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Measuring the gravitational constant in a university laboratory

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Contents

Abstract. A setup for measuring the gravitational constant in a university laboratory is described. The setup includes a torsion pendulum which swings under the action of gravitational attraction from test masses whose positions are made to change periodically in phase with pendulum oscillations by a special device. The gravity constant is calculated from the amplitude of steady-state oscillations. The experimental and calculation procedure is discussed and measurement errors are estimated.

1. Introduction

It is well known that measurements of the so-called world constants (the speed of light, the electron charge, the Planck constant, the gravitational constant) are extremely important and methods and devices allowing an increase of the precision of the measurements are continuously upgraded. The basic methods of these measurements are discussed in the courses of general physics, and some of them are included in university laboratory experiments. These are, for example, setups reproducing Millican's experiments on measuring the electron charge, the Frank and Hertz experiments on measuring the Planck constant, and some laboratory methods of measuring the speed of light.

Measurements of the Newton gravitational constant G have their own specialty. Although the first rather precise measurements of the gravity constant using a 'torsion balance' were performed by Cavendish as early as at the close of the 18th century and many other more precise experiments were carried out later on (see, for example, Ref. [1]), until recently no devices adapted for a university laboratory experiment had been proposed. This is explained first of all by the very high sensitivity the measuring device should have to measure the fairly small gravity forces, which makes the device vulnerable to the different external noises unavoidable in a student laboratory. In particular, in the torsion balance a suspension wire with a very small torsion modulus D is required, so various shocks, air fluxes, etc., induce fluctuations of the initial position (or, in terms of the measurement technique, the 'null drift' typical for high-sensitivity devices) and make the adjustment procedure very complicated. It is also clear that the method of measuring G by static deviations of test masses employed in Cavendish's and some other experiments demands heavy test masses and bulky setups unacceptable in a university laboratory. Starting from Heyl (1930), the gravity constant has been precisely measured most often by a tiny change in the oscillation period when the line connecting the test masses turns relative to the torsion pendulum. In the last years, sufficiently compact setups of this type have been elaborated which can in principle be used in university laboratories. However, they require rather costly auxiliary equipment (locating the torsion pendulum in vacuum, special thin-beam lasers to measure small turns, precise control of the oscillation period, etc.) As far as the authors know, at least Russian universities have no such devices at their disposal, and current advertisements [1] provide no detailed information on the construction and parameters of the setups, which could help in reproducing them.

In this connection, we describe in this note a device elaborated by us which does not need such complications and nonetheless, over a reasonable time (of no more than two hours, including acquaintance with the device and experimental procedure), is capable of making measurements allowing the gravity constant to be determined with an acceptable accuracy for the student laboratory of at least 10 – 15%.

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2. Description of the setup and measurement method

The general view of the setup is shown in Fig. 1. Its overall dimensions do not exceed 1.4 m in height and 40 cm in breadth, so it is easily installed on a standard laboratory table. As in the experiments mentioned above, the setup basically includes the torsion balance in the form of a 'dumbbell' consisting of two identical leaden masses m (secured on a light bar of organic glass) and vertically suspended on a thin

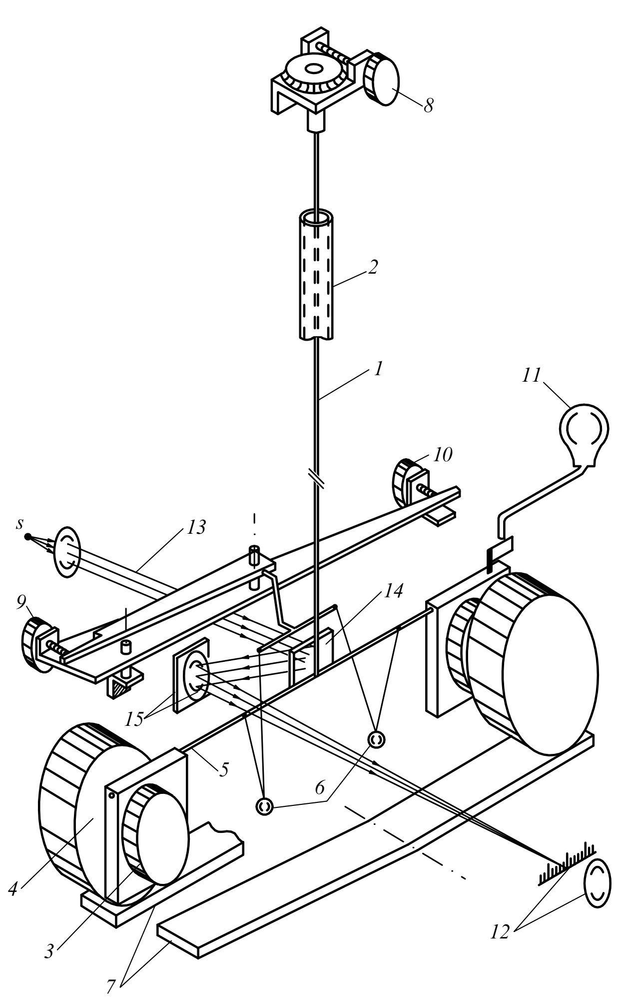

Figure 1. Scheme of the setup: *1* — pendulum suspension thread, *2* — protection tube, *3* — masses m of the pendulum, *4* — attractive masses M, *5* — the pendulum rod, *6* — additional masses Δm providing the quasi-elasticity of the system, *7* — guide tracks for rolling masses M, *8*, *9* — crude and fine null adjustment screws, *10* — pendulum oscillation period smooth regulation screw, *11* — rubber pear, *12* — ocular lens, *13* — illuminating beam, *14* — pendulum mirror, *15* — reflecting mirror with focusing lens.

thread. In the experiment, the setup allows heavier masses M (also made of lead) to approach each of the masses m alternatively from different sides, which causes resonant amplification of the oscillations. In our setup we (arbitrarily) used $m = 0.28$ kg, $M = 2.24$ kg, the length of the pendulum shoulder $l = 7.5$ cm, the distance (gap) between masses m and M at the closest approach $d = 1.2$ cm. The gravitational constant value can in principle be derived from the dynamics of amplification of the induced oscillations, i.e. from the temporal dependence of their amplitude $A_G(t)$, or, which is more simple, from the oscillation amplitude in the steady-state regime (see below). The principal advantage of such a resonant method compared to Heyl's method is that here it is not necessary to measure very small corrections to the proper oscillation period and hence there is no need to use high-precision measuring devices (at least for educational purposes).

Let us turn to Fig. 1 and note some features of the setup and measurement procedure. The pendulum deviation angle $\Theta(t)$ during oscillations is fixed by the scale *12* through a spyglass (not shown in the figure) using the illumination beam *13*. A normal lamp with collimator or something else can be used as the light source (not shown in the figure). The beam reflected from the mirror *14* is directed toward the scale *12* located on the other side from the pendulum by means of the auxiliary mirror *15* with a focusing lens. The scale grade is determined using additional calibrating devices, also not shown in the figure, so the scale grade should made known to students beforehand.

The torsion pendulum is under atmospheric pressure, although surrounded by a transparent casing of organic glass to protect from external air fluxes. The core idea in the setup construction is to overcome the 'null drift' mentioned above. Indeed, if no special care is taken, even in the absence of air fluxes the rest-tensions of the thread, small changes in temperature and humidity lead to random 'going-offs' of the pendulum. In order to eliminate them we used a suspension made of multi-thread (specially untwisted) kapron with a negligible torsion modulus; the restoring torque $M_D = -D\Theta$ determining the 'quasi-elasticity' of the setup with respect to turns around the vertical axis relative to the initial position was produced by means of additional light masses *5* with $\Delta m \ll m$ attached by kapron threads across the pendulum rod (see Fig. 1). This means is also convenient because by choosing these masses (we took $\Delta m = 1.5$ g) and adjusting their suspension by means of the screw *10* one can vary the effective torsion modulus D of the setup and consequently the period of proper oscillation of the pendulum T. It is convenient to take $T = 30-60$ s so that during the laboratory experiment one can measure a sufficient number (some tens) of periods and, in particular, to realize the steady-state regime of the induced oscillations. As a result, our setup allows one to practically avoid the zero point fluctuations and to fix it to the scale center by means of regulation screws *8* and *9* (see Fig. 1). At the same time, for small oscillation amplitudes $|\Theta(t)| \ll \pi/2$ the linear law of elasticity $M_D = -D\Theta$ holds. So the process of pendulum oscillations can be described by the usual equation of a linear oscillator with damping

$$\ddot{\Theta} + 2\delta\dot{\Theta} + \omega_0^2 = \frac{M_G(t)}{I}\,, \tag{1}$$

where I is the moment of inertia of the pendulum ($I = 2ml^2$), $\omega_0^2 = D/I$, $M_G(t) = 2lF_G(t)$ is the torque of gravitational forces F_G depending on the position of masses M, δ is the damping coefficient.

Parameters ω_0 and δ in Eqn (1) can be determined in a preliminary experiment by observing decaying free oscillations of the pendulum at $M_G(t) = 0$. To excite the latter, a tube is mounted to the setup casing (see Fig. 1), through which an air stream taking the system out of equilibrium is directed by the rubber pear *11*. As is well known, the solution of Eqn (1) at $M_G(t) = 0$ has the form [2]

$$\Theta(t) = A(t)\cos(\omega t + \varphi_0)\,, \qquad A(t) = A_0 \exp(-\delta t)\,, \tag{2}$$

where $\omega = (\omega_0^2 - \delta^2)^{1/2}$, A_0, φ_0 are the initial amplitude and phase, respectively. For sufficiently high oscillator quality ($Q = \omega_0/2\delta \gg 1$) we have $\omega \approx \omega_0 = 2\pi/T$. By watching the damping oscillations through the spyglass, it is easy to determine both their period $T = 2\pi/\omega$ and the time constant $\tau = \delta^{-1}$ [according to (2), this constant is the e-folding time of the amplitude $A(t)$]. The example of the resulting plot $A(t)$ is shown in Fig. 2; here $T = 56.3$ s, $\tau = 7.8T$, $Q = 24.5$.

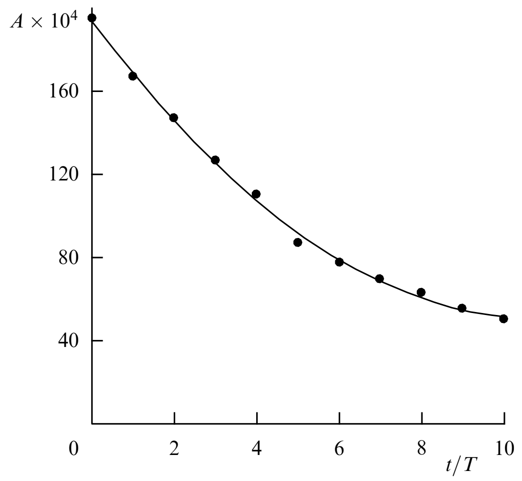

Figure 2. Plot $A_G(t)$ in the regime of damping free oscillations (experiment).

As follows from the above considerations, the main experiment consists in observing and registering the process of the resonant amplification of the torsion pendulum induced oscillations by the periodical shifting of the positions of the attractive masses M. The form of the masses is unimportant, for example, in our setup the test masses had a cylindrical shape with flat butt-ends, i.e. like thick disks or washers of radius R and thickness $H = R = 4$ cm (for heavy masses M) and $h = r = 2$ cm (for light masses m, see Fig. 1). This somewhat complicates the expression for the attraction force between masses M and m at their close approach, since the well-known Newton formula $F_G = GMmr_c^{-2}$, where r_c is the distance between the centers of the masses is not valid any more. However, having such forms allows the system sensitivity to increase as at given values of M and m and the minimal distance d between them (determined by the setup construction) the gravitational attraction force F_G increases (by 1.5 times in our case, see below). There are other advantages of such a form of masses: at $d \ll R$, the force F_G stays constant with a high accuracy for the entire range of $\Theta(t)$; moreover, masses of cylindric form are easier to manufacture than strictly spherical masses.

In order that masses M can be displaced without use of the experimentor's hands, they are installed on special light guide plates *7* (see Fig. 1), which change their slope by means of a special electro-mechanical device (not shown in the figure) such that the masses M roll from one position to another simultaneously and in the counter-phase to be alternatively near one or another mass m. The rolling time in our setup is about 2 s, which is more than one order shorter than the proper oscillation period T; a few additional seconds are taken to lift the corresponding mass before rolling down. Clearly, the most effective oscillation amplification occurs if the mass M attracts the pendulum to one side during one half period and to the opposite side during the other half period. Here the change in the mass position should be made in such a way that each time the oscillations start at the most remote position of the mass m relative to the nearby attracting mass M. The plot $M_G(t)$ then looks like a meander (Fig. 3); the oscillation amplitude $A_G(t)$ then will steadily grow and over the time interval $t \geqslant \tau = \delta^{-1}$ will reach the maximum steady-state value A_m. For example, for zero initial conditions, obviously, the

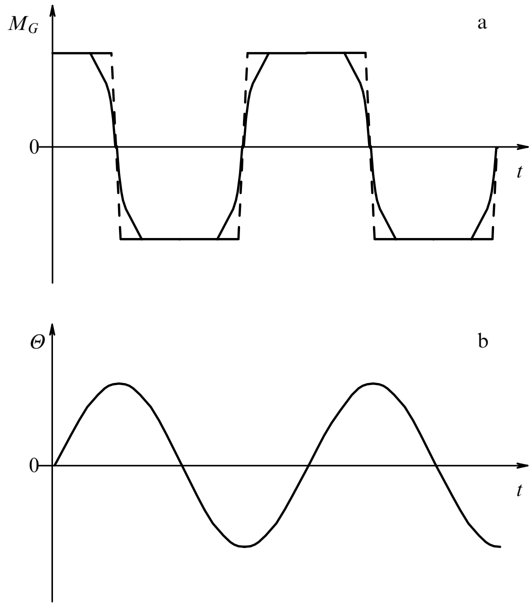

Figure 3. Plot of the torque $M_G(t)$ (a) and oscillograms of oscillations (b) in the steady-state regime.

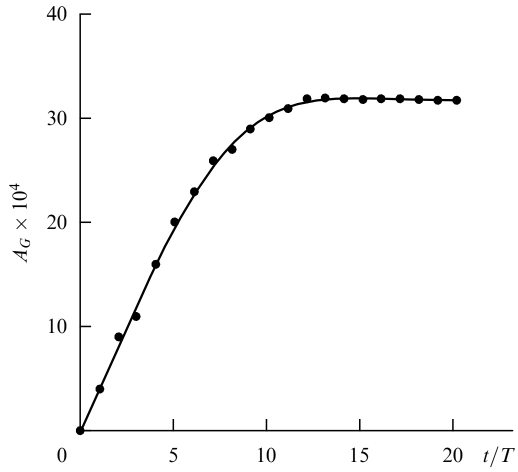

Figure 4. Plot $A_G(t)$ in the resonant amplification regime (experiment).

dependence $A_G(t)$ will have the form [2]

$$A_G(t) = A_m\left[1 - \exp(-\delta t)\right]. \qquad (3)$$

The gravitational constant G can be easily expressed through the quantity A_m (see below) which can be measured either directly (provided that the observation time $t \geqslant \tau$) or calculated by formula (3) through the measured previously value of $A_G(t)$ and the known damping coefficient $\delta = \tau^{-1}$. An example of a realistic $A_G(t)$ found in experiment is shown in Fig. 4. Note that here the steady-state oscillation amplitude A_m is about 3.2×10^{-3} rad, which is about $Q = 24.5$ times larger than the static response for the same force F_a; such a response could be only marginally detected by this device.

3. Calculating the gravitational constant

The gravitational constant G can be most easily derived from the measured induced oscillation amplitude A_m when the torque $M_G(t)$ in Eqn (1) can be approximated by a meander, i.e. $M_G = \pm M_0$ depending on the half-period (see Fig. 3). For an optimal phasing of displacement of masses M (see above) in one half-period when the pendulum turns through the angle $2A$ at the expense of work done by the force F_G, the oscillation energy increases by the amount

$$\Delta W = 2M_G A\,, \tag{4}$$

where $M_G = 2lF_G$. Over the same time interval, the relative dissipative energy losses are (see Ref. [2])

$$\frac{|\Delta W|}{W} = \delta T = \frac{2\pi\delta}{\omega}\,, \tag{5}$$

where the oscillation energy stored relates to the amplitude A by the expression $W = (1/2)DA^2 = (1/2)I\omega^2 A^2$. The energy balance in the steady-state regime then implies (with account for $I = 2ml^2$)

$$F_G = \frac{\pi}{2}\, ml\omega\delta A_{\rm m}\,. \tag{6}$$

On the other hand, the gravitation attraction force between masses M and m can be written as

$$F_G = kG\,\frac{Mm}{r_{\rm c}^2}\,, \tag{7}$$

where $r_{\rm c}$ is the distance between the disk centers and the coefficient k (form-factor) takes into account the deviation of mass forms from spheres and depends on $r_{\rm c}$ (at $H \ll r_{\rm c}$ $k \to 1$). A direct calculation using the Newton law for point-like masses for the given setup parameters ($H = R = 4$ cm, $h = r = 2$ cm, $d = 1.2$ cm, $r_{\rm c} = 4.2$ cm) yields $k = 0.66$; incidentally, the value of F_G here is as small as 1.6×10^{-8} N. Note that although here $k < 1$, the utilization of cylindrical masses is advantageous compared to spherical bodies of the same masses M and m since it allows the distance $r_{\rm c}$ between the centers of mass of the bodies to decrease. Indeed, in the last case, the equality of volumes implies that these masses would have radii $R_1 \approx 0.91R$, $r_1 \approx 0.91r$, respectively, and for the same gap $d = 1.2$ cm $r_{\rm c}$ would be $r_{\rm c1} = R_1 + r_1 + d \approx 6.6$ cm, so the attraction force, as was noted above, would be smaller (by 1.6 times in our case). As a result, by equating (6) and (7), we ultimately arrive at the expression for calculating the gravity constant from this experiment:

$$G = \frac{\pi\delta\omega l r_{\rm c}^2}{2km}\, A_{\rm m}\,. \tag{8}$$

For example, substituting $A_{\rm m}$ and ω from plots shown in Fig. 3 yields $G = 6.6 \times 10^{-8}$ cm^3 g^{-1} s^{-1}, i.e. the error in this case is less than 3%. On average, the measurement error in performing the experiment by unpracticed students is expected to be less than 10 – 15%.

The error in calculating G is first of all due to random errors of experimental determination of ω, δ, and $A_{\rm m}$ entering Eqn (8). The measurement error for ω, as is usual for devices with light spot indication, is determined by the relation of the light spot width and the indicator scale gradation value. As the value of T is determined by the moments of the pendulum crossing zero positions over many (ten and more) periods, this error does not exceed 1%. The error in measuring the amplitude $A_{\rm m}$ proves larger because this quantity is registered at the moments of maximum pendulum displacements where the angular velocity of oscillations is small; the limiting error in our setup is around 6%, but by statistical averaging over several measurements it can be reduced by about two times. The determination error of the damping coefficient δ turns out to be of the same order as it relates to dispersion of the measured values of $A_G(t)$. As for additional errors due to the deviation of the real dependence $M_G(t)$ from meander and inaccuracy in fixing moments of extreme pendulum displacements, when the relay controlling the position of masses M switches, they prove insignificant (less than 1%) because in these positions the pendulum angular velocity is small and the contribution from these factors in the energy equation (4) is also negligible. An error of about the same order appears due to geometrical factors (inaccuracy in the values of l, H, R and the pendulum moment of inertia).

4. Conclusion

From the methodical point of view, the utilization of this setup in the university laboratory is most relevant in the section "Oscillations and waves", i.e. in the 2nd year of education, when free and induced oscillations of different nature are discussed (as we did at the University of Nizhniĭ Novgorod), although the standard university courses of physics consider the gravity forces as early as in the 1st year of education. Incidentally, the contents of laboratory tasks here can be made very different, for example, by 'turning on' the induced oscillation amplification not starting from zero conditions but after some arbitrary kick, including with an amplitude exceeding that of the steady-state oscillations (this allows to somewhat decrease the rms error of A_G and G). We argue that such a laboratory work that does not include complex devices can be reproduced and successfully performed in other universities as well. Moreover, being sufficiently compact, after some modification it can also be used for lecture experiments demonstrating gravitation forces.

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