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An invariant formulation of the potential integration method for the vortical equation of motion of a material point

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Contents

Abstract. **A relativistic procedure for deriving the kinetic part of the generalized Euler equation is proposed as an argument to justify the application of the vortical equation of motion to the solution of classical discrete dynamics problems. An invariant formulation of the potential integration method for the vortical equation of motion is given for a definite class of two-dimensional motions. To demonstrate the efficiency of the method, a number of well-known theorems on the dynamics of a material point are proved. A new result of the study is the fact that zero-energy hyperelliptic motions are related to the field of 'multiplicative' type forces.**

1. Introduction

By vortical equations of motion are usually meant hydrodynamic type equations: either the Euler equation proper for an ideal liquid [1] or its multidimensional generalization to the Lamb form [2]. Unlike the Newtonian form of the equation of motion, equations of hydrodynamics belong to a totally different — *continual* — class of equations of physics, and in this respect are much closer to the field theory than to the theory of motion of the point objects. Nevertheless, these equations have been successfully applied to the discrete dynamics of Hamiltonian systems — first by I S Arzhanykh [1] and then by V V Kozlov, whose book [2] presents the results of a further in-depth analysis of mathematical corollaries to Arzhanykh's ideas and provides references to the available literature on the subject.

It should be noted that, despite the wide application of equations of hydrodynamics, the main task of the two above-mentioned authors was the integration of the canonical Hamilton equations (but with 'vortex terms' introduced into them) [1]. Therefore, the methods which were developed in the books [1, 2] belong for the most part to the so-called 'vortical methods' of integration, and will not be treated here. But conceptually, our work, as we shall see somewhat below, is rooted in this field, and in particular is closely related to the hypothesis for the existence of the so-called energy – momentum field, which I S Arzhanykh advanced in his monograph [1] based on the formal analogy between the Maxwell equations and the Euler equation.

It must be said, however, that when solving discrete dynamics problems, I S Arzhanykh [1, 3] draws this analogy having in mind the discrete form of the Euler equation, whereas the object of comparison (the Maxwell equations) remains continual. While the problems in discrete dynamics are in no way affected by this inconsistency, it must of course be eliminated if more ambitious goals are pursued. An attempt to solve this problem by relativistic methods will, among other things, help us to understand how the inherently continual Euler equation can be consistently put to use in discrete dynamics. With this in mind, we propose an invariant version of the method, which is inherently potential, of integrating the vortical equation of motion and apply it to solving the standard problems of motion of a point object in an external potential type field (a conservative system). It should be emphasized that the method we formulate is, due to its inner specifics, by no means a general one but is aimed at picking out, from all possible motions, such a class of motions whose derivation, compared with traditional nonvortical methods, becomes much simpler due solely to the direct integration of the vortical equation of motion. To this distinguished and particular class belong such motions which are, to put it mildly, very difficult to find using nonvortical formulations of mechanics if for no other reason, than because in the traditional approach these motions are obscured and not so obvious as is, for that matter, the nature of the 'multiplicative' forces that support them.

Further, since we will also touch in part on the continual stage of the problem, this will also help us to see the

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distinguishing features of our approach in comparison to the one successfully used in hydrodynamics for 'potential flows' [4].

The methodical results of this work are also augmented by the derivation of the generalized Huygens 'centrifugal' acceleration formula from the vortical equation of motion and by obtaining from this a number of well-known theorems for the dynamics of a material point. In so doing particular attention will be focused on how to reveal, utilize, and emphasize by means of comparison the methodical advantages that result directly from the coordinate invariance of the method.

2. The heuristic significance of the hypothesis for the existence of the energy – momentum density field

We start by writing down and analyzing the 'vortical' — or in other words generalized Eulerian — form of the equation of motion, used in Ref. [1] for solving the problems in discrete dynamics:

$$\frac{\partial \mathbf{p}}{\partial t} + \operatorname{grad} K - [\mathbf{V} \times \operatorname{rot} \mathbf{p}] = -m \operatorname{grad} \varphi \,, \tag{1}$$

where $\mathbf{p}$ and K are, respectively, the momentum vector and the kinetic energy of a material point of mass m moving with a velocity $\mathbf{V}$ in a generally curvilinear trajectory under the action of a field of external forces with a potential function φ and t is the time as measured by a clock in the absolute frame of reference.

Transferring the right-hand side of Eqn (1) to the left-hand side yields a different form, namely

$$\frac{\partial \mathbf{p}}{\partial t} + \operatorname{grad} H - [\mathbf{V} \times \operatorname{rot} \mathbf{p}] = 0 \,, \tag{2}$$

where H is the Hamiltonian.

Without going here into all the details, let us outline the relativistic arguments which, by analogy with the field theory, suggest the existence of the energy – momentum *field* — the hypothesis proposed by I S Arzhanykh in Ref. [1] [1].

As already noted, drawing this analogy in a consistent way requires, first of all, that Eqn (1) be treated as continual. This means that all the characteristic quantities $(\mathbf{p}, K, m)$ entering the theory should be replaced by their associated bulk densities, for which purpose it suffices to replace m by μ throughout, where μ is a function of the bulk mass density (arbitrarily distributed in space – time) of a certain substance (not necessarily a liquid) whose kinematics of 'flow' is just described by the left-hand side of the substantial equation of motion.

Now let us recall the sequence of actions which were taken by H Minkowski in the relativistic derivations of the Maxwell equations in a vacuum (see Ref. [6]).

The parent concept in this scheme, as is well known, is framed around the 4-potential, whose spacelike components are represented by the 3-vector $\mathbf{A}$ (the vector potential of the electromagnetic field), and whose timelike component is the scalar function ψ (electrostatic potential) [2].

By multiplying this 4-vector externally with the Hamilton differential 4-operator, i.e. by applying the 4-rot operator [6] to it, we produce a skew-symmetric tensor of the electromagnetic field.

It is important now to emphasize that unit vectors and differentials are transformed similarly for the Lorentz group — namely, like coordinates. Consequently, these end products of Minkowski's formalism (the external multiplication of unit vectors) and of the Poincaré – Cartan formalism of external differential forms (external multiplication of differentials) — if this latter formalism is extended to the Minkowski space — should of course be the same for the properly chosen and identical sequence of operations.

Further, using the formulae

$$\mathbf{e} = -\operatorname{grad} \psi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \,,$$
$$\mathbf{h} = \operatorname{rot} \mathbf{A}$$

which relate the field vectors to the potentials, the first set of Maxwell equations in a vacuum, viz.

$$\frac{1}{c}\frac{\partial \mathbf{e}}{\partial t} - \operatorname{rot} \mathbf{h} = 0 \,, \tag{3}$$

$$\operatorname{div} \mathbf{e} = 0 \,, \tag{4}$$

are obtained by requiring the vanishing of the 4-vector resulting from the internal multiplication of the field tensor with the Hamilton 4-operator.

The relativistic form of the left-hand side of Eqn (1) can be found in a similar way as a spacelike part of a certain 4-vector. But, for this purpose, the physical quantity we should start with is the momentum density 4-vector with components $(-\mathbf{p}', K'/c)$, where $\mathbf{p}'$ and K' are the relativistic momentum density 3-vector and the relativistic energy density, respectively, and c is the speed of light. Here too, the skew-symmetric 'field' tensor of the energy – momentum density results from the action of the 4-rot operator on the initial momentum density 4-vector. Further, proceeding much as in the field theory — but adopting a somewhat different sign arrangement — it is possible to define the vectors of this 'field' in terms of the 'potentials':

$$\mathbf{e}' = \operatorname{grad} \frac{K'}{c} + \frac{1}{c}\frac{\partial \mathbf{p}'}{\partial t} \,, \qquad \mathbf{h}' = \operatorname{rot} \mathbf{p}' \,. \tag{5}$$

However, the reduction of rank of the resulting tensor by unity should be carried out here, from dimensional considerations, by multiplying the tensor — *internally* — not with the Hamilton operator as in the field theory but rather with the velocity 4-vector (for which purpose its contravariant components are used). This gives rise to a new 4-vector with the components

$$\frac{1}{\sqrt{1-\beta^2}} \left\{ c\mathbf{e}' - [\mathbf{V} \times \mathbf{h}'], -\mathbf{V}\mathbf{e}' \right\} , \tag{6}$$

where $\beta = |\mathbf{V}|/c$, and $\mathbf{V}$ is the usual velocity 3-vector.

[1] As concerns the relativistic form of the left-hand (kinetic) side of Eqn (1), it should be noted that even though it is given in Refs [1, 3], its derivation is not provided there; at the same time, I S Arzhanykh does provide a relativistic procedure for the derivation of the field equations based on the external differential Poincaré – Cartan forms [5].

[2] The space – time metric signature $(+++-)$ used here corresponds to that adopted by Pauli [6].

It should be noted that the relativistic form of the Euler equation proper, with the function μ taken out of the differential operator signs, can be obtained in an entirely similar way by interchanging the momentum and velocity 4-vectors in the operational procedure outlined above.

If we now want to proceed to the description of kinematics of point objects, the function μ in formulae (5) should be represented in the form of a δ-function, which plays the role of the position function of a material point in the 4-dimensional space–time. Then, by integrating the 4-vector (6) over the volume and taking into account the Lorentzian shrinkage of volumes, we arrive at a new 4-vector whose spacelike part, as can easily be seen[3] from Eqns (5) and (6), reproduces the left-hand side of Eqn (1) in its relativistically generalized form, whereas the timelike component takes the scalar form

$$-\frac{\mathbf{V}}{c}\,\mathbf{e}'',$$

in which

$$\mathbf{e}'' = \frac{\partial \mathbf{p}}{\partial t} + \operatorname{grad} K,$$

where $\mathbf{p}$ and K are the relativistic expressions for the momentum and energy, respectively.

As for the timelike component of the 4-vector, it can be treated as a 'kinetic analogue' of the left-hand side of Eqn (4). However, the physical meaning of this analogy cannot be understood within the framework of classical theory, the only one to be used through the remainder of this paper.

Thus we see that the difference in the way the two transformation schemes reduce the rank of the tensor has fundamental implications, since this is the main reason for the linearity of Maxwell equations, on the one hand, and for the nonlinearity of the 'vortical theory of motion', on the other hand.

One further point to be made is of course a certain difference — which also has a fundamental nature — in the input quantities used in the analysis.

In the field theory $\mathbf{A}$ and ψ are independent but auxiliary quantities. In the 'vortical theory of motion', on the contrary, the quantities $\mathbf{p}'$ and K'/c have clear physical meaning, but both depend on velocity and therefore are not independent.

All this does not allow one to draw too close a parallel between the concepts involved in the two theories, and it is necessary to have in mind therefore that the energy–momentum 'field' in Arzhanykh's hypothesis is a fictitious field, i.e. a concept which can only be employed for heuristic purposes.

For example, the fact that in relativistic transformations the momentum density 4-vector plays the role of 4-potential suggests that it needs no redefinition in terms of other auxiliary concepts as is the case in nonvortical descriptions of the motions of point objects, where the action function S takes the role of a scalar potential:

$$\mathbf{p} = \operatorname{grad} S\,, \tag{7}$$

$$H = -\frac{\partial S}{\partial t}\,. \tag{8}$$

In general, the comparative analysis of the problems of integrating the discrete form of the vortical equation of motion and the nonvortical equation of motion deserves special attention and makes it necessary to draw a clear distinction between the two.

We have seen, for example, that in the relativistic derivation of the kinetic part of the vortical equation of motion, the momentum 3-vector is a solenoidal vector because it plays there the role of a 'vector potential'. From this point of view, formula (7) being used in Hamiltonian mechanics needs generalization, which is precisely the goal of the vortical methods for integrating the canonical equations [1, 2]. On the other hand, turning back to the vortical form, it seems at first sight that such an obvious solenoidal nature of the momentum vector of curvilinear motions compromizes to the highest degree the very idea of employing potential methods for its integration. This is evidenced by the fact that the simple substitution of Eqns (7) and (8) into Eqn (2) reduces the last to an identity independently of whether the function S has any relevance to actual motions.

Meanwhile, it is well known that the potential Hamilton–Jacobi method is the most powerful tool for integrating the canonical equations of mechanics. But here, not everything is all that smooth either, because the method depends strongly on the choice of a coordinate structure, and there are no rules to govern such a choice [5].

Proceeding now to the comparative analysis of the application of potential integration methods to the continual form of the vortical equation of motion (hydrodynamics) and to its discrete form (dynamics of point objects), the following fundamental feature should be emphasized.

In the case of ideal fluid, the existence of 'potential flows' is related to the well-known Thomson theorem [4] on the conservation of velocity circulation along the liquid contour around fluid streamlines (zero value of circulation). Clearly, in the vortical description of the motions of a point object, this theorem is not applicable because the very use of such a concept as velocity circulation around the trajectory of a point is incorrect here. We can only speak meaningfully of the velocity circulation *along* a trajectory around the origin at rest[4], with the consequence that for the curvilinear motions we once again end up with a solenoidal momentum vector.

Thus, the solenoidal nature of this vector rules out its applicability in potential integration methods for the discrete-form vortical equation of motion. This does not mean, however, that methods of this type must be totally banned here, because we can choose another — no longer solenoidal — vector to uniquely characterize the motion of a material point.

[3] Differentiating the function μ in the rot $\mathbf{p}'$ vector actually produces some additional terms in the equation. Volume integration (using the continuity equation) eliminates them only in the case of a centrally symmetric distribution of the function μ over the volume of a spherical body (the δ-function meets these requirements in the limiting case) and for an incompressible substance. All this can be avoided by starting with the Eulerian form of the equation of motion for the substance.

[4] As first shown by Cartan [7], it is this interpretation of the Stokes hydrodynamic lemma on the conservation of velocity circulation which may serve as the basis for Hamiltonian mechanics; an example of the systematic development of this idea is V I Arnol'd's approach to the construction of Hamiltonian mechanics [5].

In all circumstances, the vector we can use for this purpose is the radius vector $\mathbf{r}$ of a point, because for this vector we have the formula

$$\mathbf{r} = \mathrm{grad}\, \frac{r_0^2}{2}\,, \tag{9}$$

where r_0 is the distance from the origin to the point.

It is this fact which will be used below when formulating the invariant version of the potential integration method for Eqn (2) in the stationary case.

3. Formulation of the problem

Let us consider the vortical equation (2) in the stationary case, when the velocity of a material point does not depend on time explicitly:

$$\mathrm{grad}\, H - [\mathbf{V} \times \mathrm{rot}\, \mathbf{V}] = 0\,, \tag{10}$$

where H is the Hamiltonian normalized to the mass of the material point:

$$H = \frac{1}{2}\, \mathbf{V}^2 + \varphi\,. \tag{11}$$

The fact that we limit our consideration to the stationary case means the following.

For an arbitrarily specified dependence of the potential function on the distance to the material point — irrespective of the initial conditions of its motion but under some assumptions about the possible values of the Hamiltonian — we wish to find the complete family of the trajectories of the point, which, for any given values of the Hamiltonian, is in one-to-one correspondence with the given potential of the external forces.

Clearly, after fully establishing the geometric picture of the motion, whether it has been monitored for an infinite time (e.g., for infinite motions) or (for periodic motions) for a finite time (in both cases the time variable is excluded), it is not difficult to establish the time schedule for the motion of a particle along a preassigned family of trajectories. It suffices to impose specific boundary conditions that are compatible with this family of trajectories on the particle's motion in space and time.

The following limitation relates to the fact that we will consider, for simplicity, not all the motions possible for a given potential, but only a certain class of them, such that their family of trajectories satisfies a certain condition. Namely, we will consider plane motions along two mutually orthogonal families of geodesics for which the components of the fundamental metric tensor[5] are equal:

$$g_{11} = g_{22} \equiv g \tag{12}$$

at any point of Euclidean space.

This latter condition, as will become clear later on, is imposed in order to reduce the problem of searching for geodesics to the function theory of complex variables.

It is clear that in view of Eqn (9), the radius vector of a point in any coordinate system with origin at any point O at rest is the irrotational vector

$$\mathrm{rot}\, \mathbf{r} = 0\,,$$

and we notice also that for plane motions one has

$$\mathrm{div}\, \mathbf{r} = 2\,. \tag{13}$$

In the Cartesian coordinate system, the vector $\mathbf{r}$ can be expanded in terms of the unit vectors $\mathbf{x}_{01}$, $\mathbf{x}_{02}$ as follows

$$\mathbf{r} = x_1(\eta, \xi)\mathbf{x}_{01} + x_2(\eta, \xi)\mathbf{x}_{02}\,, \tag{14}$$

where the functions $x_{1,2}$ in any orthogonal, cylindrical type coordinate system (η, ξ, x_3) should, as a consequence of Eqns (9) and (13), satisfy the equation

$$\Delta r_0^2 = 4\,, \tag{15}$$

where Δ is the Laplacian in the curvilinear coordinate system used.

For the systems which comply with relations (12), so that $g_{11} = g_{\eta\eta}$, $g_{22} = g_{\xi\xi}$, Eqn (15) is satisfied if the functions $x_{1,2}(\eta, \xi)$ are harmonic.

This means that the Cartesian components of the vector $\mathbf{r}$ must meet the Cauchy – Riemann conditions

$$\frac{\partial x_1}{\partial \eta} = \frac{\partial x_2}{\partial \xi}\,, \qquad \frac{\partial x_1}{\partial \xi} = -\frac{\partial x_2}{\partial \eta}\,, \tag{16}$$

from which it follows that a plane vector $\mathbf{r}$ can be considered as an analytical function

$$W = x_1(\eta, \xi) + \mathrm{i}x_2(\eta, \xi) \tag{17}$$

in the complex plane[6] $\theta = \eta + \mathrm{i}\xi$. Hence, we should try to reduce the problem of integrating the equation (10) in *partial derivatives* to one of finding the analytical function W of a *single variable* θ.

Let us see how — and under what conditions — we can implement this idea.

By and large, the problem is to map the vector equation (10) onto the complex plane W and to obtain there quadratures with respect to this complex function that analytically maps — in accordance with Eqns (14) and (17) — the infinite families ($\eta = \mathrm{const}$, $\xi = \mathrm{const}$) of all the trajectories possible under these circumstances.

We will thus represent Eqn (10) in curvilinear, orthogonal, cylindrical type coordinate systems, first, which satisfy condition (12); second, for which $g_{33} = 1$, and, third, in which any unit vector is expressed in terms of the two others by the formula

$$[\boldsymbol{\eta}_0 \times \boldsymbol{\xi}_0] = \mathbf{x}_{03}\,, \tag{18}$$

subject to the cyclic permutation rule.

The velocity vector $\mathbf{V}$ in Eqn (10) in the general case is expanded in terms of the unit vectors of the plane of motion as

$$\mathbf{V} = V_1 \boldsymbol{\eta}_0 + V_2 \boldsymbol{\xi}_0\,.$$

[5] We emphasize once again that this work is aimed not at searching for some general solutions to the equations of dynamics but rather at separating from them certain special types of motion, which are very conveniently described in the 'vortical theory' framework.

[6] The mapping of plane motions onto the complex plane was, as it is known, employed by B Bolin (see Ref. [8]), but he limited himself to those solutions of Newton's equation which obey the law of areas — a limitation which we will try to avoid here.

However, of fundamental importance for our discussion is the fact that the coordinate lines $\eta, \xi = \text{const}$ will be considered here as geodesics, i.e. as the only lines along which the material point can move. It then follows that — not, of course, in general but for specified values of the Hamiltonian — we can encounter only two types of mutually orthogonal motions along the lines $\xi = \text{const}$ and $\eta = \text{const}$, respectively:

$$\mathbf{V} = V_1 \boldsymbol{\eta}_0 \,, \tag{a}$$

$$\mathbf{V} = V_2 \boldsymbol{\xi}_0 \,. \tag{b}$$

The vortical equation (10) is considerably simplified under these circumstances. Before writing it down in full, let us write separately its first and second terms for the motions (a) and (b). Using the conditions (12) and Eqns (18) we find

$$\operatorname{grad} H = \frac{1}{\sqrt{g}} \left\{ \frac{\partial H}{\partial \eta} \boldsymbol{\eta}_0 + \left[\frac{\partial (V_1^2/2)}{\partial \xi} + \frac{\partial \varphi}{\partial \xi} \right] \boldsymbol{\xi}_0 \right\}, \tag{19a}$$

$$[\mathbf{V} \times \operatorname{rot} \mathbf{V}] = -\frac{V_1}{g} \frac{\partial(\sqrt{g}\, V_1)}{\partial \xi} [\boldsymbol{\eta}_0 \times \mathbf{x}_{03}]$$
$$= \frac{1}{\sqrt{g}} \left[\frac{1}{2} V_1^2 \frac{1}{g} \frac{\partial g}{\partial \xi} + \frac{\partial (V_1^2/2)}{\partial \xi} \right] \boldsymbol{\xi}_0 \,, \tag{20a}$$

$$\operatorname{grad} H = \frac{1}{\sqrt{g}} \left\{ \frac{\partial H}{\partial \xi} \boldsymbol{\xi}_0 + \left[\frac{\partial (V_2^2/2)}{\partial \eta} + \frac{\partial \varphi}{\partial \eta} \right] \boldsymbol{\eta}_0 \right\}, \tag{19b}$$

$$[\mathbf{V} \times \operatorname{rot} \mathbf{V}] = \frac{V_2}{g} \frac{\partial(\sqrt{g}\, V_2)}{\partial \eta} [\boldsymbol{\xi}_0 \times \mathbf{x}_{03}]$$
$$= \frac{1}{\sqrt{g}} \left[\frac{1}{2} V_2^2 \frac{1}{g} \frac{\partial g}{\partial \eta} + \frac{\partial (V_2^2/2)}{\partial \eta} \right] \boldsymbol{\eta}_0 \,. \tag{20b}$$

Substituting Eqns (19a), (19b), (20a), and (20b) into Eqn (10) yields the following two sets of scalar equations

$$\frac{\partial H}{\partial \eta} = 0 \,, \tag{21a}$$

$$\frac{1}{2} V_1^2 \frac{1}{g} \frac{\partial g}{\partial \xi} = \frac{\partial \varphi}{\partial \xi} \,, \tag{22a}$$

$$\frac{\partial H}{\partial \xi} = 0 \,, \tag{21b}$$

$$\frac{1}{2} V_2^2 \frac{1}{g} \frac{\partial g}{\partial \eta} = \frac{\partial \varphi}{\partial \eta} \,. \tag{22b}$$

Equations (21a), (21b) express the existence of energy integrals when moving along geodesics. Indeed, from Eqn (21a), for example, it follows that in motion along the lines $\xi = \text{const}$ the Hamiltonian may only depend on the variable ξ [i.e. on precisely along which line (from the infinite set $\xi = \text{const}$) the point moves], and its value is fixed, $H = h(\xi)$, and remains unchanged during the particle's entire motion. The same is true of Eqn (21b), the only difference being that here $H = h(\eta)$.

Further, Eqns (22a) and (22b) in fact yield the normal (with respect to the trajectory) acceleration of the point and generalize the Huygens formula for centripetal (or centrifugal) acceleration. Indeed, it can be shown [4] that

$$\frac{1}{2g\sqrt{g}} \frac{\partial g}{\partial \xi} = \frac{1}{R_1} \,, \qquad \frac{1}{2g\sqrt{g}} \frac{\partial g}{\partial \eta} = \frac{1}{R_2} \,,$$

where $R_{1,2}$ are the local radii of curvature of the corresponding geodesics, $\xi = \text{const}$ and $\eta = \text{const}$. Hence Eqns (22a) and (22b) can be rewritten in the Huygens form

$$a_{\text{n}1,2} = \frac{\mathbf{V}_{1,2}^2}{R_{1,2}} \,,$$

where $a_{\text{n}1,2}$ are the normal components of the acceleration the material point experiences when moving along curvilinear trajectories that belong to corresponding families.

We can at once say, however, that this form is extremely inconvenient for integration, so we must make use of Eqns (22a), (22b), in which the local radius of curvature is factorized by functions that also relate to local curvature; but this approach is the most convenient for integration and it enables a transfer to the complex plane W.

Thus, from the existence of energy integrals, and in view of Eqn (11), we have

$$h(\xi) = \frac{1}{2} \mathbf{V}_1^2 + \varphi \,, \tag{23a}$$

$$h(\eta) = \frac{1}{2} \mathbf{V}_2^2 + \varphi \,. \tag{23b}$$

From these equations we may obtain the expression for the 'kinetic energy' $(1/2)\mathbf{V}_{1,2}^2$, which when substituted into Eqns (22a) and (22b) yields the compatibility conditions for the equations of each set:

$$h(\xi) \frac{\partial g}{\partial \xi} = \frac{\partial (g\varphi)}{\partial \xi} \,, \tag{24a}$$

$$h(\eta) \frac{\partial g}{\partial \eta} = \frac{\partial (g\varphi)}{\partial \eta} \,, \tag{24b}$$

which are none other than the integrability conditions for Eqn (10) within the restrictions imposed.

These are precisely the basic equations whose mapping onto the complex plane W we must now obtain.

4. Zero-Hamiltonian motions. Multiplicative potentials and hyperelliptical motions

From the point of view of transfer to the complex plane W, Eqns (24a) and (24b) possess the necessary symmetry, and the main point here is that on the right-hand sides of these equations we differentiate the product of the unknown function g and the given function φ. It is this fact which is the prerequisite for relating the potential φ to the function W on the complex plane W and for mapping the orthogonal families of plane mechanical trajectories onto it.

What makes this possible is the fact that all the quantities entering Eqns (24a) and (24b) can be expressed in terms of the function W.

Indeed, note first that in view of Eqns (14) and (17), the square of the distance from the origin to any point on the plane is given by

$$r_0^2 = W\overline{W} \tag{25}$$

(hereinafter a bar above the symbol denotes the operation of complex conjugation). Second, the square of the distance from a point with Cartesian coordinates (a_1, a_2) to any other point on the plane is also expressed in terms of the unknown function W:

$$r^2 = (W - a)(\overline{W} - \bar{a}), \tag{26}$$

where $a = a_1 + \mathrm{i}a_2$.

Next, $g_{\eta\eta}$ and $g_{\xi\xi}$ by definition are none other than

$$g_{\eta\eta} = \left(\frac{\partial x_1}{\partial \eta}\right)^2 + \left(\frac{\partial x_2}{\partial \eta}\right)^2, \quad g_{\xi\xi} = \left(\frac{\partial x_1}{\partial \xi}\right)^2 + \left(\frac{\partial x_2}{\partial \xi}\right)^2. \tag{27}$$

In particular, it is seen that the condition (12) is equivalent to the Cauchy – Riemann conditions (16).

Now the first derivative of the required function on a complex plane θ can be represented in two equivalent forms

$$W' = \frac{\mathrm{d}W}{\mathrm{d}\theta} = \frac{\partial x_1}{\partial \eta} + \mathrm{i}\,\frac{\partial x_2}{\partial \eta}, \quad W' = \mathrm{i}\left(\frac{\partial x_1}{\partial \xi} + \mathrm{i}\,\frac{\partial x_2}{\partial \xi}\right). \tag{28}$$

Comparison of Eqns (27) and (28) shows that

$$g = W'\overline{W}'. \tag{29}$$

Relation (26) also makes it possible to relate the known function φ to the unknown function W.

If the center of forces is displaced from the origin, then φ is a function of the distance r:

$$\varphi = \pm F(r),$$

where F is a generally arbitrary function determined by the conditions of the problem.

Because r is related to W by Eqn (26), we arrive at

$$\varphi = \pm F\left(\sqrt{W - a}\,\sqrt{\overline{W} - \bar{a}}\right), \tag{30}$$

where a can also be zero if the center of external forces coincides with the origin.

We note that the quantities related to the geometric characteristics of the particle's trajectories relative to the origin [Eqns (25) and (29)] and to the center of forces [Eqn (26)] are given by factorized expressions with complex conjugate multipliers.

Let us impose a similar requirement on the potential function

$$\varphi = \pm F\left(\sqrt{W - a}\right) F\left(\sqrt{\overline{W} - \bar{a}}\right) = \pm F\overline{F}. \tag{31}$$

While this requirement clearly puts serious constraints on the analytical form of the potentials, it provides a very simple way to make a transfer to the complex plane.

At the same time, the most interesting case of power functions satisfies the above requirement, thus enabling us to write down

$$g\varphi = \pm Z\overline{Z}, \tag{32}$$

where

$$Z = FW'.$$

Let us first consider the simplest possible case of motions with a zero Hamiltonian.

V I Arnol'd, when generalizing [8] Bolin's theorem on the distinctive duality of the Newton and Hooke laws of attraction, obtained, in particular, the function W for this case, too. The essential point to be made here is that the proof of the Bolin theorem heavily relies on the law of areas and the algebraic properties of the Joukowski function and therefore involves unnecessary constraints which can well be disposed of, if a more systematic method is applied. This is always very important because otherwise some exceptions (to be considered below) to the theorems may emerge.

From Eqns (11) it is apparent that motions with a zero Hamiltonian are achieved only in the case of negative potentials, and these consequently correspond to attractive forces for negative powers, and to repulsive forces for positive powers.

The transfer to the complex plane W from Eqns (24a), (24b) with their left-hand sides both being zero $[h(\xi) = 0, h(\eta) = 0]$, but with $\partial g/\partial\xi, \partial g/\partial\eta \neq 0$, does not take much effort and leads, as it should, to the same equation for both types of motions:

$$FW' = C, \tag{33}$$

where C is a constant.

In the absence of external forces ($F = \mathrm{const}$), Eqn (33) gives the solutions

$$W = C\theta \quad (C = C_1 + \mathrm{i}C_2) \tag{34}$$

with constant g, and Eqns (24a) and (24b) are satisfied for arbitrary nonzero values of the Hamiltonian.

Since Eqn (34) implies that

$$x_1 = C_1\eta + C_2\xi, \quad x_2 = C_2\eta - C_1\xi,$$

it follows that the geodesics of an empty space are two mutually orthogonal families of straight trajectories. For a fixed nonzero energy level, the point moves with a constant velocity along these trajectories. Hence, the Galilean law of inertia is contained in the vortical equation of motion.

We turn now to motions with zero Hamiltonian and $g \neq \mathrm{const}$.

For power and normalized potential functions with the force center at the origin, namely

$$\varphi = -r_0^{\alpha},$$

where α is any real number, by reference to Eqn (25) we find

$$F = W^{\alpha/2}.$$

From this, setting $(1 + \alpha/2)C$ equal to unity in Eqn (33), we obtain the equation

$$W^{\alpha/2}\,\mathrm{d}W = \frac{\mathrm{d}\theta}{1 + \alpha/2},$$

from which we have for $\alpha \neq -2$:

$$W = \theta^{2/\alpha+2}. \tag{35}$$

For $\alpha = -2$, we find separately that

$$W = \exp\theta. \tag{36}$$

Note that this is precisely the 'exception' which appears in the Bolin – Arnol'd theorem for zero-Hamiltonian motions in the field of central forces with the potential displaying a power-law dependence on distance [8] and which is thus incorporated into a unified scheme thanks to a more systematic approach to the integration of the equation of motion (importantly, the vortical type of equation).

Further integration reduces to quadratures and is in principle identical to the integration of straight-line motions.

Let us illustrate this operation of integration by taking motions along geodesics $\xi = \text{const}$ [case (a)] as an example.

Since

$$V_1 \equiv \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}\eta}\,\frac{\mathrm{d}\eta}{\mathrm{d}t} = \sqrt{g}\,\frac{\mathrm{d}\eta}{\mathrm{d}t}\,, \tag{37}$$

where s is a natural parameter (the length of an arc of the trajectory), it follows from Eqn (22a) that the equality holds true:

$$V_1 = \sqrt{\frac{2g\,(\partial\varphi/\partial\xi)}{\partial g/\partial\xi}} \tag{38}$$

with g given by Eqn (29).

Substituting Eqn (38) into Eqn (37) and integrating over η (ξ being a fixed parameter) yields an equation for determining the time of motion as a function of the current value of this coordinate:

$$t = \int \sqrt{\frac{2\,(\partial\varphi/\partial\xi)}{\partial g/\partial\xi}}\;\mathrm{d}\eta + C_1\,,$$

where the constant C_1 is determined by the initial conditions.

Let us now consider in detail Eqn (35) for the case of attractive forces ($\alpha < 0$).

For the Newton potential ($\alpha = -1$) we have, as we should, the orthogonal families of confocal parabolas, because

$$W = \theta^2 \quad (x_1 = \eta^2 - \xi^2\,,\; x_2 = 2\eta\xi)\,.$$

Generally, the function W performs a conformal mapping of the plane θ, which is caused by external forces and the result of which is that the straight geodesics of an empty space (i.e. the coordinate lines of the plane θ) transform into the curvilinear (and not necessarily infinite) plane geodesics of the Euclidean space (the coordinate lines of the plane W) with the potential φ residing there.

One further point to note is that for $\alpha \in [0, -2)$ the motions remain infinite (as they should in this case), and the infinitely distant points of the plane θ remain infinitely distant on the plane W as well. The fall onto the center along straight trajectories takes place along the geodesics $\eta = 0$ and $\xi = 0$ (a parabola degenerates into a ray). As the power α approaches -2, the 'parabolas' are flattened, and for $\alpha = -2$ we arrive at the limiting formula (36) with

$$g = \exp 2\eta \qquad (x_1 = \exp\eta\cos\xi\,,\; x_2 = \exp\eta\sin\xi)\,.$$

Here $\partial g/\partial\xi = 0$, implying that zero-Hamiltonian motions proceed only along the rays $\eta = \text{const}$ and end up with the fall onto the center (with the 'parabolas' being totally flattened into rays for any, not only zero, values of the parameter ξ).

For $\alpha \in (-2, -\infty]$, all curvilinear trajectories become finite and involve a fall onto the center. The only trajectories which remain infinite are straight ones terminating with a fall onto the center along the rays (their images on the θ plane are always geodesics passing through its origin), and they will not be pursued further.

For finite motions closing in the force pole (the zero point of the plane W), the origin of the plane W becomes the image of the infinitely distant points of the plane θ. For negative integer powers ($\alpha < -2$), the following features are observed in the behavior of these trajectories.

For $\alpha = -3$, the families of trajectories are, as already noted, cardioid-like curves closed in the center.

The case $\alpha = -4$ corresponds to Newton's theorem on the attractive force being inversely proportional to the fifth power of the distance, when trajectory families consist of circumferences touching the force center [5, 8].

By further decreasing α in Eqn (35), fractional powers appear. Accordingly, the trajectories will exhibit several center-closed branches which are gradually 'flattened' as their number increases. However (considering the fall onto the center as an elastic impact), the motions here will remain periodic as before. It is only for nonintegral α from the interval $\alpha \in (-2, -\infty]$ that motions, while remaining finite, will not be periodic and will, within an infinite period of time, everywhere densely fill the circular region around the force pole.

Let us look at some other possibilities that arise from the fact that the method employed does not rely on the law of areas.

Potentials which are not subject to this law but which again satisfy formula (31) (the necessary condition here) and depend only on distances can be constructed immediately as the *product* of several central type potentials with their poles shifted with respect to each other. Therefore we will call such potentials 'multiplicative'. Let us consider one such example, one with two centers:

$$\varphi = -\frac{1}{(r_1 r_2)^m}\,, \tag{39}$$

where $r_{1,2}$ are the distances from any point on the plane to the two force poles, which are conjugate to each other by the condition of the mirror symmetry of their coordinates with respect to the origin, and $m > 0$.

We set m equal to unity and place the conjugate centers themselves on the axis Ox_1, at distances a on the right and on the left from the origin. The image of the potential (39) on the plane W will be the function

$$F = \frac{1}{\sqrt{W^2 - a^2}}\,.$$

Assuming for convenience that the constant C in Eqn (33) is imaginary, $C = \mathrm{i}/a$, we have after performing the integration that

$$\arccos\frac{W}{a} = \theta\,,$$

or finally

$$W = a\cos\theta\,. \tag{40}$$

From this we conclude that 'the multiplicative potential' of attractive forces, given by Eqn (39) with $m = 1$ and with the

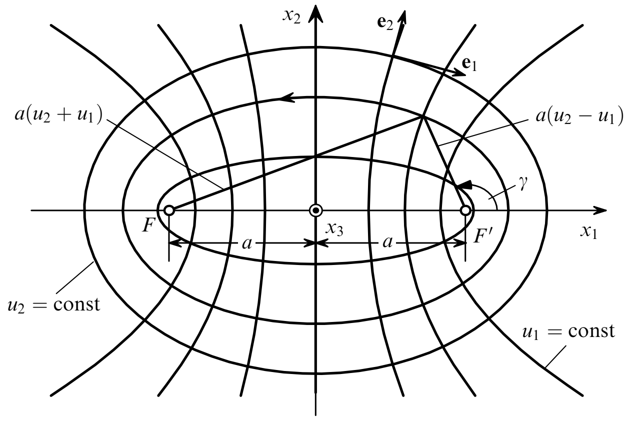

Figure 1. Families of mechanical trajectories mapped by the analytical function $W = a\cos\theta$.

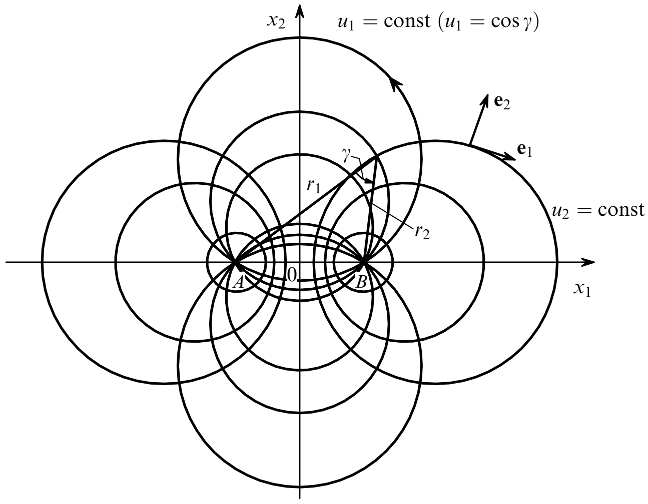

Figure 2. Families of mechanical trajectories mapped by the analytical function $W = a\tan\theta$.

force centers at the two foci $(-a, a)$ of the ellipse, is capable of maintaining the motion of a material point at a zero-energy level along elliptic trajectories (the analysis of a Lyapunov stability of orbits is omitted in this paper).

Because the potentials of this type cannot be viewed as central, this conclusion is not in disagreement with the well-known Bertrand theorem [9] which states that of all the central potentials only two — Newtonian and Hookian — have relevance to elliptic motions.

Figure 1 shows the families of trajectories for the motion of a point, which result from the analytical properties of the function (40) (in accordance with Eqns (17) and (40), the Cartesian coordinates of the point are given by

$$x_1 = a\cos\eta\cosh\xi\,, \qquad x_2 = -a\sin\eta\sinh\xi\,,$$

implying that the geodesics $\xi = \text{const}$ are the ellipses).

The notation we used in the figure is as follows

$$u_1 = \cos\eta\,, \qquad u_2 = \cosh\xi\,,$$

$$\mathbf{e}_1 = \boldsymbol{\eta}_0\,, \qquad \mathbf{e}_2 = \boldsymbol{\xi}_0\,, \qquad r_{1,2} = a(u_2 \pm u_1)\,.$$

The angle γ (true anomaly) and the eccentric anomaly η are related by the well-known formula [10]

$$\tan\frac{\gamma}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{\eta}{2}\,,$$

where e is the eccentricity ($e = u_2^{-1}$).

We next set $m = 2$ and $C = -\mathrm{i}/a$ in Eqns (39) and (33). Integrating Eqn (33) we obtain

$$W = a\tan\theta\,.$$

By separating the real and imaginary parts in this expression one can show that one type of zero-Hamiltonian motions in the field of forces with the potential (39) for $m = 2$ reduces to closed finite motions without falling onto the center, which occur along the circumferences $u_2 = \text{const}$ ($\xi = \text{const}$) of the bipolar coordinate system [11] (Fig. 2). The notation in Fig. 2 is the same as in Fig. 1, the only difference being that here $\gamma = \eta$. From this figure we can see that for this type of motions one of the force poles is always external in reference to these circumferences and that the motions proceed around the second pole — but with the central symmetry being disrupted.

The motions of the second type occur along the circumferences $u_1 = \text{const}$ ($\eta = \text{const}$), with the poles touching twice (see Fig. 2). It is not difficult to show that with respect to this type of motion the 'multiplicative potential' (39) and the central potential $\varphi = -r_{1,2}^{-4}$ are interchangeable (as indeed they should be due to the Newton theorem mentioned above).

Finally, let us consider one more integrable example of this kind with two pairs of conjugate force centers; the coordinates of the second pair $(-b, b)$ are again brought in coincidence with the axis Ox_1, and it is assumed that $b > a$. The image of the potential

$$\varphi = -\frac{1}{r_1 r_2 r_3 r_4}\,, \tag{41}$$

all the singularities of which are integrable, on the complex plane W takes the form

$$F = \frac{1}{\sqrt{(W^2-a^2)(W^2-b^2)}}\,.$$

Introducing the parameter $k = a/b$, Eqn (33) becomes

$$\frac{\mathrm{d}w}{b\sqrt{(1-w^2)(1-k^2w^2)}} = C\,\mathrm{d}\theta \qquad \left(w = \frac{W}{a}\right).$$

Setting $C = 1/b$ and integrating, we arrive at the elliptical integral of the first kind, namely

$$\theta = \int_0^w \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}\,,$$

and it is by inverting this integral that we can express the required function in terms of the elliptic Jacobian function:

$$W = a\,\mathrm{sn}\,(\theta, k)\,. \tag{42}$$

It can be shown that in conformity with the properties of the function (42) (the details of the corresponding conformal mapping can be found in the book by H Bateman and

A Erdélyi [12]), the straight geodesics of an empty space with the potential (41) appearing in it transform — for zero-Hamiltonian motions — to a biorthogonal system of the fourth-order confocal bicircular curves.

It should be stressed that the elliptic function (42) by no means maps elliptic motions [as is the case with the trigonometric function (40)], but already maps hyperelliptic — and even periodic — motions.

It is not difficult to recognize that a system approach to integrating the vortical equation of motion (10) can theoretically provide an infinite number of examples of such motions in the force field of 'multiplicative potentials' (with integrable singularities at the poles)

$$\varphi = -\prod_{i=1}^{N} \frac{1}{r_{1i} r_{2i}} ,$$

where r_{1i}, r_{2i} are the distances to the point from two force centers of the ith pair of conjugate force poles, and N is the number of such pairs.

Clearly, the function W can emerge here from the inversion of the 'normalized' hyperelliptical integrals

$$\theta = \int_0^w \left[(1 - x^2) \prod_{i=1}^{N-1} (1 - k_i^2 x^2) \right]^{-1/2} \mathrm{d}x , \tag{43}$$

where $k_i = a/b_i, (-a, a), (-b_i, b_i)$ are the coordinates of the pair closest to the common center and those of the ith pair of poles, respectively. Notice that for an arbitrary angular orientation of axes of the conjugate pairs of force centers relative to the axis Ox_1, their coordinates b_i may also assume complex values here.

The inversion of the integral (43) is equivalent to using the generalized Jacobi functions which, unlike the usual elliptic functions in Eqn (42), will depend not on a single parameter k (its absolute value) but on several k_i at once. It is interesting to emphasize that the generalized 'hyperelliptic functions' of this kind can map 'hyperelliptical motions' along geodesics in the form of polycircular curves of order higher than four. It is worthwhile to note, therefore, that there exists a relation between motions in the 'multiplicative' and central force fields, which reveals itself due to the fact that function (40), obtained here for zero-Hamiltonian motions and for those in the field of 'multiplicative type' forces, is at the same time the solution for motions with a nonzero energy level in the field of gravitational and elastic forces.

5. Nonzero energy level motions. The Bertrand theorem

The integration of Eqn (10) for $H \neq 0$ presupposes the same sequence of actions as before, the first and most important stage being the determination of the function W from the integrability conditions for Eqns (24a) and (24b).

Let us represent them in the following form

$$h(\xi) = \frac{\partial(g\varphi)/\partial\xi}{\partial g/\partial\xi} , \tag{44a}$$

$$h(\eta) = \frac{\partial(g\varphi)/\partial\eta}{\partial g/\partial\eta} . \tag{44b}$$

We next take advantage of the fact that motions along the geodesics $\xi = \text{const}$ and $\eta = \text{const}$ are executed conserving the total energy, which can be expressed by the equations

$$\frac{\partial h(\xi)}{\partial\eta} = 0 , \qquad \frac{\partial h(\eta)}{\partial\xi} = 0 .$$

From this, substituting the right-hand sides of Eqns (44a) and (44b) and changing to differentiating on the complex plane $W(\overline{W})$ with respect to $\theta\,(\bar\theta)$, we find using Eqn (32) and the results

$$\frac{\partial g}{\partial\xi} = \mathrm{i}(\overline{W}'W'' - W'\overline{W}'') \neq 0 ,$$

$$\frac{\partial g}{\partial\eta} = (\overline{W}'W'' + W'\overline{W}'') \neq 0$$

that the following equations hold:

$$\pm\left\{ \frac{(\overline{Z}Z'' - Z\overline{Z}'')}{(\overline{W}'W'' - W'\overline{W}'')} - \frac{(\overline{Z}Z' - Z\overline{Z}')}{(\overline{W}'W'' - W'\overline{W}'')^2} (\overline{W}'W''' - W'\overline{W}''') \right\} = 0 , \tag{45a}$$

$$\pm\,\mathrm{i}\left\{ \frac{(\overline{Z}Z'' - Z\overline{Z}'')}{(\overline{W}'W'' + W'\overline{W}'')} - \frac{(\overline{Z}Z' + Z\overline{Z}')}{(\overline{W}'W'' + W'\overline{W}'')^2} (\overline{W}'W''' - W'\overline{W}''') \right\} = 0 . \tag{45b}$$

From these it follows that the existence condition common to both interrelated motions is the compatibility condition for the solutions of two equations:

$$\overline{Z}Z'' - Z\overline{Z}'' = 0 , \tag{46}$$

$$\overline{W}'W''' - W'\overline{W}''' = 0 . \tag{47}$$

Hence, under the restrictions imposed earlier, these are precisely the equations we could rely on when transferring to the complex plane W.

Let us first consider the second of these equations because it has no relation to external forces while, on the other hand, immediately placing restrictions on the geometric shape of possible trajectories.

The structure of Eqn (47) clearly shows that it is satisfied identically only if

$$W' = a\mu'(\lambda, \theta) ,$$

$$W''' = \frac{a}{\lambda^2}\, \mu'''(\lambda, \theta) ,$$

with the function μ being **one** of the particular solutions of the equation

$$\mu'' + \lambda^2 \mu = 0 , \tag{48}$$

where a and λ are certain constants. Notice that it makes no sense to ascribe to the last constant any other value than unity because, otherwise, a particular solution of the equation (48):

$$\mu = \cos(\lambda\theta)$$

is reduced to the form

$$\mu = \cos\theta$$

anyway by simply calibrating the variable whose real part has no relation initially to any angular coordinate $\gamma \in [0, 2\pi]$.

Thus, we see that for motions with a nonzero energy level our earlier restriction (12) has a consequence that any other *curvilinear* motions — apart from elliptic ($\xi = \text{const}$) and hyperbolic ($\eta = \text{const}$) — are altogether ruled out [7].

It is easy to see that Eqns (46) and (47) are entirely similar in structure. Therefore, transferring to the complex plane W, we make the same substitutions as before for the solutions to Eqn (46), namely

$$Z \equiv FW' = q\zeta(\sigma, \theta)\,, \qquad Z'' = \frac{q}{\sigma^2}\,\zeta''(\sigma, \theta)\,, \tag{49}$$

where q is a constant, and ζ is *one* of the particular solutions of the equation

$$\zeta'' + \sigma^2\zeta = 0\,.$$

In other words, as a function ζ we can take either

$$\zeta = \cos\sigma\theta\,, \tag{50}$$

or

$$\zeta = \sin\sigma\theta\,, \tag{50$'$}$$

and the particular choice, as well as the determination of the constant σ, should be made in a unique way from Eqn (49), because its left-hand side contains the image of the potential function F.

At the same time, we have already seen that the function W is none other than $a\cos\theta$. This restriction [which emerged from Eqn (47)] imposes conditions both of its own and as fully determined by Eqn (49) on the selection of possible combinations of the functions ζ and F. Indeed, from all that has been said it follows that Eqn (49) can now be represented in the form

$$-aF\sin\theta = q\zeta(\sigma, \theta)\,. \tag{51}$$

From this last equation and Eqns (50), (50$'$) it follows that the only possible choice for F is a *trigonometric* function whose multiplication by $\sin\theta$ using trigonometric rules would produce the right-hand side of Eqn (50) or (50$'$) with a value of σ admissible here.

We see that the restrictions placed on and hence the algorithm of selecting the function F supporting elliptic and hyperbolic motions with a nonzero energy level are here related exclusively to the transformation rules for trigonometric functions — i.e. to elementary operations — which, from the methodical point of view, is undoubtedly a success of this approach.

After these remarks it is not difficult to recognize that there exist generally only three trigonometric functions F capable of satisfying Eqn (51) with the right-hand side in the form of the function (50) or (50$'$). Two of these functions are related to the Newtonian potential (with poles at the foci of the ellipse):

$$F_1 = \frac{1}{\sqrt{W-a}} = \frac{1}{\mathrm{i}\sqrt{a}\sqrt{1-\cos\theta}} = \frac{-\mathrm{i}}{\sqrt{2a}\,\sin(\theta/2)}\,,$$

$$F_2 = \frac{1}{\sqrt{W+a}} = \frac{1}{\sqrt{2a}\,\cos(\theta/2)}\,,$$

and for them we have from Eqn (51) that

$$q_1 = \mathrm{i}\sqrt{2a}\,, \qquad \zeta_1 = \cos\frac{\theta}{2}\,, \qquad \sigma = \frac{1}{2}\,,$$

$$q_2 = -\sqrt{2a}\,, \qquad \zeta_2 = \sin\frac{\theta}{2}\,, \qquad \sigma = \frac{1}{2}\,.$$

The third function, quite naturally, is none other than the image of the Hookian potential for elastic forces:

$$F_3 \equiv W = a\cos\theta\,, \qquad q_3 = -\frac{a^2}{2}\,, \qquad \zeta_3 = \sin 2\theta\,, \qquad \sigma = 2\,.$$

Thus, employing the vortical form of the equation of motion, we have proven the well-known Bertrand theorem already mentioned above, because no functions F other than those indicated are capable of satisfying the trigonometric equation (49) for any value of the parameter σ except $\sigma = 1/2, 2$.

Comparison of the above proof with that given by H Alfven (and whose reproduction takes several pages in Ref. [9]) shows that, in addition to the criterion of simplicity, the vortical equation approach is clearly superior in laconism.

Finally, as another illustration of the applicability of the method, we consider one special case of the periodic motion of a point in the field of two gravitating centers at rest.

The general problem of determining the mechanical trajectories for a point moving in the field of such forces was first considered by Euler and, later, in more detail, by Legendre (see book [13]). According to the classification made by C Charlier [13], the motion to be discussed belongs to the category of periodic 'planetary motions'. These motions, unlike periodic 'satellite motions' (exemplified by motions along the geodesics $u_2 = \text{const}$, shown in Fig. 2), involve both attracting centers and proceed along elliptic orbits.

From the practical point of view, this question is of interest in, for example, including the effect of Earth's oblateness on the motion of artificial satellites [5]. Other practical examples of the application of this question to celestial mechanics are given in the book by Charlier [13].

The most important initial stage of integrating the vortical equation here, too, reduces to elementary trigonometric operations. Indeed, it is necessary first to prove that the motions will be elliptic, but they will not, however, obey the second Kepler law as the integration of formula (38a) will later show.

To solve the first problem, it is necessary again to address the integrability conditions (44a) and (44b) and to substitute the potential function

$$\varphi = -\frac{p_1}{r_1} - \frac{p_2}{r_2} \tag{52}$$

into them; here, $p_{1,2} = GM_{1,2}$, $M_{1,2}$ are the masses of the two gravitating centers located at the foci of the ellipses (see Fig. 1), and G is the gravitational constant.

[7] Equation (48) does not rule out the trivial case of straight-line motions ending with a particle's falling onto the center if we take $\lambda^2 = -1$, but we do not take them into account here.

Here, the quantity $g\varphi$ in Eqns (44a) and (44b) has the form

$$g\varphi = -(Z_1\bar{Z}_1 + Z_2\bar{Z}_2)\,,$$

where

$$Z_{1,2} = F_{1,2}W'\,,$$

and $F_{1,2}$ are the images of the Newtonian potential of the gravitating poles:

$$F_{1,2} = \sqrt{\frac{p_{1,2}}{W \pm a}}\,.$$

It can be shown that the result of differentiating Eqns (44a) and (44b) in the complex plane again reduces to compatibility conditions for the solutions of two equations, which this time are

$$\begin{aligned} &\sum_{i=1,2} (\bar{Z}_i Z_i'' - Z_i \bar{Z}_i'') = 0\,, \\ &\bar{W}'W''' - W'\bar{W}''' = 0\,, \end{aligned} \qquad (46^*)$$

and which differ only in the first equation from the pair (46), (47).

For periodic 'planetary motions' ($W = a\cos\theta$) we have

$$Z_1 = F_1 W' = -\sqrt{2ap_1}\,\sin\frac{\theta}{2}\,,$$
$$Z_2 = F_2 W' = \mathrm{i}\sqrt{2ap_2}\,\cos\frac{\theta}{2}\,.$$

Equation (46^*) for this kind of motion is obeyed due to the fact that each of the terms vanishes separately, which was to be shown.

6. Conclusions

Let us summarize briefly the main results of this work and examine likely prospects for the future.

First, with regard to the theoretical justification for applying the Euler equation to the motion of point objects we note the following.

The fact that this equation (or, more precisely, its left-hand kinetic side) is the limiting form of the spacelike part of a certain 4-vector resulting from relativistically invariant operations on the momentum density 4-vector (which has clear physical meaning) — this fact, in our opinion, brings some clarity to the situation. It is another matter that in doing so we should account for the fact that real objects are by no means pointlike. And questions also remain as to, for example, how and under what conditions the spatially nonuniform distribution of mass density over volume will contribute to their inertia. We should also account for the effects due to the fact that real solids are not at all absolutely rigid, etc.

Generally, the continual form of the equation of motion of a 'substance' as a starting point for developing equations of motion for objects with a finite and not necessarily constant volume can impart some momentum to this field. Problems of this kind — ones involving continual methods, but in application to the Lagrange and Hamiltonian versions of mechanics — have already been treated in the scientific literature (see, e.g., monograph [14]).

With regard to the general purposes and specific results of the present work, we should say the following.

Conceptually, this work does not belong to the group of works that traces the hydrodynamics of Hamiltonian systems in either the narrow or broad sense of the word (variational calculus). This can already be seen from the fact that this work is concerned with the development of potential (rather than vortical) integration methods and that it treats the Euler equation directly as the basic equation of motion. One may even say, therefore, that this work is in some opposition to the methods of Hamiltonian mechanics — but only in the sense that hydrodynamic equations are not fundamental there [1, 2].

By and large, of course, the problem boils down to what kind of conveniences this or that method offers and precisely where. In this sense, even though the results of our work relate to a narrow enough area of research, they show that at least in this narrow area the direct integration of the Euler equation yields something.

In this connection, there arises yet another stratum of comparisons because the direct integration of the stationary Euler equation state is in fact the concern of the hydrodynamics of steady-state flows.

Comparisons of this kind show that a difference in the subject-matters creates significant distinctions both in the methodology and in the physical interpretation of the results.

Indeed, unlike hydrodynamics, in which the subject-matter is a *really* existing velocity field in a fluid and which, in addition, always involves pressure, the integration of the Euler equation as applied to the motions of point objects (with pressure 'turned off') offers the function W as the solution. The domain of definition of this function is the *entire* plane of motion of the object, which itself occupies no volume. Therefore, the function W maps not real motions but rather the everywhere dense, infinite families of *virtual* trajectories, which acquire reality as a result of 'reduction', occurring immediately after the particular boundary conditions of motion in space and time have been specified.

Note that independent of the object being compared the convenient and simple manner in which the key features of particle motions are expressed here is due primarily to the fact that it proves possible to formulate — and to apply to the vortical equation of motion — an *invariant* version of the potential integration method. True, thus far this can only be achieved for a certain class of motions, a class which by no means includes finite nonclosed trajectories with a nonzero energy level. But then this class includes — and in a quite systematic way — those motions which are supported by 'multiplicative potentials'. It is of great importance here that all these motions are, in principle, integrable and might be represented in terms of generalized hyperelliptic functions obtainable by inverting the corresponding Abel integrals.

In methodical terms, the sum total of this work is the highlighting of the vortical form of the equation of motion as a means for developing an invariant version of potential integration methods for the motions of point objects.

Here is what V I Arnol'd said on this item in his book [5], referring also to the Hamilton–Jacobi method — which is potential in nature and which is well known to be rooted in the nonvortical form of the equation of motion:

"Turning to the apparatus of generating functions, I can say that it is depressingly noninvariant and heavily relies on the coordinate structure in the phase space $\{(P, Q)\}$. We must therefore employ the apparatus of partial derivatives — but this is an object whose very name suggests ambiguity.''

Transfer to the complex plane clearly does solve this problem, and here the method itself starts looking for ways to find the most convenient coordinate system — thereby displaying its invariance. This, however, is achieved at the expense of employing the condition (12) which narrows the class of motions being examined. To change to other classes of motions, while remaining within the 'vortical theory' framework, one should abandon the condition (12). That this is, in principle, possible may be implied by an example of centrally symmetric circular motions — an example which, on the one hand, is already not subject to the condition (12), but, on the other hand, from the viewpoint of integrating Eqns (24a), (24b), is too elementary to deserve special treatment.

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