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Counting vacua in supersymmetric Yang – Mills theory

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Abstract. The recent progress concerning the Witten index in $N=1$ supersymmetric Yang – Mills theory is reviewed. Since 1982 there has been a controversy in counting vacua. The original calculation of the Witten index conflicted with expectations based on the chiral symmetry breaking picture. The controversy was resolved by Witten in 1998 who discovered additional disconnected components in the space of classical vacua in Yang – Mills theory compactified on a three-dimensional torus. We review the resolution of the controversy, describe those additional vacua and the corresponding gauge fields. We also discuss how the Witten index feels the physics of confinement and chiral symmetry breaking.

1. Introduction

Although supersymmetry (SUSY) in our world is broken (known particles do not form a SUSY representation), no doubt it is one of the most fundamental issues in our knowledge. There is a magic in SUSY. It suggests solutions for many serious problems we meet in our attempts to understand Nature. Consider e.g. the vacuum energy, that is, the sum of the ground state energies $\sum_i (\omega_i/2)$ of the sea of oscillators substituting a field theory. Being unobservable in the absence of gravity, it becomes of prime importance when gravity is turned on. Young W Pauli in the middle of the twenties estimated that due to the vacuum energy, the radius of our Universe would be less than the distance between the Earth and Moon. However, in a theory with unbroken SUSY, the vacuum energy of bosonic oscillators exactly cancels the vacuum energy of fermionic oscillators, so the total vacuum energy is exactly zero.

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The other impressive example concerns the problem of ultraviolet divergences. In principle, this problem has been solved without SUSY, in the sense that there is a large class of theories (renormalizable theories) with a working recipe how to treat the divergences and with complicated theorems proving consistency of the procedure. It is, nevertheless, very inspiring that in theories with SUSY the ultraviolet divergences, due to cancellations between bosonic and fermionic contributions, are always more harmless, and sometimes are just absent, like in the maximally supersymmetric Yang – Mills theory. We shall however not deal with the ultraviolet divergences here.

The algebraic explanation why a SUSY vacuum has zero energy is clear. A Hilbert space $\mathcal{H}$ of a SUSY theory consists of bosonic and fermionic states, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. There is a supercharge Q acting in the Hilbert space of the theory. It turns bosonic states into fermionic ones and vice versa. The Hamiltonian operator H reads (see, e.g. a review [1])

$$H = Q^2 \,. \tag{1}$$

Being the square of a Hermitian operator, the Hamiltonian is positive definite. No state with negative energy is possible in the theory. If a state in the Hilbert space is supersymmetric, that is, is annihilated by Q, then it is also annihilated by H. Using positive definiteness of the norm in the Hilbert space, one easily proves the inverse statement, namely, that any zero energy state is annihilated by Q. Thus, given unbroken SUSY, the lowest energy states necessarily have zero energy.

Another remarkable property of SUSY is that in such theories it is often possible to obtain exact results on the places where in theories without SUSY one can only make a conjecture or use an approximation or a simplified model. Chiral condensates and exact β-functions are worth-while examples of that kind (see, e.g., a review [2]).

One of the efficient tools in SUSY theories is the Witten index. It was introduced in Ref. [3] especially for the purpose of counting vacua in SUSY theories and thus deciding whether the SUSY in a given theory is broken or not. The definition of the Witten index is very simple. It is just given by the trace of the operator $(-1)^{\mathrm{F}}$ over zero energy states (supersymmetric vacua) of the theory. That operator, by definition, is equal to 1 on bosonic states and -1 on fermionic states, so one has

$$I_{\mathrm{W}} = \mathrm{tr}_{\mathcal{H}_0}(-1)^{\mathrm{F}} = n_{\mathrm{B}} - n_{\mathrm{F}}\,, \tag{2}$$

where $\mathcal{H}_0$ is the space of zero energy states, and n_{B} and n_{F} are the number of bosonic and fermionic vacuum states correspondingly. Thus, if the Witten index is nonzero, there are supersymmetric vacua in the theory and supersymmetry is not broken. If it is zero, it does not necessarily mean that SUSY is broken, since it may be zero due to $n_{\mathrm{B}} = n_{\mathrm{F}} \neq 0$. One can in this case refine the index, e.g., take the trace over a supersymmetric subspace of $\mathcal{H}_0$. In some instances one can involve an additional argument to count n_{B} and n_{F} separately.

The power of the Witten index is that it is invariant under many kinds of deformations of the theory. The invariance is based on the simple observation that, because of the relation (1), states with nonzero energy $H \neq E$ are paired by the action of Q, so that for a given bosonic state $|a\rangle$ there is a fermionic state $|b\rangle$, such that

$$|\,b\rangle = \frac{Q}{\sqrt{E}}\,|a\rangle,\ |a\rangle = \frac{Q}{\sqrt{E}}\,|b\rangle\,. \tag{3}$$

Thus, if in the course of deformation, say, a nonzero energy bosonic state has its energy decreasing and, at a point, enters the vacuum sector of the theory, it should be necessarily followed by a fermionic state, the Witten index being unchanged. Inversely, if at a point of deformation, say, a bosonic vacuum state gets a positive energy and thus leaves the space of vacua, it should be necessarily followed by a fermionic vacuum state, the Witten index being again unchanged. In this kind of argument one should only be careful about vacua flowing to (or flowing from) infinity in the configuration space. It remains a zero energy state until it disappears from the total Hilbert space, so it can break the invariance of the Witten index. This situation is excluded when the potential energy grows at infinity.

As a digression, notice that due to the pairing Eqn (3), one can rewrite Eqn (2) as follows:

$$I_{\mathrm{W}} = \mathrm{tr}_{\mathcal{H}}(-1)^{\mathrm{F}}\exp(-\beta H)\,, \tag{4}$$

where the trace is now taken over the total Hilbert space $\mathcal{H}$ [contributions of bosonic and fermionic states of nonzero energy cancel each other in Eqn (4)]. The right hand side of Eqn (4) looks very much like the partition function, the only difference being the presence of the operator $(-1)^{\mathrm{F}}$. It is sometimes called a *graded* partition function. It can conveniently be rewritten in the standard functional integral representation for the partition function, β being the euclidian time period, and, because the partition function is graded, fermionic fields are integrated with periodic boundary conditions in β (in the usual partition function fermionic fields are integrated with an antiperiodic boundary condition in euclidian time).

A word of warning is in order here. The transition from Eqn (2) to Eqn (4) is definitely correct in a theory with a discrete spectrum, otherwise bosonic and fermionic states with nonzero energy, though being paired, may have different densities of states resulting in a difference between Eqn (2) and Eqn (4). In that case only the $\beta \to \infty$ limit of Eqn (4) should coincide with Eqn (2) (see a thorough discussion in Ref. [4]).

To avoid that sort of complication one defines a theory on a spatial torus, so that the spectrum of the theory is discrete (it is important that SUSY is not broken on a torus; it is related to the fact that translation symmetry is not broken on a torus). All states which have zero energy in a finite volume will still have zero energy density in the infinite volume limit, so they all will be supersymmetric vacua in the infinite volume limit (provided none of them flows to infinity as we discussed above). Additional vacua may, in principle, appear in the infinite volume limit since it is possible that there is a state whose energy is positive and reaches zero asymptotically in the infinite volume limit. But this state should again be paired, so it cannot change the index.

Then one uses the invariance of the Witten index, which, in particular, should not depend on the size of the torus. One can consider a torus of small radius (much less than a typical scale of the theory, e.g. much less than $\Lambda_{\mathrm{QCD}}^{-1}$). According to the standard logic of compactification one then has a theory with two scales: soft modes are described by the fields independent of coordinates on the space torus and hard modes are described by the fields with a nontrivial dependence on those coordinates. In the limit of small size of the torus the hard modes decouple and one just has to count the number of vacua in the SUSY quantum mechanics on the space of the soft modes. Even further simplification is possible. Since the energy spectrum of the quantum mechanics on soft modes is discrete, one can quantize not the whole space of soft modes but just the space of classical vacua of those modes, and then just count all the states in the Hilbert space obtained [with the $(-1)^{\mathrm{F}}$ sign].

That was a route followed by Witten in his original paper [3]. He computed the index in $N = 1$ SUSY Yang–Mills theory, that is the theory with the Lagrangian

$$L = \frac{1}{g^2}\int \mathrm{d}^4x\,\mathrm{tr}\left(\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\bar{\lambda}\mathrm{i}\Gamma\cdot\mathrm{D}\lambda\right) + \frac{\theta}{8\pi^2}\int \mathrm{tr}\,F\wedge F\,, \tag{5}$$

where λ is a positive chirality fermion in the adjoint representation and $\bar{\lambda}$ is its conjugate. I_{W} happened to be equal to $r+1$ for any gauge group, where r is the *rank* of the gauge group.

At this point one comes to a contradiction with other facts known about $N = 1$ SUSY Yang–Mills.

Recall that in the theory there is at the classical level the so-called R-symmetry. It is a U(1) group which acts on the gluino fields by a chiral phase transformation. This U(1) group is broken by the anomaly, so that only its discrete subgroup $Z_{2\mathrm{h}}$ survives as exact quantum symmetry at the operator level (h is the dual Coxeter number, an important integer characteristic of a simple Lie group, see below). $2h$ appears because it is equal to the index of Dirac operator on gluino in the instanton background, so that instanton generated 't Hooft's vertex ([5], see also a review [6]) includes $2h$ gluino field operators of positive chirality. The positive chirality gluino has R-charge -1, so the 't Hooft vertex breaks the U(1) group but leaves $Z_{2\mathrm{h}}$ intact.

$Z_{2\mathrm{h}}$ acts as an exact symmetry in the Hilbert space of the theory, but vacua of $N = 1$ SUSY Yang–Mills are not $Z_{2\mathrm{h}}$ invariant. Instead, they are transmuted by that group. In physics terms, $Z_{2\mathrm{h}}$ is spontaneously broken to the Z_2 subgroup by a choice of vacuum state (Z_2 acts on gluino as -1 and cannot be broken because it coincides with the 2π rotation by the Lorentz group). The order parameter distinguishing different vacua is the phase of the gluino chiral condensate [7 – 10]:

$$\left\langle \epsilon^{\alpha\beta}\lambda_{\alpha}^{a}\lambda_{\beta}^{a}\right\rangle_{k} = \text{const}\ \Lambda_{\text{QCD}}^{3}\exp\left(\frac{2\pi \mathrm{i}k}{h}\right). \tag{6}$$

The meaning of Eqn (6) is that in the k-th vacuum $\lambda\lambda$ develops a k-dependent expectation value.

Thus there are h different vacua. [1] One can argue that all these vacua are bosonic [more accurately, they have the same $(-1)^{\mathrm{F}}$ number], which is more or less obvious from the fact that due to confinement all fermions are separated from vacua by a mass gap. Thus on physical grounds we have to conclude that the Witten index equals h. We are apparently at a contradiction since for many groups h is larger than $r+1$. Actually, already in the original paper [3] in 1982 Witten noticed that the $r+1$ result for I_W in the case of orthogonal groups contradicts the picture of chiral symmetry breaking.

This paradox remained unsolved for quite a long time. Its resolution was suggested quite recently by Witten himself in Ref. [11] in 1998 (see also a nice remake [12] of the original paper [3]) and triggered a progress in the classical group theory (!) [13, 14], and also led to new insights in string compactification [15, 16].

In Section 2 we review the original argument in Ref. [3]. In Section 3 we explain the resolution of the paradox. Section 4 is devoted to explicit construction of the gauge field configurations appearing in the course of resolution of the paradox. In Section 5 we discuss how the physics of confinement and chiral symmetry breaking in $N=1$ SUSY Yang – Mills is felt by (a refined version of) the Witten index. Section 6 contains conclusions. In the appendix we collected some mathematical definitions which the reader may find useful.

2. Witten's original computation

Following the discussion in the Introduction, let us consider $N=1$ SUSY Yang – Mills theory on $T^3\times R$, where R is the time axis and T^3 is a small spatial torus with all periods equal to L. Let us take the gauge condition $A_0=0$.

Boundary conditions, which one can impose on gauge potentials A on T^3, are classified by the so-called 't Hooft magnetic fluxes [17] (we discuss the fluxes in Section 5). One, however, naturally expects that the infinite volume physics should not depend on what type of boundary conditions are imposed on T^3. Let us note in passing that the way (a refined version of) the Witten index depends on magnetic and electric fluxes fits nicely into the physics of confinement and chiral symmetry breaking [12]. We shall discuss these matters in Section 5 and now proceed with periodic boundary conditions:

$$\begin{aligned} A_i(x+1,y,z) &= A_i(x,y,z)\,,\\ A_i(x,y+1,z) &= A_i(x,y,z)\,,\\ A_i(x,y,z+1) &= A_i(x,y,z)\,, \end{aligned} \tag{7}$$

where $i=1,2,3$.

[1] It is important that in the infinite volume limit there is no tunneling between vacua with different phases of the chiral condensate, that is why they give rise to spontaneous breaking of the Z_{2h} symmetry. In this respect these vacua are very different from the classical vacua classified by elements of $\pi_3(\mathrm{G})$. The latter vacua are related by instantons and do not survive as quantum vacua. Instead, they combine to a unique vacuum in every superselection sector parameterized by the famous θ-angle.

Consider the space of classical vacua. Clearly, it consists of gauge potentials with zero curvature

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = 0\,, \tag{8}$$

modulo gauge transformations periodic on T^3.

Such gauge fields in the case of connected, simply connected, compact gauge group are characterized by (conjugacy classes of) their *monodromies* (in physics terms, Wilson loops) around three independent circles on T^3:

$$\begin{aligned} \Omega_1 &= P\exp\left[\int_0^L A_1(x,0,0)\,\mathrm{d}x\right],\\ \Omega_2 &= P\exp\left[\int_0^L A_2(0,y,0)\,\mathrm{d}y\right],\\ \Omega_3 &= P\exp\left[\int_0^L A_3(0,0,z)\,\mathrm{d}z\right]. \end{aligned} \tag{9}$$

Given the zero curvature condition Eqn (8), the monodromies do not depend on the choice of the paths in Eqn (9) and commute with each other. Gauge transformations act on Ω as conjugations by group elements.

It is natural to assume that three commuting group elements Ω_i, $i=1,2,3$ can be represented as exponents of three commuting Lie algebra elements:

$$\Omega_i = \exp(2\pi \mathrm{i} C_i)\,, [C_i, C_j] = 0,\ i,j = 1,2,3\,. \tag{10}$$

C_i, $i=1,2,3$ can then be taken as coordinates on the space of classical vacua. They are obviously identified with the x, y, z-independent harmonics of the periodic gauge potentials, $A_j = (2\pi i/L)\,C_j$, $j=1,2,3$. Since these are commuting Lie algebra elements, they can be simultaneously transformed by an x, y, z-independent gauge transformation to the so-called Cartan subalgebra, that is, the maximal commuting subalgebra of the Lie algebra.

It is clear from Eqn (10) that C_i, $i=1,2,3$ actually belong to the Cartan subalgebra factorized over the so-called *coroot lattice* [since one can add to C_i any element α^v, where $\exp(2\pi \mathrm{i}\alpha^v)=1$ without changing Ω_i; equivalently, there is a set of periodic gauge transformations which are dependent on x,y,z but transform x,y,z-independent gauge potentials to x, y, z-independent ones; such a gauge transformation add an element of the type of α^v to C_i]. One says that C_i for every $i=1,2,3$ belongs to the Cartan torus which will be denoted by T_{C}.

Even after transforming gauge potentials to the Cartan torus there are some gauge transformations to be still taken into account, namely those belonging to the so-called Weyl group W, a discrete group acting in the Cartan torus. For example, in the case of $\mathrm{SU}(N)$, W acts on diagonal matrices by transposing eigenvalues. So, finally, the bosonic component of the space of classical vacua is

$$\mathcal{M} = \frac{T_{\mathrm{C}}\times T_{\mathrm{C}}\times T_{\mathrm{C}}}{\mathrm{W}}\,. \tag{11}$$

Notice that instead of considering functions on $\mathcal{M}$ it is sometimes more convenient to consider W-invariant functions on $T_{\mathrm{C}}\times T_{\mathrm{C}}\times T_{\mathrm{C}}$.

Now let us see how fermions are included. The field content of $N=1$ SUSY Yang – Mills includes gauge potentials (gluons), A_μ, which are one-forms with values in the adjoint representation, positive chirality gluinos, λ_α, $\alpha=1,2$

and their complex conjugate, $\bar{\lambda}_\alpha$, $\alpha = 1, 2$, all with values in the adjoint representation. It is obvious by supersymmetry that the fermionic part of the space of classical vacua is spanned by the x, y, z-independent λ_α, $\alpha = 1, 2$ and $\bar{\lambda}_\alpha$, $\alpha = 1, 2$ with values in the Cartan subalgebra. Notice, however, that the space spanned by both λ_α, $\alpha = 1, 2$ and $\bar{\lambda}_\alpha$, $\alpha = 1, 2$ is rather a phase space, not a Lagrangian space, because $\bar{\lambda}_\alpha$ is canonically conjugated to λ_α. In quantization one should consider only functions, say, of λ_α, $\alpha = 1, 2$. Also, as a consequence of gauge symmetry, one should consider only W-invariant functions of λ_α, $\alpha = 1, 2$. Obviously, due to the Grassmann nature of λ_α, any function of λ_α is actually a polynomial, so one should consider W-invariant polynomials.

There is an obvious invariant

$$v = \varepsilon^{\alpha\beta}\delta_{ab}\,\lambda_\alpha^a\,\lambda_\beta^b\,, \tag{12}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor and δ_{ab} is the restriction of the Killing form onto the Cartan subalgebra. Polynomials in v provide $r + 1$ W-invariant functions, where r is the rank of the gauge group, which is, by definition, the dimensionality of the Cartan subalgebra. One can prove that there are no other W-invariant functions of λ_α, $\alpha = 1, 2$. This is more or less clear: higher rank symmetric invariant tensors, like d^{abc}, do not help because of the Grassmann nature of λ_α, for a rigorous proof see, e.g., appendix in Ref. [13].

Restricting the Hamiltonian of $N = 1$ SUSY Yang–Mills to the space of x, y, z-independent A_j, λ_α, $\bar{\lambda}_\alpha$ taking values in the Cartan subalgebra one obtains simply

$$H \propto \frac{\partial}{\partial A_j^a}\,\frac{\partial}{\partial A_j^a}\,, \tag{13}$$

that is just the Laplacian operator on $T_{\mathrm{C}} \times T_{\mathrm{C}} \times T_{\mathrm{C}}$. It comes out of the E^2 (E-electric field) term in the standard Yang–Mills Hamiltonian. Since the restriction is onto the space of zero energy, and fermions enter Lagrangian with only the first-order time derivative, no fermions are left in Eqn (13). The first derivative term in the Lagrangian defines the canonical anticommutator

$$\{\bar{\lambda}_{\dot{\alpha}}^a,\,\lambda_\alpha^b\} = \delta_{\dot{\alpha}\alpha}\,\delta^{ab}\,, \tag{14}$$

so that one can treat λ_α as creation operators and $\bar{\lambda}_\alpha$ as annihilation operators.

A word of warning is in order here. In fact what one needs is not just restriction of the Hamiltonian onto the space of zero energy, but computation of the effective Hamiltonian in the framework of the Born–Oppenheimer approximation. In the case at hand these two procedures give the same result — Eqn (13). This is not the case when chiral matter is included, see the thorough discussion in Refs [18, 19].

Now one understands that a vacua wave function ψ would be a polynomial in v with A_j^a-dependent coefficients obeying the equation

$$\frac{\partial}{\partial A_j^a}\,\frac{\partial}{\partial A_j^a}\,\psi = 0\,. \tag{15}$$

The only solution of the Laplacian equation on the torus is constant, which means the coefficients ψ are, in fact, A_j^a-independent, and one is left with polynomials in v with constant coefficients. So, there are $r + 1$ bosonic vacua [more accurately, they have the same $(-1)^{\mathrm{F}}$ number since v in Eqn (12) is a bosonic operator].

Thus, according to this consideration, the Witten index is equal to $r + 1$ (more accurately, it is equal to $r + 1$ up to a sign, which is discussed in Section 5). As was explained in the introduction, this value contradicts the other facts known about the theory. So, in fact, something is lost.

Looking at the argument above one easily finds a potential loophole: there are singularities in $\mathcal{M}$ in Eqn (11). Indeed, there are submanifolds on $T_{\mathrm{C}} \times T_{\mathrm{C}} \times T_{\mathrm{C}}$ where the Weyl group acts trivially. For example, in the SU(N) case these are the submanifolds where n-th and $(n + 1)$-th eigenvalues coincide in all three diagonal matrices A_j. These submanifolds can also be characterized as those where the unbroken gauge symmetry becomes nonabelian [generically it is $\mathrm{U}(1)^r$]. At these submanifolds some additional bose and fermi modes move down to zero energy and the effective theory described above breaks down. That direction was one of the main lines of attack in the attempts to resolve the paradox.

One can, however, argue that these singularities are not responsible for the paradox.

As one of the possible arguments, at least in the case of SU(N), one can compute the index using a different type of boundary conditions, which exclude that type of singularity. Namely, one can compute the index introducing a unit 't Hooft's magnetic flux in, say, the xy-direction on the torus. On general grounds one expects the index to be independent of the choice of magnetic flux (a quite convincing argument was given in Ref. [12], we discuss it in Section 5). Counting vacua in this case is technically different from the counting above. There is no continuous space of classical vacua, there are just $r + 1$ isolated classical vacua, which, anyway, shows that the Witten index is equal to $I_N = r + 1$. The moral is that the counting above, at least in the case of SU(N), gave the correct result in spite of singularities.

A more direct argument is based on recent developments (see Refs [4, 20–28] and also the not quite recent papers [29]) concerning the Witten index in a slightly different problem. Consider SUSY Yang–Mills dimensionally reduced to $(0 + 1)$ dimensions. The dimensional reduction assumes that fields are just taken to be independent of space coordinates. Then one obtains a quantum mechanics with noncompact directions (no factorization over the coroot lattice!). In this case the continuous spectrum begins from zero energy. Nevertheless one can wonder whether there is a normalized zero energy state in the problem. In the maximally supersymmetric SU(N) Yang–Mills such a state is necessary for consistency of M-theory — IIA string duality [39–43]. Some of those recent developments hint that there are no normalized zero energy states in the case of $N = 1$ Yang–Mills with any gauge group. In application to our problem, this indicates that there are no vacuum wave functions localized near the singularities discussed above. This also indicates that the vacua wave functions described above do not fail to be normalizable near those singularities as one could in principle expect (see discussion on pp 46–47 of Ref. [12]).

What, still, is the resolution of the paradox? As was discovered in Ref. [11], in the cases when $n > r + 1$, the space of classical vacua has additional disconnected components which give an additional contribution to the index. The natural assumption (10) appears to be wrong. There might be triples of group elements which are commuting but cannot be simultaneously conjugated to the Cartan torus (equivalently, cannot be represented as exponents of commuting Lie algebraic elements).[2] In the next section we review construction of these triples and explain how the sum of contributions of different components of the space of classical vacua gives the correct value of the Witten index.

3. Additional components in the space of classical vacua and their contributions to the Witten index

Let us return to the beginning of the previous section. Everything goes through up to Eqn (10). At that point we should realize that there are different types of triples which cannot be deformed to each other, and, correspondingly, different components of the space of classical vacua. A reader may keep in mind a picture with potential energy reaching zero at different disconnected regions in the space of all fields. Then we should count the number of vacua at every component. The simplest and the biggest one is characterized by the property (10). Its bosonic part is described by Eqn (11). Its contribution to the index is $r+1$.

Before describing the general situation with the additional components, let us give an example of the nontrivial triple and how it helps in solving the paradox.

The simplest group when such a triple exists is the Spin (7) group [Spin (7) is the simply connected cover of SO (7); one takes Spin (7) instead of SO (7) to exclude the possibility that the monodromies commute up to a center element; one can equivalently work with SO (7) requiring zero magnetic flux in any direction]. For the Spin (7) group there is a unique (up to conjugation) nontrivial triple. It can be chosen in the form

$$\begin{aligned} \Omega_1 &= \gamma_{1234}\,, \\ \Omega_2 &= \gamma_{1256}\,, \\ \Omega_3 &= \gamma_{1357}\,, \end{aligned} \tag{16}$$

where we use the notation

$$\gamma_{ijkl\ldots} = \gamma_i \gamma_j \gamma_k \gamma_l \cdots \tag{17}$$

and γ_i, $i = 1, \ldots 7$ stand for the gamma-matrices. Ω in Eqn (16) mutually commute and cannot be conjugated to the Cartan torus (cannot be represented as exponents of mutually commuting Lie algebraic elements, see, e.g. Ref. [31] for a simple proof). This triple breaks Spin (7) completely, in the sense that none of the generators of Spin (7) commutes with all three Ω. This triple represents an additional isolated classical vacuum, so it contributes 1 to the index. Thus we get $I_{\mathrm{W}} = r + 1 + 1 = 5$ in total, which is equal to h for Spin (7).

The existence of such triples and the example above was pointed out in Ref. [30]. Their relevance for the resolution of the paradox was realized in Ref. [11], where also the correct counting of the index for the case of $\mathrm{SO}(N)$ groups was done. The G_2 case was analyzed in Ref. [31]. A complete classification of such triples for any simple group and correct counting of vacua was given in Ref. [13] and also in Ref. [32]. Those results were derived in a different way and were extended in some respects (essentially, to triples commuting up to center elements) in Ref. [14].

[2] As we discuss below, two group elements can always be conjugated to the Cartan torus. By conjugation we mean the following group action: $a \to gag^{-1}$, where g is a group element.

Let us now turn to the general theory of such triples. Notice first that two commuting group elements in a connected, simply connected compact gauge group can always be simultaneously conjugated to a Cartan torus. This is due to the Bott theorem which states that a centralizer[3] of an element of a connected, simply connected, compact group is connected. Suppose that Ω_1 is in a Cartan torus (any group element can be conjugated to Cartan torus). Then Ω_2 should belong to the centralizer of Ω_1 (since they commute). It can be conjugated to the Cartan torus in the centralizer. Then, since the Cartan torus of the whole group belongs to the centralizer, and since the centralizer is connected, both Ω_1 and Ω_2 appeared to be conjugated to the Cartan torus of the whole group.

Consider now triples. Take Ω_3 in the Cartan torus. Ω_1 and Ω_2 should belong to the centralizer of Ω_3, which, by the Bott theorem, is connected. The centralizer consists of a product of a number of U(1) group and of a semisimple part G_{Ω_3}. Generators of U(1) belong to the Cartan subalgebra of the whole group, so those U(1)-factors are obviously irrelevant for construction of the nontrivial triples. Now, if the semisimple part G_{Ω_3} is simply connected, then, as two commuting elements in a connected, simply connected compact group, Ω_1 and Ω_2 can be conjugated to the Cartan torus, and we do not get a nontrivial triple.

So we have to take such Ω_3 that the semisimple part of the centralizer G_{Ω_3} is not simply connected. It can be viewed as a factor of a simply connected group $\tilde{G}_{\Omega_3}$ over a subgroup D of its center. The nontrivial triple appears when Ω_1 and Ω_2, as elements of $\tilde{G}_{\Omega_3}$, commute to an element of D:

$$\Omega_1 \Omega_2 = \varepsilon \Omega_2 \Omega_1\,, \quad \varepsilon \in D\,. \tag{18}$$

The two such elements are called *the Heisenberg pair*. Clearly, they commute in G_{Ω_3}, and hence Ω_1, Ω_2 and Ω_3 commute in the original group. They cannot be conjugated to the Cartan torus since Ω_1 and Ω_2 cannot be conjugated to the Cartan torus in the centralizer of Ω_3.

We shall now describe what choice of Ω_3 gives a non simply connected semisimple part of the centralizer. To do this we need a bit of machinery of the group theory. We are going to introduce some of that. A reader not acquainted with group theory should accept it formally. All necessary facts will be informed. A reader may also find it useful to see some definitions in Appendix.

First we need extended Dynkin diagrams with dual Dynkin labels (see Figure). Nodes of the usual Dynkin diagrams are in one-to-one correspondence with the so-called simple roots, which form a basis in the root space (the root space is a space of linear forms on the Cartan subalgebra). An extended Dynkin diagram also includes a node corresponding to the minus highest root, so in total it has $r+1$ nodes, where $r = \mathrm{rank}(G)$. There is a natural map from the root space to the Cartan subalgebra which maps roots to coroots. The dual Dynkin labels on the diagram are coefficients in the linear expansion of the coroot of the highest root over simple coroots. The dual Dynkin label on the node, corresponding to the coroot of the highest root itself, is prescribed to be 1. Notice, by the way, that the dual Coxeter number h is equal to sum of all dual Dynkin labels. One immediately sees from Figure that for $\mathrm{SU}(N)$ and for $\mathrm{SP}(N)$ cases $h = r + 1$. In other cases $h > r + 1$ since the same dual

[3] The centralizer of a group element Ω is a subgroup commuting with Ω.

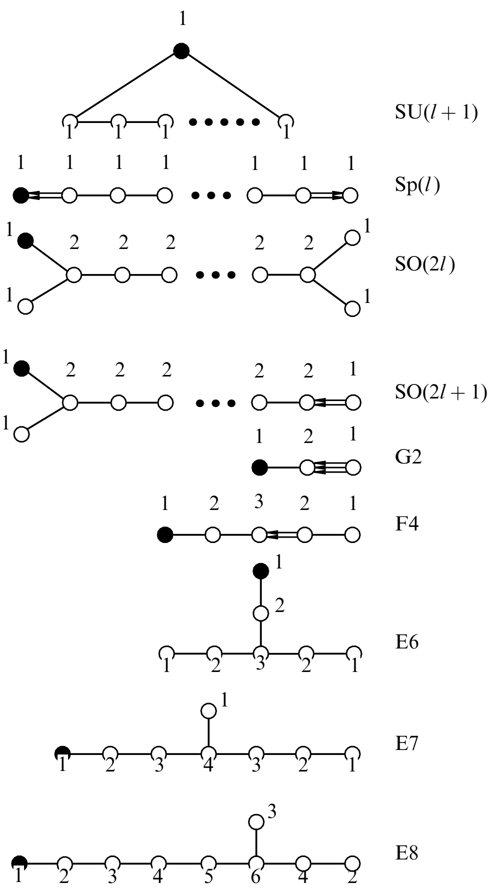

Figure. Extended Dynkin diagrams with dual Dynkin labels. Nodes corresponding to highest roots are bold-faced.

Dynkin labels have nontrivial divisors. These nontrivial divisors are at the heart of the nontrivial triples.

The proper choice of Ω_3 to have the nontrivial triple is

$$\Omega_3 = \exp\left[2\pi\mathrm{i}\sum_{j=1}^{\mu(m)} C_j\,\omega_j\right], \tag{19}$$

where sum runs over nodes with dual Dynkin labels having a common divisor m, and $\mu(m)$ is the number of such nodes; ω_j is the so-called fundamental coweight corresponding to j-th node. Fundamental coweights form a basis in the Cartan subalgebra, defined by the property that

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij}\,, \quad i, j = 1, ..., r\,, \tag{20}$$

where α_i are simple roots and $\langle\,,\,\rangle$ is the pairing between the root space and the Cartan subalgebra. The coefficients C_j, $C_j \geqslant 0$, are subject to the condition

$$\sum_{j=1}^{\mu(m)} a_j\,C_j = 1\,, \tag{21}$$

where a_j are the Dynkin labels, that is, coefficients in the expansion of the highest root over simple roots. The element (19), constructed on the nodes with a common division m, was called m-exceptional in Ref. [13].

For a generic choice of the coefficients C_j, Ω_3 has a centralizer

$$U(1)^{\mu(m)-1} \times \frac{\mathrm{SU}(N_1)\times\ldots\times\mathrm{SU}(N_l)}{D}\,, \tag{22}$$

where D is a subgroup in the center,

$$D \subset Z_{N_1}\times\ldots\times Z_{N_l}\,. \tag{23}$$

There is nice description of the semisimple part of the centralizer. One can prove [33] that its Dynkin diagram is obtained from the extended Dynkin diagram (see Figure) by crossing out the nodes included in the sum (19). For example, crossing out the nodes divisible by 4 from the E_8 extended Dynkin diagram leaves a Dynkin diagram corresponding to the $\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{SU}(4)$ group.

The structure of the center subgroup D can be understood from the general fact that $\pi_1(G) = P^{\vee}(G)/Q^{\vee}(G)$, where $P^{\vee}(G)$ is the so-called coweight lattice and $Q^{\vee}(G)$ is the coroot lattice. For example, it is quite straightforward to see that π_1 of the centralizer of Ω_3 is $Z^{\mu(m)-1}$ times a cyclic group of the order of m. $Z^{\mu(m)-1}$ belongs, of course, to the abelian subgroup of the centralizer while the cyclic group of the order of m is nothing but D. It is also quite straightforward to see that D is such that when one forgets about all Z_{N_j} but one in Eqn (23), D projects onto that one surjectively.

As was explained above, in order to obtain a nontrivial triple, the two other monodromies, Ω_1 and Ω_2 are taken in the form of a Heisenberg pair (18). When ε in Eqn (18) is a generating element[4] of D, such a pair is unique (up to conjugations) and breaks the semisimple part of the centralizer in Eqn (22) completely (that is, none of the generators of the semisimple part compute with Ω_1 and Ω_2). Thus the triple Ω_1, Ω_2 and Ω_3 generically breaks the original group to $U(1)^{\mu(m)-1}$. Such a triple — the one built on nodes with dual Dynkin labels divisible by m — was called m-exceptional in Ref. [13].

One now understands that the moduli space of the m-exceptional triples is a $\mu(m)-1$ dimensional space parameterized by the coefficients C_j in Eqn (19). The same logic as in Section 2 brings us to the conclusion that the moduli space is actually

$$\frac{T^{\mu(m)-1}\times T^{\mu(m)-1}\times T^{\mu(m)-1}}{Z_{\mu(m)}}\,, \tag{24}$$

where $Z_{\mu(m)}$ appears because it is a part of the Weyl group of the original gauge group acts on $U(1)^{\mu(m)-1}$. Then the same counting as in Section 2 shows that this component of the space of classical vacua gives rank $+\,1 = \mu(m)$ quantum vacua.[5]

Let us now verify that taking into account all those additional contributions gives exactly h vacua for any gauge group. The number of m-exceptional triples is equal to $P(m)$, where $P(m)$ stands for the number of naturals less than m and co-prime with m. $P(m)$ appears because it is the number of generating elements in the center group D (notice that the triples with the Heisenberg pairs commuting to a non-generating element of D will appear in m-exceptional triples with a smaller m). As we saw above, m-exceptional triples

[4] An element of a cyclic group is called a generating element if its powers provide all the elements of the cyclic group; in Z_l with a prime l all elements are the generating ones.

[5] Of course, at a specific choice of the coefficients C_j in Eqn (19) one can break the original group to a subgroup of rank $\mu(m)-1$ larger than $U(1)^{\mu(m)-1}$. This, however, should not affect the counting of vacua; see the discussion at the end of Section 2. Note also that we silently assume all the vacua to be bosonic, which, in principle, requires some extra analysis (see Section 5).

contribute $\mu(m)$ quantum vacua, where, recall, $\mu(m)$ is the number of dual Dynkin labels divisible by m. Finally, summing all the contributions, one obtains the proper number of vacua:

$$\sum_m P(m)\,\mu(m) = \sum_i a_i^\vee = h\,. \tag{25}$$

The latter equality is just the definition of h and the former one due to the fact that for any natural number a

$$a = \sum_{m\,|\,a} P(m)$$

(the notation $m|a$ means that m is a divisor of a).

In principle, this completes the story with the paradox. A physicist might, however, be interested in what are the gauge potentials in the nontrivial vacua. This issue has been addressed in Refs [35, 36] and is reviewed in the next section. Those not interested in the explicit constructions may skip it and proceed directly to Section 5.

4. Gauge fields in the additional vacua

The logic of consideration in Ref. [36] was as follows: one understands that the gauge potentials are essentially nontrivial in the particular $\mathrm{SU}(N_j)$ factor in Eqn (22), so one describes a $\mathrm{SU}(N_j)$ gauge potential configuration and the way it is embedded into the original group. It resembles the instanton construction, where one describes the SU(2) instanton and the way it is embedded into the whole group.

The monodromy Ω_3 decomposes as an element of that $U(1)^{\mu(m)-1}$ and a center element in the product of $\mathrm{SU}(N_j)$ in Eqn (22). Actually, one can see that the center element is trivial (equal to 1) in all $\mathrm{SU}(N_j)$ but the one containing the coroot of the highest root, where it is a generating element of the center. Looking at the Dynkin diagrams, one straightforwardly checks that in the case of m-exceptional element Ω_3, the subgroup of the centralizer containing the coroot of the highest root is always $\mathrm{SU}(m)$. Thus, in terms of the product (22), omitting the irrelevant $U(1)^{\mu(m)-1}$ part, Ω_3 is decomposed as follows:

$$\left(1, 1, \ldots, 1, \exp\left(2\pi\mathrm{i}\,\frac{q}{m}\right)\right), \tag{26}$$

where the last term in the brackets belongs to the $\mathrm{SU}(m)$ containing the highest coroot, and q is an integer prime to m, $q < m$, and the units belong to other $\mathrm{SU}(N_j)$ factors in Eqn (22).

Correspondingly, the Heisenberg pair Ω_1, Ω_2 is taken in the form

$$\begin{aligned} \Omega_1 &= \left(\Omega_1^{(1)}, \Omega_1^{(2)}, \ldots\right), \\ \Omega_2 &= \left(\Omega_2^{(1)}, \Omega_2^{(2)}, \ldots\right), \end{aligned} \tag{27}$$

so that Ω_1, Ω_2 commute to an element $\varepsilon \in D$ in Eqn (22):

$$\varepsilon = (\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots)\,, \tag{28}$$

where $\Omega_1^{(j)}$, $\Omega_2^{(j)}$ and $\varepsilon^{(j)}$ belong to the j-th $\mathrm{SU}(N)$ factor in Eqn (22).

This way we see that the problem can be solved in each $\mathrm{SU}(N)$ factor separately. The most complicated case is when all the three monodromies are nontrivial, which happens only in the $\mathrm{SU}(m)$ containing the coroot of the highest root. So we consider the $\mathrm{SU}(m)$ case and construct a gauge potential with monodromies Ω_1, Ω_2, Ω_3 such that

$$\Omega_1\Omega_2 = \exp\left(\frac{2\pi\mathrm{i}}{m}\right)\Omega_2\Omega_1 \tag{29}$$

and

$$\Omega_3 = \exp\left(\frac{2\pi\mathrm{i}}{m}\right) \tag{30}$$

[we consider only primitive roots of unit in Eqns (29), (30); generalization of the construction below to other elements of the center group Z_m is straightforward].

It was shown in Ref. [17] that any Heisenberg pair in $\mathrm{SU}(N)$ satisfying Eqn (29) can be conjugated to

$$\begin{aligned} \Omega_1 = P = \exp\mathrm{i}\delta_P &\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & \epsilon & 0 & 0 & \ldots \\ 0 & 0 & \epsilon^2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \\ \Omega_2 = Q = \exp\mathrm{i}\delta_Q &\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \ldots \end{pmatrix}. \end{aligned} \tag{31}$$

Now, the zero curvature gauge fields can be represented in the form

$$A_i = U^{-1}\,\partial_i U\,, \tag{32}$$

and we search for a gauge group matrix $U(x, y, z)$ obeying the boundary conditions:

$$\begin{aligned} U(x+1) &= PU(x)\,P^{-1}\,, \\ U(y+1) &= QU(y)\,Q^{-1}\,, \\ U(z+1) &= \exp\left(\frac{2\pi\mathrm{i}}{m}\right)U(z)\,, \end{aligned} \tag{33}$$

where the dependence on 'irrelevant' variables (y, z in the first line, etc.) is not displayed and the periods of the torus are set to unity. The case when $\Omega_3 = 1$ corresponds to U, periodical in z direction. Such $U(x, y, z)$ represents an element of $\pi_3(\mathrm{SU}(m))$.

Apparently, Eqns (32), (33) give the proper monodromies, but non-periodic gauge potentials:

$$\begin{aligned} A_i(x+1, y, z) &= PA_i(x, y, z)\,P^{-1}\,, \\ A_i(x, y+1, z) &= QA_i(x, y, z)\,Q^{-1}\,, \\ A_i(x, y, z+1) &= A_i(x, y, z)\,. \end{aligned} \tag{34}$$

Turning back to the original group, we thus eventually obtain gauge potentials, which are periodic in the z direction and are conjugated by Ω_1 and Ω_2 gauge group elements at one period shifts in x and y directions. Recall now that any two commuting elements in the original group can be conjugated to the Cartan torus. Hence there is a non-periodical gauge transformation, which makes the gauge potentials periodic. That gauge transformation, of course, does not change monodromies. So, we proceed with Eqn (33).

We start constructing $U(x,y,z)$ obeying Eqns (33) with the ansatz

$$U = \exp\left[2\pi i z\, T(x,y)\right], \tag{35}$$

where $T(x,y)$ is a Hermitian traceless matrix conjugated to the matrix

$$T_0 = \frac{1}{m}\,\mathrm{diag}(1,\ldots,1,1-m). \tag{36}$$

Apparently, $U|_{z=0} = 1$ and $U|_{z=1} = \varepsilon$, so the third condition of Eqn (33) is satisfied. The other two conditions translate as the following conditions on $T(x,y)$:

$$T(x+1) = PT(x)\,P^{-1},$$
$$T(y+1) = QT(y)\,Q^{-1}. \tag{37}$$

It is rather easy to satisfy these conditions for SU(2). If $P = i\sigma_3$ and $Q = i\sigma_1$ (any Heisenberg pair in SU(2) can be conjugated to this form), one can choose, for example,

$$T(x,y) = \frac{1}{2}\,\frac{\sigma_1\cos(\pi x) + \sigma_3\cos(\pi y) + \sigma_2\cos[\pi(x+y)]}{\sqrt{\cos^2(\pi x) + \cos^2(\pi y) + \cos^2[\pi(x+y)]}}, \tag{38}$$

where the square root factor is inserted for proper normalization. It is difficult, however, to generalize the solution (38) to the case of higher rank. To solve Eqns (37) for arbitrary rank, we first notice that the matrices conjugated to T_0 form the $CP^{m-1} = \mathrm{SU}(m)/[\mathrm{SU}(m-1)\times \mathrm{U}(1)]$ orbit of SU(m). They are conveniently parameterized as follows:

$$T_{ij}(x,y) = \frac{1}{m}\,\delta_{ij} - \psi_i(x,y)\,\psi_j^\dagger(x,y), \tag{39}$$

where ψ_i is a m-component complex column normalized to unity:

$$\psi^\dagger\psi = 1. \tag{40}$$

Now, ψ is an element of the fundamental representation of SU(m), and the parameterization (39) of the orbit $\mathrm{SU}(m)/[\mathrm{SU}(m-1)\times\mathrm{U}(1)]$ may be called fundamentalization. A traceless Hermitian matrix $T(x,y)$ from Eqn (39) has $2m-2$ real parameters [m complex parameters in the column ψ_i minus one real parameter for the normalization Eqn (40) and minus one real parameter for the irrelevant common phase of ψ_i in Eqn (39)], which is equal to the dimension of $\mathrm{SU}(m)/[\mathrm{SU}(m-1)\times\mathrm{U}(1)]$ space.

The boundary condition (37) is reduced to

$$\psi(x+1) = \exp\left[i\alpha(x,y)\right]P\psi(x),$$
$$\psi(y+1) = \exp\left[i\beta(x,y)\right]Q\psi(y), \tag{41}$$

where real functions $\alpha(x,y)$ and $\beta(x,y)$ should be chosen to compensate the nontrivial commutant (18) of P and Q and to make $\psi(x+1,y+1)$ uniquely defined. The latter self–consistency condition implies

$$\exp\left[-i\alpha(x,y)\right]\exp\left[-i\beta(x+1,y)\right]\exp\left[i\alpha(x,y+1)\right] \times \exp\left[i\beta(x,y)\right] = \omega_1 = \epsilon, \tag{42}$$

and we make a choice

$$\alpha(x,y) = \frac{2\pi i y}{m}, \qquad \beta(x,y) = 0. \tag{43}$$

The phases $\alpha(x,y)$, $\beta(x,y)$ can be interpreted as vector potentials $A_{x,y}$ of an auxiliary constant Abelian magnetic field with the flux $\Phi = 1/m$ on the 2-torus.

In words, Eqn (41) means that we need to construct a global section of a $\mathrm{SU}(m)\times\mathrm{U}(1)/Z_m = \mathrm{U}(m)$ bundle over T^2 with C^m as a typical fiber ($\exp[i\alpha(x,y)]\,P$ and $\exp[i\beta(x,y)]\,Q$ are the transition matrices). The first Chern class of the bundle is

$$c_1 = \frac{1}{2\pi}\int \mathrm{Tr}\{F\} = m\Phi = 1. \tag{44}$$

This is a problem which the Jacobi Θ functions with rational characteristics (see e.g. Ref. [34]) are tailor-made for.

Notice that Q acts on the column ψ by cyclically shifting its elements one step up so that the second condition in Eqn (41) simply fixes all the components ψ_j in terms of ψ_1:

$$\psi_{1+j}(x,y) = \psi_1(x,y+j) \tag{45}$$

and requires thereby periodicity of ψ_1 when y is shifted by m,

$$\psi_1(x,y+m) = \psi_1(x,y). \tag{46}$$

All other components ψ_j also enjoy this property. In view of Eqns (45), (31), (43), the first condition in Eqn (41) is reduced to

$$\psi_1(x+1,y) = \exp\left(\frac{2\pi i y}{m}\right)\psi_1(x,y). \tag{47}$$

The conditions (46), (47) are obviously satisfied by the choice

$$\psi_1(x,y) = N(x,y)\sum_{n\in Z}\exp\left[-\pi\left(n+\frac{y}{m}\right)^2 + 2\pi i x\left(n+\frac{y}{m}\right)\right], \tag{48}$$

where $N(x,y)$ is a periodical function of x and y with period 1. Other ψ_j are defined via Eqn (45). The factor $N(x,y)$ should be chosen so that the normalization condition (40) is satisfied. For N to be well defined we need to check that ψ_j, $j = 1,\ldots,m$ do not have a common zero. To this end it is convenient to express ψ_j in terms of Jacobi Θ-functions. Using the definition of the theta functions $\Theta_{l/m,\,j/m}(z,\tau)$ with rational characteristics l/m, j/m (see Ref. [34]):

$$\Theta_{l/N,\,m/N}(z,\tau) = \sum_{n\in Z}\exp\left[i\pi\tau\left(n+\frac{l}{N}\right)^2 + 2\pi i\left(n+\frac{l}{N}\right)\left(z+\frac{m}{N}\right)\right], \tag{49}$$

one straightforwardly verifies that

$$\psi_j(x,y) = N(x,y)\exp\left[-\pi\left(\frac{y}{m}\right)^2 + 2\pi i x\,\frac{y}{m}\right] \times \Theta_{(j-1)/m,\,0}\left(x + i\frac{y}{m},\, i\right). \tag{50}$$

Now, $\Theta_{l/m,\,j/m}(z,\tau)$ have zeros at $z = (l/m + p + 1/2)\,\tau + (j/m + q + 1/2)$, $p,q \in Z$, so ψ_j, $j = 1,\ldots,m$ have no common zero. Thus the factor N is well defined and equals

$$N(x,y) = \frac{\exp\left[\pi(y/m)^2\right]}{\sqrt{\sum_{l=0}^{m-1}\left|\Theta_{l/m,\,0}[x + i(y/m), i]\right|^2}}. \tag{51}$$

Substituting Eqn (50) into Eqns (39), (35), and (32), we obtain the gauge fields we were seeking for. All other solutions to the boundary conditions (33) are related to this particular solution under gauge transformations, including transformations representing $\pi_3(\mathrm{SU}(m))$.

Let us now compute the Chern – Simons (CS) number of this field,

$$N_{\mathrm{CS}} = \frac{1}{8\pi^2} \int_{T^3} \mathrm{Tr}\left(A\, \mathrm{d}A + \frac{2}{3} A^3 \right), \tag{52}$$

which is normalized so that the CS number differs by 1 on two flat gauge fields related by an instanton. Since in our case the connection is flat, we actually need to compute the integral

$$N_{\mathrm{CS}} = -\frac{1}{8\pi^2} \int \mathrm{d}\mathbf{x}\, \mathrm{Tr}\left\{ (\partial_x U^{-1} \partial_y U - \partial_y U^{-1} \partial_x U)\, U^{-1} \partial_z U \right\}. \tag{53}$$

When the spatial manifold is S^3, N_{CS} is an integer. The same holds for the normal untwisted torus. But in the twisted case the situation is different.

To find (53), notice first that $U^{-1}\, \partial_z U = 2\pi \mathrm{i} T$. To find the factors $\partial_{x,y} U^{-1}$, $\partial_{x,y} U$, it is convenient to represent U and U^{-1} as follows:

$$U = \exp\left(\frac{2\pi \mathrm{i} z}{N} \right) \left\{ 1 + \left[\exp(-2\pi \mathrm{i} z) - 1 \right] \Pi \right\}, \tag{54}$$

with $\Pi_{ij} = \psi_i \psi_j^+$, $\Pi^2 = \Pi$. Then Eqn (53) is reduced to

$$\begin{aligned} N_{\mathrm{CS}} &= \frac{1}{\pi \mathrm{i}} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \sin^2(\pi z)\, \mathrm{Tr}\Big\{ \left[(\partial_x \Pi)\, (\partial_y \Pi) \right. \\ &\quad \left. - (\partial_y \Pi)\, (\partial_x \Pi) \right] \Pi \Big\} \\ &= \frac{1}{2\pi i} \int \mathrm{d}x\, \mathrm{d}y \left[\partial_x (\psi^\dagger \partial_y \psi) - \partial_y (\psi^\dagger \partial_x \psi) \right]. \end{aligned} \tag{55}$$

The last integral involves full derivatives and can be readily computed using the boundary conditions (41). The result depends only on the 'Abelian vector potentials' $\alpha(x, y)$, $\beta(x, y)$ and coincides with the flux of the corresponding auxiliary magnetic field, so

$$N_{\mathrm{CS}} = \frac{1}{m}. \tag{56}$$

It is clear that U^p gives rise to a configuration with $\Omega_3 = \exp(2\pi \mathrm{i} p/m)$ and with $N_{\mathrm{CS}} = p/m$. In particular, U^m gives an element of $\pi_3(\mathrm{SU}(m))$.

5. Witten index and the physics of confinement and chiral symmetry breaking

In this section we would like to discuss that the Witten index happens to contain information about such dynamical issues as confinement and chiral symmetry breaking(!). These matters have been thoroughly analyzed in Ref. [12].

To probe confinement, one should consider a refined version of the Witten index, $I_{\mathrm{W}}(e, m)$, which is the Witten index in the sector with magnetic flux m and electric flux e. The fluxes m and e can be introduced [17] in the case when the gauge group is not simply connected, which, however, is not very unusual. Consider, e.g., the adjoint form of $\mathrm{SU}(N)$ group, $\mathrm{SU}(N)/Z_N$ which has $\pi_1(\mathrm{SU}(N)/Z_N) = Z_N$. The magnetic fluxes on the 'space-like' T^3 appear when the gauge potentials are periodic only up to gauge transformations:

$$A(x_i + L) = \Omega_i^{-1} A(x_i)\, \Omega_i\,, \tag{57}$$

and the matrices Ω_i and Ω_j, specifying the boundary conditions in i-th and j-th directions, considered as $\mathrm{SU}(N)$ matrices, commute to an element of the center group:

$$\Omega_i \Omega_j = \exp\left(\frac{2\pi \mathrm{i} m_{ij}}{n} \right) \Omega_j\, \Omega_i\,. \tag{58}$$

The integers m_{ij} represent the configuration of magnetic fluxes.

The definition of electric flux is clearer in the case of the two-dimensional 'space' torus T^2. Consider a gauge transformation on T^2, that is, a map from T^2 to the gauge group G. Restricting it to the nontrivial cycles on T^2 defines a homomorphism from the fundamental group of the torus, $\pi_1(T^2)$, to the fundamental group of G, $\pi_1(G)$. The gauge transformation g can be deformed to unity if and only if that homomorphism is trivial. Let W_0 be the group of gauge transformations which can be deformed to unity, and W be the group of all gauge transformations. In quantizing a gauge theory one should impose only W_0 invariance, since only W_0 invariance is assumed by the Gauss law. The factor $\Gamma = W/W_0 = \mathrm{Hom}\,(\pi_1(T^2), \pi_1(G))$ may act in the Hilbert space of the theory, $\mathcal{H}$. So the Hilbert space decomposes in characters of Γ:

$$\mathcal{H} = \bigoplus_e \mathcal{H}_e\,, \tag{59}$$

where the characters e are multiplicative functions on Γ with values in $\mathrm{U}(1)$, $e \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$. If we work with the Hilbert space $\mathcal{H}_m$ of a given magnetic flux m, then

$$\mathcal{H}_m = \bigoplus_e \mathcal{H}_{e,m}\,. \tag{60}$$

In the case of a three-dimensional torus, T^3, the factor-group $\Gamma = W/W_0$ is nontrivial not only due to $\pi_1(G)$, but also due to $\pi_3(G)$, prominent in the instanton physics [6]; Γ contains a subgroup isomorphic to $\pi_3(G)$. Since in quantizing a gauge theory one usually restricts to the θ-sector, one should consider only those characters of Γ which on a gauge transformation, representing the generating element of $\pi_3(G)$, act as $\exp(\mathrm{i}\theta)$.

Consider first the case with zero magnetic flux, $m = 0$. Because of confinement, the ground state of the theory should have zero electric flux (or, equivalently, it should be Γ-invariant). Thus in $N = 1$ SUSY Yang – Mills theory one expects that

$$I_{\mathrm{W}}(e, 0) = 0\,, \quad \text{if} \quad e \neq 0\,. \tag{61}$$

It was explained in Ref. [12] that Eqn (61) implies the Witten index I_{W} [the one discussed in previous sections, not $I_{\mathrm{W}}(e, m)$] should not depend on the type of the boundary conditions imposed on the gauge fields on the space T^3.

The situation is more interesting in the presence of magnetic flux, $m \neq 0$. Recall first the idea of oblique confinement [17]. Normal confinement is due to condensation of monopoles. Oblique confinement is due to condensation of dyons. Under a 2π increase of the θ-parameter, monopole acquires electric charge [37]. Normal confinement turns to an oblique confinement. It turns out that at such a shift of the θ-parameter, $\theta \to \theta + 2\pi$, the electric flux e changes by an amount depending on m:

$$e \to e + \Delta(m) \tag{62}$$

[it was explained above, that e is a function on Γ with values in U(1); in Eqn (62) it is assumed that e is a phase of a U(1)-element); correspondingly, $\Delta(m)$ is defined up to a prime number]. The value $\Delta(m)$ was called in Ref. [12] the 'spectral flow'. It was shown in Ref. [12] that $\Delta(m)$ is equal to the fractional part of the instanton charge (which is typically fractional when $m \neq 0$).

Notice that upon the shift $\theta \to \theta + 2\pi$ we should find ourselves in the same theory, but, maybe, in a different vacuum state. If originally we were in the phase of 'normal' confinement, then the vacuum state was Γ-invariant, that is, it had zero electric flux. Upon the shift $\theta \to \theta + 2\pi$ it flowed to the state with the electric flux (62). Thus, if $I_{\mathrm{W}}(0, m) \neq 0$, then $I_{\mathrm{W}}(\Delta(m), m) \neq 0$, so the physics of confinement imply that

$$I_{\mathrm{W}}(e, m) = 0\,, \text{ unless } e \text{ is a multiple of } \Delta(m)\,. \tag{63}$$

Both Eqn (61) and Eqn (63) are nontrivial in the assumption of confinement in the theory and they were also verified in Ref. [12] by computations along the lines described in Sections 2, 3.

Let us now turn to the chiral symmetry breaking. As we mentioned in the Introduction, in the infinite volume subgroup Z_h, the Z_{2h} symmetry group transmutes vacua described by different phases of the chiral condensate. One may consider linear combinations of those vacua having definite chiral charge (the chiral charge is defined mod $2h$, because U(1) is broken to Z_{2h}). One expects to see that the vacua described in Section 3 are characterized by the same set of chiral charges as in the infinite volume. In Ref. [12] it was shown to be the case (!).

Recall that in Section 3 the vacua were obtained by quantizing about disconnected components of the space of classical vacua. Every component contributes a state described by a wave function which is just a constant and a number of states obtained by applying the $\lambda\lambda$ operator to that 'unit' wave function. Clearly, the $\lambda\lambda$ operator changes the chiral charge by 2. A nontrivial point is to understand what is the chiral charge of the 'unit' state on every component. Consideration of that is similar to the one in the study of (fractional) fermionic charge on solitons. The chiral charge of those 'unit' states was expressed in terms of CS invariants of the nontrivial triples. This analysis of the fermionic charge of the 'unit' states was also necessary to make sure that the disconnected components contribute with the same sign to Eqn (25) [12, 14].

This concludes our discussion of how the Witten index feels the physics of confinement and chiral symmetry breaking.

6. Conclusions

We reviewed the resolution of the long-standing controversy counting vacua in $N = 1$ SUSY Yang – Mills. The correct counting, compatible with the infinite volume physics, was restored by discovering additional disconnected components in the space of classical vacua in Yang – Mills theory on a torus. Those additional components are characterized by such triples of monodromies (Wilson loops) of the zero curvature gauge potentials which cannot be simultaneously conjugated to the Cartan torus in the gauge group. Remarkably, the new physics insights triggered a progress in the classical group theory — classification of such triples had been absent.

One can wonder whether higher-dimensional generalizations of triples — n-tuples — will find their place in physics. Notice that 2-tuples (which can always be conjugated to the Cartan torus) appear in 3d SUSY Yang – Mills (at a particular value of the θ-parameter, see Ref. [38]) and give $I_{\mathrm{W}} = 1$, 3-tuples appear in 4d SUSY Yang-Mills and give $I_{\mathrm{W}} = h$. Can one extend the sequence? The question is open.

We would also like to mention here another role n-tuples can play in physics. Consider the Kaluza – Klein type compactification to an n-torus. It is well known that the original gauge group can be broken by monodromies of the gauge potentials over the torus. The monodromies should be commuting because of the condition that the curvature components along the torus are zero. If they can be conjugated to the Cartan torus (which is usually assumed), the original group can be broken only to a subgroup of the same rank. This is not the case for nontrivial n-tuples. As is clear from Section 3, nontrivial triples can break, say, E_8 completely. A reader can convince himself that there are many ways to choose the coefficients C_j in Eqn (19) to break E_8 to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

The gauge potentials corresponding to the nontrivial triples of monodromies are described in Section 4. It is challenging to find Yang – Mills solutions interpolating between a trivial and a nontrivial, or between two different nontrivial classical vacua. There are indications that self-dual solutions of this type on $T^3 \times R$ should exist.

One can hope that Section 5, devoted to the question of how information about confinement and chiral symmetry breaking is encoded in the Witten index, may convince practitioners-skeptics that formal fancy things can still be useful.

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7. Appendix. Glossary

We start this glossary with some definitions from the group theory. They are given in the logical (vs. alphabetical) order.

A Cartan subalgebra is (up to some subtleties) a maximal commutative subalgebra in a semisimple Lie algebra. For example, in the case of $\mathrm{SU}(N)$, the Cartan subalgebra consists of traceless diagonal matrices.

The rank r is the dimensionality of the Cartan subalgebra.

Roots are linear forms on the Cartan subalgebra (that is, given a root α and an element a of the Cartan subalgebra, one can obtain a number $\alpha(a)$ which is linear in a and α), labeling eigen-subspaces of the action of elements of Cartan subalgebra in the Lie algebra. One says that a Lie algebraic element e_α belongs to the root subspace corresponding to the root α if it obeys

$$[a, e_\alpha] = \alpha(a) e_\alpha \tag{A.1}$$

for any element a of the Cartan subalgebra. For any root α, $-\alpha$ is also a root, that is, there is a Lie algebraic element $e_{-\alpha}$ such that

$$[a, e_{-\alpha}] = -\alpha(a)\, e_\alpha\,. \tag{A.2}$$

The set of all roots can be divided into the set of positive roots and the set of negative roots.

Simple roots are the roots that cannot be represented as a sum of positive roots. They form a basis in the linear space of all roots.

The highest root is such a root that its sum with any positive root is no longer a root. There is only one highest root in the case of a simple Lie algebra.

The root lattice is the lattice spanned by linear combinations of the simple roots with integer coefficients. Warning: not all elements of the root lattice are roots!

The Killing form is the canonical invariant symmetric nondegenerate bilinear form on a Lie algebra. In the case of a simple Lie algebra it is unique up to normalization. It can be viewed as a metric form. Its restriction to the Cartan subalgebra gives a bilinear form on the Cartan subalgebra. Physicists use it in the form of $\mathrm{tr}(ab)$, where a, b belong to a Lie algebra.

Coroots are elements of the Cartan subalgebra which correspond to roots in the following way. Let us agree to look at Cartan subalgebra elements as vectors with upper indices. The roots can be viewed as vectors with lower indices. The Killing form allows one to transform roots to coroots and vice versa. Thus, given a root α_j, one constructs the corresponding coroot $\alpha^{\vee j}$

$$\alpha^{\vee j} = \frac{2g^{jk}\alpha_k}{g^{mn}\alpha_m\alpha_n}\,, \tag{A.3}$$

where g^{ij} is (inverse) to the Killing form restricted to the Cartan subalgebra. For any coroot $\alpha^\vee$, the property

$$\exp(2\pi i \alpha^\vee) = 1 \tag{A.4}$$

holds. Coroots corresponding to the simple roots form a basis in the Cartan subalgebra.

The coroot lattice is the lattice spanned by linear combinations of the coroots corresponding to the simple roots.

Cartan torus — the maximal compact commutative subgroup in a Lie group. Elements of the Cartan torus are obtained as exponentials of the elements of the Cartan subalgebra. Due to the property (A.4) the Cartan torus can be viewed as the factor of the Cartan subalgebra over the coroot lattice.

Fundamental coweights ω^j, $j = 1, \ldots, r$ are introduced as elements of the Cartan subalgebra which form a basis dual to the basis α_j, $j = 1, \ldots, r$ of simple roots in the space of roots, that is

$$\alpha_j(\omega^k) = \delta_j^k\,. \tag{A.5}$$

The integer span of the fundamental coweights form the *weight lattice* for the adjoint form of the Lie group.

The Weyl group can be introduced as the group acting in the space of roots by reflections relative to hyperplanes orthogonal to roots:

$$W_\alpha : \beta \to \beta - \alpha\, \beta(\alpha^\vee)\,, \tag{A.6}$$

where W_α is the element of the Weyl group, corresponding to the root α. Via the Killing form, the Weyl group action can be defined on the Cartan subalgebra. Notice that any Lie group acts on itself and on its Lie algebra by conjugations. Most of the conjugations move an element of the Cartan subalgebra out of this subalgebra. Some of the conjugations transform the Cartan subalgebra to itself. Such conjugations form a group acting in the Cartan subalgebra and this group coincides with the Weyl group as defined in Eqn (A.4).

This ends our excursion into group theory. Below are definitions of some other mathematical terms used in the text.

The centralizer of a group G element Ω is a subgroup of G consisting of elements commuting with Ω.

A monodromy of a connection A around a circle C in physics terms means a Wilson loop $P\exp(\int_C A)$ of a gauge potential A over contour C.

Surjectively means 'on', that is, if a set X is mapped to a set Y surjectively then every element of Y can be obtained as an image of some element of X under that mapping.

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