METHODOLOGICAL NOTES

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Formulating Fermat's principle for light traveling in negative refraction materials

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Abstract. The formulation of Fermat's principle for electromagnetic waves traveling in materials with a negative refractive index is refined. It is shown that a formulation in terms of the minimum (or extremum) of wave travel time between two points is not correct in general. The correct formulation involves the extremum of the total optical length, with the optical length for the wave propagation through left-handed materials taken to be negative.

A team of researchers from the University of California at San Diego has reported [1, 2] the practical realization of composite materials whose unusual electrodynamic properties are adequately explained by regarding their refractive index as being negative. The negative values of the refractive index can be used to characterize isotropic materials in which the phase and group velocities are antiparallel. Such a situation occurs in particular for materials where both the permittivity ε and permeability μ are negative scalars [3].

This implies that in the expression for the refractive index n, namely

$$n = \pm\sqrt{\varepsilon\mu}\,, \tag{1}$$

the plus (minus) sign corresponds to the positive (negative) values of both ε and μ.

It should be noted that the very antiparallelism of phase and group velocities has long been realized in, for example, electronic devices, and is usually described by the term 'a negative group velocity'. However, devices of this kind cannot be characterized by definite — and less still scalar — values of ε and μ.

Materials with a negative refractive index behave in a manner not entirely consistent with some basic laws of optics — in particular, with the way Snell's law and the Doppler and Cherenkov effects usually manifest themselves [3]. For example, a light ray refracted on the border between vacuum and a medium with a negative refractive index $n < 0$ is deflected in a direction opposite to that for the usual $n > 0$ case.

This situation is fully described by Snell's law

$$\sin(\varphi) = \sin(\psi)\,n\,, \tag{2}$$

if the value of n is taken to be negative.

The Doppler and Cherenkov effects also exhibit a reversal. It was also shown in Ref. [3] that convex and concave lenses in a sense 'interchange' and that an ordinary plane-parallel plate can, under certain conditions, function similar to a convergent lens, as shown in Fig. 1. A complete enough treatment of such a plane lens is given in Ref. [5]. All these phenomena can be described by the well-known formulas of geometrical optics, provided the sign of the refractive index n in them is taken negative.

Although paper [3] provided a fairly complete description of electrodynamical properties of negative refraction materials, no such materials were available to experimenters at the time. It was suggested, in particular, that magnetic semiconductors might be candidates for both negative ε and μ, but this idea did not lead to success — primarily due to technological problems in manufacturing such materials.

The breakthrough came only recently, when a UC-San Diego team [1, 2] synthesized an artificial composite material which may have widely different — and, in particular, negative — effective values of ε and μ over a range of centimeter wavelengths. In this composite, the matrix is a dielectric material, and the inclusions are the metallic elements with the size less than the wavelength of the incident radiation. These elements are realized in two varieties. The first variety are simply thin metallic rods. These are in fact antennae interacting with the electric component of the incident field. The second variety are miniature rings with openings — actually small antennae interacting with the magnetic component of the field. Both these varieties of

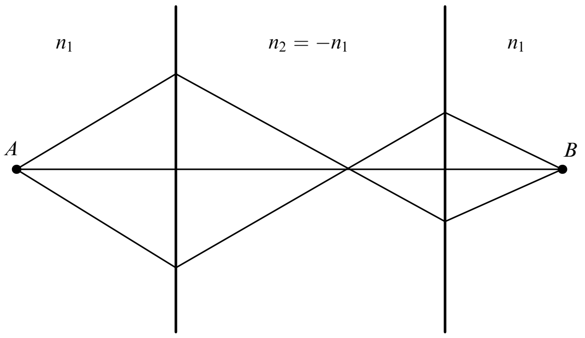

Figure 1. Light transmission from object A to image B through a plane-parallel lens made of the material with a negative refractive index $n_2 = -n_1$.

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elements are arranged spatially in a specific, strictly regular pattern and form a kind of lattice whose period is also less than the incident radiation wavelength.

By proper choice of all the parameters of such a lattice one can obtain an artificial medium with various — in particular, negative — refractive indices. The experiment conducted in Ref. [2] provided convincing evidence that the refraction of an electromagnetic wave at the border between vacuum and such a composite medium obeys Snell's law with n negative. Hence the basic ideas of work [3] may be considered experimentally proven.

Publications [1, 2] have spurred studies on the properties and practical applications of negative refraction materials. The reader is referred to the UC-San Diego website for scanning the results of these studies [4].

Materials that possess a negative refractive index were called 'left-handed materials' in Refs [1, 2]. While this term sounds good in English, it has no agreeable Russian translation, so that in Russian it is perhaps better to call the materials with $n < 0$ as 'negative refraction materials' (or NRMs in the abbreviated form). Ordinary materials with $n > 0$ can accordingly be denoted as PRMs. The term NRM corresponds to some extent to the term 'negative refraction', widely used currently in English-language publications on the subject.

The advent of NRMs did not in fact lead to any bizarre phenomena but, as already noted, in the case of NRMs some optical laws look different than in the PRM case we are accustomed to.

To the sequence of the phenomena and effects listed in Ref. [3] one more important law, or more precisely the principle, Fermat's principle, should be added. Although the formulations of this principle vary in the literature[1], all of them can be reduced to the following two.

1. Light travels from one point in space to another along the shortest path possible. (Here the term 'shortest' implies that the time spent on the travel along this path is a minimum.)

2. Light travels from one point in space to another along the trajectory corresponding to the minimum length of optical path. (The term 'optical path' refers to the distance which light would cover in vacuum during the time it takes the light to travel from one point in space to the other or, alternatively, the whole number of wavelengths in a given path.)

Apart from these differences, it is often stated — quite correctly — that the term 'minimum' in relation to the path length or travel time should in some cases be replaced by the term 'maximum' or even simply 'extremum'.

Returning now to the two formulations above, it is clear that both of them are equally valid for light traveling only

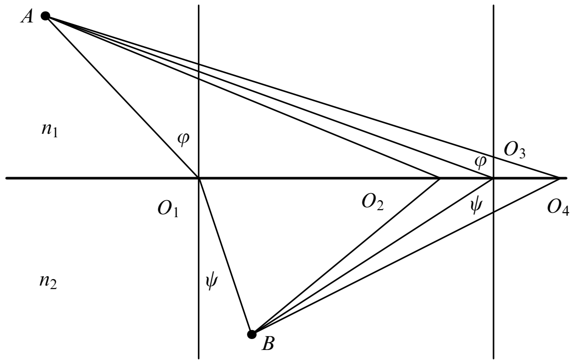

Figure 2. Light transmission from point A to point B through the plane boundary between two media with the refractive indices n_1 and n_2. *Case* $n = n_2/n_1 > 0$: light travels along the path AO_1B. *Case* $n = n_2/n_1 < 0$: light travels along AO_3B; AO_2B and AO_4B are virtual light travel paths for this case.

through PRMs, but none applies to the case of light traveling through a NRM, at least somewhere along its path. This is easily seen by referring to Fig. 2 which shows possible paths for a ray crossing a plane boundary between two media with the refractive indices n_1 and n_2, respectively.

If both n_1 and n_2 are positive (i.e. both media are composed of PRMs), then the ray takes the path AO_1B, and the angles φ and ψ satisfy Snell's law

$$\sin(\varphi)\, n_1 = \sin(\psi)\, n_2\,. \tag{3}$$

The optical length L for this path is

$$L = n_1(AO_1) + n_2(O_1B)\,. \tag{4}$$

It is readily seen that Snell's law (1) holds if and only if the variation δL of the optical path (4) vanishes:

$$\delta L = \delta\{n_1(AO_1) + n_2(O_1B)\} = 0\,. \tag{5}$$

Note that for the actual path AO_1B the quantity L itself is then a minimum and has a positive sign.

If both n_1 and n_2 are negative (NRMs reside both above and below the interface), the course of the rays will be the same as in the preceding case but with one important difference. In the first case, the wave vector in either medium is aligned with the rays, i.e. is directed from A to B, whereas in the second case the wave vector is opposite to the direction of the rays, i.e. goes from B to A [3]. The optical length L in this case turns out to be negative, and is a maximum for the actual path AO_1B.

Both of the above cases corresponded to the positive value of the quantity $n = n_2/n_1$, the relative refractive index for the second medium with reference to the first.

The situation changes significantly when the quantity $n = n_2/n_1$ becomes negative. This happens when PRM is located on one side of the interface, while NRM on the other. In this case a ray from the first medium to the second will take the path AO_3B, and the angles φ and ψ will again obey Snell's law, but this time for the negative value of ψ. For the actual travel path the relation

$$\delta L = \delta\{n_1(AO_3) + n_2(O_3B)\} = 0 \tag{6}$$

[1] The formulation of Fermat's principle as given in the *Encyclopedic Dictionary of Physics* (Moscow: Sov. Entsiklopediya, 1983), article "Fermat's principle": "The simplest form of Fermat's principle is the statement that light traveling between two points in space takes the path along which its travel time is shorter than along any other path connecting these points". In the *Encyclopaedia Britannica* (http://www. britannica.com) Fermat's principle is understood as the "statement that light traveling between two points seeks a path such that the number of waves (the optical length between the points) is equal, in the first approximation, to that in neighboring paths. Another way of stating this principle is that the path taken by a ray of light in traveling between two points requires either a minimum or a maximum time" (see the article "Fermat's principle").

will hold. This relation replaces Eqn (5) on substitution of a negative value of n for the NRM, for example, $n_2 < 0$. This ensures the extremum optical length condition for the actual path of light, the optical length being defined in terms of the refractive index with account for its sign. However, in this case one cannot assert *a priori* that the actual path of light corresponds precisely to the maximum or the minimum of the optical length. The type of extremum in this case depends on the geometry of the problem and on the specific values of n_1 and n_2.

A very important point to be made is that the actual path from point A to point B is not the shortest one in terms of the time of travel. The virtual path AO_2B will be travelled by light in less time, and AO_4B in longer time than it takes light to travel the actual path AO_3B.

Thus, the formulation of Fermat's principle in terms of the time of light travel is not correct in general. The correct formulation of this principle must necessarily be given in terms of the extremum of the length of the optical path:

the actual path of light travel in a medium corresponds to a local extremum of the length of the optical path.

The term 'local' here suggests that the problem may involve a number of possible optical paths such that conditions (3) and (5) are fulfilled for them.

The length L of optical path for light travel between points A and B, in the most general case where the refractive index n varies from point to point, is given by the integral

$$L = \int_A^B n \, \mathrm{d}l \, . \qquad (7)$$

Because the quantity n entering into Eqn (7) can also be negative, it is clear that the length L of the optical path (which is actually the eikonal) may have any sign and any value. It is negative if the light passes through an NRM only. Sometimes it can be zero. This is precisely the case for the length of the optical path between the object and its image in a lens made of NRM (see Fig. 1) [5]. The concept of an optical path is related to the total phase incursion along the course of a ray and is determined by the refractive index n — a quantity which determines the phase velocity of light, not its group velocity. The frequently used definition of the length of the optical path in terms of the time of light travel in fact identifies the phase velocity with the group velocity — something which is incorrect in general and leads to grave errors in the case of NRMs in particular.

The difference between the group and phase velocities for the case of the lens shown in Fig. 1 leads to one more effect. The times of light travel along the central ray and peripheral rays turn out to be different in this device, even though the optical lengths for all the rays are the same. As a result, ultrashort light pulses will be distorted when passing such a lens. Usual lenses made of PRMs do not have (ideally) this drawback.

It is important to note that for many specialists the essential conclusions of Refs [2, 3] are hard to swallow. For example, Ref. [6] argues that the laws of refraction are different for the phase and group velocities. The result, the authors believe, is that the phase and group velocities are at a certain angle to each other in an NRM. The authors are not embarrassed by the fact that the existence of such an angle is a characteristic feature of optically anisotropic media, which cannot in principle be described by a scalar refractive index n. Pendry and Smith [7] explained very convincingly this misconception.

The reader interested in the properties of NRMs is referred to the Los Alamos electronic archive (http://www.lanl.gov/) where some more papers on this subject were posted on 1 July 2002 in the cond-mat section. An abridged version of the present paper can be found on the Internet [8, 9].

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