# Supersymmetry - $\mathbf{3 0}$ years ago 

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#### Abstract

A personal history of the creation of the first fourdimensional supersymmetric model is presented. It also covers the author's memoir of his cooperation with Yu A Gol'fand, which pioneered supersymmetry studies. The preprint of the author's 1971 paper is published as the appendix.


These notes, based on my talk at the Conference 'Thirty Years of Supersymmetry' held in Minneapolis on 13-15 October 2000 , tell the story of how the first supersymmetric model in four dimensions came into being [1]. I briefly review the articles on this matter published in the early 1970s and discuss some minor results which have been mentioned only in my thesis for Candidate of Physicomathematical Sciences [2]. Also, some misconceptions and missed opportunities will be addressed.

Early in 1968 I graduated from the Physics Department of the M V Lomonosov Moscow State University (MSU), with first class honors and already a co-author of two publications $[3,4]$ on the possible effects of the violation of space-charge symmetry in atomic nuclei. I dreamt of solving larger-scale problems, which of course implied post-graduate studies, but Yuriĭ Mikhaĭlovich Shirokov, who had directed my graduate work, failed to secure me a post-graduate position either at the V A Steklov Mathematical Institute of the USSR Academy of Sciences where he worked or at the MSU Physics Department where he taught. He therefore recommended me to Yuriĭ Abramovich Gol'fand, then at the Theoretical Department of the P N Lebedev Physics Institute of the USSR Academy of Sciences.

That was the place where the prominent Soviet physicists - I E Tamm, V L Ginzburg, and A D Sakharov - worked in the late 60 s and early 70s. Yu A Gol'fand was only at the stage of writing his thesis for doctorate at the time. He was a rather short person which, allied with his quick gait and warm and friendly smile, somehow leveled off our age and status differences. He showed me several permutation relations between the momentum and angular momentum operators and some spinors and explained me that the consistency of these relations was verified by the Jacobi identities. It was these permutation relations which laid the foundation for what six years later came to be known as 'supersymmetry' [5].

My task at the first stage of research was to establish whether the proposed algebra was a unique one or whether there was an alternative to it. To get an answer required

[^0]solving a system of equations in algebra structural constants which resulted from the Jacobi identities. I restricted myself to a set of four complex spinor charges and came to the conclusion that there are four varieties of such algebras, two of which are now known as $N=1$ and $N=2$ superalgebras, and in two others the momentum does not commute with spinor charges (analogous to the de Sitter algebra). The results obtained late in 1968 were not published until later $[6,7]$ because their physical relevance was not at all clear at the time. Of these algebras, the simplest, $N=1$, algebra was selected for further analysis.

Twice a week, before the Theoretical Department seminar was to start, I reported to Gol'fand on my progress in calculations and a discussion of my results followed. There were textbooks on my work table on various aspects of group theory applications in physics, but as to recent original papers on the proposed subject, I found none. Nor was I lucky later on, when reviewing the appropriate scientific literature for my thesis. (In this connection, I have recently read with great interest the historical study by M S Marinov in the book [8]). Whether Gol'fand knew about F A Berezin's and some other similar works or not, I do not know. He might probably consider me to be a person to whom he had communicated all the essential information required for the work. His whole behavior seemed to show to me - and possibly to some others as well - that he and I were the only people whose work really mattered. Still, as far as I saw it, he was in equable relations with all his colleagues from the Theoretical Department. He was very fond of joking and laughing - sometimes with the gaps in my professional training as the target.

The central problem I had to solve was to establish the relation between quantum field theory and the algebra to be constructed. Nobody knew whether such a relation existed at all and if it did, whether the algebra representations would be finite. The reason I keep saying 'algebra' rather than 'group' is as follows. Gol'fand and I considered a group with Grassmann variables as parameters and were able to establish how supercoordinates transform under spinor translations. It did not occur to us, however, that we could expand the superfield in a power series of the Grassmann variables and thus establish its relation to the set of ordinary fields entering the supermultiplet. Accordingly, the superfield - as a concept that did not after all yield a mechanism for constructing a superinteraction - was not mentioned in our early publications and was only reflected in my thesis. In retrospect, the route to success proved less elegant and hence more laborconsuming. My idea was to search for the algebra representations in terms of the creation and annihilation operators for the particles involved.

As long as we were dealing with algebra operators alone, our main concern had been with properly treating the gamma matrices and carefully changing the sign at certain places in the Jacobi identities when working with anticommutators. Having changed to field theory, it was very hard to get used to
the fact that one and the same multiplet contained both bosons and fermions, and harder still to conceive how they transformed one into another as a result of spinor translations. Surely there was nowhere to crib from, unlike in my practice on Marxist-Leninist philosophy examinations. "Not God but man makes pot and pan", Gol'fand used to say to encourage me. The only thing which was clear from the very first was that the particles in a supermultiplet all have the same mass - not much to be upbeat about either.

Finally, late in 1969, the superspin operator and two irreducible algebra representations (a chiral multiplet with spins 0 and $1 / 2$, and a vector multiplet with spins $0,1 / 2$, and 1) were constructed and the following general properties of the representations were established:

- First, in each irreducible representation the maximum spin differs from the minimum one by no more than unity. This was found by expanding the group (not superfield) operator as a Taylor series in the Grassmann variables of spinor translations. The expansion was truncated - hence the representations were finite!
- Second, the numbers of boson and fermion degrees of freedom in each multiplet are equal, so that the vacuum energies of the boson and fermion states add up to zero.

While I was very keen to publish the results immediately, Gol'fand did not feel they would attract much attention. Besides, he did not perhaps wanted me to waste time on preparing a manuscript. The result was that the foundations of the supersymmetric free-field theory were issued only in 1971 [9] (see Preprint No. 41 below). In 1969, in the meantime, I addressed myself to the construction of the supersymmetric interaction between the multiplets obtained.

At the time, a task was assigned to post-graduate students to analyze and assess the inventions that were mailed to the Academy of Sciences. In a rocket design I had to examine within that program, a liquid flowed within a closed tube inside the rocket: along a straight line in one direction and in a zigzag manner in the opposite, and the idea was that a force arising from relativistic effects might propel the rocket. The mere argument that this is at odds with the conservation of momentum would of course be lost on the author who was most definitely unaware of the law. It took me quite a while to debug the poor devil's reasoning, and to repeat his mistake and sin against a certain conservation law has been my perpetual fear since then.

While the psychological barrier due to the Fermi-Bose medley was overcome, some technical difficulties arose. The point is that in the interaction construction technique which was employed the spinor generators of an algebra were expressed not only in terms of the second power of the fields, but also of the third, after which a Hamiltonian commuting with the generators of spinor translations was calculated algebraically. The result was a set of equations in which the number of unknown constants was about a dozen depending on the field combinations.

Today, I routinely solve systems of hundreds of nonlinear equations, and computer problems only arise when I go beyond a thousand - a far cry from 1970, when my ballpen and rolled sheets of waste draft blueprints were all I had at my disposal. It is an easy matter solving an exercise problem when you know for sure that a solution exists and all you need is to find it . But in those days, when one of my equations was inconsistent with the others, I did not know whether my arithmetic was wrong or whether the problems had no solution at all. In the meantime, the completion date of my
post-graduate studies was approaching. It was time to think about the defense of my thesis, and there was the problem of employment hovering in the distance, so I considered it as nothing short of a miracle when I saw that the constants calculated from some of my equations satisfy the others. Furthermore, it turned out that the unknown constants of the fourth powers of the fields in spinor translation operators could be set equal to zero. The system of equations was solved, and the first supersymmetric interaction - now known as massive supersymmetric electrodynamics - was constructed.

So setting the results in order began. I thought of one big comprehensive paper, but Gol'fand decided otherwise. Because preparing a paper for print usually took quite a time in our journals, his idea was to publish a short note [1] in Pis'ma Zh. Eksp. Teor. Fiz. (JETP Lett.). He therefore cut my manuscript ruthlessly to squeeze it into the standard letter volume, allocating the cuts to other publications [6, 7, 9]. At the same time I was busy constructing the self-action for a vector multiplet. I succeeded in constructing only the trilinear part of the interaction, however, which is the reason why this result was reflected only in my dissertation [2].

The title of the dissertation was so weird that it took the academic secretary a great deal of effort to read it at the thesis defense session in September 1971. This was probably in line with how poorly we ourselves understood the problem: we believed, mistakenly, that constructing a supersymmetric interaction by expanding group generators as a power series in the coupling constant was dictated by the specifics of the supersymmetry. Nevertheless, summarizing what had been done in those three years under the constant support from Gol'fand I feel safe to say that I did solve the problem he had formulated - that is, I did demonstrate a quantum-field realization of supersymmetry for a specific case. Admittedly, neither our seminar reports nor the defense of the dissertation generated much interest in scientific circles. As a matter of fact, even the huge bunch of flowers my supervisor brought to the banquet was not for me, a pathfinder and hero - it was for my wife who stayed at home to take care of our infant in arms.

Now there was no working position for me at FIAN, and it was not until late in 1971, after half a year of job hunting, that I was given a job in the Physics Department at the All-Union Institute of Scientific and Technical Information. It was quite a problem for me to find time there for continuing my research into the Fermi - Bose symmetry - a situation which even prompted one theoretician to compare me with Einstein. As to Gol'fand, he found himself in even greater straits at the time: indeed, his dismissal from FIAN's Theoretical Department in 1972 left him only casual earnings to live on. Such - shall I say - recognition of our work showed how much the ideas of new symmetry were at odds with the scientific world outlook of those then at the helm of Russian - then Soviet - physics. It was only in 1989 that things changed, and upon the recommendation of L B Keldysh - head of FIAN's Theoretical Department and of FIAN as a whole - we were awarded the I E Tamm Prize for Theoretical Physics by the Russian Academy of Sciences for our work on supersymmetry.

I wonder now, what was it that made Gol'fand address the problem in the first place? Clearly not any specific experimental finding - the constancy of the speed of light, for example, or the approximate equality of the neutron and proton masses. Nor was it about removing any internal
inconsistencies in a theory, like those involved in developing GR or posing the Weinberg-Salam model. What actually inspired him I think were the numerous achievements which the use of various symmetry principles had produced in physics throughout the twentieth century altogether.

With the thesis defended and the employment issue resolved, I was now able to take a closer look at what had been done and it occurred to me that the cancellation of vacuum energy singularities - the result I had demonstrated for free fields - might have some relevance to interacting fields as well. Knowing the field interaction constants involved in the supersymmetric electrodynamics model had been obtained, it was not difficult to see that in the one-loop approximation the boson field mass singularities are not quadratic but - like those in a fermion field - also logarithmic. This result did not inspire any enthusiasm in Gol'fand: he considered it to be just a freak of chance. Perhaps his intuition let him down, or maybe other - totally nonscientific problems - filled his head at the time. The analysis of the effects of spinor translations on the $S$ matrix led me to the conclusion that the cancellation of singularities was no accident and should occur in higher approximations as well. But - alas - the singularities did not disappear completely. If only the nonrenormalized model would become renormalizable! Unfortunately, the model building technique I used did not allow me to study nonrenormalizable models.

In 1974, at a suggestion of I V Tyutin, I read with great interest the paper of Wess and Zumino [5]. The linear representations resulting from the presence of auxiliary fields, and the covariant derivatives with respect to Grassmann variables - all this was very beautiful and allowed easy construction of a superinteraction. Salam and Strathdee [10] integrated over the Grassmann variables - a procedure whose postulates were formulated by F A Berezin in 1965 [11] - thereby putting both types of superfield arguments once and for all on equal footing. Using this technique, I developed two models, one with Abelian [12] and the other with non-Abelian [13] massive vector fields, and showed them to be renormalizable. I was also able to show that renormalizability admits an alternative proof, one using the standard mechanism of spontaneous violation of gauge symmetry. The question of whether there is some link between supersymmetry and high-energy physics remained an open one.

Stiff competition and the absence of support from any quarters had the consequence that after publishing these two papers I actually gave up my work on supersymmetry and started seeking a research area of my own. The most important lessons I drew from my cooperation with Gol'fand were: few assumptions, a simple construction, and nontrivial - if at first sight unrealistic - results. On the other hand, my PC skills now allow me to address problems of uncertain solvability. From what I know, it is a highly risky business to seek a black cat in a dark room - especially if there is none there. Still, what I am currently doing is the numerical analysis of the Born-Infeld model of electrodynamics for a finite-size membrane with a bounding string - a situation in which the divergences of electrical energies cancel those of magnetic ones. This cancellation is not the goal in itself, though. It turns out that, on the one hand, a certain combination of mass, magnetic moment and electrical charge is independent of two unknown dimensional constants of the model; on the other hand, this combination relates to a wellknown observable quantity - the fine-structure constant. As
of now, the solutions I obtain converge rather poorly as the number of lattice cells is increased - indicating perhaps that the problem has no solutions; or that I should find a better way to stretch the lattice over space; or, finally, that some unknown symmetry in the equations of motions should be found to enable a preliminary analytical treatment.

In concluding, I would like first of all to express my gratitude to the organizers of the 'Thirty Years of Supersymmetry' Conference for their financial support and for the opportunity they gave me to present my own view of the events happened thirty years ago. And finally, and most importantly, I see Yuriĭ Abramovich Gol'fand as a person who not only taught me a profession but who also developed in me a taste for the risky business of following unbeaten tracks.

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## Irreducible representations of the bispinor generator extension of the algebra of Poincaré group generators

## E P Likhtman

## 1. Introduction

In Ref. [1], a specific extension of the algebra $\mathcal{P}$ of Poincaré group generators by introducing spinor translation generators $W_{\alpha}$ and $\bar{W}_{\beta}$ has been considered:

$$
\begin{align*}
& {\left[M_{\mu v}, M_{\sigma \lambda}\right]_{-}=\mathrm{i}\left(\delta_{\mu \sigma} M_{v \lambda}+\delta_{v \lambda} M_{\mu \sigma}-\delta_{\mu \lambda} M_{v \sigma}-\delta_{v \sigma} M_{\mu \lambda}\right),} \\
& {\left[P_{\mu}, P_{v}\right]_{-}=0, \quad\left[M_{\mu v}, P_{\lambda}\right]_{-}=\mathrm{i}\left(\delta_{\mu \lambda} P_{v}-\delta_{v \lambda} P_{\mu}\right),} \\
& {\left[M_{\mu v}, W\right]_{-}=\frac{\mathrm{i}}{4}\left[\gamma_{\mu}, \gamma_{v}\right] W, \quad \bar{W}=W^{+} \gamma_{0},}  \tag{1a}\\
& {[W, \bar{W}]_{+}=\gamma_{\mu}^{+} P_{\mu}, \quad[W, W]_{+}=0, \quad\left[P_{\mu}, W\right]_{-}=0,}  \tag{1b}\\
& \stackrel{ \pm}{\gamma_{\mu}}=\stackrel{ \pm}{s} \gamma_{\mu}, \quad \stackrel{ \pm}{s}=\frac{1}{2}\left(1 \pm \gamma_{5}\right), \quad \gamma_{5}^{2}=1,
\end{align*}
$$

where we have omitted spinor indices. Hereinafter it will be understood that $d_{\mu} d_{\mu}$ stands for $d_{0} d_{0}-d_{1} d_{1}-d_{2} d_{2}-d_{3} d_{3}$. In the same paper a specific implementation of this algebra, with a Hamiltonian describing the interaction of quantized fields, was built up. That example has shown that algebra (1) imposes severe limitations on the types of interaction between quantized fields. In constructing the example, two linear irreducible representations of algebra (1) were employed, whose derivations were not given in the paper. In this work we present these derivations as well as constructing - and examining the properties of - other representations of algebra (1).

## 2. Space of states and invariant subspaces

In order to identify in which space the representations of algebra (1) act, note that algebra $\mathcal{P}$ is a subalgebra of that defined by Eqns (1). Accordingly, any representation of algebra (1) is also a representation of $\mathcal{P}$, and the spaces where these representations act coincide. However, an irreducible representation of algebra (1) will be a reducible one for $\mathcal{P}$. This reducible representation breaks down into a series of irreducible representations, each of which may occur several times. We will label these irreducible representations with numbers $\chi$ to distinguish between them.

Of physical interest are those of the representations of algebra (1) which are reduced along irreducible representations of $\mathcal{P}$ and are characterized by mass and spin. Therefore, the basis vector of the space in which to build representations of algebra (1) can be written as

$$
\begin{equation*}
\left|\chi, p_{\lambda}, j, m, \chi\right\rangle \tag{2}
\end{equation*}
$$

where $\varkappa$ is the mass, $p_{\lambda}$ is the three-dimensional momentum, $j$ is the spin, $m$ the projection of the spin onto axis $z$, and the number $\chi$ labels irreducible representations of an algebra $\mathcal{P}$.

In a space with basis vectors (2) there are subspaces invariant under the action of the operators of algebra (1). According to the Schur lemma [2], in order to find invariant subspaces one must search out invariant operators which, by definition, commute with all the operators of algebra (1). It is easily seen that the operator $P_{\mu}^{2}$ has this property. Therefore, a space whose basis vectors correspond to particles with one and the same mass $x$ will be an invariant subspace. The spins of vectors in this invariant subspace cannot be all the same because the square of the spin operator is not an invariant operator of algebra (1):

$$
\left[\Gamma_{\mu}^{2}, W\right]_{-} \neq 0, \quad \Gamma_{\mu}=\frac{1}{2} \varepsilon_{\mu \nu \lambda \sigma} M_{\nu \lambda} P_{\sigma}
$$

An invariant operator in algebra (1) will be the operator $D_{\mu}^{2}$ introduced according to

$$
D_{\mu}=\Gamma_{\mu}+\frac{1}{2}\left(\bar{W} \gamma_{\mu} W-\frac{P_{\mu} P_{v}}{P_{\sigma}^{2}} \bar{W} \gamma_{v} W\right), \quad P_{\sigma}^{2}=\varkappa^{2}>0 .
$$

There seem to be no other invariant operators in algebra (1). We do not know a priori, exactly which vectors (2) form the basis of an irreducible representation of algebra (1) since we do not know the properties of the operator $D_{\mu}^{2}$. However, in addition to their having equal masses, it is also possible to argue that the difference between the maximum and minimum spins in the irreducible representation of algebra (1) does not exceed unity. In other words, there are no more than three different spins in an irreducible representation of


Facsimile of the title page of Preprint No. 41.
algebra (1). Otherwise, acting by operators of algebra (1) successively on a state vector with spin $j_{1}$, one could obtain a vector whose projection onto the state vectors with spin $j_{2}$ for $\left|j_{1}-j_{2}\right|>1$ would be nonzero. To see that this is not possible, we notice that the successive action of operators of algebra (1) can generally be presented as the action of a polynomial in powers of the elements of algebra (1), which in view of the permutation relations (1) we represent as
where $C_{\mu v, \ldots, \ldots \ldots}^{\alpha, \ldots ; \ldots}$ are numerical coefficients.
The product of an arbitrary number of operators $M_{\mu \nu}$ and $P_{\lambda}$ does not change a spin of the state. The number of operators $W$ (and $\bar{W}$ ) in each term may only equal unity or two: the product of a larger number of operators $W$ (and $\bar{W}$ ) vanishes because of the anticommutation relations in Eqn (1). For the same reason, the product of two operators, $W_{\alpha} W_{\beta}$, is nonzero only for $\alpha \neq \beta$ - but this operator does not change a spin of the state. And only a single operator $W$ (or $\bar{W}$ ) changes the spin by a half, whereas the product $W_{\alpha} \bar{W}_{\beta}$ can change the spin by unity. Thus we conclude that an invariant subspace can contain only the spins $j, j+\frac{1}{2}, j+1$. To build an irreducible representation of algebra (1), it is necessary to find the transition matrix elements of the operators of algebra (1), taken between these states.

## 3. Equations for reduced matrix elements

The matrix elements of the ordinary translation operator are written out clearly as

$$
\begin{align*}
& \left\langle\varkappa, p_{\lambda}, j, m, \chi\right| P_{\mu}\left|\varkappa, p_{\lambda}^{\prime}, j^{\prime}, m^{\prime}, \chi^{\prime}\right\rangle \\
& \quad=\left(\delta_{\mu 0} \sqrt{\varkappa^{2}+p_{\lambda}^{2}}+\delta_{\mu \lambda} p_{\lambda}\right) \delta\left(p_{\lambda}-p_{\lambda}^{\prime}\right) \delta_{j j^{\prime}} \delta_{m m^{\prime}} \delta_{\chi \chi^{\prime}} \tag{3}
\end{align*}
$$

where $\lambda=1,2,3$. Also, the operator $M_{\mu \nu}$ will have its usual form. We shall be interested in the matrix elements of the operators $W_{\alpha}$ and $\bar{W}_{\beta}$, diagonal in $\chi$ and $p_{\lambda}$ in view of relations (1). It is readily seen that the operators $\bar{s} W$ and $\bar{W}_{S}^{+}$obey only trivial permutation relations, so we restrict the discussion to those representations ${ }^{1}$ for which $\bar{s} W=\bar{W} s=0$. Without any loss of generality we choose a $\gamma$-matrix representation with a diagonal $\gamma_{5}$. The operators $W_{\alpha}$ and $\bar{W}_{\beta}$ then become two-component. Next, knowing spinor transformation properties under Lorentz transformations and assuming that $x>0$, we go over to the frame of reference with $p_{\lambda}=p_{\lambda}^{\prime}=0$. In this coordinate system we can apply the Wigner-Eckart theorem [4], according to which

$$
\begin{aligned}
& \langle\varkappa, 0, j, m, \chi|{ }^{+} W_{\alpha}\left|\varkappa, 0, j^{\prime}, m^{\prime}, \chi^{\prime}\right\rangle \\
& =\left(\begin{array}{ccc}
j & \frac{1}{2} & j^{\prime} \\
m & \alpha & -m^{\prime}
\end{array}\right)(-1)^{j^{\prime}-m^{\prime}} \sqrt{(2 j+1)\left(2 j^{\prime}+1\right)}\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle,
\end{aligned}
$$

$$
\langle\varkappa, 0, j, m, \chi| \bar{W} \bar{s}_{\beta}\left|\chi, 0, j^{\prime}, m^{\prime}, \chi^{\prime}\right\rangle
$$

$$
=\left(\begin{array}{ccc}
j^{\prime} & \frac{1}{2} & j  \tag{4}\\
m^{\prime} & \beta & -m
\end{array}\right)(-1)^{j-m} \sqrt{(2 j+1)\left(2 j^{\prime}+1\right)}\langle j \chi| f^{+}\left|j^{\prime} \chi^{\prime}\right\rangle,
$$

where

$$
\left(\begin{array}{ccc}
j & \frac{1}{2} & j^{\prime} \\
m & \alpha & -m^{\prime}
\end{array}\right)
$$

are the Wigner symbols, $\left|j-j^{\prime}\right|=1 / 2,\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle$ are the reduced matrix elements, and

$$
\sqrt{(2 j+1)\left(2 j^{\prime}+1\right)}
$$

is a convenient normalization factor. This representation (4) secures correct commutations with the momentum operator and three-dimensional rotation operators. In order to satisfy the remaining relations of algebra (1), we substitute Eqns (3) and (4) into Eqn (1b) and take account of the following formulas for summing $3 j$-symbols over spin projections [4]:

$$
\begin{aligned}
& \sum_{m^{\prime}}(-1)^{j^{\prime}+m}\left(\begin{array}{ccc}
j_{1} & j_{2} & j^{\prime} \\
m_{1} & m_{2} & m^{\prime}
\end{array}\right)\left(\begin{array}{ccc}
j_{3} & j_{4} & j^{\prime} \\
m_{3} & m_{4} & -m^{\prime}
\end{array}\right) \\
& \quad=\sum_{J M}(-1)^{2 j_{4}+J+M}(2 J+1) \\
& \quad \times\left\{\begin{array}{ccc}
j_{1} & j_{2} & j^{\prime} \\
j_{3} & j_{4} & J
\end{array}\right\}\left(\begin{array}{ccc}
j_{3} & j_{2} & J \\
m_{3} & m_{2} & M
\end{array}\right)\left(\begin{array}{ccc}
j_{1} & j_{4} & J \\
m_{1} & m_{4} & -M
\end{array}\right),
\end{aligned}
$$

where

$$
\left\{\begin{array}{ccc}
j_{1} & j_{2} & j^{\prime} \\
j_{3} & j_{4} & J
\end{array}\right\}
$$

[^1]is the $6 j$-symbol. The result is as follows
\[

$$
\begin{align*}
& \sum_{J M j^{\prime} \chi^{\prime}}(-1)^{2 j^{\prime \prime}+J+M}(2 J+1) \\
& \times\left\{\begin{array}{ccc}
j & j^{\prime} & J \\
\frac{1}{2} & \frac{1}{2} & j^{\prime}
\end{array}\right\}\left(\begin{array}{ccc}
\frac{1}{2} & \frac{1}{2} & J \\
-\beta & \alpha & M
\end{array}\right)\left(\begin{array}{ccc}
j & j^{\prime \prime} & J \\
m & -m^{\prime \prime} & -M
\end{array}\right) \\
& \times(-1)^{m+\alpha-j^{\prime}} \sqrt{(2 j+1)\left(2 j^{\prime \prime}+1\right)\left(2 j^{\prime}+1\right)^{2}} \\
& \times\left(\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f^{+}\left|j^{\prime \prime} \chi^{\prime \prime}\right\rangle\right. \\
& \left.\quad+(-1)^{J-j+j^{\prime \prime}}\langle j \chi| f^{+}\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f\left|j^{\prime \prime} \chi^{\prime \prime}\right\rangle\right)=\chi \delta_{j j^{\prime \prime}} \delta_{m m^{\prime \prime}} \delta_{\chi \chi^{\prime \prime}} \tag{5}
\end{align*}
$$
\]

(A similar formula is obtained by substituting Eqn (4) into the relation $[W, W]_{+}=0$.) We now proceed by using the values of the $6 j$-symbols [4]:

$$
\begin{aligned}
& \left\{\begin{array}{ccc}
\begin{array}{cc}
j_{1} & j_{2} \\
\frac{1}{2} & j_{3} \\
\frac{1}{2} & j_{3}-\frac{1}{2}
\end{array} j_{2}+\frac{1}{2}
\end{array}\right\} \\
& \quad=(-1)^{j_{1}+j_{2}+j_{3}}\left[\frac{\left(j_{1}+j_{3}-j_{2}\right)\left(j_{1}+j_{2}-j_{3}+1\right)}{\left(2 j_{2}+1\right)\left(2 j_{2}+2\right)\left(2 j_{3}\right)\left(2 j_{3}+1\right)}\right]^{1 / 2}, \\
& \\
& =(-1)^{j_{1}+j_{2}+j_{3}}\left[\frac{\left(j_{1}+j_{2}+j_{3}+1\right)\left(j_{2}+j_{3}-j_{1}\right)}{\left(2 j_{2}\right)\left(2 j_{2}+1\right)\left(2 j_{3}\right)\left(2 j_{3}+1\right)}\right]^{1 / 2}
\end{aligned}
$$

Equation (5) then becomes

$$
\begin{align*}
& \sum_{j^{\prime} \chi^{\prime}}\left(\frac{2 j^{\prime}+1}{2}\right)\left(\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f^{+}\left|j^{\prime \prime} \chi^{\prime \prime}\right\rangle\right. \\
& \left.\quad+\langle j \chi| f^{+}\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f\left|j^{\prime \prime} \chi^{\prime \prime}\right\rangle\right)=\varkappa \delta_{j j^{\prime \prime}} \delta_{\chi \chi^{\prime \prime}},  \tag{6a}\\
& \sum_{j^{\prime} \chi^{\prime}}(-1)^{j^{\prime}}\left(\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f^{+}\left|j \chi^{\prime \prime}\right\rangle\right. \\
& \left.\quad-\langle j \chi| f^{+}\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f\left|j \chi^{\prime \prime}\right\rangle\right)=0 \quad(\text { for } j \neq 0),  \tag{6b}\\
& \sum_{j^{\prime} \chi^{\prime}}\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle\left\langle j^{\prime} \chi^{\prime}\right| f\left|j^{\prime \prime} \chi^{\prime \prime}\right\rangle=0 \\
& \quad\left(\text { except for the case } j=j^{\prime \prime}=0\right) . \tag{6c}
\end{align*}
$$

These are precisely the desired equations whose solution will give us the explicit form of the operators ${ }^{+} W$ and $\bar{W} \bar{s}$. Notice that the solutions to Eqns (6) describe the representations in which the invariant operator $D_{\mu}^{2}$ may or may not be a multiple of the unit operator, i.e. they generally describe the reducible representations of algebra (1).

## 4. The number of particles in algebra (1) representations, and some solutions of Eqns (6)

First of all, we will employ formula (6) to derive restrictions on the number of particles in a representation of algebra (1). For this purpose we multiply Eqn (6a) by $(-1)^{2 j}(2 j+1) / 2$ and sum the product over $j=j^{\prime \prime}$ and $\chi=\chi^{\prime \prime}$. The left-hand side of the equation then vanishes. To make sure that this is
the case, it is sufficient to permute the cofactors under the spur sign in the second term and to take advantage of the fact that $(-1)^{2 j}=(-1)^{2 j^{\prime}+1}$ [see formulas (4)]. Then the second term will differ from the first only in sign. The right-hand side of Eqn (6a) also must be zero, leading to the following restriction on the number $n_{j}$ of particles of spin $j$ in an algebra (1) representation:

$$
\begin{equation*}
\sum_{j}(-1)^{2 j}(2 j+1) n_{j}=0 . \tag{7}
\end{equation*}
$$

In relativistic quantum field theory, it is well known [5] that reducing the operator of a particle free energy to its normal form gives rise to an infinite term interpreted as vacuum energy. It is also recognized that the sign of this term is different for particles obeying Bose and Fermi statistics. According to Eqn (7), an algebra (1) representation involves particles with different statistical properties, with boson and fermion states always present in equal numbers. Then it follows that in any algebra (1) representation the infinite positive energy of the boson states is canceled by the infinite negative energy of the fermion states.

After these preliminary remarks we proceed directly to the solution of Eqns (6). Let us try to find representations involving particles with only two spin states. In this case $j^{\prime}$ in formulas (6) assumes only one value, and there is in fact no summation over $j_{\prime^{\prime}}^{\prime}$. Making use of this fact, we multiply Eqn (6b) by $(-1)^{j^{\prime}}\left(2 j^{\prime}+1\right) / 2$ and add equation (6a) to the result. In the right-hand side of the resulting equation we will have a nonsingular matrix operating in the space of the variable $\chi$ (at $j, j^{\prime}$ and $j^{\prime \prime}$ fixed). The matrix on the left will be nonsingular only when $j \neq 0$ [see Eqn (6b)], so that a two-spin algebra (1) representation can involve only spins 0 and $1 / 2$. In this case the simplest solution of the system (6) with $n_{0}=2$ $(\chi=1,2)$ and $n_{1 / 2}=1(\chi=1)$ is easily shown to be as follows

$$
\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle=\sqrt{\chi}\left(\begin{array}{cc|c}
0 & 0 & 1  \tag{8}\\
0 & 0 & 0 \\
\hline 0 & 1 & 0
\end{array}\right),
$$

where the matrix on the right acts on the state

$$
\left(\begin{array}{l}
a \\
b \\
- \\
c
\end{array}\right),
$$

with the amplitudes $a$ and $b$ describing particles with spin 0 , and $c$, with $1 / 2$.

For representations with three spins, viz. $j, j+1 / 2$, and $j+1$, the smallest spin $j$ is arbitrary. The simplest solution for $n_{j}=1(\chi=1), n_{j+1 / 2}=2(\chi=1,2)$, and $n_{j+1}=1(\chi=1)$ can be written in a form analogous to formula (8):

$$
\langle j \chi| f\left|j^{\prime} \chi^{\prime}\right\rangle=\sqrt{\chi}\left(\begin{array}{r|rr|r}
0 & 0 & 1 & 0  \tag{9}\\
\hline 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline 0 & 0 & -1 & 0
\end{array}\right) .
$$

Representations (8) and (9) are irreducible. This can be seen without even calculating the eigenvalues of the invariant operator $D_{\mu}^{2}$. Suffice it to note that no representations with a fewer number of particles satisfy the necessary condition (7). Whether there are irreducible representations of algebra (1) other than those we have found is a question which needs further consideration.

## 5. Second quantized relativistic representations of algebra (1)

In this section we will show how representations (8) and (9) that operate in the space with basis vectors (2) can be transformed to their relativistically covariant form operating in the space of the occupation numbers. Algebra operators in such representations must be expressed in terms of second quantized free fields with equal masses but different spins. Because a relativistically invariant equation for spin- $1 / 2$ particles describes both particles and antiparticles, it is necessary that antiparticles also be introduced into representation (7). The algebra operators will then be expressed in terms of non-Hermitian free scalar fields $\varphi(x), \omega(x)$ and the spinor field $\psi_{1}(x)$. Let us show that the operator

$$
\begin{equation*}
W^{0}=\stackrel{+}{s} W^{0}=\frac{1}{\mathrm{i}} \int\left[\varphi^{*}(x) \stackrel{\leftrightarrow}{\partial}_{0}^{+} \stackrel{+}{\psi} \psi_{1}(x)+\omega(x) \stackrel{\leftrightarrow}{\partial}_{0} \stackrel{+}{s} \psi_{1}^{\mathrm{c}}(x)\right] \mathrm{d}^{3} x \tag{10}
\end{equation*}
$$

(where the superscript ' 0 ' indicates that the operator is bilinear in field operators, and ' $c$ ' denotes charge conjugation) satisfies the permutation relations (1) ${ }^{2}$. Thus, for example, one finds

$$
\begin{align*}
& {\left[W^{0}, \bar{W}^{0}\right]_{+}} \\
& =\mathrm{i} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\varphi^{*}(x){\stackrel{\leftrightarrow}{\partial_{x_{0}}}}_{\gamma_{\mu}}^{+} \mathrm{i}_{x_{\mu}} D(x-y) \overleftrightarrow{\partial}_{y_{0}} \varphi(y)\right] \\
& +\mathrm{i} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\omega(x){\left.\stackrel{\leftrightarrow}{\partial_{x_{0}}}{ }_{\gamma}{ }_{\mu} \mathrm{i}_{x_{\mu}} D(x-y) \overleftrightarrow{\partial}_{y_{0}} \omega^{*}(y)\right]}\right. \\
& +\mathrm{i} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\bar{\psi}_{1}(y) \bar{s} \times \overleftrightarrow{\partial}_{y_{0}} D(y-x){\stackrel{\leftrightarrow}{\partial_{x_{0}}}}^{+} \stackrel{+}{s} \psi_{1}(x)\right] \\
& +\mathrm{i} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\bar{\psi}_{1}^{\mathrm{c}}(y) \bar{s} \times{\stackrel{\leftrightarrow}{\partial_{y_{0}}}} D(y-x){\stackrel{\leftrightarrow}{\partial_{x_{0}}}}^{+} \psi_{1}^{\mathrm{c}}(x)\right] \\
& =-\int \mathrm{d}^{3} x\left[\varphi^{*}(x) \mathrm{\partial}_{\mu} \stackrel{\leftrightarrow}{\partial}_{0} \varphi(x)\right] \times \stackrel{+}{\gamma}_{\mu} \\
& -\int \mathrm{d}^{3} x\left[\omega^{*}(x) \mathrm{\partial}_{\mu} \stackrel{\leftrightarrow}{\partial}_{0} \omega(x)\right] \times \stackrel{+}{\gamma}_{\mu} \\
& +\frac{i}{2} \int \mathrm{~d}^{3} x\left[\bar{\psi}_{1}(x) \stackrel{\leftrightarrow}{\partial}_{0} \bar{\gamma}_{\mu} \psi_{1}(x)\right] \times \stackrel{+}{\gamma}_{\mu} \\
& +\frac{\mathrm{i}}{2} \int \mathrm{~d}^{3} x\left[\bar{\psi}_{1}^{\mathrm{c}}(x) \stackrel{\leftrightarrow}{\partial}_{0} \bar{\gamma}_{\mu} \psi_{1}^{\mathrm{c}}(x)\right] \times{ }^{+}{ }_{\mu} \\
& =\int \mathrm{d}^{3} x T_{\mu 0}(x) \times \stackrel{+}{\gamma}_{\mu} . \tag{11}
\end{align*}
$$

In doing the calculations, the Fierz identities, equations of motion, and permutation relations for free fields have been used. Note also that the energy - momentum tensor $T_{\alpha \beta}$ is not generally symmetric in the case of a spinor field. If, however, one of the indices is zero, then we obtain
$T_{\mu 0}=T_{0 \mu}$,
and the integral in the right-hand side of Eqn (11) becomes the energy - momentum operator for the fields $\varphi(x), \omega(x)$, and $\psi_{1}(x)$. The remaining relations in the set (1) are proved in a similar fashion. The action of the operator $W^{0}$ on field
${ }^{2}$ The operator $W$ is defined to within a phase factor (see Ref. [3]).
operators reduces to a linear transformation of these fields. This transformation can be written schematically as follows

$$
\begin{aligned}
& \varphi(x) \rightarrow \psi(x) \rightarrow \omega(x) \rightarrow 0, \\
& \omega^{*}(x) \rightarrow \psi^{\mathrm{c}}(x) \rightarrow \varphi^{*}(x) \rightarrow 0 .
\end{aligned}
$$

We now proceed to generalize representation (9) to include the case of quantized fields. We restrict our considerations to the case in which the smallest spin $j$ is zero, and two spin- $1 / 2$ particles may be considered coupled by the charge conjugation operation. Then the operators of algebra (1) in this representation are expressed in terms of the Hermitian scalar field $\chi(x)$, the Hermitian transverse vector field $A_{\mu}(x)$, and the spinor field $\psi_{2}(x)$. This irreducible representation may be distinguished from the irreducible representation (10) by the mass of the particles, and must differ by the eigenvalues of the invariant operator $D_{\mu}^{2}$. The operator $W^{0}$ in this representation has the form
$W^{0}=\stackrel{+}{s} W^{0}=\frac{1}{\mathrm{i} \sqrt{2}} \int\left[\chi(x) \stackrel{\leftrightarrow}{\partial_{0}} \stackrel{+}{s} \psi_{2}(x)+A_{\mu} \stackrel{\leftrightarrow}{\partial}_{0}{ }_{\gamma}{ }_{\mu} \psi_{2}(x)\right] \mathrm{d}^{3} x$.

The verification of formula (12) is done along the same lines as in the case of formula (10):

$$
\begin{align*}
& {\left[{ }^{+} W^{0}, \bar{W}^{0-}{ }_{s}\right]_{+}} \\
& =-\frac{1}{2 \mathrm{i}} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\bar{\psi}_{2}(y) \bar{s} \times \overleftrightarrow{\partial}_{y_{0}} D(y-x){\stackrel{\leftrightarrow}{\partial_{x_{0}}}}^{\stackrel{s}{s} \psi_{2}(x)}\right] \\
& +\frac{1}{2 \mathrm{i}} \iint \mathrm{~d}^{3} x \mathrm{~d}^{3} y\left[\psi_{2}(y){ }_{\gamma}{ }_{\mu} \times \overleftrightarrow{\partial}_{y_{0}}\left(\delta_{\mu \nu}+\frac{1}{\mu^{2}} \partial_{y_{\mu}} \partial_{y_{v}}\right)\right. \\
& \left.\times D(y-x){\stackrel{\leftrightarrow}{x_{0}}}_{x_{\mu}} \stackrel{+}{\mu}^{\prime} \psi_{2}(x)\right] \\
& -\frac{1}{2} \int\left[\chi(x) \stackrel{\leftrightarrow}{\partial_{0}} \partial_{\mu} \chi(x)\right] \mathrm{d}^{3} x \times \stackrel{+}{\gamma}_{\mu} \\
& -\frac{1}{2} \int\left[A_{\mu}(x) \overleftrightarrow{\partial}_{0} \partial_{\alpha} A_{\nu}(x)\right] \mathrm{d}^{3} x \times \stackrel{+}{\gamma}_{\mu} \bar{\gamma}_{\mu} \stackrel{\rightharpoonup}{\gamma}_{\mu}=\stackrel{+}{\gamma}{ }_{\mu} P_{\mu}, \tag{13}
\end{align*}
$$

where $\mu \neq 0$ is the mass of the fields $\chi(x), A_{\mu}(x)$, and $\psi_{2}(x)$. In this representation, the action of the operator $W^{0}$ on free fields can be patterned schematically as follows

$$
\psi_{2}^{\mathrm{c}(x)} \searrow_{A_{\mu}(x)}^{\nearrow} \begin{aligned}
& \chi(x) \\
& \nearrow
\end{aligned} \psi_{2}(x) \rightarrow 0
$$

In the intermediate results involved in the derivation of formula (13), the field mass $\mu$ occurs in the denominator and hence cannot be set equal to zero. A passage to the case of zero mass is performed by abandoning the condition of a transverse character of the vector field and changing to the diagonal pairing:

$$
\left[A_{\mu}(x), A_{v}(y)\right]_{-}=-\frac{1}{\mathrm{i}} \delta_{\mu v} D(x-y) .
$$

The field $\psi_{2}(x)$ in this case becomes two-component ( ${ }_{s}^{+} \psi_{2}=0$ ), the first term is no longer needed in derivations leading to Eqn (13), and there is therefore no need to introduce a scalar field $\chi(x)$. For $\mu=0$, the operator $W^{0}$
takes the form

$$
\begin{equation*}
W^{0}=\stackrel{+}{s} W^{0}=\frac{1}{\mathrm{i} \sqrt{2}} \int\left[A_{\mu}(x) \stackrel{\leftrightarrow}{\partial_{0}} \dot{\gamma}_{\mu} \psi_{2}(x)\right] \mathrm{d}^{3} x . \tag{14}
\end{equation*}
$$

The analysis of the properties of nonzero-mass representations in Sections 2-4 has shown that the numbers of the fermion and boson states in an algebra (1) representation are equal, implying that the operator $P_{\mu}^{0}$ is automatically represented in its normal form. This is also seen from the fact that the action of the operators $W^{0}$ and $\bar{W}^{0}$ on vacuum always gives zero and that

$$
P_{\mu}^{0}=\operatorname{Sp}\left(\bar{\gamma}_{\mu}\left[W^{0}, \bar{W}^{0}\right]_{+}\right) .
$$

Therefore, representation (14) also possesses this property, and the massless vector and spinor particles can only reside in two states with opposite spirality.

## 6. Conclusions

We have found several irreducible representations for algebra (1), in which ordinary fields unite into certain multiplets. The question arises whether these multiplets can be identified with some observable particles. The main difficulty in answering this question is that the particles in the multiplet all have the same mass, while at the same time differing in spin. At present, therefore, algebra (1) and its realizations should be viewed as a certain Hamiltonian formulation of quantum field theory.

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[^1]:    ${ }^{1}$ Desisting from this requirement requires introducing an indefinite metrics [3].

