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An example illustrating the potentiality and peculiarities of a variational approach to electrostatic problems

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Contents

Abstract. The advantages and peculiarities of the variational method of solving electrostatic problems are clearly illustrated using the ground resistance as a relatively simple example.

1. Introduction

Looking through the manuals on classical electrodynamics, listed in the comprehensive bibliography at the end of the well-known encyclopedic type collection of problems [1], one immediately notices that the variational principles of magnetostatics are not discussed in any of them, and just a few occasionally mention the variational principles of electrostatics. The problem book itself [1] gives none of the examples of using the variational principles of electrostatics and magnetostatics. Neither can such examples be found in modern courses on electricity and magnetism [2, 3].

Such a situation in the monographs and textbooks on classical electrodynamics is rather surprising, since in the classical treatise of James Clerk Maxwell [4] we find clear-cut formulations of the variational principles of Dirichlet and Thomson, which in particular allow us to find upper and lower bounds for the energy parameters, such as the conductor capacitance, interelectrode resistances, etc. The same book, with a reference to Rayleigh's works and in particular to his *The Theory of Sound* [5], indicates that the formula for calculating the resistance R of a straight wire of variable cross section $S(l)$, made of a homogeneous material with resistivity ρ, viz.

$$R = \int \frac{\rho\,\mathrm{d}l}{S}\,,$$

where the integration is performed with respect to the length l of the wire, leads in reality to underestimated values, i.e. the following inequality holds true:

$$R \geqslant \int \frac{\rho\,\mathrm{d}l}{S}\,. \tag{1}$$

It is interesting that this fact is overlooked by the authors of university courses [2, 3], who treat expression (1) as an exact formula. In Ref. [3, p. 224], formula (1) is used for calculating the resistance of a homogeneous truncated cone with end contacts.

It is hard to understand the reasons why the variational principles of electrostatics failed to find their way into the extensive scientific literature on electrodynamics, since variational methods in physics have always been regarded as highly efficient. Quite possibly that Rayleigh's variational estimates for the resistance R of a round conductor of radius a and length L, made from the material with resistivity ρ and being in contact at one of its ends with a conducting homogeneous half-space characterized by resistivity ρ', quoted by Maxwell (see Sects 306 – 308 in his monograph [4]), namely

$$1 + \frac{\rho'}{\rho}\,\frac{a}{L}\,\frac{\pi}{4} < \frac{\pi a^2}{\rho L}\,R < 1 + \frac{\rho'}{\rho}\,\frac{a}{L}\,\frac{8}{3\pi}\,, \tag{2}$$

are rather a demonstration of Rayleigh's skill than a proof of the universality of variational methods. We believe that inequalities (2) deserve to be included in all textbooks on classical electrodynamics.

It is also likely that variational principles of electrostatics have been ignored in the books on electrodynamics because the application of variational principles to the solution of specific problems requires, as seen from examples given by Rayleigh and cited by Maxwell [4], the development of appropriate mathematical methods, which sometimes may be quite sophisticated. Certain typical features of such methods can be seen in the problems on the capacitance of a cube and capacitor with round plates, studied by G Polya and G Szego in their monograph entitled *Isoperimetric Inequal-*

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ities in Mathematical Physics [6]. However, the electrostatics-devoted part of this book, with the exception of a few examples, is not quite consistent with the theme of the book announced in its title. We also believe that the intent of the authors to give isoperimetric formulations of the electrostatic problems leads to the distortion of the physical crux of the matter. So one might guess that the physicists were slow in the development of variational methods in electrostatics because of mathematical difficulties, while the mathematicians because of the incomplete understanding of the physical side of the problem.

For two decades the author has been working on the construction of electrostatics based on the variational principles. At first it seemed that this task can be completed by unifying the known methods with the variational approach and demonstrating the new possibilities for solving the different classes of electrostatic problems. Gradually, however, it was becoming evident that the construction of analytical electrostatics requires the introduction of new concepts that are organically inherent in the theory.

The fact is that the main task associated with the application of variational principles to the solution of a particular problem consists in the selection of the test (approximating) fields. With the proper selection one may hope to obtain exact estimates for the energy parameters, like those obtained by Rayleigh [see Eqn (2)]. By tacit assumption, the adequate selection of test fields depends on the skill of the researcher, on his physical intuition. If we accept this point, then the variational principles of electrostatics ought to be realized in numerous variational inequalities similar to Rayleigh's inequalities (1) and (2). And such a set of inequalities actually does exist.

The best known Poincaré – Faber – Szego inequality [6] states that the sphere has the minimum capacitance among the conducting bodies of the same volume, and is essentially isoperimetric. Other interesting inequalities [7] indicate that the formulas for the capacitance of flat and cylindrical capacitors, found in all textbooks and manuals [2, 3], viz.

$$C = \frac{\varepsilon\varepsilon_0 S}{d}\,, \qquad C = \frac{2\pi\varepsilon\varepsilon_0 l}{\ln(b/a)}\,, \tag{3}$$

always underestimate the actual capacitance. The known formula for the single-layer solenoid inductance [2, 3]

$$L = \mu\mu_0\, n^2 S l \tag{4}$$

is shown in paper [8] to give exaggerated values of the true inductance. The inequality $L < \mu\mu_0\, n^2 S l$ seems to be the first inequality found with the aid of the variational principles of magnetostatics. Observe that the variational inequalities (1), (3) and (4) indicate that even though the variational approach may be not acknowledged expressly, it is present in the approximate formulas of electrostatics and magnetostatics.

However, considerably more opportunities for implementation of the variational principles become available when the selection of test fields is governed not by the intuition of the researcher but rather a certain procedure. The development of such procedures was facilitated by the introduction of new concepts: higher conductor polarizabilities [9], characteristic multipoles [10], the magnetic quadrupole polarizability [11] as well as the more general interpretation of conventional concepts [12, 13]. With the aid of such new notion it was possible to settle the problem of forming the Lagrange function of a conducting body in an electrostatic field [10] and the problem of moments (inverse problem) in electrostatics [9, 10], to analyze the problem of stability of a diamagnetic particle in the quadrupole magnetic field [11], and to proceed from the expressions of particular inequalities similar to Rayleigh's inequalities (1) and (2) to the study of whole classes of variational inequalities [14].

Today, when the reality of analytical electrostatics is beyond doubt, we would like to illustrate its capabilities by a relatively simple example which could be included in the textbooks on electrodynamics. This example is based on the known problem [2, 3] of the ground resistance of an ideally conducting sphere placed in a homogeneous conducting half-space. Let us separate out the basic steps in the solution of this problem by the variational method.

2. Variational formulations of the problem

Assume that the radius of the earthing sphere is a, its center coincides with the origin of coordinates, and the Earth surface is represented by the plane $z = h > a$. The resistivity of the homogeneous soil surrounding the sphere we denote by ρ.

To solve the problem of the ground resistance we need to find the electric potential $\varphi_0(\mathbf{r})$ in the spatial domain $V = [z < h; r > a]$ that will satisfy the Laplace equation in this domain. This potential assumes a constant value U on the surface of the sphere with $r = a$; the partial derivative of $\varphi_0(\mathbf{r})$ with respect to z must become zero on the Earth surface $(z = h)$. The ground resistance can be expressed via the power functional $P_V(\varphi_0)$ of the heat release with the aid of the relations

$$\frac{U^2}{R} = P_V(\varphi_0) = \frac{1}{\rho}\int_V (\nabla\varphi_0)^2\,\mathrm{d}V = -\frac{U}{\rho}\int_{r=a} \partial_r\varphi_0\,\mathrm{d}S = UI\,. \tag{5}$$

Basic variational principle. Let us compare with $\varphi_0(\mathbf{r})$ the continuous piecewise smooth potentials $\varphi(\mathbf{r})$ in the volume V, whose value at the surface of the sphere is

$$\varphi(\mathbf{r})\Big|_{r=a} = U\,. \tag{6}$$

Consider the directly verified identity

$$P_V(\varphi) = P_V(\varphi_0) + P_V(\varphi - \varphi_0) + \frac{2}{\rho}\int_V \nabla\varphi_0\nabla(\varphi - \varphi_0)\,\mathrm{d}V\,,$$

in the right-hand side of which the integral is zero because the value of $\varphi_0(\mathbf{r})$ on the sphere coincides with $\varphi(\mathbf{r})$, $\partial_z\varphi_0(\mathbf{r}) = 0$ at $z = h$, and in the domain V we have $\Delta\varphi_0 = 0$. Given that the functional $P_V(\varphi)$ is positive definite, we use this identity to arrive at the inequality

$$\frac{U^2}{R} = P_V(\varphi_0) \leqslant P_V(\varphi)\,, \tag{7}$$

which allows the lower bounds for the ground resistance R to be found using the potentials $\varphi(\mathbf{r})$ from the class defined above.

When $\varphi(\mathbf{r})$ is a harmonic function whose sources lie at the surface of the sphere with $r = a$ and the plane $z = h$, it may be convenient to use in place of $P_V(\varphi)$ the functional

$$P(\varphi) = \frac{1}{\rho}\int_{R^3 - S_p} (\nabla\varphi)^2\,\mathrm{d}V\,, \tag{8}$$

where S_p is the point set of the plane $z = h$. The admissible potentials $\varphi(\mathbf{r})$ are continuous on the surface of the sphere with $r = a$, and their derivatives with respect to z are continuous at the plane $z = h$. It is obvious that

$$\frac{U^2}{R} = P_V(\varphi_0) \leqslant P_V(\varphi) \leqslant P(\varphi)\,. \tag{9}$$

Dual variational principle. Now let us compare the solenoidal fields $\mathbf{j}(\mathbf{r})$ with the actual distribution of the current density $\mathbf{j}_0(\mathbf{r}) = -\rho^{-1}\nabla\varphi_0$. Observe that

$$\rho\int_V \left[\mathbf{j}(\mathbf{r}) - \mathbf{j}_0(\mathbf{r})\right]^2 \mathrm{d}V = P_V(\varphi_0) - Q_V(\mathbf{j}) \geqslant 0\,,$$

$$Q_V(\mathbf{j}) = 2U\int_{r=a} j_r\,\mathrm{d}S - \rho\int_V \mathbf{j}^2\,\mathrm{d}V\,,$$

if

$$j_z\Big|_{z=h} = 0\,. \tag{10}$$

In this way we arrive at the inequality

$$\frac{U^2}{R} \geqslant Q_V(\mathbf{j})\,, \tag{11}$$

which allows the upper bounds for the ground resistance to be found using the solenoidal fields $\mathbf{j}(\mathbf{r})$ that satisfy the boundary condition (10).

When $\mathbf{j} = -\rho^{-1}\nabla\psi$, and the sources of the potential ψ lie on the sphere with $r = a$ and the plane $z = h$, it is convenient to use in place of $Q_V(\mathbf{j})$ the functional

$$Q(\mathbf{j}) = 2U\int_{r=a} j_r\,\mathrm{d}S - \rho\int_{R^3 - S_p} \mathbf{j}^2\,\mathrm{d}V\,. \tag{12}$$

Then $\psi(\mathbf{r})$ is continuously extended into the volume of the sphere with $r < a$, and the extension into a domain $z > h$ is accomplished so as to preserve the continuity of $\partial_z\psi$. It is obvious that $Q_V(\mathbf{j}) \geqslant Q(\mathbf{j})$, and therefore one finds

$$\frac{U^2}{R} \geqslant Q_V(\mathbf{j}) \geqslant Q(\mathbf{j})\,. \tag{13}$$

Error estimates. A very attractive feature of the variational approach to electrostatic problems is that the energy variable (in our case, the conductivity $Y = R^{-1}$) is estimated from both sides. In other words, along with the approximate value, for which it is natural to take

$$\tilde{Y}_V = \frac{1}{2U^2}\left[P_V(\varphi) + Q_V(\mathbf{j})\right], \tag{14}$$

one can also evaluate the accuracy of this approximation. As follows from relations

$$|Y - \tilde{Y}_V| = \frac{1}{2}\left|\frac{P_V(\varphi)}{U^2} - Y + \frac{Q_V(\mathbf{j})}{U^2} - Y\right|$$

$$\leqslant \frac{1}{2}\left(\left|\frac{P_V(\varphi)}{U^2} - Y\right| + \left|\frac{Q_V(\mathbf{j})}{U^2} - Y\right|\right)$$

$$= \frac{1}{2}\left(\frac{P_V(\varphi)}{U^2} - Y + Y - \frac{Q_V(\mathbf{j})}{U^2}\right) = \frac{1}{2U^2}\left[P_V(\varphi) - Q_V(\mathbf{j})\right],$$

the absolute error of such an approximation by modulus does not exceed $\left[P_V(\varphi) - Q_V(\mathbf{j})\right]/2U^2$. Accordingly, the relative error of approximating the variable Y by the quantity $\tilde{Y}_V$ will not exceed

$$\Delta_V = \frac{P_V(\varphi) - Q_V(\mathbf{j})}{P_V(\varphi) + Q_V(\mathbf{j})}\,. \tag{15}$$

If the estimates are made with the aid of the functionals $P(\varphi)$ and $Q(\mathbf{j})$, then the corresponding approximation for conductivity and the accuracy of this approximation must be calculated by formulas (14) and (15), where the subscript V is dropped out.

Fields $-\nabla\varphi$ and $\rho\mathbf{j}$ may be treated as the fields approximating the actual electric field $-\nabla\varphi_0$. For such an approximation it is advisable to take $(\rho\mathbf{j} - \nabla\varphi)/2$. The standard deviation of the approximating field from the actual field is given by

$$\frac{1}{4\rho}\int_V \left[\rho(\mathbf{j} - \mathbf{j}_0) + \nabla(\varphi_0 - \varphi)\right]^2 \mathrm{d}V$$

$$= \frac{1}{4}\left[\int_V \rho(\mathbf{j} - \mathbf{j}_0)^2\,\mathrm{d}V + \frac{1}{\rho}\int_V \left(\nabla(\varphi_0 - \varphi)\right)^2 \mathrm{d}V\right]$$

$$= \frac{1}{4}\left[P_V(\varphi) - P_V(\varphi_0) + Q_V(\mathbf{j}_0) - Q_V(\mathbf{j})\right]$$

$$= \frac{1}{4}\left[P_V(\varphi) - Q_V(\mathbf{j})\right].$$

Taking into consideration this result, the relative accuracy of approximation may be expressed as

$$\delta_V = \sqrt{\frac{1}{2}\,\Delta_V}\,. \tag{16}$$

To conclude this section, let us note that all these findings can be extended to the problem of determining the resistance of earthing performed by a conductor of an arbitrary shape, by replacing the sphere domain with that of the conductor without any additional modifications. In other words, here we have provided the general variational formulation of the problem of ground resistance.

Inequalities (9) and (13) as well as formulas (15) and (16) for calculating the errors will be used as the basis for our variational method of calculating the ground resistance.

It should also be emphasized that the variational scheme described here will generally apply to other classes of electrostatic problems, for example, to the calculation of the capacitance coefficients for a system of conductors.

3. Construction of the sequence of lower bounds for ground resistance

The potentials admissible in the functionals $P_V(\varphi)$ and $P(\varphi)$ must assume one and the same constant value U on the earthing conductor. This condition is satisfied by introducing the potentials

$$\varphi_0(\mathbf{r}) = U\Phi_0(\mathbf{r}) + \sum_{k=1}^{N} q_k G(\mathbf{r}, \mathbf{r}_k)\,, \tag{17}$$

where q_k are the constants that have not been defined yet; $\Phi_0(\mathbf{r})$ is the potential created by the charges in the solitary earthing conductor, when its self-potential is 1; $G(\mathbf{r}, \mathbf{r}_k)$ is the Green function of the Dirichlet problem for the region external with respect to the domain of the earthing conductor, and $\mathbf{r}_k$ are the radius vectors of points located in the region $z > h$. In the problem under consideration, one obtains

$$\Phi_0(\mathbf{r}) = \frac{a}{r}\,,$$
$$G(\mathbf{r}, \mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{R}|} - \frac{a}{\sqrt{r^2R^2 - 2a^2\mathbf{r}\cdot\mathbf{R} + a^4}}\right). \tag{18}$$

The terms in sum (17) are the potentials of point charges q_k located at points $\mathbf{r}_k$ in the presence of grounded conducting sphere with $r = a$. In this way, the true potential $\varphi_0(\mathbf{r})$ is approximated here by the sum of potentials of the point charges (17).

Substituting potential (17) into the functionals (5) and (8), we find

$$P_V(\varphi) = \frac{U^2}{R_0}\left(\frac{R_0}{R_0^{(i)}} + 2\mathbf{a}\cdot\mathbf{q}' + \mathbf{q}'\cdot\hat{A}\cdot\mathbf{q}'\right),$$
$$P(\varphi) = \frac{U^2}{R_0}\left(1 + 2\mathbf{b}\cdot\mathbf{q}' + \mathbf{q}'\cdot\hat{B}\cdot\mathbf{q}'\right). \tag{19}$$

Here

$$R_0 = \left(\frac{1}{\rho}\int_{R^3}(\nabla\Phi_0)^2\,\mathrm{d}V\right)^{-1}$$

is the resistance of the earthing conductor immersed into the homogeneous medium with the resistivity ρ, and in addition

$$\frac{1}{R_0^{(i)}} = \frac{1}{R_0} + \frac{1}{\rho}\int_{z=h}\Phi_0\,\partial_z\Phi_0\,\mathrm{d}S\,,$$

$$\mathbf{q}' = \frac{R_0}{\varepsilon_0\,\rho U}(q_1, q_2, \ldots, q_N)\,, \qquad \mathbf{a} = (a_1, a_2, \ldots, a_N)\,,$$

$$a_i = \varepsilon_0\int_{z=h}G(\mathbf{r}, \mathbf{r}_i)\,\partial_z\Phi_0\,\mathrm{d}S\,;$$

$\hat{A}$ is the quadratic matrix whose elements are

$$A_{ij} = \frac{\rho}{R_0}\,\varepsilon_0^2\int_{z=h}G(\mathbf{r}, \mathbf{r}_i)\,\partial_z G(\mathbf{r}, \mathbf{r}_j)\,\mathrm{d}S\,,$$

$$b_i = -\Phi_0(\mathbf{r}_i)\,, \qquad B_{ij} = \frac{\rho\varepsilon_0}{R_0}\left[G_n(\mathbf{r}_i, \mathbf{r}_j) - G(\mathbf{r}_i, \mathbf{r}_j)\right],$$

$$G_n(\mathbf{r}, \mathbf{R}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{R}|} + \frac{1}{|\mathbf{r}-\mathbf{R}+2\mathbf{k}(\mathbf{k}\cdot\mathbf{R}) - 2h\mathbf{k}|}\right), \tag{20}$$

where, in turn, $G_n(\mathbf{r}, \mathbf{R})$ is the Green function of the Neumann problem for the plane $z = h$, and $\mathbf{k}$ is the unit vector of the z-axis. Observe that by choosing the natural (in the context of this problem) units of measurement of the resistance R_0 and charge $\varepsilon_0\rho U/R_0$ we were able to ensure that only dimensionless variables are found in parentheses in the right-hand sides of formulas (19).

When the earthing conductor is a sphere, and the charges q_i are located on the z-axis at points $z_i > h$, we have

$$R_0 = \frac{\rho}{4\pi a}\,, \qquad \frac{R_0}{R_0^{(i)}} = 1 - \frac{a}{4h}\,, \qquad a_i = -\frac{a}{2z_i}\left(1 - \frac{az_i}{2hz_i - a^2}\right),$$

$$A_{ij} = \frac{1}{2}\left(\frac{a}{z_i + z_j - 2h} - \frac{a^3}{2hz_iz_j - a^2(z_i + z_j)}\right),$$

$$b_i = -\frac{a}{z_i}\,, \qquad B_{ij} = \frac{a^2}{z_iz_j - a^2} + \frac{a}{z_i + z_j - 2h}\,.$$

Minimizing the right-hand sides of Eqns (19), we arrive at the following estimates of the ground resistance:

$$R > R_0\left(\frac{R_0}{R_0^{(i)}} - \mathbf{a}\cdot\hat{A}^{-1}\cdot\mathbf{a}\right)^{-1},$$
$$R > R_0\left(1 - \mathbf{b}\cdot\hat{B}^{-1}\cdot\mathbf{b}\right)^{-1}. \tag{21}$$

Let us denote these estimates corresponding to different N by $R_N^{(i)}$ and $\tilde{R}_N^{(i)}$. They are the functions of z_i, and the optimization over them could lead to best possible estimates for a given N. Such an optimization, however, requires the elaboration of cumbersome patterns of calculation, so we take

$$z_1 = 2h\,, \qquad z_{k+1} = 2h - \frac{a^2}{z_k}\,. \tag{22}$$

This choice corresponds to the image method in electrostatics [4], which allows, according to the modern theory [15], the features of the electrostatic field analytically extended inside the sphere to be found.

In particular, for $h = a$, when one expects the highest inaccuracy of variational estimates, we have

$$\frac{R_0^{(i)}}{R_0} = \frac{4}{3}\,, \qquad \mathbf{q}' = 0\,,$$

$$\frac{R_1^{(i)}}{R_0} = \frac{36}{25} = 1.44\,, \qquad \mathbf{q}' = \left(\frac{2}{3}\right),$$

$$\frac{R_2^{(i)}}{R_0} = \frac{88}{61} = 1.44262295\,, \qquad \mathbf{q}' = \left(\frac{10}{11}\,, -\frac{5}{22}\right),$$

$$\frac{R_3^{(i)}}{R_0} = \frac{5400}{3743} = 1.44269303\,, \qquad \mathbf{q}' = \left(\frac{44}{45}\,, -\frac{21}{54}\,, \frac{14}{135}\right),$$

$$\frac{\tilde{R}_0^{(i)}}{R_0} = 1\,, \qquad \mathbf{q}' = 0\,,$$

$$\frac{\tilde{R}_1^{(i)}}{R_0} = \frac{10}{7} = 1.42857143\,, \qquad \mathbf{q}' = \left(\frac{3}{5}\right),$$

$$\frac{\tilde{R}_2^{(i)}}{R_0} = \frac{75}{52} = 1.44230769\,, \qquad \mathbf{q}' = \left(\frac{22}{25}\,, -\frac{1}{5}\right),$$

$$\frac{\tilde{R}_3^{(i)}}{R_0} = \frac{1348}{1073} = 1.44268406\,, \qquad \mathbf{q}' = \left(\frac{750}{774}\,, -\frac{285}{774}\,, \frac{70}{774}\right).$$

Judging from the internal convergence of the lower bounds, it would seem that the third variational estimate deviates from the exact value only in the fifth decimal figure. One cannot be certain, however, without knowing the appropriate upper bounds.

4. Construction of the sequence of upper bounds for ground resistance

In the evaluation of upper bounds for ground resistance, in the functionals $Q(\mathbf{j})$ and $Q_V(\mathbf{j})$ one should use the solenoidal fields of the current density $\mathbf{j}$ that satisfy the condition of impenetrability for the current (10) on the Earth surface. Such fields of the current densities include

$$\mathbf{j}(\mathbf{r}) = -\frac{1}{\rho}\,\nabla\sum_{k=1}^{N} q_k G_n(\mathbf{r}, \mathbf{r}_k)\,, \tag{23}$$

where $\mathbf{r}_k$ are the radius vectors of the point charges q_k located in the spatial domain of the earthing conductor, and $G_n(\mathbf{r}, \mathbf{R})$ is the Green function of the Neumann problem for the half-space $z < h$, defined by formula (20). We shall later perform maximization of the functionals

$$\begin{aligned} Q_V(\mathbf{j}) &= \frac{U^2}{R_0}\left(2\mathbf{e}\cdot\mathbf{q}' - \mathbf{q}'\cdot\hat{C}\cdot\mathbf{q}'\right), \\ Q(\mathbf{j}) &= \frac{U^2}{R_0}\left(2\mathbf{e}\cdot\mathbf{q}' - \mathbf{q}'\cdot\hat{D}\cdot\mathbf{q}'\right) \end{aligned} \tag{24}$$

over the so far undefined values of charges q_k. Above we utilized the notation

$$\mathbf{e} = (1; 1; \ldots; 1)\,, \qquad \mathbf{q}' = \frac{R_0}{\varepsilon_0\rho U}(q_1; q_2; \ldots; q_N)\,,$$

$$C_{ij} = -\frac{\rho}{R_0}\,\varepsilon_0^2\int_S G_n(\mathbf{r}, \mathbf{r}_i)\mathbf{n}\cdot\nabla G_n(\mathbf{r}, \mathbf{r}_j)\,\mathrm{d}S\,,$$

$$D_{ij} = \frac{\rho\varepsilon_0}{R_0}\left[G_n(\mathbf{r}_i, \mathbf{r}_j) - G(\mathbf{r}_i, \mathbf{r}_j)\right],$$

where S is the surface area of the conductor; $\mathbf{n}$ is the outward-directed unit vector normal to the conductor surface at the point $\mathbf{r}$, and $G(\mathbf{r}, \mathbf{R})$ is the Green function of the Dirichlet problem for the domain occupied by the conductor.

When the earthing conductor is a sphere, $G(\mathbf{r}, \mathbf{R})$ can be found using formula (18). The symmetry of the problem prompts us that the charges q_i ought to be arranged along the z-axis; we denote their coordinates by z_i ($|z_i| < a$). According to our calculations, we obtain

$$\begin{aligned} C_{ij} = {} & \frac{1}{2}\,\frac{1}{1-\alpha_i\alpha_j} - \frac{1}{2}\,\frac{1}{1-\beta_i\beta_j} + \frac{1}{4}\left(\frac{1}{\sqrt{\alpha_i\alpha_j}}\ln\frac{1+\sqrt{\alpha_i\alpha_j}}{1-\sqrt{\alpha_i\alpha_j}}\right. \\ & + \sqrt{\frac{\beta_i}{\alpha_j}}\ln\frac{1+\sqrt{\alpha_j\beta_i}}{1-\sqrt{\alpha_j\beta_i}} + \sqrt{\frac{\beta_j}{\alpha_i}}\ln\frac{1+\sqrt{\alpha_i\beta_j}}{1-\sqrt{\alpha_i\beta_j}} \\ & \left. + \sqrt{\beta_i\beta_j}\ln\frac{1+\sqrt{\beta_i\beta_j}}{1-\sqrt{\beta_i\beta_j}}\right), \end{aligned}$$

$$\alpha_i = \frac{z_i}{a}\,, \qquad \beta_i = \frac{a}{2h - z_i}\,,$$

$$D_{ij} = \frac{a^2}{a^2 - z_i z_j} + \frac{a}{2h - z_i - z_j}\,.$$

Maximization of $Q_V(\mathbf{j})$ and $Q(\mathbf{j})$ with respect to $\mathbf{q}'$ leads to the following estimates for the ground resistance:

$$R < R_0\left(\mathbf{e}\cdot\hat{C}^{-1}\cdot\mathbf{e}\right)^{-1}, \qquad R < R_0\left(\mathbf{e}\cdot\hat{D}^{-1}\cdot\mathbf{e}\right)^{-1}. \tag{25}$$

We denote these estimates by $R_N^{(s)}$ and $\tilde{R}_N^{(s)}$, in the order of increasing N. They are functions of z_i, whose values could be used for carrying out the optimization. This is, however, a cumbersome process, so we put

$$z_1 = 0\,, \qquad z_{k+1} = \frac{a^2}{2h - z_k}\,, \tag{26}$$

using the method of constructing the images in electrostatics. In particular, for $h = a$ one finds

$$\frac{R_1^{(s)}}{R_0} = \frac{4}{3} + \frac{1}{8}\ln 3 = 1.47065987\,, \qquad \mathbf{q}' = \left(\frac{24}{32 + 3\ln 3}\right),$$

$$\frac{R_2^{(s)}}{R_0} = 1.44330464\,, \qquad \mathbf{q}' = (0.93768642;\ -0.24483200)\,,$$

$$\frac{\tilde{R}_1^{(s)}}{R_0} = 1.5\,, \qquad \mathbf{q}' = \left(\frac{2}{3}\right),$$

$$\frac{\tilde{R}_2^{(s)}}{R_0} = \frac{13}{9} = 1.4(4)\,, \qquad \mathbf{q}' = \frac{1}{13}(12;\ -3)\,,$$

$$\frac{\tilde{R}_3^{(s)}}{R_0} = \frac{189}{131} = 1.442748909\,, \qquad \mathbf{q}' = \frac{1}{189}(186;\ -75;\ 20)\,,$$

$$\frac{\tilde{R}_4^{(s)}}{R_0} = \frac{642}{445} = 1.44269629, \quad \mathbf{q}' = \frac{1}{642}(640;\ -300;\ 140;\ -35).$$

5. Comparison of estimates, accuracy calculation, and selection of analytical formulas

The important advantage of the variational method over other approaches in electrostatics is that it allows evaluation of the error that is inevitable in any calculations. In other words, the variational method allows one to make calculations with a controlled accuracy. Indeed, comparing, for example, $\tilde{R}_4^{(s)}$ and $R_3^{(i)}$, we find

$$\varDelta = 1.25\times 10^{-6}\,, \qquad \delta = 7.9\times 10^{-4}\,.$$

Whence we conclude that the approximation of the electric field by the combination of test fields (17) and (23) with $N = 3$ and $N = 4$, respectively, is accomplished with a mean square error of 0.079%.

The method of variational inequalities allows one to perform only those computations that are necessary for achieving the calculation accuracy required. With this purpose one makes simultaneous estimates from above and from below and compares them. For example, if we need to find the distribution of the electric field around the earthing conductor for $h = 2a$ with a mean square error less than 1%, then by calculating

$$\frac{R_0^{(i)}}{R_0} = \frac{8}{7}\,, \qquad \frac{\tilde{R}_1^{(s)}}{R_0} = \frac{5}{4}\,, \qquad \frac{R_1^{(i)}}{R_0} = 1.245940\,,$$

$$\frac{\tilde{R}_2^{(s)}}{R_0} = 1.245977\,,$$

we see that the approximation accuracy required is given by the fields corresponding to the first lower bound and the second upper bound. In this situation the higher order estimates are not necessary.

The expressions for variational estimates appear here as a set of approximate analytical formulas for ground resistance. Let us quote some of them:

$$\frac{\tilde{R}_1^{(s)}}{R_0} = 1 + \frac{1}{2H}\,,$$

$$\frac{R_1^{(s)}}{R_0} = 1 + \frac{1}{2H} - \frac{1}{2(4H^2-1)} + \frac{1}{8H}\ln\frac{2H+1}{2H-1}\,,$$

$$\frac{\tilde{R}_2^{(s)}}{R_0} = \frac{H(16H^4+8H^3-12H^2-2H+3)}{(4H^2-1)(4H^3-2H+1)}\,,$$

$$\frac{R_0^{(i)}}{R_0} = \frac{4H}{4H-1}\,, \qquad \frac{\tilde{R}_1^{(i)}}{R_0} = \frac{2H(4H^2+2H-1)}{8H^3-2H+1}\,,$$

$$\frac{R_1^{(i)}}{R_0} = \frac{4H(4H^2-3)(4H^2-1)^2}{(4H-1)(4H^2-3)(4H^2-1)^2-2(2H^2-1)(4H^2-2H-1)^2}\,.$$

We employed in the foregoing the notation $H = h/a$. The error in the value of ground resistance calculated by the formula

$$R = \frac{1}{2}\left(R_1^{(i)} + \tilde{R}_2^{(s)}\right) \tag{27}$$

will not exceed $\Delta = 0.0015$. In reality, the error is much smaller, as seen from the comparison of the right-hand side of Eqn (27) with the higher order estimates, and does not exceed 0.033% everywhere over the region of variation of $H \geqslant 1$. Expression (27) can be recommended for calculating the ground resistance of a conducting sphere.

6. Conclusions

The example discussed above gives a good illustration of the capabilities of the variational approach to the calculation of electrostatic fields and their energy characteristics.

The peculiarities of the variational approach depend on the fact that it significantly extends the scope of concepts used in electrostatics. The direct and dual variational principles, the energy functionals, the classes of fields admissible for comparison, and the upper and lower bounds of the energy parameters — such are the new concepts that are necessary for the formulation of a variational approach to electrostatics.

With respect to the conventional methods, the variational approach stands in a dual perspective. On the one hand, it relies on the known methods and uses them as subsidiary ones. In the example described above, such is the electric image method. On the other hand, the variational modifications of the known methods may acquire new features. For instance, in the problem of ground resistance, the charges q_i could be located at the points that do not coincide with the positions of the images. Such an extension of the image method from the standpoint of the variational approach ought to be referred to as the method of approximating the electric field by the fields of point charges.

We believe that chapters devoted to variational methods of electrostatics and magnetostatics will sooner or later be included in the books on electrodynamics.

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References

1. Batygin V V, Toptygin I N *Sbornik Zadach po Elektrodinamike* (Collection of Problems in Electrodynamics) (Moscow: Nauka, 1970) [Translated into English: *Problems in Electrodynamics* 2nd ed. (London: Academic Press, 1978)]
2. Sivukhin D V *Obshchiĭ Kurs Fiziki* (General Course in Physics) Vol. 3 *Elektrichestvo* (Electricity) (Moscow: Nauka, 1977)
3. Matveev A N *Elektrichestvo i Magnetizm* (Electricity and Magnetism) (Moscow: Vysshaya Shkola, 1983)
4. Maxwell J C *A Treatise on Electricity and Magnetism* 3rd ed. (Oxford: Clarendon, 1892) [Translated into Russian: Vol. 1 (Moscow: Nauka, 1989) Gl. 4, 6]
5. Rayleigh J W S *The Theory of Sound* 2nd ed. (New York: Dover Publ., 1945) [Translated into Russian: Vol. 2 (Moscow: GITTL, 1955)]
6. Pólya G, Szegö G *Isoperimetric Inequalities in Mathematical Physics* (Princeton: Princeton Univ. Press, 1951) [Translated into Russian (Moscow: Fizmatgiz, 1962)]
7. Kazantsev V P *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (8) 96 (1984)
8. Kazantsev V P *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (7) 70 (1987)
9. Kazantsev V P *Dokl. Ross. Akad. Nauk* **361** 469 (1998) [*Dokl. Phys.* **43** 465 (1998)]
10. Kazantsev V P *Teor. Mat. Fiz.* **119** 441 (1999)
11. Kazantsev V P *Dokl. Ross. Akad. Nauk* **369** 617 (1999) [*Dokl. Phys.* **44** 796 (1999)]
12. Kazantsev V P *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (10) 66 (1999)
13. Kazantsev V P *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (10) 53 (2000)
14. Kazantsev V P *Dokl. Ross. Akad. Nauk* **373** 29 (2000) [*Dokl. Math.* **62** 113 (2000)]
15. Kyurkchan A G, Sternin B Yu, Shatalov V E *Usp. Fiz. Nauk* **166** 1285 (1996) [*Phys. Usp.* **39** 1221 (1996)]