Physics – Uspekhi **45** (6) 637 – 644 (2002)

METHODOLOGICAL NOTES

PACS numbers: **44.35.+c**, 47.27.Fe, 92.60.Ek

'Negative heat capacity' in stratified fluids

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DOI: 10.1070/PU2002v045n06ABEH001185

Contents

Abstract. **In 'doubly nonequilibrium' fluids (e.g., in temperature- and admixture-concentration-stratified solutions in the gravity field) situations may occur in which adding (extracting) heat can lower (raise) the temperature of the medium. Several simple examples of this phenomenon are considered.**

1. Introduction

In recent years certain paradoxical results were reported in the geophysical literature of in-situ measurements [1, 2]: "Readings of instruments lowered into the well (in the Antarctic shelf ice — *L I*) revealed an unexpected paradox: the seawater beneath the ice has a higher temperature in winter than in summer." [2]. In other words, in a certain sense one may speak of the negative heat capacity of the given geophysical medium.

Of course, such results call for careful verification. In any case, however, it would be reasonable to ask whether such effects are in principle feasible in the relatively simple hydrodynamic systems that are usually considered in geophysical hydrodynamics. In this paper we give several simple examples to demonstrate that such effects are indeed possible in fluids stratified with respect to the temperature, especially in the two-component 'doubly nonequilibrium' media — for example, in water stratified with respect to temperature and salinity, or in air stratified with respect to temperature and humidity. To the best of our knowledge, this phenomenon has so far escaped attention (with the exception of a few papers [3]), which is why the above-mentioned experimental results may seem paradoxical.

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Received 21 January 2002, revised 13 March 2002
Uspekhi Fizicheskikh Nauk **172** (6) 691 – 699 (2002)
Translated by A S Dobroslavskiĭ; edited by S M Apenko

2. Example of an exact solution: convection caused by a vertically extended heat source

Consider an unbound volume of solution whose density ρ in the conventional approximation [4 – 6] is a linear function of the temperature T and the concentration of admixture s:

$$\rho = \rho_0\left[1 - \alpha(T - T_0) + \beta(s - s_0)\right] . \qquad (1)$$

Here α is the coefficient of thermal expansion of the medium, β is the corresponding coefficient for the concentration of admixture (known in oceanography as the salinity compression coefficient). The subscript '0' denotes the 'reference' values of the variables. The simplest models of stratified media assume that the temperature, the admixture concentration and the density of the medium are linear functions of the vertical coordinate z (directed counter to gravity):

$$\bar{\rho}(z) = \rho_0\left[1 + (-\alpha\gamma_T + \beta\gamma_s)\, z\right] , \qquad (2)$$

where $\gamma_T = \mathrm{d}\bar{T}/\mathrm{d}z$, $\gamma_s = \mathrm{d}\bar{s}/\mathrm{d}z$ are constant gradients [1], and the bar denotes the 'background' values of the variables (to distinguish from the perturbations considered below that are associated with heat release and the resulting convection).

For the sake of simplicity we confine ourselves to the case when the density stratification of the medium is neutral (stratification of the temperature and that of the admixture concentration balance out in the density field):

$$-\alpha\gamma_T + \beta\gamma_s = 0 , \quad \bar{\rho}(z) = \rho_0 = \text{const} . \qquad (3)$$

Consider the simple problem of convection in such a medium — the response of the medium to the instantaneous release of heat uniformly distributed along the vertical axis z [7]. The expression for the heat source is

$$Q(r, t) = Q_0\, \frac{\delta(r)\, \delta(t)}{2\pi r} , \qquad (4)$$

[1] If the gradients γ_T, γ_s are constant, one may encounter the problem of an unrestricted rise or fall of the temperature, admixture concentration and density of the medium at $z \to \pm\infty$. This formal difficulty, however, is inessential, since in real situations we are dealing with layers of finite thickness.

where δ is the delta function, r is the distance from the z-axis, and Q_0 has the meaning of the amplitude of the source. Since the source does not depend on the coordinate z, we seek a solution that does not depend on z. The validation and limits of applicability of such 'one-dimensional regime' of convection are discussed, for example, in Refs [8, 9].

The system of equations of hydrodynamics, transport of heat and admixture we consider in the approximation that is often used for problems of convection — the Boussinesq approximation [4–6]. In this approximation the compressibility of the medium is assumed negligible, but the density is considered as a function of temperature and admixture concentration. In this approximation the continuity equation is $\operatorname{div} \mathbf{v} = 0$, where $\mathbf{v}$ is the velocity vector. Hence, together with the condition $\partial/\partial z = 0$, follows the absence of radial motions (perpendicular to the z-axis). Then the set of equations in question is (for the one-component medium it has been derived in greater detail, for example, in Ref. [9])

$$\frac{\partial w}{\partial t} = \nabla(\nu \nabla w) + g(\alpha T' - \beta s')\,, \qquad (5)$$

$$\frac{\partial T'}{\partial t} + \gamma_T w = \nabla(\varkappa \nabla T') + Q\,, \qquad (6)$$

$$\frac{\partial s'}{\partial t} + \gamma_s w = \nabla(\chi \nabla s')\,. \qquad (7)$$

Here w is the velocity component along the z-axis (there are no other velocity components in this configuration), t is the time, accent denotes deviations from the background state described above. Emphasize that the complete set of equations of hydrothermodynamics and admixture transport has reduced to a rather simple system (5)–(7) (linear as long as the exchange coefficients ν, $\varkappa$, χ do not depend on the velocity and other unknowns) solely by virtue of symmetry of the problem, without any assumptions regarding the smallness of perturbation amplitudes.

For the boundary conditions we assume first of all that perturbations are damped out away from the source (as $r \to \infty$). In addition, from symmetry considerations at $t > 0$ we have

$$\frac{\partial w}{\partial r} = \frac{\partial T'}{\partial r} = \frac{\partial s'}{\partial r} = 0\,, \quad \text{if} \quad r = 0\,. \qquad (8)$$

The problem in such a statement is especially easy solved when the exchange coefficients for different substances are constant and equal to each other: $\nu = \varkappa = \chi = \text{const} = K$ (this simplification is often used in geophysical applications, when dealing with the effective coefficients of turbulent exchange). We multiply equation (6) by α, equation (7) by β, and subtract the second from the first. Using equation (3), we get the following equation for the dimensionless buoyancy $b = \alpha T' - \beta s'$:

$$\frac{\partial b}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r} r \frac{\partial b}{\partial r} + \alpha Q\,. \qquad (9)$$

This is the standard equation of diffusion (heat conductivity), whose solution for the instantaneous source (4) is

$$b = \frac{\alpha Q_0}{4\pi K t} \exp\left(-\frac{r^2}{4Kt}\right). \qquad (10)$$

The quantity b up to a coefficient is the source in equation (5). The solution of the latter, as is easily proved, can be expressed in the form

$$w = gtb = \frac{\alpha g Q_0}{4\pi K} \exp\left(-\frac{r^2}{4Kt}\right). \qquad (11)$$

Now let us find the temperature perturbation. We rewrite equation (6) with due account for (11):

$$\frac{\partial T'}{\partial t} = \frac{K}{r}\frac{\partial}{\partial r} r \frac{\partial T'}{\partial r} - g\gamma_T tb + Q\,. \qquad (12)$$

The solution of equation (12) can be represented as the sum of two terms corresponding to the two sources on the right-hand side of (12). One of these terms corresponds to the source Q — this is the familiar Green function of the equation of heat conductivity. The other source $(-g\gamma_T tb = -\gamma_T w)$ is easily proved to give the following contribution to the solution

$$-\frac{1}{2} g\gamma_T t^2 b = -\frac{1}{2}\gamma_T t w\,.$$

We see that the temperature perturbation associated with the heat release is

$$T' = \frac{Q_0}{4\pi K t}\exp\left(-\frac{r^2}{4Kt}\right) - \frac{1}{2}\frac{\alpha g \gamma_T t Q_0}{4\pi K}\exp\left(-\frac{r^2}{4Kt}\right)$$
$$= \frac{Q_0}{4\pi K t}\exp\left(-\frac{r^2}{4Kt}\right)\left(1 - \frac{1}{2}\alpha g \gamma_T t^2\right). \qquad (13)$$

In a similar way we can find the expression for the perturbation of the admixture concentration [which will only contain a term similar to the second term in (13)].

The first term on the right-hand of (13) does not call for explanations: it describes the conventional diffusion of heat released at $t = 0$ at the z-axis. The second term describes the temperature variations associated with the advection of heat — the transfer of heat by the rising convective vertical movements in the stratified medium. The special feature of the second term is that its amplitude, unlike that of the first term, increases with time. The waxing response to arbitrarily small initial perturbation is a sign of an unstable system. In this sense we may speak of a new type of instability that has not apparently been discussed before.

Another feature of the second term in (13) is that its sign can be (at $\gamma_T > 0$) opposite to the sign of the initial heat release.

Figure 1 shows the time dependence of the temperature perturbation on the z-axis at $\gamma_T > 0$. At small times, the first term in (13) prevails — we are dealing with the conventional 'diffusion smearing' of the released heat. Later on, however, the temperature perturbation changes its sign — in response to the release of heat the system exhibits negative temperature variation (supply of cooler liquid from below). In this context one may speak of the effective negative heat capacity of the medium under consideration.

This example of a convection problem may seem very specialized and speculative (an infinitely long heat source in an unbound medium). This example, however, has been used here only for simplicity and for getting the exact analytical solution. It is easy to see that for 'doubly nonequilibrium' media (stratified with respect to both the temperature and the admixture concentration) similar results can be obtained for other well-known forms of convection from local sources — convection jets and isolated thermals [10]. The physical mechanism is quite transparent. The volume of the medium that has received an additional amount of heat will float up; it may become colder than the surrounding medium (since it

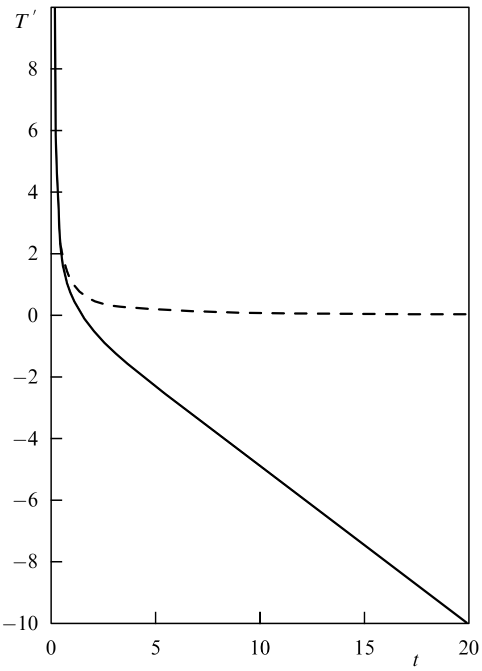

Figure 1. Evolution of the temperature perturbation on the axis $z = 0$. The temperature is normalised to $Q_0(\alpha g\gamma_T)^{1/2}/4\pi K$, the time to $(\alpha g\gamma_T)^{-1/2}$. The dashed line shows a similar evolution for a nonstratified medium.

carries a cooler liquid), but will continue to float up (because of the buoyancy due to the negative perturbation of the admixture concentration). In this way, in response to the supply of additional heat to the liquid system it exhibits a negative variation of the temperature! In Ref. [11], where the problem of convection near an infinite vertical plate in a two-component mixture is solved, it is shown that similar effects are also possible in the case of a stable stratification of density.

3. Rayleigh–Benard type problems

Let us now consider the problems of convection from flat horizontal sources — for example, convection in a horizontal layer. Such problems are known to be highly nontrivial and are often extremely complicated even in the case of single-component media (see, for example, [5, 8]). For the description of turbulent convection in problems of such geometry, a theoretical scheme was proposed in Ref. [12] that allows an analytical treatment. This nonlinear model is based on dimension and similarity considerations, and also builds on the semiempirical theory of turbulence. Subsequently it was developed theoretically and to some extent experimentally validated [13–16]. It is not difficult to extend this approach to the case of a two-component medium [17].

As a rule, we shall consider situations when the density stratification is sufficiently unstable, so that the Rayleigh number is large, and the convection is turbulent. It is not precluded, however, that the stratification with respect to the temperature or the admixture concentration (further on for the sake of definiteness we shall refer to salinity) can be stable. In other words, the contributions of the temperature and the admixture concentration to the vertical density gradient may have opposite signs.

According to the theoretical scheme of Ref. [12], the vertical convective turbulent transfer of heat (buoyancy) is described by a quasilinear heat conduction type equation:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(K\,\frac{\partial T}{\partial z}\right). \tag{14}$$

(In this section T, s and ρ describe the corresponding values averaged with respect to turbulent pulsations.) The effective coefficient of turbulent exchange K, which describes the convective transfer of heat (buoyancy), depends on the degree of convective instability — that is, on the vertical density gradient $\partial\rho/\partial z$. From dimensional and similarity considerations, or from the turbulent energy balance equation [18], we find

$$K = l^2\sqrt{\frac{g}{\rho_0}\,\frac{\partial\rho}{\partial z}}\,, \quad \frac{\partial\rho}{\partial z} \geqslant 0\,. \tag{15}$$

Here ρ_0 is a certain average ('reference') value of the density of the medium, l is the scale of turbulence. In the commonly used approximation [4, 5, 12]

$$\frac{\partial\rho}{\partial z} = -\rho_0\alpha\,\frac{\partial T}{\partial z}\,,$$

whence

$$K = l^2\sqrt{-g\alpha\,\frac{\partial T}{\partial z}}\,, \tag{16}$$

where we assume that $\partial T/\partial z \leqslant 0$.

In order to extend this model to the case of a two-component mixture, we must first of all add the equation of admixture transport, similar to (14):

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial z}\,K\,\frac{\partial s}{\partial z}\,. \tag{17}$$

The density gradient in this case is comprised of two components (thermal and salinity):

$$\frac{\partial\rho}{\partial z} = \rho_0\left(-\,\alpha\,\frac{\partial T}{\partial z} + \beta\,\frac{\partial s}{\partial z}\right). \tag{18}$$

From (18) and (15) we find

$$K = l^2\sqrt{g\left(-\,\alpha\,\frac{\partial T}{\partial z} + \beta\,\frac{\partial s}{\partial z}\right)}\,, \tag{19}$$

where we assume that

$$-\alpha\,\frac{\partial T}{\partial z} + \beta\,\frac{\partial s}{\partial z} \geqslant 0\,. \tag{20}$$

The set of equations (14), (17), (19) describes the nonlinear interaction of the fields of temperature and admixture concentration through their influence on the convection (on the coefficient K). In order to close this system we need an additional hypothesis concerning the scale of turbulence l. In the simplest case there are reasons to assume l to be proportional to the thickness of the turbulized layer [19, 12]. Sometimes a more detailed description is required, which takes into account the decrease in l on approaching the boundary of the convection region.

As the simplest example, let us consider the stationary (on average) turbulent convection in the horizontal layer of the two-component medium. By Q_T and Q_s we denote the mean steady fluxes of heat and admixture across the layer under consideration $0 < z < H$:

$$c_p \rho_0 K \frac{dT}{dz} = -Q_T \,, \tag{21}$$

$$\rho_0 K \frac{ds}{dz} = -Q_s \,, \tag{22}$$

where c_p is the specific heat capacity of the medium. The linear combination of the last two equations gives

$$\rho_0 K \left(\beta \frac{ds}{dz} - \alpha \frac{dT}{dz} \right) = \frac{\alpha Q_T}{c_p} - \beta Q_s \,. \tag{23}$$

In accordance with (20), the left-hand side of (23) is assumed to be non-negative. This implies a restriction on the relation of heat and admixture fluxes:

$$Q_s \leqslant \frac{\alpha Q_T}{\beta c_p} \tag{24}$$

(otherwise the density stratification would be steady, and the model would be not applicable).

From (23) and (19) we get

$$\beta \frac{ds}{dz} - \alpha \frac{dT}{dz} = \left(\frac{\alpha Q_T / c_p - \beta Q_s}{\rho_0 g^{1/2} l^2} \right)^{2/3} , \tag{25}$$

$$K = \left[\frac{l^4 g}{\rho_0} \left(\frac{\alpha Q_T}{c_p} - \beta Q_s \right) \right]^{1/3} . \tag{26}$$

Then from (21) and (22) we have

$$\frac{dT}{dz} = -\frac{Q_T / c_p}{\left[g \rho_0^2 l^4 (\alpha Q_T / c_p - \beta Q_s) \right]^{1/3}} \,, \tag{27}$$

$$\frac{ds}{dz} = -\frac{Q_s}{\left[g \rho_0^2 l^4 (\alpha Q_T / c_p - \beta Q_s) \right]^{1/3}} \,. \tag{28}$$

Equations (26) – (28) contain enough information about the vertical profiles of K, T and s, assuming that we know the profile $l(z)$. In particular, if near the lower boundary of the layer $z = 0$ we assume that $l \sim z$, we get the well known asymptotics [18]

$$\frac{dT}{dz} \sim z^{-4/3} \,, \quad \frac{ds}{dz} \sim z^{-4/3} \,, \quad K \sim z^{4/3} \,.$$

The temperature drop in the layer can be represented as [2]

$$\Delta T \equiv T|_{z=0} - T|_{z=H} = -\int_0^H \frac{dT}{dz} \, dz$$

$$= \frac{Q_T / c_p}{\left[g \rho_0^2 (\alpha Q_T / c_p - \beta Q_s) \right]^{1/3}} \int_0^H \frac{dz}{l^{4/3}(z)} \,. \tag{29}$$

The analysis of this last expression is made easier by rewriting it in the dimensionless form

$$\hat{T} = \frac{\hat{Q}_T}{(\hat{Q}_T - \hat{Q}_s)^{1/3}} \,, \tag{30}$$

where we use the dimensionless variables

$$\hat{T} = \alpha \Delta T \,, \quad \hat{Q}_T = \frac{\alpha Q_T}{c_p \rho_0 w} \,, \quad \hat{Q}_s = \frac{\beta Q_s}{\rho_0 w} \,,$$

$$w^{-1} = \left[\frac{1}{g} \left(\int_0^H \frac{dz}{l^{4/3}} \right)^3 \right]^{1/2} = \left[\frac{H^3}{g (\overline{l^{4/3}})^3} \right]^{1/2} .$$

The bar denotes averaging with respect to the thickness of the layer under consideration.

Figure 2 shows the relation [expressed by equation (30)] between the temperature drop across the layer and the heat flux for different signs of stratification of the admixture concentration and in the absence of such stratification. According to condition (20), all three functions are defined in the domains to the right of the values of $\hat{Q}_T$ in Fig. 2 corresponding to the neutral stratification of density. In the one-component medium (curve *2*) we have $\Delta T \sim Q_T^{2/3}$. The shapes of curves *1* and *2* in Fig. 2 agree well with the intuitive expectations that the heat fluxes must be monotonic functions of temperature differences, and vice versa. In this respect the behavior of curve *3* is nontrivial, exhibiting a downward portion in the interval $\hat{Q}_s < \hat{Q}_T < (3/2)\,\hat{Q}_s$ (at the right-hand end of this interval expression (30) as a function of $\hat{Q}_T$ assumes the minimum value of $3(\hat{Q}_s/2)^{2/3}$). This curve corresponds to the case when the convection takes place 'despite' the stable stratification of the admixture, because of heating of the liquid layer from below, or cooling from above.

Assume, for example, that the temperature on the lower boundary of the layer is fixed. How does the temperature on the upper boundary change if, other conditions being equal, we increase the removal of heat from the upper boundary (thus increasing the heat flux Q_T through the layer)? At first glance it seems obvious that the increased heat sink must lead to a reduction of temperature on the upper boundary. The results obtained above indicate, however, that in the interval $\hat{Q}_s < \hat{Q}_T < (3/2)\,\hat{Q}_s$ the temperature will change in the opposite sense — it will increase! Indeed, in this interval the

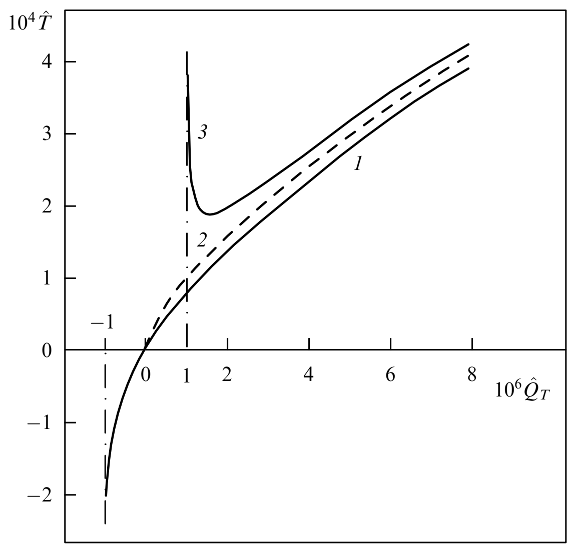

Figure 2. Relation between the vertical temperature difference in the horizontal layer and the heat flux. Curves *1*, *2*, *3* correspond, respectively, to the unstable ($\hat{Q}_s = -10^{-6}$), neutral ($\hat{Q}_s = 0$) and stable ($\hat{Q}_s = 10^{-6}$) stratifications of the impurity concentration. The dot-and-dash lines are vertical asymptotes of curves *1* and *3*.

[2] Expression (29) does not take into account the temperature difference in the thin laminar layers near the horizontal boundaries of the region under consideration. Since the contribution from these layers to the overall vertical temperature difference can be very large, it must be kept in mind that we are only referring to the temperature drop across the turbulent layer. A more consistent model would include all the above-mentioned layers and describe their interaction. Here we consider it expedient, however, to use the simplest possible model and focus on the qualitative features of the solution.

temperature difference decreases with increasing Q_T. Therefore, as Q_T increases (that is, with the increased removal of heat on the upper boundary), the temperature of the upper boundary must change towards the temperature of the lower boundary — that is, increase.

Being somewhat unexpected, this result can be readily explained from the following physical considerations. As the removal of heat from the upper boundary increases, the convection and mixing increase in the layer under consideration. This, therefore, increases the supply of heat upwards from the lower boundary, whose temperature is fixed. In this interval of Q_T, such effect turns out to be more important than the straightforward cooling owing to the removal of heat from the upper boundary. This finding may be interpreted in terms of the effective 'negative heat capacity' of the medium under consideration in the indicated range of parameters.

Another interesting feature of the result is the feasibility of non-unique solutions. One and the same vertical temperature drop ΔT on curve *3* corresponds (with the exception of one point) to two different values of flux $\hat{Q}_T$, and hence two different regimes of turbulent convection. The issue of the feasibility of each of these regimes cannot apparently be resolved without analyzing the relevant nonstationary problems. The possible non-uniqueness of regimes in the case of turbulent convection was recently noted in the context of a different problem in Ref. [20].

Another example of a nonstationary problem was considered in Ref. [17] — the development of turbulent convection in stratified saline water with a cooled surface. Nontrivial effects are possible in this case too if the contributions of temperature and salinity to the background density stratification have opposite signs. Flash cooling of the surface will off course reduce its temperature at first. This negative temperature variation fades off with time. However, cooling of the surface may initiate the Rayleigh–Taylor instability. The associated convective mixing of the upper layer of stratified liquid is accompanied by the change of the temperature of the surface that depends on the initial temperature stratification. Rather than fading off, such a temperature variation may increase with time. What is more, the sign of this change depends on the sign of γ_T (the initial vertical temperature gradient); in particular, it can be positive. In other words, the liquid medium may respond to the flash cooling of the surface with a progressive increase of temperature of the surface! This is possible in situations when the contributions of temperature and admixture concentration to the initial density stratification have opposite signs.

4. Currents in a medium inhomogeneously heated from above

The arising of convection is often associated with heating of the medium from below (or cooling from above). However, horizontally nonuniform cooling from below or heating from above also lead to convection even in situations when the density everywhere decreases with the height. The rising currents are caused by the horizontal pressure gradients. Such problems have many applications in geophysics, where these currents are more commonly known not as 'convection' but as 'thermal circulations'. The studies of such currents comprise a vast body of literature. As far as we know, however, it was only recently that attention was paid to certain peculiar features of such problems for two-component media — for example, saline water or moist air [21, 22].

Consider a semi-bound layer of water stratified with respect to temperature and salinity in such a way that the hydrodynamic equilibrium is stable (emphasize that the temperature or the salinity stratifications may each be unstable, but the overall density stratification is stable). The squared Brent–Vaisala frequency (the buoyancy frequency) [5, 8] is consist of the 'thermal' and 'salinity' components:

$$N^2 = N_T^2 + N_s^2 \,, \qquad (31)$$

where $N_T^2 = \alpha g \gamma_T$, $N_s^2 = -\beta g \gamma_s$.

For describing the small amplitude perturbations caused by the steady nonuniform heating and/or 'salinization' at the surface $z = 0$, we use the linearized stationary set of equations of hydrodynamics, and equations of heat and admixture transport in the Boussinesq approximation [5, 6]:

$$0 = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{v} + g(\alpha T - \beta s)\,\mathbf{e}_z \,, \quad \nabla\mathbf{v} = 0 \,,$$

$$\gamma_T\,\mathbf{v}\cdot\mathbf{e}_z = \varkappa\nabla^2 T \,, \quad \gamma_s\mathbf{v}\cdot\mathbf{e}_z = \chi\nabla^2 s \,. \qquad (32)$$

Here T and s are the perturbations of temperature and salinity, respectively; $\mathbf{v}$ is the three-dimensional vector of perturbation of the velocity field; and $\mathbf{e}_z$ is the unit vector in the direction of z-axis.

We assume that on the surface of the water $z = 0$ the perturbations of the vertical fluxes of heat and salt are specified, which are harmonic functions of the horizontal coordinates x and y:

$$\left.\begin{aligned} c_p\rho_0\varkappa\,\frac{\partial T}{\partial z} &= Q_T \cos k_x x \cos k_y y \\ \rho_0\chi\,\frac{\partial s}{\partial z} &= Q_s \cos k_x x \cos k_y y \,, \end{aligned}\right\} \quad \text{at} \quad z = 0 \,, \qquad (33)$$

where the meaning of the parameters Q_T, Q_s, k_x, k_y is straightforward. We neglect the deformations of the water surface and the tangential stresses:

$$w = 0 \,, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0 \,, \qquad (34)$$

where u, v, w are the components of velocity along the axes x, y and z, respectively. We assume that away from the surface (at $z \to -\infty$) all perturbations fade out.

We seek solutions in the form of harmonic functions of horizontal coordinates:

$$\begin{aligned} w(x,y,z) &= W(z) \cos k_x x \cos k_y y \,, \\ u(x,y,z) &= U(z) \sin k_x x \cos k_y y \,, \qquad (35) \\ v(x,y,z) &= V(z) \cos k_x x \sin k_y y \end{aligned}$$

etc. Eliminating from (32) all unknowns except w, we get the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right)^3 W = k^6 S W \,, \qquad (36)$$

where $k^2 = k_x^2 + k_y^2$,

$$S = \frac{1}{k^4\nu}\left(\frac{N_T^2}{\varkappa} + \frac{N_s^2}{\chi}\right) = R_T + \frac{R_s}{\tau} \,, \quad \tau = \frac{\chi}{\varkappa} \,, \qquad (37)$$

$$R_T = \frac{N_T^2}{\nu\varkappa k^4} \,, \quad R_s = \frac{N_s^2}{\nu\varkappa k^4}$$

are the analogs of the Rayleigh number. The dimensionless parameter S has the meaning of a certain generalized Rayleigh number. It is non-negative for the stable background states.

The solution of equation (36) is found in the standard way in the form of a linear combination of exponentials $\exp(q_i kz)$, where q_i are the roots of the characteristic equation

$$(q^2 - 1)^3 = S\,.$$

In the general case, the solution is rather cumbersome [21]. Here we shall only reproduce the most interesting asymptotics for large values of the generalized Rayleigh number S (this corresponds to the important role of stable density stratification). At $S \gg 1$,

$$w \approx \frac{Q_T k S^{1/6}}{2c_p \rho_0 \gamma_T}\,\frac{(1+P/\tau)}{(1+N_s^2/\tau N_T^2)}\left[\exp(S^{1/6}kz) - 2\exp\left(\frac{1}{2}\,S^{1/6}kz\right)\cos\left(\frac{\sqrt{3}}{2}\,S^{1/6}kz - \frac{\pi}{3}\right)\right] \times \cos k_x x \cos k_y y\,, \tag{38}$$

$$T \approx \frac{Q_T}{c_p \rho_0 \varkappa k(1+N_s^2/\tau N_T^2)}\left\{\frac{1}{\tau}\left(\frac{N_s^2}{N_T^2} - P\right)\exp(kz) + \frac{(1+P/\tau)}{2S^{1/6}}\left[\exp(S^{1/6}kz) + 2\exp\left(\frac{1}{2}\,S^{1/6}kz\right) \times \cos\left(\frac{\sqrt{3}}{2}\,S^{1/6}kz\right)\right]\right\}\cos k_x x \cos k_y y\,, \tag{39}$$

$$s \approx \frac{\alpha Q_T}{\beta c_p \rho_0 \chi k(1+N_s^2/\tau N_T^2)}\left\{\left(\frac{N_s^2}{N_T^2} - P\right)\exp(kz) - \frac{N_s^2}{N_T^2}\,\frac{(1+P/\tau)}{2S^{1/6}}\left[\exp(S^{1/6}kz) + 2\exp\left(\frac{1}{2}\,S^{1/6}kz\right) \times \cos\left(\frac{\sqrt{3}}{2}\,S^{1/6}kz\right)\right]\right\}\cos k_x x \cos k_y y\,, \tag{40}$$

where

$$P = -\frac{\beta Q_s}{\alpha Q_T/c_p} = -\chi\beta\,\frac{\partial s}{\partial z}\left(\varkappa\alpha\,\frac{\partial T}{\partial z}\right)^{-1}$$

is the dimensionless ratio of perturbations of the 'salinity' and 'thermal' flows of buoyancy on the surface.

The solution involves a number of dimensionless parameters, in particular S, $N_s^2/N_T^2 = R_s/R_T$, P, τ (only the analog of the Rayleigh number remains when we disregard the binary nature of the medium). We readily see that the effects of the admixture (stratification of the salinity) give rise to essentially new and nontrivial properties of the solution as compared with the single-component hydrodynamic system.

If the exchange coefficients for heat and salt are markedly different (as we know [5], in the case of molecular exchange $\tau \sim 10^{-2}$), we first of all notice the unequal 'weight' of the temperature and salinity contributions to the effects of buoyancy. Although the overall density stratification is proportional to $N_T^2 + N_s^2$, the solutions, as a rule, involve not this sum but rather the quantity $N_T^2 + N_s^2/\tau$ (in particular, the generalized Rayleigh number S is proportional to this quantity). In other words, the contribution of the salinity to the background density stratification (with the value of τ of the indicated order of magnitude) is greater than the contribution from the thermal term by two orders of magnitude. The destabilizing role of the effects of 'differential diffusion' is well known [5]. From this solution we see that, depending on the sign of γ_s, the salinity stratification can be a stabilizing factor, and a much more efficient one than the temperature stratification. Then, if we compare the effects on the solution from the 'salinity' and 'thermal' perturbations on the boundary (parameters Q_s and Q_T), we also find the 'priority' of the former: the solution in some cases will contain the combination $1 + P/\tau$. This implies that in the solution the relative weight of the salinity-related buoyancy is τ^{-1} — that is, they disturb the system much more than similar buoyancy flows of thermal origin.

As follows from (38), vertical motions (as well as the perturbations of horizontal velocity, density and pressure, penetrate the water to the depth of H_s of the order of $k^{-1}S^{-1/6} \ll k^{-1}$. This is a understandable and expected extension of the result for the one-component medium, stratified only with respect to the temperature [23]. In such a medium, the linear perturbations of all variables go down into the water to a depth of the order of

$$H_T = \left(\frac{\nu\varkappa}{k^2 N_T^2}\right)^{1/6} = k^{-1}R_T^{-1/6}\,. \tag{41}$$

At $\tau \ll 1$, the S number exhibits a much stronger dependence on $\beta\gamma_s$ than on $\alpha\gamma_T$ — that is, the stable salinity stratification is a much stronger obstacle for the penetration of stationary perturbations into the medium than thermal stratification (with equal vertical density gradients).

These results can be easily interpreted from the following physical considerations. At $\tau \ll 1$, the relaxation of salinity perturbations is much slower than that of thermal perturbations. Because of this, the salinity-related effects of buoyancy and stratification may be much more pronounced than their 'thermal' counterparts.

Even in the absence of effects of 'differential diffusion' (at $\tau = 1$), we can point to at least two nontrivial properties of the solution.

Expressions (39), (40) both involve kz in the exponent, which has no counterpart in the case of the one-component medium [23]. The appearance of this term indicates that the perturbations of temperature and salinity now have components that penetrate to a depth of the order of the horizontal scales of the perturbation $L = k^{-1}$. For comparison we might note that for large values of Rayleigh number the depth of penetration of perturbations into the one-component medium, according to (41), is much less than L. At first glance, if we also add the stable salinity stratification to the stable temperature stratification, this will only add to the overall stability of the system, and reduce the penetration depth and the amplitude of all perturbations. As follows from the solution, however, this conclusion only applies to some extent to the perturbations of velocity, pressure and density. By contrast, the penetration depth of steady thermal perturbations increases dramatically (with the exception of certain cases with special boundary conditions). Moreover, the amplitude of this 'additional' slowly decaying exponential can considerably exceed all other terms in (39) (because the latter contain a large quantity $2S^{1/6}$ in the denominator) — that is, we also observe a sharp increase in the amplitude of thermal perturbations. However, the amplitudes of perturbations of temperature and salinity in (39), (40) stand in such relation to each other that the density perturbations only exist in a relatively thin layer of thickness $LS^{-1/6}$, where the density perturbations, as a rule, are a small difference between the thermal and the salinity components.

The properties of the solution that relate to the possible sign of temperature perturbations are highly nontrivial. At first glance it may seem obvious that this sign must be the same as the sign of Q_T (at least when $Q_s = 0$): the additional inflow of heat to the medium (such perturbation of the flow we refer to as positive, $Q_T > 0$) should work to increase the temperature of the corresponding region of the medium. As follows from solution (39), however, the temperature response of the surface layer, depending on the values of the parameters, may have the opposite sign! Indeed, as follows from (39), we have

$$T|_{z=0} \approx \frac{Q_T}{c_p \rho_0 \varkappa k (1 + N_s^2/\tau N_T^2)} \times \left[\frac{1}{\tau} \left(\frac{N_s^2}{N_T^2} - P \right) + \frac{3(1 + P/\tau)}{(2S^{1/6})} \right] \cos k_x x \cos k_y y \, . \tag{42}$$

For example, assume for simplicity that the perturbation of the salinity flux on the surface is zero ($Q_s = 0$, $P = 0$); the temperature stratification is stable, and the salinity stratification is unstable ($N_T^2 > 0$, $N_s^2 < 0$), and the negative number $N_s^2/\tau N_T^2$ is less than one in absolute value but still of the order of unity. When the value of S is large enough, the main term in brackets in (42) is the first term, because the denominator of the second term is $\sim 2S^{1/6}$. As we have said, the first term is negative — that is, the sign of the temperature response is opposite to the sign of the variation of the inflow of heat! Here we are speaking of situations when the coefficient of the first exponential in (39) is negative, and is greater in magnitude than the coefficients in front of all other exponentials.

Figure 3 gives examples of the profiles of variations of temperature, density and vertical velocity caused by the positive change of the heat flux on the surface ($Q_T > 0$, $Q_s = 0$, $x = y = 0$). We use the parameters corresponding to the usual values for the turbulized subsurface seawater layer [21]: $\tau = 1$, $\varkappa = \nu = 2 \times 10^{-4}$ m^2 s^{-1}, $k = 0.47$ m^{-1} ($L \equiv 2\pi/k \approx 13$ m). We see that the perturbations of velocity and density at $S \gg 1$ penetrate into the water to a depth which is much less than the characteristic horizontal scale of the perturbation L. The structure of flows is similar to that in the corresponding problem for the one-component medium [23] — a sequence of simple circulation cells that rapidly fade off with the depth, so that one or two topmost cells are more or less articulate in Fig. 3. The amplitude and the depth of penetration of the dynamic perturbations will be even less in the presence of effects of 'differential diffusion' ($\tau \ll 1$) and stable salinity stratification (this follows from (38), since with decreasing τ the absolute values of the dimensionless numbers $N_s^2/\tau N_T^2$ and S increase). The effects related to the two-component nature of the medium do not lead to any qualitative changes in the profiles of perturbations of velocity and density (curves marked with '2' in Fig. 2 correspond to the absence of these effects).

We could have anticipated such behavior of the dynamic perturbations. The situation with the temperature perturbations, however, is entirely different. We see that the case of the one-component medium, studied earlier in detail (curve T_2) is highly specific, and the inclusion of stratification of the admixture (salinity) completely alters the situation: the depth of penetration and the amplitude of perturbations at $S \gg 1$ increase dramatically. Assume, for example, that in addition to the stable background thermal stratification we have a stable salinity stratification. It seems intuitively clear that the amplitude and the penetration depth of perturbations in such a case (other conditions being equal) must decrease. Comparing the curves T_2 and T_3 in Fig. 3, however, we see the opposite — the perturbations are reinforced and deepened! Observe once again that the reinforcement of perturbations related to the two-component nature of the medium occurs even when we depart from the region of instability.

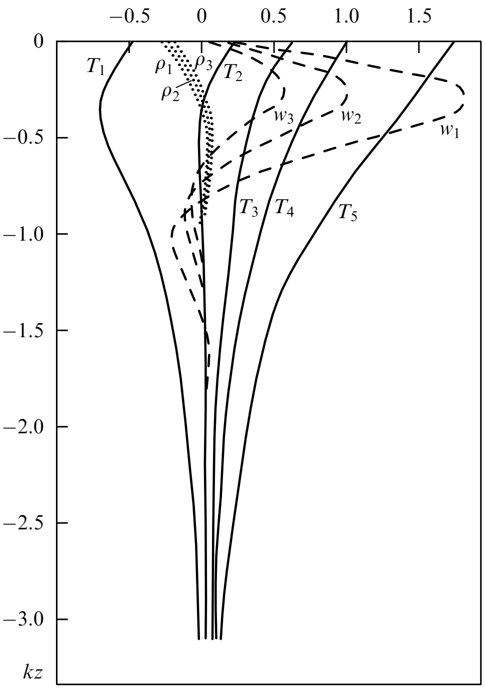

Figure 3. Example of dimensionless vertical profiles beneath the heat source on the surface of the medium ($Q_T > 0$, $x = y = 0$) at $\tau = 1$. Perturbations of the temperature are normalised to $Q_T/c_p\rho_0\varkappa k$ and shown by solid lines, perturbations of the temperature are normalised to $\alpha Q_T/c_p\varkappa k$ and shown by dotted lines, profiles of the vertical velocity are normalised to $2.5kQ_T/c_p\rho_0\gamma_T$ and shown by dashed lines. Curve numbers correspond to the following background states: '1' — $N_T^2 = 10^{-4}$c^{-2}, $N_s^2 = -0.5 \times 10^{-4}$ c^{-2}, $S \approx 0.25 \times 10^5$; '2' — $N_T^2 = 10^{-4}$ c^{-2}, $N_s = 0$, $S \approx 0.5 \times 10^5$; '3' — $N_T^2 = N_s^2 = 10^{-4}$ c^{-2}, $S \approx 10^5$. The temperature profiles numbered '4' and '5' correspond to '2' and '1' with the 'inverted' temperature and salinity stratifications: '4' — $N_T^2 = 0$, $N_s^2 = 10^{-4}$ c^{-2}, $S \approx 0.5 \times 10^5$, '5' — $N_s^2 = -2N_T^2 = 10^{-4}$ c^{-2}, $S \approx 0.25 \times 10^5$.

The most unexpected result, however, is the sign of the temperature variations in the region of parameters where the stable temperature stratification is partly compensated by the unstable stratification of the admixture. In response to the additional inflow of heat, the upper layer of the water becomes cooler (curve T_1 in Fig. 3).

This last result can be interpreted as follows. If the salinity stratification is not taken into account, the supply of heat to some portion of the surface increases the temperature in the subsurface region, and gives rise to ascending motions in this region. These upward motions carry the colder water from below. This transfer (term $\gamma_T w$ in the heat transport equation) partially compensates for the supply of heat from above. Therefore, the positive variation of the temperature in the absence of effects of admixture stratification is small, and occurs in a relatively thin layer of thickness $\Delta z \sim H = L/R_T^{1/6}$. The presence of an unstable salinity gradient works to reinforce the transport of cold water from below. As a result, the water in the surface layer is even cooler than the background. In spite of this, the variation of density is negative, because the water coming from below has a lower salinity, and so the ascending motion is sustained in this case too.

It is interesting to note that the profiles of density perturbations ρ_1, ρ_2, ρ_3 (see Fig. 3) for the three background states are very close. The differences in the corresponding velocity profiles are large, but only quantitative. At the same time, the perturbations in the temperature field T_1, T_2, T_3 may differ qualitatively (have different signs).

Observe also that the same flows of heat on the surface may lead to different changes in the media that have similar density stratifications but different salinity and thermal stratifications. This is clear from the comparison of profiles T_4 and T_2, or T_5 and T_1.

We see that in a stably stratified two-component medium that is non-uniformly heated from above (or cooled from below), flows may arise that even in the linear approximation may produce a temperature response of unexpected sign. The temperature may fall in response to the additional supply of heat, or vice versa. There also are some other nontrivial effects related to the two-component nature of the medium. In particular, the presence of stable stratification of the admixture in addition to the stable temperature stratification may lead (contrary to the intuition) to a considerable increase in the amplitude and depth of penetration into the medium of thermal perturbations associated with the horizontal instabilities of heat flows on the surface. Given the considerable difference in the transfer coefficients for heat and salt, the salinity stratification may have a much greater stabilizing effect on the layer of water than a similar stable density stratification of thermal origin.

5. Conclusion

We see that stratified two-component fluids in a field of gravity may exhibit nontrivial effects, examples of which could be extended. In particular, as already mentioned, in Ref. [11] the solution was found of the problem of the convection jet that arises when the medium is heated by an immersed vertical plate. In this case, the flow of heat directed into the liquid may lower its temperature instead of increasing it. In addition, when we go from a one-component medium to a two-component medium with stable stratification, we may encounter a considerable increase in the amplitude and penetration depth of perturbations. Such effects are especially conspicuous when the temperature and the concentration of admixture give contributions of opposite signs to the density stratification of the medium.

One of the results is that in the two-component medium the thermal response to a relatively small release of heat may be rather strong (and increasing with time). This is associated with the fact that the two-component medium may lack the feedback loop, which is necessarily present in the medium that has stable stratification only with respect to the temperature. Vertical motions in the latter are prevented by the stable stratification of density, so that the transfer of heat by vertical motions resulting from the release of heat is limited. In other words, in the expression for the convective heat flow $-\gamma_T w$ in a one-component medium the two factors usually cannot both be large at the same time: if $\gamma_T > 0$ is large, then w is small (sufficiently stable stratification will suppress vertical motions). In the two-component medium, however, the stable temperature stratification may be compensated (partially or completely) by the unstable stratification of the admixture. Because of this, the conditions for vertical motions can be much more advantageous; γ_T and w can both be large in magnitude at the same time. This creates favorable conditions for the development of intensive vertical convective flows of heat, and therefore for temperature perturbations of large amplitude.

The sign of these flows and thermal perturbations depends on the sign of the background vertical temperature gradient. Accordingly, in the two-component medium this sign may be negative or positive (in a one-component medium that is only stratified with respect to the temperature we may only speak of one sign of γ_T, since there is no reason to consider the background states with unstable density gradients). Thus, situations are possible in the two-component medium when the sign of the temperature response is opposite to the sign of the heat source — heat release leads to negative variation of the temperature.

This work was supported by RFBR (Projects 98-05-64527 and 02-05-64203) and ISTC (Project G-553).

References

1. Nicholls K W *Nature* **388** 460 (1997)
2. Silkin B I *Meteorolog. Gidrolog.* (4) 127 (1998)
3. Vul'fson N I *Dokl. Akad. Nauk SSSR* **151** 1089 (1963)
4. Gershuni G Z, Zhukhovitskiĭ E M *Konvektivnaya Ustoĭchivost' Neszhimaemoĭ Zhidkosti* (Convective Stability of Incompressible Fluid) (Moscow: Nauka, 1972)
5. Turner J *Buoyancy Effects in Fluids* (Cambridge: Univ. Press, 1973) [Translated into Russian (Moscow: Mir, 1977)]
6. Landau L D, Lifshitz E M *Gidrodinamika* (Fluid Mechanics) (Moscow: Nauka, 1986) [Translated into English (Oxford: Pergamon Press, 1987)]
7. Ingel' L Kh *Pis'ma Zh. Eksp. Teor. Fiz.* **72** 753 (2000) [*JETP Lett.* **72** 527 (2000)]
8. Gebhart B et al. *Buoyancy-Induced Flows and Transport* (Washington, DC: Hemisphere Publ. Corp., 1988) [Translated into Russian (Moscow: Mir, 1991) Section 7.2, p. 435]
9. Ingel' L Kh *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana* **26** 794 (1990) [*Izv. Atmos. Ocean. Phys.* **26** 582 (1990)]
10. Ingel' L Kh *Izv. Ross. Akad. Nauk, Fiz. Atmos. Okeana* **37** 639 (2001) [*Izv. Atmos. Ocean. Phys.* **37** 592 (2001)]
11. Ingel' L Kh *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza* (3) 92 (2002)
12. Vul'fson A N, Ingel' L Kh *Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana* **18** 233 (1982)
13. Barenblatt G I, Benilov A Yu *Okeanolog.* **23** 743 (1983)
14. Benilov A Yu, Rozovskiĭ M Kh *Okeanolog.* **23** 944 (1983)
15. Dikarev S I, Zatsepin A G *Okeanolog.* **23** 950 (1983)
16. Vul'fson A N *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza* (3) 90 (2001)
17. Ingel' L Kh *Izv. Ross. Akad. Nauk, Fiz. Atmos. Okeana* **38** (2002) (in press)
18. Monin A S, Yaglom A M *Statisticheskaya Gidrodinamika: Teoriya Turbulentnosti* (Statistical Hydromechanics: Theory of Turbulence) Vol. 1 (St. Petersburg: Gidrometeoizdat, 1992) [Translated into English: *Statistical Fluid Mechanics: Mechanics of Turbulence* Vol. 1 (Cambridge, Mass.: MIT Press, 1971)]
19. Barenblatt G I *Podobie, Avtomodel'nost', Promezhutochnaya Asimptotika* (Similarity, Self-similarity, Intermediate Asymptotics) (Leningrad: Gidrometeoizdat, 1982) [Translated into English: *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge: Cambridge Univ. Press, 1996)]
20. Ingel' L Kh *Meteorolog. Gidrolog.* (12) 47 (2000) [*Russ. Meteorology Hydrology* (12) 33 (2000)]
21. Ingel' L Kh *Okeanolog.* **41** 502 (2001) [*Oceanology* **41** 478 (2001)]
22. Ingel' L Kh *Dokl. Ross. Akad. Nauk* **379** 381 (2001) [*Dokl. Earth Sci.* **379** 662 (2001)]
23. Lineĭkin P S, Maderich V S *Teoriya Okeanicheskogo Termoklina* (Theory of Oceanic Thermocline) (Leningrad: Gidrometeoizdat, 1982)