes the uniform distribution.

The Pegg – Barnett distribution (3.5)

$$P(\theta) = \frac{1}{2\pi}\left\{1 + (2p - 1)\cos\theta\right\}$$

will be uniform at $p = 1/2$. For the same state, the distribution

$$P_Q(\phi) = \frac{1}{2\pi}\left\{1 + \frac{\sqrt{\pi}}{2}\left[p - (1 - p)\,\frac{3}{2\sqrt{2}}\right]\cos\phi\right\}$$

will be uniform at $p = 1/(1 + 2\sqrt{2}/3) \neq 1/2$.

The corresponding Wigner phase distribution

$$P_{\mathrm{W}}(\phi) = \frac{1}{2\pi}\left\{1 + \sqrt{\frac{\pi}{2}}\cos\phi\right\}$$

is negative at $\cos\phi < -\sqrt{2/\pi}$.

The phase distribution calculated via the operator $\hat{C}$ does not coincide with any of the above.

Graphic illustrations of the discrepancies between the phase distribution functions in different approaches in the case of superposition and mix of coherent states are given in Refs [46, 31]. For coherent states, the phase distributions according to Pegg – Barnett and according to Wigner differ at the maximum of $P(\theta)$ approximately by 10% at $|\alpha| = 1$, and by 5% at $|\alpha| = 2$.

6. Operational approach. Methods of phase measurement

We know the necessary conditions of exact measurement of quantum observables: the instrument must only receive the information about the observable of interest. Simultaneous reception of information about non-commuting observables is not allowed. The sufficient condition for such measurement is the interaction of the system with the instrument, when the Hamiltonian of the interaction is proportional to the observable in question. Such a scheme of measurements is known as the standard scheme of quantum measurements.

The realization of the standard scheme of phase measurement has not yet been invented (and will probably never be) since it requires such a physical interaction at which the energy of interaction would depend on the phase but not on the amplitude of oscillations. There are no direct phase measurement methods even in the domain of classical (quasiclassical) states. Measurements of phase, like almost all other measurements, are concerned with the coordinate interactions — that is, interactions whose energy is the function of the generalized coordinates: position, charge, field strength.

In the classical domain, obtaining any information about the oscillator requires just knowing the realizations of the time evolution of the coordinate $x(t)$. Most methods of phase measurement rely on the conversion of the signal from quantum level to the classical level.

Measurement of phase by the method of amplification. Using a phase-insensitive quantum (laser, maser) or parametric amplifier, the initial microscopic field is reinforced to the macroscopic level, and the known classical interference techniques are then employed for measuring the phase of the amplified field. In fact, it is possible to measure simultaneously the amplitude r of the field, and find the joint distribution in the phase plane $w(r, \varphi)$. In the process of amplification, the noise of the amplifier is added to the input signal.

The estimates of phase and amplitude of the input field based on the measured phase and amplitude of the amplified field can only be approximate. The accuracy of estimates corresponds to the accuracy of joint measurements of non-commuting observables. The phase distribution

$$P(\varphi) = \int_0^\infty w(r, \varphi)\, r\, \mathrm{d}r$$

obtained by averaging $w(r, \varphi)$ will be broader than the actual phase distribution.

In the case of strong amplification, the function $w(r, \varphi)$ is related to the $Q(\alpha)$-function of the initial field [47 – 49]

$$w(\rho, \varphi) = G^{-1} Q(G^{-1/2}\alpha)\,, \qquad (6.1)$$

where G is the intensity gain. Since $Q(\alpha) \geqslant 0$, this function can be interpreted as the true probability (by contrast to the Wigner function).

As shown in Ref. [50], the measurement of phase by the method of amplification corresponds to the measurement of the Hermitian operator

$$\hat{A} = \frac{1}{\pi} \int \varphi |\alpha\rangle\langle\alpha|\, r\, \mathrm{d}r\, \mathrm{d}\varphi\,.$$

According to (6.1), the $Q(\alpha)$-function is obtained simply by changing the scale of the measured distribution $w(r, \varphi)$. Thus, the method of phase measurement by amplification consists in measuring the Q-function of the signal field. An example of experimental realization is described in Ref. [51].

Heterodyning. The signal at frequency $\nu_0 + \Delta\nu$ is mixed in the beam splitter with a strong coherent field of the heterodyne at frequency ν_0, and is sent to the photodetector. Then the quadrature demodulator is used for measuring the amplitudes y_1 and y_2 of the quadrature components $y_1 \sin \Delta\nu t$ and $y_2 \cos \Delta\nu t$ of photocurrent oscillations [52].

The procedure of measurement, like in the case of amplification, adds noise to the signal. The current oscillations at frequency $\Delta\nu$ include a contribution from the vacuum field at mirror frequency $\nu_0 - \Delta\nu$. It has been proved that the distribution $w(y_1, y_2)$ is nothing else but the $Q(\alpha)$-function for the signal field with $\alpha = y_1 + \mathrm{i}y_2$. Hence, the heterodyne scheme of measurement is physically equivalent to the scheme with amplification.

Homodyning. The conventional homodyne scheme has two inputs and two outputs (ports), to one of which the signal field with frequency ν is applied, and to the other the strong coherent field of the same frequency. The two beams formed after the beam splitter, in which the signal field is mixed with the reference field, fall on two photodetectors. The readings of the photodetectors are used for evaluating one of the quadrature amplitudes of the signal field.

In Ref. [53] an eight-port scheme was proposed (four inputs and four outputs) for a homodyne detector, which provides for simultaneous measurement of both quadrature amplitudes. A 50 : 50 splitter divides the signal into two beams, which are each sent to a homodyne detector. The reference beams are obtained by splitting a strong coherent beam, and are phase-shifted by $\pi/2$ with respect to one another. The two quadrature amplitudes of the signal field are evaluated simultaneously.

Such joint measurement of non-commuting observables, like the measurement schemes described above, are approximate. Fluctuations in the form of vacuum fields penetrate through the two spare ports on the splitters of the signal and reference beams. As shown in Ref. [54], in this homodyne scheme the measured joint distribution of quadratures is also a Q-function of the signal field. (The quantum analysis of the eight-port scheme is carried out in Section 7.5.)

Optical homodyne tomography. The distribution function of phase (like that of other observables) can be calculated from the experimentally determined Wigner function. This can be accomplished with the aid of optical homodyne tomography [55]. Optical homodyne tomography is based on the scheme of homodyne detection, which serves for measuring the quadrature components of the field.

By definition, the Wigner function is

$$W(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \langle x + x' |\, \hat{\rho}\, | x - x' \rangle \exp(-2\mathrm{i}px')\, \mathrm{d}x'\,,$$

where $\hat{\rho}$ is the operator of density, x is the coordinate, and p is the momentum. In the case of the optical mode

$$\hat{x} = \frac{1}{\sqrt{2}}\,(\hat{a} + \hat{a}^{\dagger})\,, \qquad \hat{p} = \frac{1}{\sqrt{2}}\,(\hat{a} - \hat{a}^{\dagger})\,, \qquad [\hat{x}, \hat{p}\,] = \mathrm{i}\,.$$

The quadrature amplitudes $\hat{x}_\theta$ and $\hat{p}_\theta$, defined with respect to the reference phase θ, are

$$\hat{x}_\theta = \hat{x}\cos\theta + \hat{p}\sin\theta\,, \qquad \hat{p}_\theta = \hat{p}\cos\theta - \hat{x}\sin\theta\,.$$

The distribution $P_\theta(x_\theta)$ of the quadrature amplitude x_θ can be represented by the integral over p_θ of the Wigner function [56]:

$$P_\theta(x_\theta) = \int_{-\infty}^{\infty} W(x_\theta\cos\theta - p_\theta\sin\theta,\; x_\theta\sin\theta + p_\theta\cos\theta)\,\mathrm{d}p_\theta\,.$$

The concept of optical homodyne tomography is based on the feasibility of a unique transform into the Wigner function of the set of functions $P_\theta(x_\theta)$, corresponding to the consecutive values of θ from 0 to π [56]. In the case of a finite series of θ values this can be accomplished by the method of Radon transform [57], which is known in the theory of tomographic imaging. Given the Wigner function $W(x, p)$, the Fourier transform can be used for finding the density matrix of the mode under investigation:

$$\langle x + x'|\,\hat{\rho}|x - x'\rangle = \int_{-\infty}^{\infty} W(x, p)\exp(2\mathrm{i}px')\,\mathrm{d}p\,.$$

Having defined the density matrix, one can calculate the distributions of phase and other observables.

The results of experimental studies of the mode in the vacuum state and in the state of compressed vacuum (obtained by compressing the vacuum state with respect to one of the quadrature components) are presented in Ref. [55]. Phase distributions calculated according to the Pegg – Barnett method via the experimentally determined Wigner function, and the Wigner distribution agree well with the theoretical findings.

A characteristic feature of the methods of measurement described above is that they supply information about the phase distribution in the input ensemble, but do not reduce it into states where the uncertainty of phase is less.

Measurement of phase with the aid of a null detector of the coordinate. This method is based on the idea of measuring the phase of oscillations by detecting the instant of transition of the coordinate through its zero value [59, 60]. Such measurement presumes the coordinate interaction of an oscillator with the device in a very narrow region near the zero. In this case the instant of interaction of the oscillator with the instrument corresponds to the phase of oscillations 0 or π. This ambiguity can be eliminated if the null-detector is also capable of detecting the direction of motion.

The interaction of the oscillator with the null-detector is similar to the interaction of a particle with a potential barrier. The more accurate the measurement of the time of crossing the zero, the higher the potential barrier should be. As a result, the increasing accuracy of measurement increases the probability of reflection and prevents the coordinate from crossing the zero [60].

At the same time, the above method of phase measurement is the method of preparation of state with a relatively well-defined phase. The measurement of the time when the coordinate turns to zero leads to the compression of the oscillator state with respect to one quadrature component, and the perturbation of the other. The posterior uncertainty of the phase in the course of compression decreases until the compressed state overlaps with the vacuum state. At $\langle n\rangle \gg 1$ the minimum uncertainty of phase of the compressed state is [61]

$$\Delta\theta_{\mathrm{sqv}} = \frac{1}{2\langle n\rangle}\,\sqrt{\ln\langle n\rangle}\,.$$

The possible realizations of null detectors for mechanical and electromagnetic oscillators are described in Ref. [59].

7. Quantum phase difference

The study of phase is more of academic than of practical interest. In actual phase-sensitive experiments the outcome depends on the phase difference of two (or more) field modes. In most cases one of the modes is taken in the quasiclassical state with a well defined phase, and serves as a reference wave. The measurement of phase difference between the signal wave and the reference wave in such case is actually the measurement of phase of the signal wave. In other situations, however, the two modes may occur in essentially nonclassical and entangled states. Similarly to the single-mode situation, there are several approaches to the phase-dependent properties of the two-mode field.

7.1 Phase difference as the difference of phases of individual modes

The phase difference θ_- in the two-mode field can be represented as the phase difference of individual modes $\theta_- = \theta_2 - \theta_1$. Assuming that the phase state of the single-mode field is vector (3.1), by analogy with relation (3.6) we find the joint distribution of phases θ_1 and θ_2 [9, 62]:

$$P(\theta_1, \theta_2) = \langle\theta_2|\langle\theta_1|\,\hat{\rho}|\theta_1\rangle|\theta_2\rangle\,, \tag{7.1}$$

where $\hat{\rho}$ is the density operator of the two-mode field. The joint phase distribution can be translated into the distribution of the phase difference

$$P(\theta_-) = \int_{2\pi} P(\theta_1, \theta_1 + \theta_-)\,\mathrm{d}\theta_1\,. \tag{7.2}$$

In the case of an arbitrary pure state of the two-mode field

$$|\,ff\,\rangle = \sum_{n_1, n_2} C_{n_1 n_2}|n_1\rangle|n_2\rangle\,,$$

where $|n_1\rangle$ and $|n_2\rangle$ are the energy states of the first and second modes, from (7.1) we get

$$P(\theta_1, \theta_2) = \frac{1}{4\pi^2}\left|\sum_{n_1, n_2} C_{n_1 n_2}\exp\left[-\mathrm{i}(n_1\theta_1 + n_2\theta_2)\right]\right|^2. \tag{7.3}$$

The corresponding distribution of phase difference is

$$P(\theta_-) = \frac{1}{2\pi}\sum_{n_1, n_2}\sum_{n_1'} C^*_{n_1 n_2} C_{n_1'(n_1 - n_1' + n_2)}\exp\left[\mathrm{i}(n_1' - n_1)\theta_-\right]. \tag{7.4}$$

This distribution of phase difference is known as the Pegg–Barnett distribution. If at least one of the modes — for example, mode 2 — is in the energy state $|n_j\rangle$, then (7.4) gives the anticipated result: $P(\theta_-) = 1/2\pi$, which agrees with the notion of the complete uncertainty of the mode's phase in such a state.

Obviously, the distribution of the phase difference $P(\theta_-)$ is invariant with respect to the simultaneous equal shift of the two phases. In other words, the distributions of phase difference are the same in the states $\hat{\rho}$ and $\exp(\mathrm{i}\phi_0(\hat{N}_1 + \hat{N}_2))\,\hat{\rho}\,\exp(-\mathrm{i}\phi_0(\hat{N}_1 + \hat{N}_2))$. This means that the phase difference and the total number of photons in the two-mode field are compatible observables. Consequently, states are possible for which the phase difference and the total number of photons are simultaneously well defined. The complementary observable to the phase difference is the difference of numbers of photons in the modes $\hat{N}_2 - \hat{N}_1$. Therefore, the process that reduces the uncertainty of the phase difference also causes perturbation of the difference of numbers of photons, but does not affect the total number of photons.

The compatibility of the phase difference and the total number of photons allows representation of relation (7.2) in the following form (the subscript '–' is dropped) [62]:

$$P(\theta) = \sum_{n=0}^{\infty} \langle \theta^{(n)} |\, \hat{\rho} | \theta^{(n)} \rangle\,, \tag{7.5}$$

where the vector

$$|\theta^{(n)}\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n_1=0}^{n} \exp(\mathrm{i}n_1\theta)\,|n_1\rangle |n - n_1\rangle \tag{7.6}$$

lies in the subspace with the total number of photons n.

The function

$$P(n,\theta) = \langle \theta^{(n)} |\, \hat{\rho} | \theta^{(n)} \rangle \tag{7.7}$$

is the joint distribution of the phase difference and the total number of photons. The compatibility of the phase difference and the total number of photons can be expressed as

$$P(\theta) = \sum_{n=0}^{\infty} P(n,\theta)\,, \qquad P(n) = \int P(n,\theta)\,\mathrm{d}\theta\,, \tag{7.8}$$

where $P(n)$ is the probability of having n photons in the two-mode field. Thus it is possible to study the phase-difference properties of any state of the field separately in every finite-dimensional subspace H_n without loss of information.

Each function $P(n,\theta)$ has a certain amount of redundancy. It has been proved that the function $P(n,\theta)$ is not arbitrary [62]. In accordance with (7.7) and (7.6), the most general form of this function is

$$P(n,\theta) = \sum_{k=-n}^{n} c_k \exp(\mathrm{i}k\theta)\,, \qquad c_k = c_{-k}^{*}\,. \tag{7.9}$$

The function $P(n,\theta)$ depends on $2n+1$ parameters: it can be completely defined by its values at $2n+1$ appropriately selected points. These can be, for example,

$$\theta_r^{(n)} = \frac{2\pi r}{2n+1} \tag{7.10}$$

with integer-valued r from $-n$ to n.

Inverting (7.9), the coefficients c_k can be expressed in terms of $P(n,\theta)$ at points $\theta = \theta_r^{(n)}$:

$$c_k = \frac{1}{2n+1} \sum_{r=-n}^{n} P(n,\theta_r^{(n)}) \exp\left(-\frac{\mathrm{i}2\pi kr}{2n+1}\right).$$

The values of the function $P(n,\theta)$ at points $\theta = \theta_r^{(n)}$ are given by

$$P(n,\theta_r^{(n)}) = \langle \theta_r^{(n)} |\, \hat{\rho} | \theta_r^{(n)} \rangle\,,$$

where $|\theta_r^{(n)}\rangle$ is the subset of vectors $|\theta^{(n)}\rangle$. As a result, the function $P(n,\theta)$ becomes

$$P(n,\theta) = \frac{1}{2n+1} \sum_{k,r=-n}^{n} P(n,\theta_r^{(n)}) \exp\left[-\mathrm{i}k(\theta - \theta_r^{(n)})\right]. \tag{7.11}$$

Now the mean value of the function $f(\theta)$ can be represented by the relation

$$\langle f(\theta) \rangle = \sum_{n=0}^{\infty} \frac{2\pi}{2n+1} \sum_{r=-n}^{n} P(n,\theta_r^{(n)})\, f^{(n)}(\theta_r^{(n)})\,. \tag{7.12}$$

Here $f^{(n)}(\theta_r^{(n)})$ is the function obtained from $f(\theta)$ by removing all Fourier frequencies above n.

7.2 Phase difference operator

Similarly to the case of the single-mode field (one oscillator), the exponential operator of phase difference, defined by analogy with the classical relation $\exp[\mathrm{i}(\phi_1 - \phi_2)] = \exp(\mathrm{i}\phi_1)\exp(-\mathrm{i}\phi_2)$ in the form of the product $\hat{E}_1\hat{E}_2^{\dagger}$ of the Susskind–Glogower operators for individual modes, will not be unitary. An approach that permits defining the unitary operator corresponding to the phase difference was proposed in Ref. [63]. It is based on the polar decomposition of the complex amplitudes of the two modes in the form

$$\hat{a}_1\hat{a}_2^{\dagger} = \hat{E}_{12}\sqrt{\hat{N}_1(\hat{N}_2 + 1)}\,, \tag{7.13}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators of the corresponding modes.

Similarly to the one-dimensional case, the polar expansion does not completely define the exponential of phase operator. In the present case, the matrix element $\langle n_1, 0|\hat{E}_{12}|0, n_2\rangle$ is undefined, and so the operator $\hat{E}_{12}$ cannot be uniquely determined solely from the criterion of being unitary. The additional conditions are given by relations

$$[\hat{E}_{12}, \hat{N}_1 + \hat{N}_2] = 0\,, \tag{7.14}$$

$$\left[\hat{E}_{12}, \frac{1}{2}(\hat{N}_1 - \hat{N}_2)\right] = \hat{E}_{12}\,, \tag{7.15}$$

derived from the principle of correspondence between quantum commutators and classical Poisson brackets. Observe that the relations (7.14) and (7.15) also admit a nonunitary solution for $\hat{E}_{12}$ — for example, the operator $\hat{E}_1\hat{E}_2^{\dagger}$.

To obtain the unitary solution of (7.13) the equation is reduced to the form

$$\hat{S}_- = \hat{E}_{12}\sqrt{\hat{S}_+\hat{S}_-}\,, \qquad \hat{S}_{\pm} = \frac{1}{2}(\hat{S}_1 \pm \mathrm{i}\hat{S}_2)\,. \tag{7.16}$$

Here $\hat{S}_j$ are the Hermitian Stokes operators, corresponding to the measured classical Stokes parameters [64]:

$$\hat{S}_0 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\,, \qquad \hat{S}_1 = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\,, \\ \hat{S}_2 = \mathrm{i}\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\,, \qquad \hat{S}_3 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\,. \tag{7.17}$$

Operators (7.17) satisfy the commutation relations for the Lie algebra of the 3D rotation group SU(2):

$$[\hat{S}_i, \hat{S}_j] = 2\mathrm{i}\epsilon_{ijk}\hat{S}_k\,, \qquad [\hat{S}_i, \hat{S}_0] = 0\,. \tag{7.18}$$

Up to a factor of 2, the operators $\hat{S}_i$ $(i = 1, 2, 3)$ coincide with the operators of the angular moment, while $\hat{S}_0$ is the operator of the total number of photons.

Further steps for defining the unitary operator $\hat{E}_{12}$ are justified by its commutativity with the operator of the sum of photons (7.14). This property allows study of the restrictions of operator $\hat{E}_{12}$ at first separately on each of the subspaces H_n with the number of photons n. The solution of equation (7.16) is found as the unitary exponential phase operator [65, 66]:

$$\hat{E}_{12}^{(n)} = \sum_{m=1}^{n} |m-1, n-m+1\rangle\langle m, n-m| \\ + \exp\left[\mathrm{i}(n+1)\phi^{(n)}\right]|n,0\rangle\langle 0,n|\,, \tag{7.19}$$

where $\phi^{(n)}$ is the arbitrary phase.

The unitarity of the operator is ensured by the additional term based on the finiteness of the number of states. Accordingly, in each subspace H_n there are $n+1$ orthonormal states, such that

$$\hat{E}_{12}^{(n)}|\phi_r^{(n)}\rangle = \exp(\mathrm{i}\phi_r^{(n)})|\phi_r^{(n)}\rangle\,, \qquad r = 0, 1, \ldots, n\,. \tag{7.20}$$

In the n-basis the state (7.20) is

$$|\phi_r^{(n)}\rangle = \frac{1}{\sqrt{n+1}}\sum_{n_1=0}^{n} \exp(\mathrm{i}n_1\phi_r^{(n)})|n_1, n-n_1\rangle\,, \tag{7.21}$$

and for one and the same 2π-window in all subspaces we have

$$\phi_r^{(n)} = \phi_0 + \frac{2\pi r}{n+1}\,. \tag{7.22}$$

Operator $\hat{E}_{12}$ for the complete space is

$$\hat{E}_{12} = \sum_{n=0}^{\infty}\hat{E}_{12}^{(n)} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}|\phi_r^{(n)}\rangle\exp(\mathrm{i}\phi_r^{(n)})\langle\phi_r^{(n)}|\,. \tag{7.23}$$

Since the operator $\hat{E}_{12}$ is unitary, it defines the Hermitian operator of the phase difference:

$$\hat{\Phi}_{12} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}|\phi_r^{(n)}\rangle\,\phi_r^{(n)}\,\langle\phi_r^{(n)}|\,. \tag{7.24}$$

Observe that formally relation (7.19) is similar to the representation (3.15) of the unitary operator of the single-mode field $\exp(\mathrm{i}\hat{\phi}_s)$ in the Pegg – Barnett H_s-space, in which the additional term $\exp\left[\mathrm{i}(s+1)\theta_0\right]|s\rangle\langle 0|$ ensures that the space is a cyclic one.

The situation in our present case is similar. Under the action of operator (7.19) the state $|m, n-m\rangle$ becomes $|m-1, n-m+1\rangle$ if $m \geqslant 1$, and the state $|0, n\rangle$ transforms into the state $|n, 0\rangle$. Accordingly, the transfer of excitation of modes from one end of the range to the other is allowed. Operator $\hat{\Phi}_{12}$ has discrete eigenvalues. In each subspace H_n there are $n+1$ values evenly distributed on the interval 2π. The Pegg – Barnett phase operator $\hat{\Phi}_s$ (3.16) has similar discrete values at $s = n$. The importance of discreteness is greater when n is small, but in the case of $n = 0$ the requirement of unitarity leads to the uniform distribution of phase.

The eigenstate (7.21) of operator $\hat{E}_{12}^{(n)}$ is the state with the number of photons equally distributed between the modes. This is the expected result. In the single-mode field the phase is conjugate to the number of photons, and the phase state is the state with a uniform distribution of the number of photons. In the two-mode field the observable conjugate to the phase difference is the difference between the number of photons in the modes, and it would be logical to expect that in the state with the preset phase difference the distribution of the difference of the number of photons in the modes will be uniform. Not so obvious is the discreteness of the values of phase difference.

The discrete values of phase difference $\theta_r^{(n)}$ are also observed in the Pegg – Barnett approach [see (7.10)], where they are not associated with the discrete eigenvalues of the observable, but are simply a set of points where the values of the continuous function $P(n, \theta)$ with a finite width are defined. In case of the phase difference operator, the values $\phi_r^{(n)}$ are the eigenvalues of the Hermitian operator. Accordingly, if a measurement were possible that corresponds to the projector-valued probability operator measure $|\phi_r^{(n)}\rangle\langle\phi_r^{(n)}|$, the outcome of this measurement would be the discrete values (7.22).

There is some doubt, however, as to whether the discreteness of phase is a physically sound result. In the first place, aside from the discrete term the formula (7.22) for $\phi_r^{(n)}$ also contains an arbitrary continuous term ϕ_0. Secondly, the change of phase difference by an arbitrary amount θ [for example, under the action of operator $\exp(\mathrm{i}\theta\hat{n}_1)$] leads to a new set of orthonormal vectors $\exp(\mathrm{i}\theta\hat{n}_1)|\phi_r^{(n)}\rangle$, which may be obtained from the old set by replacing ϕ_0 with $\phi_0 + \theta$. Accordingly, the change of the phase difference can be continuous. Thirdly, from the formal standpoint, the Hermitian phase difference operator with discrete eigenvalues could also be obtained in the Pegg – Barnett approach by choosing from the set of vectors (7.6) (which in the general case are non-orthogonal) a subset of orthogonal vectors with the corresponding discrete values of θ. This, however, would not in any way affect the interpretation of the phase difference as the difference of the continuous values of phases of individual modes.

The experimental proof of the discreteness of phase difference is often supposed to be the discreteness of readings in the known interferometer schemes, in which the last stage is the photon counter. This proof is, however, not convincing, because the discreteness of readings is associated with the discreteness of the photon count. Aside from this, the interferometer schemes measure the sine or the cosine of the phase difference rather than the phase difference itself. Although the eigenvalues of the operators of cosine and sine of phase difference in the H_n-space are also discrete, they do not coincide with the values of $\phi_r^{(n)}$ (see Section 7.3).

From (7.24) it also follows that the mean value of the function $f(\hat{\Phi}_{12})$ in the state $|\psi\rangle$ is

$$\langle\psi|\,f(\hat{\Phi}_{12})\,|\psi\rangle = \sum_{n=0}^{\infty}\sum_{r=0}^{n}P(n, \phi_r^{(n)})\,f(\phi_r^{(n)})\,, \tag{7.25}$$

$$P(n, \phi_r^{(n)}) \equiv \left|\langle\psi|\phi_r^{(n)}\rangle\right|^2\,.$$

Expression (7.25) differs from its counterpart (7.12) in the Pegg – Barnett approach only in the limits of summation and the normalization coefficient.

In the limiting transition in operator $\hat{E}_{12}^{(n)}$, when at least one of the modes (for example, mode 2) is in a highly excited state, we get the same pattern of violation of operator algebra as in the case with the Pegg – Barnett phase operator [63]. The limit of operator $\hat{E}_{12}$ as $n \to \infty$ is the Susskind – Glogower nonunitary exponential phase difference operator $\hat{E}_1 \hat{E}_2^{\dagger}$.

The mean value of the periodic function of phase difference $f(\hat{\Phi}_{12})$ in the same limit in the state $|\psi\rangle = |\psi_1, \psi_2\rangle$ is given by

$$\langle\psi|\, f(\hat{\Phi}_{12})|\psi\rangle \simeq \int_{\theta_0}^{\theta_0+2\pi} \mathrm{d}\theta \int_{\phi_0}^{\phi_0+2\pi} P_1(\phi+\theta)\, P_2(\theta) f(\phi)\, \mathrm{d}\phi\,, \tag{7.26}$$

where $P(\theta) = \left|\langle\theta|\psi\rangle\right|^2$ is the distribution function of phase of the single-mode field.

Example 1. *Coherent states of modes*. If both modes occur in highly excited coherent states $\alpha_i = |\alpha_i| \exp(\mathrm{i}\phi_i)$, from (7.26) we get the same result as in case of the semiclassical analysis:

$$\langle\Phi_{12}\rangle = \phi_1 - \phi_2\,, \qquad \left\langle(\Delta\Phi_{12})^2\right\rangle = \frac{1}{4|\alpha_1|^2} + \frac{1}{4|\alpha_2|^2}\,. \tag{7.27}$$

Example 2. *State of two-mode compressed vacuum*. This nonclassical state of light with strong quantum correlations between the modes [67] can be represented as the result of action of operator of two-mode compression on the vector of the two-mode vacuum field:

$$|\zeta\rangle = \exp\left(\zeta \hat{a}_1^{\dagger}\hat{a}_2^{\dagger} - \zeta \hat{a}_1 \hat{a}_2\right)|0,0\rangle\,. \tag{7.28}$$

In the n-basis this state corresponds to the expression

$$|\zeta\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n s}{\cosh s} \exp(\mathrm{i}n\xi)\,|n,n\rangle\,, \tag{7.29}$$

where the parameter of compression is $\zeta = s\exp(\mathrm{i}\xi)$. The total number of photons is even. The projection onto the H_{2n} space is the exact state with the preset difference of photon numbers. Accordingly, one could anticipate a uniform phase difference distribution.

From (7.29) it follows that

$$\langle\Phi_{12}\rangle = \phi_0 + \pi\left\langle\frac{\hat{N}}{\hat{N}+1}\right\rangle,$$
$$\left\langle(\Delta\Phi_{12})^2\right\rangle = \frac{4\pi^2}{3}\left\langle\frac{\hat{N}(\hat{N}+1/2)}{(\hat{N}+1)^2}\right\rangle - \pi^2\left\langle\frac{\hat{N}}{\hat{N}+1}\right\rangle^2, \tag{7.30}$$

where $\hat{N}$ is the operator of the total number of quanta. In the limit of large s, as expected, $\left\langle(\Delta\Phi_{12})^2\right\rangle \to \pi^2/3$. However, as $s \to 0$, when the state tends to $|0,0\rangle$ (the vacuum state), we have $\left\langle(\Delta\Phi_{12})^2\right\rangle \to 0$, which does not agree with the notion of the continuous phase difference.

7.3 Operators of sine and cosine of phase difference

The classical $\cos(\theta_2 - \theta_1)$ and $\sin(\theta_2 - \theta_1)$ correspond to the Hermitian operators [8]

$$\hat{C}_{12} = \frac{1}{2}\left(\hat{E}_{1-}\hat{E}_{2+} + \hat{E}_{1+}\hat{E}_{2-}\right), \tag{7.31}$$

$$\hat{S}_{12} = \frac{1}{2\mathrm{i}}\left(\hat{E}_{1-}\hat{E}_{2+} - \hat{E}_{1+}\hat{E}_{2-}\right), \tag{7.32}$$

which commute with the operator of the total number of photons. As a consequence, there are eigenfunctions of operators that exhibit a fixed principal number n. Since the number of such states is finite, the wave function is normalizable. Therefore, the spectra of operators $\hat{C}_{12}$ and $\hat{S}_{12}$ are discrete, as opposed to the spectra of operators $\hat{C}$ and $\hat{S}$.

The commutator

$$[\hat{S}_{12}, \hat{C}_{12}] = \frac{1}{2\mathrm{i}}(\hat{P}_0^{(1)} - \hat{P}_0^{(2)})\,, \qquad \hat{P}_0^{(j)} = |n_j = 0\rangle\langle n_j = 0|\,,$$

is diagonal in the states with the given total number of excitations:

$$\left[[\hat{S}_{12}, \hat{C}_{12}], \hat{N}\right] = 0\,. \tag{7.33}$$

The eigenfunctions of operator $\hat{C}_{12}$, which at the same time are the eigenfunctions of operator $\hat{N}$, are

$$|n, \cos\theta_{nr}\rangle = \left(\frac{2}{n+2}\right)^{1/2} \sum_{n_1=0}^{n} \sin\left((n+1)\theta_{nr}\right)|n_1, n-n_1\rangle\,, \tag{7.34}$$

where

$$\theta_{nr} = \pm\frac{\pi r}{n+2}\,, \qquad r = 1, 2, \ldots, n+1\,. \tag{7.35}$$

The scalar product of eigenfunctions is

$$\langle\cos\theta_{nr}, n\,|\,n, \cos\theta_{ms}\rangle = \delta_{nm}\delta_{rs}\,. \tag{7.36}$$

The state $|0,0\rangle$ corresponds to the eigenfunction of operator $\hat{C}_{12}$ with the zero eigenvalue.

The eigenfunctions of operators $\hat{S}_{12}$ and $\hat{N}$ are

$$|\,n, \sin\varphi_{nr}\rangle = \left(\frac{2}{n+2}\right)^{1/2} \sum_{n_1=0}^{n} (-\mathrm{i})^{n_1} \sin\left[(n+1)\varphi_{nr}\right] \times |n_1, n-n_1\rangle\,. \tag{7.37}$$

The corresponding eigenvalues (angles)

$$\varphi_{nr} = \theta_{nr} - \frac{\pi}{2}$$

lie in the interval

$$-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\,.$$

When the phase difference is changed by an arbitrary value θ, the vector $\exp(\mathrm{i}\theta\hat{n}_1)|n, \cos\theta_{nr}\rangle$ ceases to be the eigenvector of operator $\hat{C}_{12}$. A similar situation occurs with the eigenvector of operator $\hat{S}_{12}$.

The values of phase difference θ_{nr} and φ_{nr} are discrete, and in this respect they are similar to $\phi_r^{(n)}$. However, while the discreteness of the phase difference $\phi_r^{(n)}$ is $2\pi/(n+1)$, the discreteness of the phase difference θ_{nr} and φ_{nr} is $\pi/(n+2)$.

Summary. After comparison of the different approaches to the description of phase difference of the two-mode field, one can answer the question 'what is the actual phase difference?'. In the author's opinion, this question is incorrect from the outset. There is no uniquely defined and observable phase difference that would correspond to its classical counterpart. In quantum theory everything depends on what is being measured — that is, what probability operator measure is realized in this particular measurement.

If it is the probability operator measure corresponding to the Pegg – Barnett approach, then the results of the measurement will be continuous. If the probability operator measure corresponds to the operators $\hat{\Phi}_{12}$ or $\hat{C}_{12}$ and $\hat{S}_{12}$, the readings will be discrete.

7.4 Phase difference and *Q*-function

As indicated above, practically all realistic methods of measuring the phase of the single-mode field reduce to the definition of the Q-function of the signal field. The phase distribution is obtained as the marginal distribution after integration with respect to the radial variable. In a similar way, one could define the simultaneous distribution of phases of the two-mode field as the marginal distribution of the Q-function of the two-mode field after integration with respect to the radial variables of each of the modes.

By definition, the Q-function of the two-mode field is

$$Q(\alpha_1, \alpha_2) = \frac{1}{\pi} \langle \alpha_1, \alpha_2 | \rho | \alpha_1, \alpha_2 \rangle \,, \tag{7.38}$$

where $\alpha_{1,2} = r_{1,2} \exp(\mathrm{i}\theta_{1,2})$, $|\alpha_1, \alpha_2\rangle = |\alpha_1\rangle |\alpha_2\rangle$.

The corresponding simultaneous phase distribution is

$$P(\theta_1, \theta_2) = \int_0^\infty \int_0^\infty Q\big(r_1 \exp(\mathrm{i}\theta_1),\, r_2 \exp(\mathrm{i}\theta_2)\big)\, r_1 r_2 \, \mathrm{d}r_1 \, \mathrm{d}r_2 \,. \tag{7.39}$$

According to Ref. [62], after some algebra we get

$$P(\theta) = \sum_{n=0}^{\infty} P(n, \theta) \,.$$

Here

$$P(n, \theta) = \frac{1}{2\pi} \sum_{n_1, n_1' = 0}^{n} G_{n_1 n_1'}^{(n)} \exp\big[\mathrm{i}(n_1 - n_1')\theta\big] \times \langle n_1', n - n_1' | \, \hat{\rho} \, | n_1, n - n_1' \rangle \,, \tag{7.40}$$

$$G_{n_1 n_1'} = \frac{\Gamma\big((n_1 + n_1')/2 + 1\big)\, \Gamma\big((n + 1 - (n_1 + n_1')/2)\big)}{[n_1! n_1'! (n - n_1)! (n - n_1')!]^{1/2}} \,.$$

The structure of equation (7.40) is similar to that of (7.7) and differs only in the values of the coefficients $G_{n_1 n_1'}^{(n)}$ (in the Pegg – Barnett expression these coefficients are equal to one). This difference in the coefficients reflects the fact that expression (7.5) corresponds to the distribution that can be obtained through the exact measurement of phase difference, whereas (7.7) relates to the inexact measurement, characteristic of the simultaneous measurement of amplitude and phase of oscillations [68, 69].

7.5 Operational definition of phase difference

An original scheme of measurement of phase difference, the corresponding operators and experimental results are reported in Refs [53, 71]. The scheme involves four beam splitters (BS) and four photodetectors (D), as well as four input and four output ports (Fig. 4). The signals under investigation only occupy two of the input ports, the other two receive vacuum fields (01 and 02). The output modes 3 – 6 fall on the photodetectors D_3 – D_6. One of the arms of interferometer introduces a phase shift by one-fourth of the wavelength.

Analysis from the standpoint of classical optics gives the following relations between the registered energies W_j and

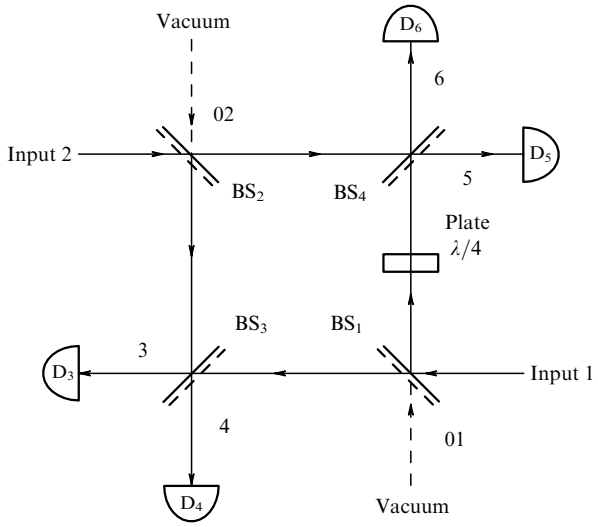

Figure 4. Eight-port scheme of measurement of phase difference [53].

the cosine and sine of the phase difference of the input modes:

$$\frac{W_4 - W_3}{\sqrt{W_1 W_2}} = \cos(\phi_2 - \phi_1) \,,$$

$$\frac{W_6 - W_5}{\sqrt{W_1 W_2}} = \sin(\phi_2 - \phi_1) \,,$$

$$W_j \equiv \alpha \int_t^{t+T} I_j(\xi) \, \mathrm{d}\xi \,, \qquad j = 1, 2, \ldots, 6 \,.$$

Here I_j is the intensity of modes, T is the time interval of the measurements, and α is the quantum efficiency of the detectors.

The transition to the quantum description consists in the replacement of the continuous values of W_j by the discrete readings of photodetectors, which are subsequently replaced with the operators of the number of quanta in the modes. Some freedom is allowed in the replacement of $W_1 W_2$ with

$$(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2 \,.$$

The measured operators are represented in the form

$$\hat{C}_{\mathrm{M}} = \frac{\hat{n}_4 - \hat{n}_3}{\big[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2\big]^{1/2}} \,, \tag{7.41}$$

$$\hat{S}_{\mathrm{M}} = \frac{\hat{n}_6 - \hat{n}_5}{\big[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2\big]^{1/2}} \,, \tag{7.42}$$

$$\hat{n}_4 - \hat{n}_3 = \frac{1}{2}(\hat{a}_1^\dagger + \eta \hat{a}_{10}^\dagger)(\hat{a}_2 + \eta \hat{a}_{20}) + \mathrm{H.c.} \,,$$

$$\hat{n}_6 - \hat{n}_5 = \frac{1}{2}(-\mathrm{i}\hat{a}_1^\dagger + \mathrm{i}\eta \hat{a}_{10}^\dagger)(\hat{a}_2 + \eta \hat{a}_{20}) + \mathrm{H.c.} \,,$$

where η is the phase coefficient that takes care of the phase shift in beam splitters.

When the operators are defined in the form (7.41) and (7.42), the experiments in which simultaneously $n_4 - n_3 = 0$ and $n_6 - n_5 = 0$ were excluded from the statistical treatment, and the normalization was adjusted appropriately. The distribution of phase difference was calculated via the

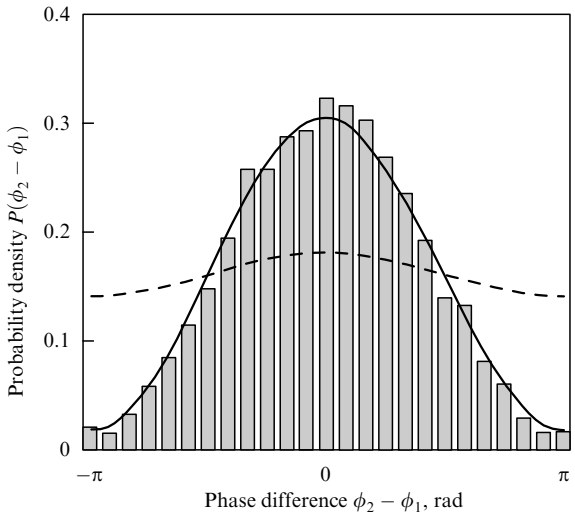

Figure 5. Experimental and theoretical distributions obtained by direct measurement of phase difference between weak ($|\alpha_1|^2 = 0.047$, $|\alpha_2|^2 = 0.076$) input modes in the coherent states: the solid curve shows the theoretical distribution of phase difference according to the method of Ref. [71], the dashed curve is the distribution according to the Pegg–Barnett formula (7.5).

characteristic function

$$C(x) = \left\langle (\hat{C}_{\mathrm{M}} + \mathrm{i}\hat{S}_{\mathrm{M}})^x \right\rangle .$$

Two different experimental techniques were used for studying the distribution of the phase difference of two weak coherent fields $|\alpha_1\rangle$ and $|\alpha_2\rangle$ for $|\alpha_1|^2 = 0.047$ and $|\alpha_2|^2 = 0.076$.

Figure 5 shows experimental and theoretical distributions obtained by direct measurement of the phase difference between two weak input modes. The two input modes were obtained by splitting the coherent laser beam with an additional beam splitter. The results of the experiment agree well with the calculations according to the above formulas, but differ dramatically from the calculations by the Pegg–Barnett formula (7.5), and contradict the notion of the large uncertainty of phase in weak fields.

The results of the experiment were so unexpected that it was quite reasonable to ask whether there is a unique phase difference operator for two quantum fields [71]. In our opinion, the observed narrow phase difference distribution is associated with the fact that the processing of experimental data was performed discarding the zero readings that disguise information about the phase. Such a treatment has the right to exist, but it is not certain to what extent its results may be interpreted as the distribution of phase difference between coherent fields.

In the second case the distribution of phase difference between weak modes was measured in an indirect way. First the distributions of phase difference between each of the weak modes and the strong coherent field were found experimentally (in the scheme of measurement one of the weak modes was replaced with the strong field). In this case the distributions of phase difference were actually the distributions of phases of the weak waves. The phase difference distribution of the weak waves was then calculated as the distribution of the difference of two independent random quantities. The resulting distribution agrees well both with the theoretical distribution according to Ref. [71], and with the Pegg–Barnett distribution.

8. Detection and evaluation of phase shift and phase difference

8.1 Errors in detection and evaluation

The problem of detection of phase shift is a special case of the general problem of verification of hypotheses [17]. This problem consists in using the results of measurement for making the decision as to in which of the two possible states the object occurred: in the state $\hat{\rho}_0$, or in the phase-shifted state $\hat{\rho}_1$. The wrong decision may be either a false alarm (solution H_1 for the state $\hat{\rho}_0$), or a miss (solution H_0 when the actual state is $\hat{\rho}_1$).

The mean probability of error in distinguishing between the two states is [17, 19]

$$P_{\mathrm{ed}} = \zeta_0 P(H_1|\,\rho_0) + \zeta_1 P(H_0|\,\rho_1)\,,$$

where ζ_0 and ζ_1 are the *a priori* probabilities of the states, and $P(H_1|\,\rho_0)$ and $P(H_0|\,\rho_1)$ are the probabilities of false alarm and miss, respectively. The probability of error depends on the states that are being distinguished, on what is being measured, and on the criterion used for making the decision.

The method of optimal measurement that gives the least probability of error is the subject of the theory of optimal detection. It gives the general equation of optimal measurement and the lower limit of the mean probability of error in distinguishing between the two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ [17, 19]:

$$P_{\mathrm{ed}} = \frac{1}{2}\left[1 - \sqrt{1 - 4\zeta_0\zeta_1|\gamma|^2}\,\right], \qquad \gamma = \langle\psi_0|\psi_1\rangle\,. \tag{8.1}$$

Only the orthogonal states can be distinguished without mistake ($\gamma = 0$).

Also used for the measure of the distinguishing error is the quantity

$$D = \sqrt{1 - 4\zeta_0\zeta_1|\gamma|^2}\,, \tag{8.2}$$

which is known as the distinguishability [73]. Error-free distinguishing between the states corresponds to $D = 1$.

Another measure of the reliability of distinguishing is the so-called measured distinguishability

$$D_m = \sum_i \bigl(P_0(X_i) - P_1(X_i)\bigr)\,. \tag{8.3}$$

Here X_i are the results of measurement of the observable $\hat{X}$ selected from certain considerations, and $P_0(X_i)$ and $P_1(X_i)$ are their distributions in the corresponding states. If the values of X_i are continuous, the sum in (8.3) is replaced with the integral. Obviously, $D_m \leqslant D$. They will be equal only if $\hat{X}$ is the optimal observable.

When the decision is made based on the method of highest plausibility, the *a priori* probabilities of the states are not taken into account. Accordingly, in (8.1) and (8.3) one must set $\zeta_0 = \zeta_1 = 1/2$. In this case

$$D = \sqrt{1 - |\gamma|^2}\,. \tag{8.4}$$

The problem of evaluation of a physical quantity consists in obtaining the estimate for the variable in question. While in the problem of detection the phase shift is given, in the evaluation problem its value is not known. Since the phase

shift does not change the form of the distribution, but only changes its parameter (the mean value), the problem of evaluation of phase shift pertains to the quantum problem of evaluating the parameter of shift (displacement) [17, 19].

Similarly to the problem of detection, the quantum theory of evaluation deals with the problem of optimal measurement — that is, the measurement that allows one to get the best estimate with respect to the selected criterion. If the evaluation is based on the previously obtained results of the experiment, the evaluation problem becomes classical: the lower limit of dispersion of the evaluation is defined by the Rao – Kramer inequality. This classical limit will not exceed the quantum limit only if the measurement is optimal.

The optimal measurement corresponds to the quantum analog of the Rao – Kramer inequality [17 – 19]. For the evaluation of the parameter of displacement θ of a state $|\psi_0\rangle$ to the state $|\psi(\theta)\rangle = \exp(\mathrm{i}\theta\hat{N})|\psi_0\rangle$ this inequality has the form

$$\Delta^2\tilde{\theta} \geqslant \left[4k\langle\psi(\theta)|\left(\frac{\partial\hat{\rho}}{\partial\theta}\right)^2|\psi(\theta)\rangle\right]^{-1}, \tag{8.5}$$

where $\tilde{\theta}$ is the estimate of the parameter θ, and $\hat{\rho} = |\psi(\theta)\rangle\langle\psi(\theta)|$. If $\hat{N}$ is the operator of the number of quanta, then θ is the parameter of displacement of phase of the oscillator. Since a displacement by 2π brings one back to the initial state, the phase shift can only be detected within 2π. Accordingly, the dispersion of the estimate must also fall within this interval. If we disregard this circumstance, from (8.5) we get

$$\Delta^2\tilde{\theta} \geqslant \frac{1}{4k\Delta^2 n}, \tag{8.6}$$

where $\Delta^2 n$ is the dispersion of the number of quanta in the initial state, and k is the number of independent measurements. Obviously, relation (8.6) only holds when $\Delta n \gg 1$. As $\Delta n \to 0$ we must have $\Delta\tilde{\theta} \to \pi/\sqrt{3}$, in accordance with the uniform phase distribution.

According to (8.6), the dispersion of the optimal estimate of phase shift of a pure state depends not on the form of the initial state, but only on the dispersion of the number of quanta in this state. The choice of optimal measurement depends on the initial state. In the case of optimal evaluation of the phase shift, the optimal measurement may be not the measurement of the phase, but rather the measurement of a phase-dependent combination of observables [19].

8.2 Condition of reliable distinction of phase shift

The scalar product of the initial $|\psi_0\rangle$ and phase-shifted $|\psi(\theta)\rangle = \exp(\mathrm{i}\theta\hat{N})|\psi\rangle$ vectors can be represented as the mean value of the operator $\exp(\mathrm{i}\theta\hat{N})$ in the state $|\psi_0\rangle$

$$\gamma = \langle\psi_0|\exp(\mathrm{i}\theta\hat{N})|\psi_0\rangle = \sum_{n=0}^{\infty}\exp(\mathrm{i}\theta n)P_0(n),$$

where $P_0(n)$ is the distribution of the number of photons in the initial state.

There are many states that satisfy the condition of orthogonality $\gamma = 0$. In Refs [74, 19] the problem of the minimum value P_{ed} is solved for a given value of Δn. It is demonstrated that

$$|\gamma|^2_{\min} = \begin{cases} \cos^2(\Delta n\,\theta), & \Delta n\,\theta < \pi/2, \\ 0, & \Delta n\,\theta > \pi/2. \end{cases} \tag{8.7}$$

The corresponding optimal state $|\psi_0\rangle$ at $\Delta n\,\theta \leqslant \pi/2$ is the state in which

$$P_0(n) = \frac{1}{2}\left[\delta(n - n_0 - \Delta n) + \delta(n - n_0 + \Delta n)\right]. \tag{8.8}$$

(At $\Delta n\,\theta > \pi/2$ the optimal state will be a different one [74].)

It follows that one can reliably detect the deterministic phase shift by

$$\theta \geqslant \frac{\pi}{2\Delta n} \geqslant \frac{\pi}{2n_0} = \frac{\pi}{2\langle n\rangle}. \tag{8.9}$$

This formula gives *the quantum limit of the reliably distinguished deterministic phase shift*. (The quantity $\pi/(2\langle n\rangle)$ is close but not equal to the Heisenberg limit of uncertainty of the phase (3.12).) A relation similar to (8.9) was obtained in Ref. [73].

Relation (8.7) assumes both optimal initial state and optimal measurement, which (as follows from the above arguments concerning the phase states) is not the measurement of phase. The measurement of phase (that is, the application of the probability operator measure $\hat{\Pi}(\theta) = |\theta\rangle\langle\theta|$) is not optimal even in the case of distinguishing between the Pegg – Barnett orthogonal states $|\theta_s\rangle$ [73].

The optimal measurement corresponds to the probability operator measure [17, 19]

$$\hat{\Pi}(X) = \sum_{\eta_k > 0}|\eta_k\rangle\langle\eta_k|.$$

Here $|\eta_k\rangle$ are the eigenvectors of the operator

$$\hat{\rho}_1(\theta) - l\hat{\rho}_0 = \sum_k \eta_k|\eta_k\rangle\langle\eta_k|,$$

where $\hat{\rho}_0$ and $\hat{\rho}_1(\theta)$ are the statistical operators of the initial and the displaced states, and l is a real number that defines the threshold for making the decision concerning the displacement of the state. When the criterion of maximum plausibility is used, $l = 1$. The decision that the phase has shifted is taken if we get $\eta_k > 0$ as a result of the measurement. In Ref. [73] the distinguishability of phase shift of a number of known states is studied.

A coherent state allows detection with a distinguishability $D < 1$ of a phase shift of

$$\theta = \arccos\left(1 - \frac{1}{2\langle\hat{n}\rangle}\ln\frac{1}{1 - D^2}\right). \tag{8.10}$$

In order to obtain the desired distinguishability, the mean number of photons must be not less than $(1/4)\ln\left[1/(1-D^2)\right]$. When $\langle\hat{n}\rangle$ is large enough, the asymptotic value of (8.10) is

$$\theta = \left(-\frac{1}{\langle n\rangle}\ln(1 - D^2)\right)^{1/2}. \tag{8.11}$$

The exponential state,

$$|\psi\rangle = A\sum_{n=0}^{s}\frac{1}{n+1}\exp(\mathrm{i}\phi n)|n\rangle,$$

(where A is the normalization coefficient), proposed as the state for the most accurate observation of phase shift [75, 76] does not hold up to the expectations. Even as $s \to \infty$ the

exponential state provides for a distinguishability not higher than $D = \sqrt{3}/2$ for $\theta = \pi$.

The phase-coherent state,

$$|\psi\rangle = \sqrt{1-x^2}\sum_{n=0}^{\infty} x^n \exp(\mathrm{i}\beta n)|n\rangle\,,$$

where x and β are real numbers, and $-1 < x < 1$, at $\langle n\rangle = x^2/(1-x^2) \gg 1$ allows one to distinguish a phase shift

$$\theta = \sqrt{\frac{D^2}{1-D^2}}\,\frac{1}{\langle n\rangle}\,.$$

8.3 Distinguishability of phase shift in the two-mode states The errors in the distinguishability of the two-mode states

$$|\psi_0\rangle = \sum_{n_1=0}^{n} c_{n_1}|n_1, n-n_1\rangle\,,$$

$$|\psi(\theta)\rangle = \sum_{n_1=0}^{n} \exp(\mathrm{i}\theta n_1)c_{n_1}|n_1, n-n_1\rangle$$

are defined by the same relations (8.1) and (8.3) as in the case of the single-mode states.

The necessary condition of reliable distinguishability of the shift of phase difference is satisfied for the orthogonal states (7.21), (7.34) and (7.37). The least reliably distinguishable shift of phase difference of the state (7.21) with the total number of photons n is

$$\frac{2\pi}{n+1} = \frac{\pi}{\langle n_1\rangle + 1/2} \simeq \frac{\pi}{\langle n_1\rangle}\,, \tag{8.12}$$

where $\langle n_1\rangle$ is the mean number of photons in one mode. In the states (7.34) and (7.37) at the same value of n this shift is reduced almost by half:

$$\frac{\pi}{n+2} \simeq \frac{\pi}{2\langle n_1\rangle}\,.$$

From the formal standpoint both solutions are correct. Each variant corresponds to its own optimal measurement — that is, the optimal probability operator measure. The physical feasibility of the corresponding measurement is not guaranteed. The minimal shift of phase difference for a given distinguishability for known nonorthogonal two-mode states was calculated in Ref. [73].

If the two-mode field is formed by splitting a *coherent* field with a 50% beam splitter (whose other input is the vacuum field), then it will occur in the state

$$|\psi\rangle = \left|\frac{\alpha}{\sqrt{2}}\,, \frac{\mathrm{i}\alpha}{\sqrt{2}}\right\rangle.$$

As a result of the phase shift of one of the modes we get the state

$$|\psi(\theta)\rangle = \exp(\mathrm{i}\theta\hat{n}_1)\,|\psi\rangle\,.$$

In this case the minimal phase shift detectable with distinguishability $D < 1$ coincides with the relation (8.10) if by $\langle n\rangle$ we mean $\langle n_1\rangle = \langle n\rangle/2$.

If a conventional Mach–Zehnder interferometer is used with the initial coherent field for the generation of the two-mode state and the detection of phase difference, then the shift of phase difference for a given measured distinguishability D_m is

$$\theta \simeq 2\left(-\frac{1}{\langle n\rangle}\ln(1-D_m)\right)^{1/2},$$

which is $\sqrt{2}$ times greater than in the case of optimal measurement. This is an illustration of the fact that in the general case the balance interferometer is not an optimal device for the measurement of phase shift [73].

In the case when the input fields of the beam splitter are *states* $|n\rangle$ and $|0\rangle$, the resulting field will be in the two-mode binomial state. The optimal distinguishability of the initial state and the state shifted by θ with respect to phase difference is given by the relation [73]

$$D = \left(1 - \cos^{2\langle n\rangle}\frac{\theta}{2}\right)^{1/2}.$$

The smallest reliably distinguishable shift of phase difference here equals π irrespective of the value of $\langle n\rangle$. If the Mach–Zehnder interferometer is used instead of the optimal instrument, the measured distinguishability for the binomial field and $\langle n\rangle \gg 1$ will be the same as for the coherent field with the same mean value $\langle n\rangle$.

For the input field of the interferometer that would allow approaching the Heisenberg limit of phase shift distinguishability, the *state* $|n, n\rangle$ was proposed ($|n\rangle$-states at each of the two inputs, the total number of photons is $2n$) [77–79]. In this case the optimal distinguishability of the shift of phase difference is [73]

$$D = \left(1 - L_n^2(\cos\theta)\right)^{1/2},$$

where $L_n(x)$ is the nth Legendre polynomial. It is possible to reliably distinguish the phase difference shift equal to $\theta \approx 4.8/\langle n\rangle$, where $\langle n\rangle = 2n$. Then in the Mach–Zehnder interferometer before the 50% beam splitter the measured distinguishability is $D_m = D^2$. The two distinguishabilities turn to unity at one and the same value of θ.

An experimental study of the distribution of the phase difference in the two-mode field formed from the input field $|1, 1\rangle$ is described in Ref. [72]. The two-mode state $|1, 1\rangle$ in the form of short pulses was generated with the aid of a spontaneous degenerate parametric transform, as a result of which the pumping photon splits into a pair of indistinguishable photons traveling in different directions. After rotation of the polarization plane of one of the modes by $\pi/2$, the modes were mixed on the polarization beam splitter in such a way that the state $|1, 1\rangle$ was created again at one of the output ports, this time on the basis of two orthogonal linearly polarized modes.

A phase shift between the modes was introduced with the liquid crystal birefringent cell. The fast axis of the cell was oriented with respect to the polarization vectors in such a way that the state $|1, 1\rangle$ was transformed into the state

$$\frac{1}{\sqrt{2}}\left(-\mathrm{i}\sin(2\xi)|0, 2\rangle + \sqrt{2}\cos(2\xi)|1, 1\rangle - \mathrm{i}|2, 0\rangle\right).$$

The parameter ξ corresponds to the angle between the fast axis of the cell and the vertical polarization vector.

The angle ξ can be selected so that the state becomes equidistributed:

$$|\psi^{(2)}\rangle = \frac{1}{\sqrt{3}}\left(-\mathrm{i}|0,2\rangle + |1,1\rangle - \mathrm{i}|2,0\rangle\right). \tag{8.13}$$

The phase shift between the modes can be controlled by varying the voltage applied to the cell. The state phase-shifted with respect to the first mode is

$$|\psi^{(2)}(\theta)\rangle = \frac{1}{\sqrt{3}}\left(-\mathrm{i}|0,2\rangle + \exp(\mathrm{i}\theta)|1,1\rangle - \mathrm{i}\exp(2\mathrm{i}\theta)|2,0\rangle\right). \tag{8.14}$$

After the liquid crystal cell the two-mode field falls on the other polarization beam splitter. Its characteristics are such that it transforms the state (8.13) back in the state $|1,1\rangle$ with spatially separated modes. In this case the photodetectors placed after the beam splitter give 100% coincident readings. The probability of coincidence decreases with increasing phase difference shift. The initial and the displaced states can be unmistakably distinguished if they are mutually orthogonal.

From (8.13) and (8.14) it follows that

$$\left|\gamma(\theta)\right|^2 = \left|\langle\psi^{(2)}|\psi^{(2)}(\theta)\rangle\right|^2 = \frac{\sin^2(3\theta/2)}{9\sin^2(\theta/2)}. \tag{8.15}$$

Accordingly, the states will be orthogonal at $\theta = \pm 2\pi/3$, which agrees with (8.12). If with such phase shift the probability of coincidence of photodetector readings is zero, then both the optimal distinguishability D and the measured distinguishability D_m in this scheme will be equal to one.

The experimental results reported in Ref. [72] as the number of coincidences as a function of phase shift are only in qualitative agreement with the theory. There is a maximum of the number of coincidences at $\theta = 0$, and minimums at $\theta = \pm 2\pi/3$. However, while the visibility of the theoretical curve is unity, the visibility of the experimental curve is 45%. Nevertheless, the impacts of various technical effects on the visibility estimated in Ref. [72] indicate that the effects of quantum interference between the modes were indeed observed in the experiment.

8.4 Symphoton states in Michelson interferometer

The measuring schemes in which the studied effect is the change of the phase of the wave are actually using waves in the coherent state. Such waves are more easily generated and maintained. However, they also have an important drawback: their resolution with respect to the phase shift is inversely proportional to $\sqrt{\langle n\rangle}$ rather than to $\langle n\rangle$, as in the Heisenberg limit. For the required sensitivity $\Delta\theta$ the power of the wave in the coherent state must be inversely proportional to $\Delta^2\theta$, and may become unacceptably large. For example, a power as large as 30 kW circulates in the interferometer in the laser gravitational antenna (LIGO Project, USA [80]).

The planned increase of sensitivity will require raising the power to 1 MW. At the same time, if we use the optimal state in place of the coherent state, the required power could be reduced to 10^{-4} W. An original method of creating such a state in a modified Michelson interferometer was proposed in Ref. [81].

The method relies on the idea of intra-resonator nonperturbing measurement of the cross-quadrature component

$$\hat{\mathcal{X}}_{\pi/2} = \mathrm{i}(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1). \tag{8.16}$$

Operator $\hat{\mathcal{X}}_{\pi/2}$ is a particular case of the family of operators

$$\hat{\mathcal{X}}_\theta = \hat{a}_1^\dagger\hat{a}_2\exp(\mathrm{i}\theta) + \hat{a}_2^\dagger\hat{a}_1\exp(-\mathrm{i}\theta). \tag{8.17}$$

These operators commute with the Hamiltonian of the two modes:

$$\left[\hat{\mathcal{X}}_\theta, \hbar\omega(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2)\right] = 0. \tag{8.18}$$

In other words, they are integrals of motion, and as such the potentially quantum nondemolition observables [82, 19, 20].

The basic distinction of the scheme of the nonperturbing measurement from the conventional schemes with photodetectors consists in that the measurement $\hat{\mathcal{X}}_\theta$ is at the same time the preparation of a state with a well-defined value of $\mathcal{X}_\theta$. Therefore, any initial state of the field in the resonator with the aid of such measurements can be transformed into the state $|\mathcal{X}_\theta\rangle$.

The eigenstates of the cross-quadrature operators have the form

$$|N,n\rangle = \frac{1}{\sqrt{2^N n!(N-n)!}}\left(\hat{a}_1^\dagger + \hat{a}_2^\dagger\exp(-\mathrm{i}\theta)\right)^n \times\left(\hat{a}_1^\dagger - \hat{a}_2^\dagger\exp(-\mathrm{i}\theta)\right)^{N-n}|0,0\rangle. \tag{8.19}$$

Here N is the sum of quanta in the system, and n is an integer from 0 to N. In the state (8.19) each of the N photons has equal probabilities of occurring in each of the modes. However, the amplitudes of probabilities for n quanta are orthogonal to the amplitudes of probabilities for the remaining $N-n$ quanta. Because of such entanglement between the modes, the states (8.19) are referred to as *symphotonic*. The eigenvalues of operators $\hat{\mathcal{X}}_\theta$ are discrete and equal to $n-(N-n) = 2n-N$.

The symphotonic states are sensitive to the change in phase difference between the modes. The phase shift leads to the transition between the states with different values of n. The change in the distribution of photons can be detected in the measurement of $\hat{\mathcal{X}}_{\pi/2}$. The probability of such transitions is

$$p = 1 - \left|\langle N,n|\hat{U}(\phi_1,\phi_2|N,n\rangle\right|^2 = D^2,$$

where $\hat{U}(\phi_1,\phi_2)$ is the operator of the phase difference shift by $\delta\varphi = \phi_2 - \phi_1$. When $\delta\varphi \ll 1$, the probability is

$$p = \frac{1}{4}(\delta\varphi)^2\left(N + 2n(N-n)\right).$$

If $n \simeq N/2$, the probability $p \to 1$ when the phase shift between the modes is $\delta\varphi \simeq \sqrt{8}/N$.

9. Conclusions

The general conclusion from the arguments developed above is the following. A unified theory of the phase of a quantum oscillator does not exist. The existing approaches to the problem of phase lead to results that differ not only quantitatively, but qualitatively as well. This conclusion is supported by the list of the main statements and formulas of the theory of phase.

1. The phase of the oscillator cannot be put into correspondence with a Hermitian operator. Accordingly, there are no orthogonal eigenstates of the phase, and an

exact measurement of phase is not possible even theoretically. The phase is not a dynamic observable like, for example, the coordinate. (Some physicists believe, however, that such a statement is too strong.)

2. The phase can be put into correspondence with a nonorthogonal probability operator measure (3.7). This probability operator measure is used for calculating the distribution of phase (3.6).

3. Hermitian operators, corresponding to the classical cosine and sine of the phase, do not commute. The distribution of values of cosine and sine of the phase cannot be calculated via the distribution of the phase (3.6).

4. The treatment of phase as a parameter in the phase space leads to the distribution of phase as a marginal distribution in terms of quasi-probabilities. The marginal distributions often do agree well with the experiment, but occasionally they give a negative value of the probability.

5. The treatment of phase difference in the two-mode fields as the difference of phases of individual modes leads to nonorthogonal states (7.6) with continuous values of phase difference. The distribution of phase difference in arbitrary states of the field is given by expression (7.5).

6. The treatment of phase difference as a physical observable allows one to find the Hermitian operator of phase difference (7.24). This operator commutes with the operator of the total number of photons in the two-mode field, and has discrete eigenvalues. The limit of this operator, however (when the number of excitations in one of the modes tends to infinity) is a non-Hermitian operator of phase difference.

7. Hermitian operators, corresponding to the classical cosine and sine of phase difference (7.31) and (7.32) do not commute. The discreteness of eigenvalues of these operators differs from the discreteness of eigenvalues of operator of phase difference.

So what is the answer to the question asked in the title of this review: what is the phase (and the phase difference) 'in reality? The final answer to this question does not exist. In our opinion, it would be logical to state that any property of phase predicted by the theory is physically meaningful if there is a measurement that realizes the probability operator measure corresponding to this property. The key issue is the feasibility of such a measurement. Existing methods of phase measurement consist in the simultaneous (and therefore approximate) measurements of phase and amplitude, or the simultaneous measurements of the quadrature amplitudes. The experimental distributions of phase and phase difference agree well with the distributions calculated from formulas (3.6) and (7.5).

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