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Correspondence between supersymmetric Yang – Mills and supergravity theories

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Contents

Abstract. The AdS/CFT correspondence establishes a relationship between supersymmetric gravity (SUGRA) on anti-de Sitter (AdS) space and supersymmetric Yang – Mills (SYM) theory, which is a conformally invariant field theory (CFT). The AdS space is the solution of the Einstein – Hilbert equations with a constant negative curvature. Why is this relationship important? What kind of relationship is this? How does one find it? The purpose of this paper is to answer these questions. We try to present the main ideas and arguments underlying this relationship, starting with a brief sketch of 'old' string theory results and proceeding with the definition of D-branes and a description of their main features. A demonstration of the discussed correspondence and arguments in it its favor conclude the paper.

1. Introduction

A string description of Yang – Mills (YM) theory has been a long standing problem in quantum field theory [1]. The arguments are as follows. The mathematical description of any phenomenon requires some exactly solvable approximation to be found and some small parameter to exist which one can expand over in order to approach the real situation. In the case of YM theory, a suitable approximation at large energies appears to be in terms of free vector particles. They carry quantum numbers taking values in the adjoint representation of a non-Abelian gauge group and the small parameter in question is the coupling constant g^2 [1].

However, there is a problem in this description because quantum effects cause the coupling constant g^2 to grow as one approaches large distances or small energies. Furthermore, at some distance scale the description in terms of the fundamental YM variables becomes invalid due to singularities in the perturbative theory [1]. As a result, it is unclear how to pass to low energies in YM theory. Thus the question appears: What is the approximation to YM theory that can be applicable at any energy?

The most promising approach to this problem is to consider SU(N) YM theory as $N \to \infty$ [1]. In this limit the YM perturbation series drastically simplifies [2] and the only graphs that survive look like 'triangulations' of a sphere. This is one of the hints [1] suggesting that there could be a string description of YM theory in this limit in the form of a two-dimensional theory representing these 'triangulations'. The graphs that contribute to the 'triangulations' represent a power series expansion in $g^2 N$ (which is taken to be finite as $N \to \infty$) rather than simply in powers of the YM coupling constant g^2 [2]. At the same time all graphs having topologies of torus and spheres with more than one handle are suppressed by powers of $1/N^2$. Here $1/N^2$ appears to be the small parameter over which one can expand to approach the real situation.

Why does description in terms of string theory seem to be preferable? The point is that string theory has a very well

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developed and powerful apparatus for calculating the amplitudes of various processes [1, 3, 4] and, moreover, there are hopes to solve it.

In the case of ordinary YM, nobody has yet succeeded in finding such a string description, but a considerable progress in conformally invariant supersymmetric YM (SYM) theories has recently been made [5 – 8]. It is worth mentioning that the string description of conformal YM theories is of pure academic interest since due to the conformal invariance the dynamics of these theories is known at all distances. However such a string description can potentially reveal some features of string theory for the ordinary YM.

There are several non-anomalous and self-consistent string theories which satisfy the supersymmetry (SUSY) in the target space — the space where a string evolves. The target space should be ten-dimensional, since otherwise there is no well developed apparatus for calculating the superstring amplitudes [1, 3, 4]. At the same time, the string world-sheets are two-dimensional universes swept out by the strings during their time evolution.

There are an infinite number of ways to excite the world-sheet theory to give different quantum states of the string. Each of them looks like a particle living in the target space. Among these particles, there are a finite number of massless ones, while all other particles have masses of the order of the string tension, which is usually taken to be very high. So, at distances larger than the characteristic string scale (exactly when the strings appear as point-like objects) only the massless particles survive which are described by a field theory in the target space rather than by string theory.

Among the massless *closed* string excitations, there is a symmetric tensor particle, which, due to the symmetry properties of string theory, has exactly the same number of the degrees of freedom as a graviton. The only large-scale theory (containing the lowest powers of derivatives of the fields) that could describe the graviton is Einstein – Hilbert gravity in the target space. As can be shown rigorously [1, 3, 4], it is this theory (interacting with other massless string excitations) that follows from string theory at large distances. At the same time, in the case of superstring theory one obtains SUGRA at large distances.

It is also possible to obtain SYM interacting with SUGRA by including open strings in the theory along with closed ones. This is because the massless excitation of *open* string theory is a vector particle which has the proper number of physical polarizations to be a gauge boson.

Bearing this in mind, one could say that there is a string theory for four-dimensional SYM. In the situation under consideration, superstring theory provides a regularization of SYM theory. In fact, superstring theory is finite and valid at any distances, while at large distances it leads to a theory containing SYM. But this is unsatisfactory because at the characteristic string scale, when we get such a superstring description of SYM, we also have to deal with quantum gravity, and the dimensionality of space-time is ten rather than four.

Fortunately, new ways have recently been found to add open string sectors to the closed ones. They lead to new ways of coupling SYM to SUGRA. To find them one has to add a stack of N D-branes to the closed string theory in such a way that it respects SUSY. The D-branes are multi-dimensional sub-manifolds of the target space on which open strings terminate [9], while closed strings can still live in the bulk of the target space. So, in the world-volume of a stack of N D-branes one gets a $\mathrm{U}(N)$ SYM theory at low energies [10], while in the bulk the standard SUGRA is valid.

Thus, the strings that could describe SYM theory are attached to our four-dimensional world (D3-brane world-volume) while fluctuating in the bulk of the target space [5, 9]. To specify the string description, one has to find the geometry in which the strings fluctuate, and to do this one must probe the D3-brane from the outside, i.e. from the bulk. At the same time, the theory as seen by an observer traveling further and further away from the D-brane could change uncontrollably, since we do not know the full dynamics of string theory. To overcome this difficulty, one has to force the D-branes to respect some part of the SUSY transformations of superstring theory. Let us explain why.

In quantum field theory and statistical mechanics, when one goes from small to large distances, it is necessary to average over all fluctuations in the theory with wavelengths smaller than the distance scale in question. This could lead to a change of parameters in the theory. For example, a charged source placed in a plasma is screened because the opposite charges to the source are attracted, whereas the charges of the same sign are repelled. It is the simplicity of the system that allows us to predict how the charge of the source will vary as one approaches it. However, in YM theory the situation is more complicated. In fact, how the charge of the theory varies with respect to the distance is known only up to some low-energy scale. As a result, a proper low-energy description of strong interactions is still unknown. Similar things could happen in any non-linear theory, such as gravity or string theory.

The difference in the presence of SUSY is that bosonic and fermionic degrees of freedom can be exchanged in the theory [12]. It is this symmetry that causes the cancellation between the screening and anti-screening due to fermions and bosons. The latter happens only if a source respects some part of the SUSY transformations, i.e. if it is a Bogomol'nyĭ – Prassad – Sommerfeld (BPS) state [4, 11, 12]. Although not rigorously, we hope that the reader at least has a flavor of how it works.

Thus, the presence of SUSY helps one find how the geometry is curved in the D-brane background. For example, the characteristic curvature of the D3-brane is proportional to $(g^2N)^{1/4}$ in units of the string tension.

Let us now explain the new ideas that the D-branes can provide in seeking a string description of YM theory in contrast to the 'old' string theory. First, in this case one can deal directly with four-dimensional SYM theory. Second, one can vary the regularization energy scale for YM theory living on a D-brane world-volume and make it much smaller than the string one [5]. This works as follows: After a regularization, we have suppressed the information about high-frequency modes, and the high energy theory underlying the one in question should contain this information. What happens in the D-brane case is that high enough frequency modes of the fields living on the D-brane world-volume could escape to the bulk of the target space: They could create closed strings living in the bulk [13]. However, closed strings with energies smaller than the brane curvature can not escape to infinity [14] but instead they stay in the throat region — the strongly curved part of the bulk in the vicinity of the D-brane, because they do not have enough energy to climb over the gravitational potential and escape to the flat asymptotic region.

In conclusion, we only need the theory in the throat region in order to respect unitarity [5, 6, 15]. Thus, if the limit $N \to \infty$ and $g \to 0$ is taken in such a way that $g^2 N \ll 1$, we have the full string theory in the throat regularizing the SYM on the D-brane world-volume [6], since in this limit the size of the D-brane throat is very small and only string theory can apply. However, if limits $N \to \infty$ and $g \to 0$ are taken so that $g^2 N \gg 1$ the classical gravity can be used. In fact, the size of the throat is very big in this situation. This means that in this limit the string theory for SYM is described by the *classical* superstring in the throat background, i.e. there is a string description of SYM before gravity becomes quantized. Now, in the simplest situation, SYM on the brane has $\mathcal{N} = 4$ supersymmetries, hence its β-function is zero and it is conformally invariant. In the corresponding gravity description, the geometry of the throat of the brane is AdS.

This paper is organized as follows. Two chapters devoted to string theory are included for self-consistency. In Section 2 the main ideas of string theory are presented using the example of bosonic string theory. In Section 3 we proceed with the definition of type II superstring theories and review their massless spectrum. After presenting superstring theory, the notion of D-branes is introduced in Section 4 and their relation to gravity solitons [1] and to SYM theory is shown. We conclude with the AdS/CFT-correspondence. For completeness, a discussion of the BPS states is included in the Appendix.

Unfortunately, it is impossible to give the details of these subjects even in a lengthy book, so our presentation has a rather sketchy character. We hope however that it highlights the main ideas and gives some food for thought about the matter in question. We are not trying to fully review this broad subject, and our reference list is therefore far from being complete. A more or less complete set of references can be found elsewhere [16].

2. Bosonic string theory

Only the first quantized string theory [1, 3, 4] is fully constructed at present. This is the 'quantum mechanics' of string world-sheets, which are two-dimensional spaces swept by quantum strings during their time evolution inside the target space. As for a relativistic particle, the action for a relativistic string is proportional to the area of its world-sheet:

$$S \propto \int d^2\sigma \sqrt{-\det|g_{ab}|}\,, \qquad g_{ab} = \partial_a \tilde{x}_\mu \, \partial_b \tilde{x}_\mu\,, \tag{1}$$

where σ_a $(a = 1, 2)$ are coordinates on the world-sheet and $\tilde{x}_\mu(\sigma)$ $(\mu = 0, \ldots, d-1)$ are two-dimensional functions describing the embeddings of strings into a d-dimensional flat target space.

However, the action (1) is nonlinear and, hence, difficult to quantize. To make it quadratic in $\tilde{x}_\mu$, one includes a new dynamical variable in the theory — the string intrinsic metric h_{ab} [1]. In this case, string theory is described by a two-dimensional σ-model interacting with two-dimensional gravity:

$$S_{\rm st} = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-h}\, h^{ab}\, \partial_a \tilde{x}_\mu \, \partial_b \tilde{x}_\mu + \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-h}\,, \tag{2}$$

$$h = \det|h_{ab}|\,, \qquad h^{ab} = h^{-1}_{ab}\,.$$

Here α' is the inverse string tension. Usually it is taken to be much smaller than any distance scale that has so far been probed by experiments.

On the level of classical equations of motion $h_{ab} \propto g_{ab}$. Hence, the action (2) is classically equivalent to that in Eqn (1). On the quantum level, however, these two theories are different at least naively (see, however, Ref. [1]). In fact, the functional integral of theory (1) implies a summation over all possible string world sheets, i.e. over embeddings $\tilde{x}_\mu$. In contrast, in theory (2) the sum is taken over all possible metrics on each world-sheet and over the world-sheets themselves.

From now on we will be dealing with theory (2). This theory is invariant under the reparametrization transformations $\sigma_a \to f_a(\sigma)$, which represent general covariance on the string world-sheets. Using this *two-parametric* symmetry one could get rid of two components of the metric:

$$ds^2 = h^{ab}\, d\sigma_a \, d\sigma_b = \exp\left[\varphi(z, \bar{z})\right] dz \, d\bar{z}\,,$$

where $z = \exp\left[\sigma_1 + i\sigma_2\right]$. For a world-sheet with spherical topology this can be done unambiguously, while for the torus and higher topologies, this can be done only up to a complex structure [3, 4]. We do not explain the details of the complex structure, because we are not going to use this notion (except the fact that it exists) anywhere below.

After the above reparametrization, the gauge fixing our action is still invariant under the conformal transformations:

$$z \to f(z)\,, \quad \exp\varphi(z, \bar{z}) \to \left|\partial_z f(z)\right|^2 \exp\varphi(z, \bar{z})\,, \quad \partial_{\bar{z}} f(z) = 0.$$

These transformations allow us to get rid of the internal metric giving

$$S'_{\rm st} = \frac{1}{2\pi\alpha'}\int d^2 z \, \partial_z \tilde{x}_\mu \, \partial_{\bar{z}} \tilde{x}_\mu + \text{Faddeev–Popov ghost terms}\,. \tag{3}$$

We do not discuss here what the Faddeev–Popov ghosts are because we will not use this notion (except the fact that it exists) anywhere below. A more or less complete discussion of Faddeev–Popov ghosts within the string theory framework can be found in Ref. [3].

Getting rid of the metric completely as in (3) is possible only classically. In fact, after quantization of the σ-model (2) the so called conformal anomaly [1, 3, 4] appears because the conformal symmetry is broken by quantum effects. So $\varphi(z, \bar{z})$ becomes a dynamical field. It is necessary to cancel the anomaly since otherwise it is not known how to calculate string theory correlation functions [1]. Contributions to the anomaly coming from $\tilde{x}_\mu$ $(\mu = 0, \ldots, d-1)$ and from the Faddeev–Popov ghosts cancel each other if $d = 26$.

Furthermore, there is a remnant of the reparametrization invariance on string world-sheets with higher topologies, which is referred to as modular invariance. The

[1] Here and below *all* solutions of the equations of theory of gravity with a finite tension or mass are referred to as solitons.

modular transformations act on the complex structures [3, 4]. One must also respect this invariance, because otherwise there could be problems with gravitational and gauge anomalies in the target space, and hence the unitarity would be violated.

2.1 Generating functional

Only if all these symmetries are respected can one properly define the fundamental quantity of bosonic string theory:

$$Z(G,B,\Phi,T)=\sum_{\mathcal{G}=0}^{\infty} Z_{\mathcal{G}}$$
$$=\sum_{\mathcal{G}=0}^{\infty}\int[\mathcal{D}h_{ab}]_{\mathcal{G}}\prod_{\mu=0}^{d-1}\mathcal{D}\tilde{x}_\mu \exp\left\{-\mathrm{i}S_{\mathrm{st}}(\tilde{x}_\mu,h_{ab},G_{\mu\nu},B_{\mu\nu},\Phi,T)\right\}. \tag{4}$$

Here the measure $\prod_{\mu=0}^{d-1}\mathcal{D}\tilde{x}_\mu$ is as for d scalars, while $[\mathcal{D}h_{ab}]_{\mathcal{G}}$ should be defined in accordance with the reparametrization, conformal and modular invariances [1].

The sum in this formula is over the genus $\mathcal{G}$ of the string world-sheets. This is an expansion over string loop corrections which are present in addition to the aforementioned σ-model quantum corrections. If one considers only closed strings, these corrections are represented by spheres with $\mathcal{G}$ handles, otherwise they are discs with holes and handles with the total number [2] $\mathcal{G}$.

We start with a discussion of the closed bosonic string theory. In this case the action in (4) is:

$$S_{\mathrm{st}}(\tilde{x}_\mu,h_{ab},G_{\mu\nu},B_{\mu\nu},\Phi,T)$$
$$=\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\left\{\sqrt{-h}\,h^{ab}G_{\mu\nu}(\tilde{x})\,\partial_a\tilde{x}^\mu\,\partial_b\tilde{x}^\nu\right.$$
$$\left.+\,\epsilon^{ab}B_{\mu\nu}(\tilde{x})\,\partial_a\tilde{x}^\mu\,\partial_b\tilde{x}^\nu+\alpha'\sqrt{-h}\,R^{(2)}\Phi(\tilde{x})+\sqrt{-h}\,T(\tilde{x})\right\}, \tag{5}$$

where ϵ^{ab} is the completely anti-symmetric tensor in two dimensions and $R^{(2)}$ is the two-dimensional scalar curvature for the metric tensor h_{ab}.

Now we see that the dilaton's VEV Φ_∞ gives a coupling constant for the string loop expansion:

$$Z_{\mathcal{G}}\propto\exp\left\{-\mathrm{i}\,\frac{\Phi_\infty}{2\pi}\int\mathrm{d}^2\sigma\sqrt{-h}\,R^{(2)}\right\}$$
$$=\exp\left[2(\mathcal{G}-1)\Phi_\infty\right]=g_{\mathrm{s}}^{2(\mathcal{G}-1)}, \tag{6}$$

where the index 's' distinguishes the string coupling constant from that in YM theory. Furthermore, substituting $G_{\mu\nu}=\eta_{\mu\nu}$, $B_{\mu\nu}=0$, $\Phi=0$ and $T=1$ into (5), one gets the former expression (2) for S_{st}.

The physical meaning of $Z(G,B,\Phi,T)$ is that it is the generating functional for interaction amplitudes between the *smallest mass* string states. In fact, we can obtain such amplitudes by varying the functional Z over the sources G, B, Φ, and T. We are interested only in the smallest mass states because we need to find a classical limit (large distance behavior) of string theory. It is exactly this limit where we can use what is known from our world. This is the reason why we do not include any other sources, which would correspond to massive states, in the functional integral (4), (5).

[2] It should be mentioned at this point that the open string theory contains closed strings on its loop level. In fact, the annulus amplitude (the first loop correction in the open string theory) is equivalent to the cylinder amplitude (the tree level in the closed string theory). Besides, the unitarity demands that the closed string excitations should be added to the open string ones.

2.2 Low energy spectrum

Why do the operators in (4), (5) with the sources G, B, Φ, and T correspond to the smallest mass states? First, it is necessary to explain how a two-dimensional operator is related to a string state. The action of an operator on the vacuum of the conformal theory (2) excites it. If we take a particular harmonic of a source (for example, $T=\,:\!\exp\left[\mathrm{i}\,p_\mu\,\tilde{x}_\mu\right]\!:$), it looks, from the target space point of view, like a moving string in a particular quantum state. In fact, it is a plane wave inside the target space.

Furthermore, it is not necessary to modify the functional integral (4) to describe the interactions of string states. This is one of the main differences between string theory and a field theory describing particles. It relies upon two fundamental facts: First, in contrast to the particle paths, for any disconnected set of one-dimensional manifolds it is always possible to find a two-dimensional string world-sheet that includes these manifolds in its boundary. Such a world-sheet represents a Feynman graph for a string amplitude, and the components of its boundary represent the initial and final states of some process in the string theory. Second, because of the conformal symmetry, one can always 'amputate' external 'legs' in the string amplitude. More specifically, by a conformal transformation it is possible to refract the 'external legs' and to transform each point of the one-dimensional boundary into a point on the world-sheet in which the corresponding vertex operator acts.

To show why the operators in question correspond to the smallest mass excitations, let us consider an N-point correlation function [1]:

$$\mathcal{A}_N=\int\prod_{j=1}^{N}\mathrm{d}^2\sigma_j\left\langle\mathcal{O}_1\big(\tilde{x}_\mu(\sigma_1)\big)\ldots\mathcal{O}_N\big(\tilde{x}_\mu(\sigma_N)\big)\right\rangle, \tag{7}$$

where the average $\langle\ldots\rangle$ is taken using the functional integral (4) with the action (2). $\mathcal{O}_j$ are some operators with conformal weights [3] Δ_j equal to 2, so that the integrals over $\mathrm{d}^2\sigma_j$ are conformally invariant. Appropriate operators include those present in (5), such as:

$$\mathcal{O}_G=G_{\mu\nu}\,:\!\partial_z\tilde{x}_\mu\,\partial_{\bar{z}}\tilde{x}_\nu\!:\,. \tag{8}$$

$\mathcal{O}_G$ has a well defined conformal weight if $G_{\mu\nu}=f_{\mu\nu}\,:\!\exp\left[\mathrm{i}\,p_\mu\,\tilde{x}_\mu\right]\!:$, where $f_{\mu\nu}$ is some polarization from the target space point of view.

In integral (7) there is a region where $\sigma_1\to\sigma_2$ and close to it the operator product expansion (OPE) can be used:

$$\lim_{\sigma_1\to\sigma_2}\left[\mathcal{O}_i(\sigma_1)\,\mathcal{O}_j(\sigma_2)\right]\approx\sum_k C_{ijk}|\sigma_1-\sigma_2|^{\Delta_k-\Delta_j-\Delta_i}\mathcal{O}_k(\sigma_1)\,, \tag{9}$$

where the sum in the RHS runs over a basis of local operators in the world-sheet conformal theory. Using this OPE, one

[3] The definition of the conformal weight Δ_j of an operator $\mathcal{O}_j$ is: $\mathcal{O}_j(\sigma_j)=\lambda^{-\Delta_j}\mathcal{O}_j(\lambda\sigma_j)$.

finds [1]:

$$\mathcal{A}_N = \int d^2\eta \int \prod_{j=2}^{N} d^2\sigma_j \left\langle \mathcal{O}_1(\sigma_2+\eta)\,\mathcal{O}_2(\sigma_2)\ldots\mathcal{O}_N(\sigma_N)\right\rangle$$
$$\approx \sum_k C_{12k}\int^a d^2\eta\,|\eta|^{\Delta_k-4}$$
$$\times \int \prod_{j=2}^{N} d^2\sigma_j \left\langle \mathcal{O}_k(\sigma_2)\,\mathcal{O}_3(\sigma_3)\ldots\mathcal{O}_N(\sigma_N)\right\rangle$$
$$+\,\text{less singular terms} \approx \sum_k \frac{1}{\Delta_k-2}\,\mathcal{A}_3^{(k)}\mathcal{A}_{N-1}^{(k)}$$
$$+\,\text{less singular terms}\,, \tag{10}$$

where $\mathcal{A}_3^{(k)} \propto C_{12k} \propto \langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_k\rangle$ and we have taken the integral over $|\eta|$ up to a scale a that is smaller than all other distances between σ_j's.

Now we take into account that the operator $T(\tilde{x}) = {:}\exp\left[i p_\mu \tilde{x}_\mu(\sigma)\right]{:}$ has a conformal weight equal to $\alpha' p_\mu^2/2$. One can find this weight using Wick's theorem for the two-point correlation function of this operator [1] and the propagator for $\tilde{x}_\mu$ from the action (2). Furthermore, for G proportional to ${:}\exp\left[i p_\mu \tilde{x}_\mu(\sigma)\right]{:}$ we have the conformal weight for the operator (8) equal to $\alpha' p_\mu^2/2+2$. The operators from (5) have the same conformal weight provided B and Φ are proportional to ${:}\exp\left[i p_\mu \tilde{x}_\mu(\sigma)\right]{:}$. In a similar way, for other sources [not present in (5)], also taken to be proportional to ${:}\exp\left[i p_\mu \tilde{x}_\mu(\sigma)\right]{:}$, one obtains $\alpha' p_\mu^2/2+\delta_k$, where $\delta_k > 2$ due to local operators which stand near sources like the operator ${:}\partial_z \tilde{x}_\mu\,\partial_{\bar{z}} \tilde{x}_\mu{:}$ which stands behind $G_{\mu\nu}$ in Eqn (8).

Thus, in any channel where $\sigma_i \to \sigma_j$ we have on the RHS of (10) a sum over all propagators of the string excitations, each corresponding to some operator $\mathcal{O}_k$:

$$\mathcal{A}_N = \sum_k \frac{\mathcal{A}_3^{(k)}\mathcal{A}_{N-1}^{(k)}}{p_\mu^2 + 2(\delta_k-2)/\alpha'}\,. \tag{11}$$

In conclusion, there is a relation between the conformal weights of operators $\mathcal{O}_k$ and the masses of the corresponding string states $m_k^2 = 2\,(\delta_k-2)/\alpha'$. Our observation shows that T describes a tachyonic state with $m_T^2 = -p_\mu^2 = -4/\alpha'$, because $\delta_T = 0$. At the same time G, B and Φ describe massless states ($\delta_{G,B,\Phi} = 2$), while all other operators correspond to massive ones ($\delta_k > 2$).

2.3 A relation between gravity and string theory

Bearing the above considerations in mind, we can consider string theory at distances (set by G, B, Φ, and T) much bigger than $\sqrt{\alpha'}$. First, in this case one can replace separate quanta (8) by smooth fields, as in passing from photons to radio waves. Second, in this situation massive string excitations are decoupled. This means that at these scales we should obtain a field theory rather than a string theory. In fact, a free string is equivalent to infinitely many free particles: the string propagator is just an infinite sum of particle propagators (11). Hence, forgetting about massive particles reduces the sum in (11) to a finite number of the smallest mass excitations.

In this way at the scales considered and for $d = 26$ one finds [17]:

$$Z(G,B,\Phi,T) = \frac{1}{16\pi\Gamma_{\mathrm{N}}}\int d^{26}x\,\sqrt{-G}\,\exp\left[-2\Phi\right]$$
$$\times\left[\mathcal{R} + 4G^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{12}\,H^2_{\mu\nu\lambda}\right.$$
$$\left.+\frac{1}{2}\,G^{\mu\nu}\,\partial_\mu T\,\partial_\nu T + \frac{1}{2}\,m_T^2 T^2\right] + O(\alpha', \Gamma_{\mathrm{N}})\,. \tag{12}$$

This is a 26-dimensional dilaton gravity interacting with the anti-symmetric tensor $H_{\mu\nu\lambda} \propto \partial_{[\mu}B_{\nu\lambda]}$. Here $\Gamma_{\mathrm{N}} \propto g_s^2\alpha'^{12}$ is the 26-dimensional Newton's constant; from now on $O(\alpha', \Gamma_{\mathrm{N}})$ schematically represents a two-dimensional σ-model and string loop corrections. If you like, such corrections appear due to string massive modes.

Equation (12) means that in the limit $\alpha' \to 0$ (in units of the characteristic scale given by functions G, B, Φ, and T) the Z functional gives exactly the same Feynman vertices and propagators as the leading contribution in the RHS of (12). Unfortunately, this fact can be explicitly established only for the simplest background fields G, B, Φ, and T, such as the flat metric with constant fields B and Φ. Problems appear because there are no well developed methods for quantization of the non-linear σ-model (5) with arbitrary sources G, B, Φ, and T. The best that has been established so far is that the vacua in the LHS and RHS of (12) are equivalent. Indeed, the conformal invariance of the σ-model (5) imposes conditions on the sources [1, 3]: it is necessary to have vanishing β-functions for the sources G, B, Φ, and T. These conditions are nothing but equations of motions for the action (12).

There is a way to intuitively understand why one should obtain this particular action (12) from string theory. The action (5) is invariant under infinitesimal transformations of G and B fields given by:

$$G_{\mu\nu} \to G_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}\,, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\rho_{\nu]}\,, \tag{13}$$

of which the first is nothing but the general covariance of the graviton field. It is necessary (but not sufficient) to respect these invariances to maintain the unitarity of the theory. Now the goal is to find a long-range effective action for the sources in Z which would be invariant under the transformations in question. It is easy to see that the action (12) is the only low energy one which obeys these conditions and includes interactions with the dilaton Φ. The reason why '$Z = S(\text{sources})$' rather than '$Z = \exp\left[-iS(\text{sources})\right]$' is that we are dealing with *first quantized* string theory.

2.4 Open bosonic string theory

Now let us consider open bosonic string theory. To maintain Poincaré invariance in the target space, one naively (see Section 4) could think of using only the Neumann boundary conditions on the coordinates $\tilde{x}_\mu$ of the open strings.

As we have already mentioned, open string theory contains closed string theory at the loop level. Hence, open bosonic string theory contains all the same sources in its generating functional as in (4). In addition it includes sources for its own excitations. Furthermore, at the open string ends one can add quantum (Chan–Paton) numbers (indices), taking values in the fundamental representation of a gauge group.

Thus, following the same reasoning as above, the massless open string vertex operator can be found to be a path-ordered

Wilson exponent [4]:

$$\operatorname{Tr} \mathrm{P} \exp \left\{ \mathrm{i} \int_{\text{boundary}} \mathrm{d}\tau\, \hat{a}_\mu(\tilde{x})\, \partial_t \tilde{x}_\mu \right\}, \tag{14}$$

where ∂_t is a tangential derivative to the string's boundary and τ is some parameterization of the latter. The presence of the operator (14) in (4) means that the ends of the strings are charged with respect to $\hat{a}_\mu$. It is a gauge field taking values in the adjoint representation of the gauge group. In fact, Eqn (14) is invariant under the gauge transformations: $\hat{a}_\mu \to \hat{a}_\mu + \partial_\mu \hat{\lambda} + \mathrm{i}[\hat{a}_\mu, \hat{\lambda}]$.

As for closed bosonic string theory, the open string generating functional is equivalent at large distances to 26-dimensional dilaton gravity (12) interacting with YM theory for the gauge field $\hat{a}_\mu$.

2.5 On the unification of gravity and Yang – Mills theories

Let us briefly discuss the possible relation of string theory to quantized Einstein gravity and its unification with gauge interactions. The action (12) is written using the so called string metric. From the latter one can pass to the standard Einstein metric through the rescaling $G_\mathrm{E} = G \exp[-4\Phi]$. Hence, the 26-dimensional Einstein – Hilbert action appears as a part of the large distance or classical approximation to *quantum* string theory. Moreover, both gauge and gravity theories can be treated on the same grounds: as approximations to string theory.

Furthermore, one can derive a four-dimensional theory via so called compactifications [3, 4]. In order to do that, one considers the 26-dimensional world as the product of a non-compact four-dimensional space with some very small compact 22-dimensional one. Both spaces should be solutions to the equations of motion following from (12).

All that seems to be promising. However, the closed bosonic string theory contains a tachyonic excitation T with $m_T^2 = -4/\alpha'$. This is a pathological excitation. Its presence means that during the quantization an unstable vacuum has been chosen. In fact, the tachyon is a negative mode excitation over the vacuum, moreover, in closed bosonic string theory higher self-interaction terms for the tachyon apparently do not seal this instability. So in bosonic string theory the form of the true vacuum is unknown and it is unclear even whether it exists at all.

3. Type II superstring theory

To obtain a self-consistent string theory one should consider supersymmetric generalizations of bosonic string theories [3, 12]. There are several non-anomalous types of superstring theories. Here we are going to discuss only the closed type II strings in the Neveu – Schwarz – Ramond (NSR) formalism. In this case SUSY is added to bosonic string theory via anti-commuting ψ_μ fields which are world-sheet super-partners of $\tilde{x}_\mu$. In principle one must take into account the world-sheet metric field and its super-partner as well. However, as in the case of the bosonic string, by fixing symmetries of superstring theory, we could get rid of the fields in question.

Thus, as a starting point we have an $\mathcal{N} = 1$ two-dimensional SUGRA interacting with conformally invariant matter, represented by $\tilde{x}$ and ψ [1, 3, 4]. This is a SUSY extension of the theory described by (2). Due to the presence of the conformal symmetry, the SUSY reparametrization invariance of the action is enhanced to superconformal symmetry. As we discuss below, it is necessary to do some extra work to obtain SUSY inside the target space.

[4] Note that there is also an open string tachyon which we do not consider here.

We consider here a Hamiltonian quantization of type II superstring theories [3], which is more convenient for our purposes than the functional integral approach [1]. Free superstrings are described by the action:

$$\begin{aligned} S_\text{sst} &= \frac{1}{4\pi\alpha'} \int \mathrm{d}^2 z\, (\partial_z x^\mu\, \partial_{\bar z} x_\mu + \psi^\mu\, \partial_{\bar z} \psi_\mu + \text{c.c.}) \\ &\quad + \text{Faddeev} - \text{Popov ghost terms}, \\ z &= \exp[\sigma_1 + \mathrm{i}\sigma_2], \end{aligned} \tag{15}$$

where we have eliminated the world-sheet metric field and its super-partner via SUSY reparametrization and superconformal invariances.

In theory (15) one must impose the standard periodic boundary conditions on $\tilde{x}_\mu$: $\tilde{x}_\mu(\sigma_1, \sigma_2 + 2\pi) = \tilde{x}_\mu(\sigma_1, \sigma_2)$. At the same time, to respect the aforementioned modular invariance, the quantum theory of superstrings should contain sectors with two types of possible boundary conditions for the world-sheet fermions [1, 3, 4]. The first type of boundary condition is due to Ramond:

$$\psi_\mu(\sigma_2 + 2\pi) = \psi_\mu(\sigma_2) \quad (\mathrm{R}), \tag{16}$$

while the second one is due to Neveu and Schwarz:

$$\psi_\mu(\sigma_2 + 2\pi) = -\psi_\mu(\sigma_2) \quad (\mathrm{NS}) \tag{17}$$

with the same conditions for $\bar{\psi}_\mu$ in both cases.

Therefore, there are two kinds of mode expansions for solutions of the free two-dimensional Dirac equation $\partial_{\bar z} \psi_\mu = 0$:

$$\begin{aligned} \psi^\mu(z) &= \psi_0^\mu + \sum_n \frac{b_n^\mu}{z^n} \quad (\mathrm{R}), \\ \psi^\mu(z) &= \sum_n \frac{c_{n+1/2}^\mu}{z^{n+1/2}} \qquad (\mathrm{NS}). \end{aligned} \tag{18}$$

There is a similar but independent expansion for $\bar{\psi}(\bar z)$ as well. It is the conformal invariance that allows us to treat the left (z) and right ($\bar z$) sectors independently: in a conformal field theory they do not interact with each other.

We omit the mode expansion for $\tilde{x}_\mu$ because the corresponding creation operators do not yield massless excitations in superstring theory.

3.1 Quantization and massless spectrum

To quantize superstring theory (15) one imposes the standard commutation (anti-commutation) relations on its bosonic (fermionic) fields. Then the modes b_n and $c_{n+1/2}$ with positive and negative n's become annihilation and creation operators, respectively. At the same time, the zero modes ψ_0^μ generate the algebra of Dirac γ-matrices:

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}, \tag{19}$$

where $\eta^{\mu\nu}$ is Minkowski metric.

Superstring states are constructed by multiplying the states from the left sector by the states from the right sector that satisfy a level matching condition. Thus, since the boundary conditions can be independently imposed in the left and right sectors, there are four kinds of states:

$$\begin{aligned}&\mathrm{NS}-\widetilde{\mathrm{NS}}\,, \qquad \mathrm{NS}-\tilde{\mathrm{R}}\,,\\ &\ \mathrm{R}-\widetilde{\mathrm{NS}}\,, \qquad \mathrm{R}-\tilde{\mathrm{R}}\,.\end{aligned} \tag{20}$$

In order to find the masses of excitations in this theory it is necessary to use the two-dimensional energy-momentum tensor:

$$\mathcal{T}(z)=\mathcal{T}_{11}+\mathcal{T}_{22}-2\mathrm{i}\mathcal{T}_{12}=-\frac{1}{2}(\partial_z\tilde{x}_\mu)^2+\frac{1}{2}\,\psi_\mu\,\partial_z\psi_\mu \tag{21}$$

in the left sector. The energy-momentum tensor in the right sector $\bar{\mathcal{T}}(\bar z)$ is the complex conjugate of (21). The corresponding conserved Hamiltonians are $L_0=\int \mathrm{d}z\mathcal{T}(z)$ in the left sector and similarly $\bar{L}_0$ in the right sector

$$L_0=\int \mathrm{d}z\,\mathcal{T}(z)\,, \qquad \bar{L}_0=\int \mathrm{d}\bar z\,\bar{\mathcal{T}}(\bar z)\,.$$

Hence, the total Hamiltonian is

$$H=L_0+\bar{L}_0+\text{const}\,,$$

where the constant comes from the normal ordering and has different values in the R- and NS-sectors [3, 4]. With such a Hamiltonian one finds the smallest mass states [3, 4]:

$$\begin{array}{lll}\text{mass} & \mathrm{NS} & \mathrm{R}\\ m^2=-2/\alpha' & |0\rangle & -\\ m^2=0 & c^\mu_{-1/2}|0\rangle & |0\rangle,\ \psi_0^\mu|0\rangle,\ \psi_0^\mu\psi_0^\nu|0\rangle,\dots\end{array} \tag{22}$$

and similarly in the $\widetilde{\mathrm{NS}}$- and $\tilde{\mathrm{R}}$-sectors. The vacuum $|0\rangle$ in the R-sector is defined below, while $|0\rangle$ in the NS-sector is the standard vacuum for fermions.

Furthermore, in order to maintain the modular invariance one must project both the left and right sectors to an eigenstate of the operator $(-1)^f$ [1, 3, 4]. Here f counts the world-sheet fermion number in superstring theory, i.e. this operator anti-commutes with all fermionic creation and annihilation operators. That is to say that one must take the partition function in superstring theory to be

$$Z=\mathrm{Tr}\left\{\left[(-1)^f\pm 1\right]\exp(-H)\right\}$$

with either a plus or minus sign rather than just

$$Z=\mathrm{Tr}\left[\exp(-H)\right].$$

This is the so called GSO projection.

If one includes only those states that obey

$$\left[(-1)^f+1\right]|\text{state}\rangle=0$$

then the tachyon state $|0\rangle$ in the NS-sector decouples from the spectrum, while the $c^\mu_{-1/2}|0\rangle$ state survives [5]. Thus, in the $\mathrm{NS}-\widetilde{\mathrm{NS}}$-sector we have $c^\mu_{-1/2}\bar{c}^\nu_{-1/2}|0,\bar 0\rangle$ as a massless state, whose symmetric, anti-symmetric and trace part are related to the familiar $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ excitations in superstring theory.

Let us now discuss what happens in the left R-sector (consideration of the right $\tilde{\mathrm{R}}$ sector is similar) [4]. We change the basis of the zero modes ψ_0^μ to

$$d_0^\pm=\frac{1}{\sqrt{2}}(\psi_0^1\mp\mathrm{i}\psi_0^0),\quad d_i^\pm=\frac{1}{\sqrt{2\mathrm{i}}}(\psi_0^{2i}\pm\psi_0^{2i+1}),\quad i=1,\dots,4\,. \tag{23}$$

Then from (19) one gets:

$$\{d_I^+,d_J^-\}=\delta_{IJ}\,,\qquad I,J=0,\dots,4\,. \tag{24}$$

These $d_I^\pm$ generate $2^5=32$ Ramond ground states $|s\rangle=|\pm1/2,\dots,\pm1/2\rangle$ as:

$$\begin{aligned}&d_I^\pm\left|\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right\rangle=0\,,\\ &d_I^+\left|-\frac{1}{2},\dots,s_I=-\frac{1}{2},\dots,-\frac{1}{2}\right\rangle\\ &\quad=\left|-\frac{1}{2},\dots,s_I=+\frac{1}{2},\dots,-\frac{1}{2}\right\rangle.\end{aligned} \tag{25}$$

One can verify that fixing the SUSY reparametrization gauge yields Super-Virasoro conditions on the physical states of superstring theory [3, 4]. They appear as the standard conditions of Dirac's approach to the Hamiltonian quantization. These conditions are:

$$\begin{aligned}\mathcal{T}\,|\text{state}\rangle=0\,,&\qquad \bar{\mathcal{T}}\,|\text{state}\rangle=0\,,\\ \partial_z\tilde{x}_\mu\,\psi_\mu\,|\text{state}\rangle=0\,,&\qquad \partial_{\bar z}\tilde{x}_\mu\,\bar\psi_\mu\,|\text{state}\rangle=0\,,\end{aligned} \tag{26}$$

which are nothing but the conditions of superconformal invariance of superstring theory. To cancel the anomaly in this case one should take $d=10$ rather than $d=26$.

Now from the first condition in the second row of (26) it follows that $p_\mu\psi_0^\mu|\text{state}\rangle=0$. At the same time, in the reference frame, where $p^\mu=(p^0,p^0,0,\dots,0)$, $p_\mu\psi_0^\mu=\sqrt{2}p^0d_0^+$. Hence, $s_0=+1/2$, which leaves only $s_i=\pm1/2$ $(i=1,\dots,4)$, i.e. 16 physical vacua: 8_s with an even number of $(-1/2)$ and 8_c with an odd number of $(-1/2)$ [3]. These 8_s and 8_c states compose spinor representations of the ten-dimensional Lorentz group with different chiralities [3, 4]. In fact, ψ_μ^0 generates the algebra of ten-dimensional Dirac matrices (19) and 8_c and 8_s are its two irreducible representations.

The GSO projection keeps one of these states (8_c or 8_s) and removes the other. Taking into account that there are two possibilities for the vacuum:

$$(-1)^f\left|-\frac{1}{2},\dots,-\frac{1}{2}\right\rangle=\pm\left|-\frac{1}{2},\dots,-\frac{1}{2}\right\rangle, \tag{27}$$

one concludes that there can be two types of theories. If we choose opposite signs for the vacua in the R- and $\tilde{\mathrm{R}}$-sectors, we obtain a non-chiral type IIA theory. If we choose the same sign, then we have a chiral type IIB theory.

In conclusion, in the R and $\tilde{\mathrm{R}}$-sectors the massless states (25) have target space fermionic quantum numbers [3, 4]: depending on a choice (27), they are ten-dimensional fermions of either one chirality $|\beta\rangle$ or another $|\bar\beta\rangle$. Schematically, this means that in the $\mathrm{R}-\tilde{\mathrm{R}}$ sector there are states such

[5] It is this kind of GSO projection that leads, after taking account of both left and right sectors, to the appearance of SUSY in the target space. After the projection, the off-diagonal elements in (20) yield the target-space superpartners for the diagonal ones [3, 4].

as

$$\begin{aligned} &\left(\gamma^{[\mu_1} \ldots \gamma^{\mu_{n+1}]}\right)_{\lambda\dot{\beta}} |\lambda\dot{\beta}\rangle \quad (\mathrm{IIA})\,, \\ &\left(\gamma^{[\mu_1} \ldots \gamma^{\mu_{n+1}]}\right)_{\lambda\beta} |\lambda\beta\rangle \quad (\mathrm{IIB})\,, \end{aligned} \tag{28}$$

where γ_μ are ten-dimensional Dirac matrices in the Weyl–Majorana representation. These states correspond to the bosonic tensor fields $A_{\mu_1 \ldots \mu_n}$ with the field strengths $F_{\mu_1 \ldots \mu_{n+1}} = \partial_{[\mu_{n+1}} A_{\mu_1 \ldots \mu_n]}$. Now we see that due to the chirality properties of the massless states, in the type IIA theory there are *only odd* rank A fields. At the same time in the type IIB theory *only even* rank A fields are present.

Type II string theories are invariant under *two* SUSY transformations in the target space (Q and $\tilde{Q}$), which correspond to the left and right sectors on the world-sheet, respectively [3, 4]. This is the reason why one refers to these string theories as type II.

3.2 Type IIB superstrings at large distances

Below we mostly consider type IIB string theory (type IIA theory is very similar) with its bosonic massless excitations. Besides the standard NS – $\widetilde{\mathrm{NS}}$ fields G, B, and Φ, this theory contains R – $\tilde{\mathrm{R}}$ fields which are the scalar A, two-form tensor potential $A_{\mu\nu}$, four-form tensor potential $A_{\mu\nu\alpha\beta}$ and their duals. In fact, by construction, among the fields described in (22) – (28) there are various duality relations:

$$\begin{aligned} F_{\mu_1 \ldots \mu_9} &= \epsilon_{\mu_1 \ldots \mu_{10}} \partial_{\mu_{10}} A\,, \\ F_{\mu_1 \ldots \mu_7} &= \epsilon_{\mu_1 \ldots \mu_{10}} F^{\mu_8 \mu_9 \mu_{10}}\,, \\ F_{\mu_1 \ldots \mu_5} &= \epsilon_{\mu_1 \ldots \mu_{10}} F^{\mu_6 \ldots \mu_{10}}\,. \end{aligned} \tag{29}$$

Here $\epsilon_{\mu_1 \ldots \mu_{10}}$ is the completely anti-symmetric tensor in ten dimensions.

As in bosonic string theory, the superstrings contain the target space SUGRA at large distances. In the case of the superstring theory it is known how to calculate its generating functional only if $d = 10$, when the superconformal anomaly is canceled. Thus, the bosonic part of the large distance type IIB ten-dimensional SUGRA action is [3, 4]:

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{16\pi \Gamma_{\mathrm{N}}} \int \mathrm{d}^{10}x \\ \times \sqrt{-G} \Bigg\{ \exp\left[-2\Phi\right] \left[\mathcal{R} + 4G^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi - \frac{1}{12} H^2_{\mu\nu\gamma} \right] \\ + \frac{1}{2} \Bigg[G^{\mu\nu} \partial_\mu A \, \partial_\nu A + \tilde{F}^2_{\mu\nu\gamma} + \frac{1}{2} \tilde{F}^2_{\mu_1 \ldots \mu_5} \\ + \frac{1}{2} \epsilon_{\mu_1 \ldots \mu_{10}} A_{\mu_1 \ldots \mu_4} H_{\mu_5 \mu_6 \mu_7} F_{\mu_8 \mu_9 \mu_{10}} \Bigg] \Bigg\} \\ + \text{fermions} + O(\alpha', \Gamma_{\mathrm{N}})\,, \end{aligned} \tag{30}$$

$$\begin{aligned} \tilde{F}_{\mu\nu\gamma} &= F_{\mu\nu\gamma} - A H_{\mu\nu\gamma}\,, \\ \tilde{F}_{\mu_1 \ldots \mu_5} &= F_{\mu_1 \ldots \mu_5} - \frac{1}{2} A_{[\mu_1 \mu_2} H_{\mu_3 \mu_4 \mu_5]} + \frac{1}{2} B_{[\mu_1 \mu_2} F_{\mu_3 \mu_4 \mu_5]}\,. \end{aligned}$$

Here $\Gamma_{\mathrm{N}} = 8\pi^6 g_s^2 \alpha'^4$ is the ten-dimensional Newton's constant. Furthermore, in this action one must impose the self-duality condition on the R – $\tilde{\mathrm{R}}$ four-form field as shown in the last row of (29). There are also various dual versions of the type IIB SUGRA, which are expressed through the dual tensor fields from (29).

Thus, we see that superstring theories are self-consistent and lead at large distances (the classical limit) to SUGRA theories. Now we are ready to discuss various solitons in SUGRA and string theories.

4. D-branes and SUGRA solitons

In SUGRA theory there are many different solitons [4, 19]. In ten dimensions they can be particle-like black holes or different types of branes (membranes etc.) which are multi-dimensional analogues of four-dimensional black holes. Their singularities live on multi-dimensional sub-manifolds of the ten-dimensional target space and are surrounded by multi-dimensional event horizons. They can be neutral or charged with respect to some tensor gauge fields (like $B_{\mu\nu}$ or the R – $\tilde{\mathrm{R}}$ fields discussed in the previous section) just as point-like black-holes can be charged with respect to gauge *vector* fields: one could surround the locus of a soliton by a multi-dimensional sphere and then find the flux of the corresponding tensor field.

Now, keeping in mind that string theory suggests the quantization of gravity, one can ask what are the quantum counterparts of these solitons? Besides being of academic interest, the answer to this question can reveal some features of black hole thermodynamics [4, 18]. Furthermore, as we discuss below, it gives a relation between SYM theory and SUGRA.

The problem is that in order to pass from large distance gravity to microscopic string theory, one needs to vary the parameters α' (measured with respect to some characteristic scale) and g_s in string theory. It happens that during this variation, when the background fields are turned on, the corrections $O(\alpha', \Gamma_{\mathrm{N}})$ in (30) can become more relevant than the leading large distance contribution. As we briefly discussed in the introduction, these corrections can even change the form of the background completely. First, this destroys the event horizon, which appears to be a low energy global characteristic [4, 6]. Geometrically it is seen when the size of the horizon of a soliton becomes smaller than the characteristic string scale. Second, a variation of the parameters in question can lead to an uncontrollable renormalization of the charge and tension of a soliton or even to a change of the fundamental degrees of freedom in the theory. In fact, we do not have complete knowledge of the string theory dynamics.

However, in the presence of SUSY one can control the renormalization of the low energy (large distance) action. Furthermore, there are solitons in SUSY theories for which the renormalizations of their mass and charge are under control [11]. They are referred to as BPS solitons and respect at least some part of the SUSY transformations in such theories. Note that arbitrary excitations do not respect any symmetries, while the fact that SUSY is respected imposes strong restrictions on possible dynamics [12].

Furthermore, of all BPS solitons in string theory, quantum counterparts are known only for those which are charged with respect to the R – $\tilde{\mathrm{R}}$ tensor fields. For only in the latter case does a good two-dimensional conformal field theory description exist. Although historically R – $\tilde{\mathrm{R}}$ BPS SUGRA solitons were found first [19] and only after that did their quantum D-brane description appear [9], we start our discussion with the definition of D-branes. Then we explain their relation to SUGRA solitons and to SYM theory.

4.1 Definition of D-branes

One could wonder if it is possible to consider open string sectors in closed type II superstring theories. It appears that to avoid anomalies [4] open strings in these sectors should have both Neumann (N) and Dirichlet (D) type boundary conditions on string coordinates [9]:

$$\begin{aligned} &\partial_{\mathrm{n}} x_m = 0\,, \quad \psi_m = \pm\bar{\psi}_m\,, \quad m = 0, \ldots, p \quad (\mathrm{N})\,, \\ &x_i = C_i\,, \quad \psi_i = \mp\bar{\psi}_i\,, \quad i = p+1, \ldots, 9 \quad (\mathrm{D})\,, \end{aligned} \tag{31}$$

where C_i are some fixed numbers and ∂_{n} is a normal derivative to the string boundary.

Therefore, in such a situation the ends of open strings can freely move only along directions labeled by 'm'. In fact, they are confined to $(p+1)$-dimensional sub-manifolds placed at $x_i = C_i$ in a ten-dimensional target space. These sub-manifolds, completely filling the 'p' directions and situated at $x_i = C_i$, are referred to as Dp-branes. At the same time in the bulk of the target space there are ordinary type II closed strings.

The Dp-branes have several features which are relevant to our further discussion. First, they break Poincaré invariance inside the target space

$$\mathrm{P}(10) \to \mathrm{P}(1+p) \times \mathrm{SO}(9-p)\,.$$

Hence, to maintain P(10), one should consider these Dp-branes as dynamical excitations in superstring theory. Second, to respect SUSY one must consider $p = 0, 2, 4, 6, 8$ for type IIA and $p = -1, 1, 3, 5, 7$ for type IIB theories[6] [4] (see below). Third, because of the boundary conditions (31), the Dp-branes can not respect more than a half of SUSY transformations in type II string theories. In fact, the two SUSY transformations (due to Q and $\tilde{Q}$) are related to each other: the left and right sectors on the string world-sheets are no longer independent due to the boundary conditions.

Interactions of a Dp-brane with the massless closed string excitations are described by [9]:

$$\begin{aligned} &Z(G, B, \Phi, \{A\}, a, \phi, \text{fermions}) \\ &\quad = \sum_{\mathcal{G}=0}^{\infty} \int [\mathcal{D}h_{ab}]_{\mathcal{G}}\, \mathcal{D}\tilde{x}_\mu\, \mathcal{D}(\text{fermions}) \\ &\quad \times \exp\left\{-\mathrm{i}S_{\mathrm{Dst}}(\tilde{x}_\mu, h_{ab}, G_{\mu\nu}, B_{\mu\nu}, \Phi, \{A\}, a, \phi, \text{fermions})\right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} &S_{\mathrm{Dst}}(\tilde{x}_\mu, h_{ab}, G_{\mu\nu}, B_{\mu\nu}, \Phi, \{A\}, a, \phi, \text{fermions}) \\ &\quad = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma \left\{\sqrt{-h}\, h^{ab} G_{\mu\nu}(\tilde{x})\, \partial_a \tilde{x}^\mu\, \partial_b \tilde{x}^\nu \right. \\ &\quad \left. + \epsilon^{ab} B_{\mu\nu}(\tilde{x})\, \partial_a \tilde{x}^\mu \partial_b \tilde{x}^\nu + \alpha' \sqrt{-h}\, R^{(2)} \Phi(\tilde{x})\right\} \\ &\quad + \mathrm{R} - \tilde{\mathrm{R}}\text{-fields} + \int \mathrm{d}\tau\, a_m(\tilde{x}_m)\, \partial_{\mathrm{t}} \tilde{x}_m \\ &\quad + \int \mathrm{d}\tau\, \phi_i(\tilde{x}_m)\, \partial_{\mathrm{n}} \tilde{x}_i + \text{fermions}\,. \end{aligned}$$

Here τ is some parametrization of the boundary. As usual, closed strings appear at the loop level in the open string theory. In this functional we have fixed a light-cone gauge: $\phi_m(\tilde{x}) = \tilde{x}_m$, where $\phi_\mu = (\phi_m, \phi_i)$ describes the embedding of the Dp-brane into the target space.

Let us clarify the meaning of the quantity (32). If one puts $G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0, \Phi = 0$ and all $\mathrm{R} - \tilde{\mathrm{R}}$ fields with fermions to zero, then equation (32) describes the time evolution of a quantum state in a two-dimensional conformal field theory. In fact, taking a time slice we fix the initial conditions and the boundary conditions (31) and integrate over all fields in the theory with these boundary and initial conditions. This is by definition a quantum state. Adding time gives us the time evolution of this state. In the case when all background fields are non-trivial, Eqn (32) describes interactions of the quantum state with these fields. Moreover, we show below that (32) at large distances describes the interactions of a SUGRA soliton — the classical limit of the quantum state in question — with the aforementioned SUGRA fields.

Let us explain, following [10], the origin of the sources a_m and ϕ_i in (32). As we have already noted, string theory should be invariant under the transformations described by (13). For a closed string they are respected, but when a string world-sheet has a boundary, boundary terms appear after such transformations. To cancel the first of the transformations in (13), one must add a field ϕ_i at the string boundary. It should transform as $\phi_i \to \phi_i - \xi_i/\alpha'$ to compensate (13). At the same time the boundary term appearing as a result of transformation (13) along the Dp-brane vanishes since Poincaré invariance is respected there.

Hence, ϕ_i would appear as pure gauge degrees of freedom if there were no breaking of Poincaré invariance in the presence of a Dp-brane. Furthermore, from this consideration it is clear what the physical meaning of these fields is: They represent transverse fluctuations of the Dp-branes around their positions C_i. In other words, C_i are just VEV's of the fields ϕ_i: $\phi_i + C_i \to \phi_i$.

Likewise, to maintain the second invariance in (13), the string boundaries should be charged with respect to an Abelian gauge field a_m. In this case the boundary term appearing after the second transformation (13) is compensated by a shift $a_m \to a_m - \rho_m/\alpha'$. (This shift is different from the ordinary gauge transformation $a_m + \partial_m \lambda$ of the field a_m.) The physical meaning of the fields a_m is that they describe longitudinal fluctuations of the Dp-branes.

4.2 D-branes at low energies

At energies much smaller than $1/\sqrt{\alpha'}$ the functional (32) takes the following form [20]:

$$\begin{aligned} &Z(G, B, \Phi, \{A\}, a, \phi, \text{fermions}) \\ &\quad = S_{\mathrm{II}}(G, B, \Phi, \{A\}, \text{fermions}) \\ &\quad + m_p \int \mathrm{d}^{p+1}x\, \exp[-\Phi] \sqrt{-\det(g_{mn} + b_{mn} + 2\pi\alpha' f_{mn})} \\ &\quad + Q_p \int \mathrm{d}^{p+1}x\, \epsilon_{0\ldots p} A_{0\ldots p} + \text{fermions} + O(\alpha', \Gamma_{\mathrm{N}})\,, \end{aligned} \tag{33}$$

where $\epsilon_{0\ldots p}$ is the $(0\ldots p)$-component of the $(p+1)$-dimensional totally anti-symmetric tensor, and

$$\begin{aligned} &f_{mn} = \partial_{[m} a_{n]}\,, \quad g_{mn} = G_{ij}\, \partial_m \phi_i\, \partial_n \phi_j + G_{i(m} \partial_{n)} \phi_j + G_{mn}\,, \\ &b_{mn} = B_{ij}\, \partial_m \phi_i\, \partial_n \phi_j + B_{i[m} \partial_{n]} \phi_j + B_{mn} \end{aligned} \tag{34}$$

are the field strength for a_m, the induced metric, and the B field on the Dp-brane world-volume. Note that in the action

[6] The case $p = -1$ describes the so called D-instanton, which is a D-brane whose 'world-volume' is just a point in the ten-dimensional Euclidean target space. This D-instanton is described by open strings with Dirichlet type boundary conditions in all ten directions.

(33) we maintain all powers of f_{mn} while neglecting its derivatives.

In Eqn (33) $S_{\rm II}$ is the type II ten-dimensional ($S_{\rm II} \propto \int {\rm d}^{10}x\ldots$) SUGRA action. In the case of type IIB string theory $S_{\rm II}$ is given by the leading contribution in (30). The second contribution in (33) is the so called Dirac – Born – Infield (DBI) action for non-linear $(p+1)$-dimensional ($\propto \int {\rm d}^{p+1}x\ldots$) electrodynamics. Its coefficient is the mass per unit volume of the Dp-brane and can be found to be equal [4] to

$$m_p = \frac{\pi}{g_{\rm s}}(4\pi^2\alpha')^{-(p+1)/2}\,.$$

The third term shows that Dp-branes are sources for the R – $\tilde{\rm R}$ tensor fields A. In other words, Dp-branes are charged with respect to the $(p+1)$-tensor R – $\tilde{\rm R}$ fields with charges Q_p [9]. Taking into account the special properties of R – $\tilde{\rm R}$ fields [discussed after equation (28)], it is clear why there can be only Dp-branes with $p=0,2,4,6,8$ in type IIA and $p=-1,1,3,5,7$ in type IIB theories [4].

What is important for our further discussion is that the action (33) is SUSY invariant. In fact, Dp-branes (32) respect a half of the SUSY invariance in type II string theories, and obey

$$m_p = Q_p\, l_{\rm st}^{-(p+1)}\,, \qquad l_{\rm st} \sim \sqrt{\alpha'}\,. \tag{35}$$

The force between any two equivalent and parallel Dp-branes vanishes [4]. This is because the repulsion due to the R – $\tilde{\rm R}$ tensor field compensates the gravitational attraction. This is called the 'No force condition' and is important for our further considerations.

4.3 D-branes as sources for R – $\tilde{\rm R}$ SUGRA solitons

Now let us probe a Dp-brane at large distances, when $r = \sqrt{x_i x^i} \gg \sqrt{\alpha'}$ [note that $g_{\rm s} \to 0$ to suppress the corrections $O(\alpha', \Gamma_{\rm N})$]. We reiterate that it is the SUSY invariance of the action (33) that allows us to go easily from large to small r (and vice versa) and the leading contribution in (33) does not change. Hence, when passing to large r we can just forget about the Dp-brane excitations a_m and ϕ_i. This means that a large distance observer does not feel them and one can substitute their classical values $a_m = \phi_i = 0$ in (33) if there are no sources for these fields. Thus, if there are no non-trivial background fields G, B and Φ, we have:

$$Z = S_{\rm II} + m_p\int {\rm d}^{p+1}x + Q_p\int {\rm d}^{p+1}x\,\epsilon_{0\ldots p}\,A_{0\ldots p} + O(\alpha', \Gamma_{\rm N})\,. \tag{36}$$

The second and the third terms in this equation are just sources for the curvature and the corresponding R–$\tilde{\rm R}$ field. They can be rewritten as

$$\int {\rm d}^{p+1}x\ldots \propto \int {\rm d}^{10}x\,\delta^{9-p}(x_i - C_i)\ldots\,,$$

so solutions of the classical equations of motion for (36) with these sources appear to be BPS R – $\tilde{\rm R}$ SUGRA solitons:

$$\begin{aligned} {\rm d}s^2 &= f_p^{-1/2}\,{\rm d}x_m\,{\rm d}x^m + f_p^{1/2}({\rm d}r^2 + r^2\,{\rm d}\Omega^2_{8-p})\,,\\ \exp[-2\Phi] &= f_p^{(p-3)/2}\,, \qquad A_{0\ldots p} = -\frac{1}{2}(f_p^{-1} - 1)\,, \end{aligned} \tag{37}$$

where $p=0,2,4,6,8$ in type IIA and $p=-1,1,3,5,7$ in type IIB theories [19]. These solutions are the states in SUGRA that are classical limits of the states (32) in string theory. This is how one finds a relation between the Dp-branes and R – $\tilde{\rm R}$ Dp-brane SUGRA solitons. At the same time, Eqn (36) describes low energy fluctuations around these SUGRA solutions.

All the solutions (37) are BPS and for any function f_p they preserve a half of the SUSY transformations in SUGRA theory. The equations of motion of SUGRA theory (36) (related to the closure of the SUSY algebra) imply [19] that f_p should obey

$$\Delta^{9-p} f_p(r) = m_p\,\delta^{9-p}(x_i - C_i)\,. \tag{38}$$

Here Δ^{9-p} is the Laplacian for the flat metric in the directions $p+1,\ldots,9$. Hence, one gets:

$$f_p = 1 + \left(\frac{r_p}{r}\right)^{7-p}, \qquad r_p \propto \frac{1}{m_p^{1/(p+1)}}\,, \tag{39}$$

Note that one can neglect string corrections to (36) in the case $r_p \gg \sqrt{\alpha'}$.

We now consider N Dp-branes parallel to each other and placed at $\mathbf{r}_s$, $s=1,\ldots,N$. We can do that safely because of the 'No force condition'. At low energies (large distances) such a system of N Dp-branes corresponds to a R – $\tilde{\rm R}$ Dp-brane soliton (37) with a charge $Q_p \propto N$ and

$$f_p = 1 + \sum_{s=1}^{N}\left(\frac{r_p}{|\mathbf{r} - \mathbf{r}_s|}\right)^{7-p}. \tag{40}$$

The tension of the soliton is $M_p = N m_p$. Note that when one puts all the Dp-branes on top of each other ($r_s = 0$ for all s)

$$f_p = 1 + \left(\frac{R_p}{r}\right)^{7-p}, \qquad R_p^{7-p} = N r_p^{7-p}\,. \tag{41}$$

In this case one can neglect string theory corrections to (36), (37), (41) if $R_p \gg \sqrt{\alpha'}$.

Solitons (37) are multi-dimensional analogs of the four-dimensional critical Reissner – Nordstrom black hole. Note that the event horizon of these solutions is at $r=0$.

4.4 D-branes and SYM

Now let us probe a Dp-brane at small distances $r \ll R_p$ as $g_{\rm s} \to 0$. In this case one can forget about long wavelength fluctuations of the bulk fields G, B, Φ, and $\{A\}$. Hence, these fields are equal to their classical values, i.e. to zero in the absence of external sources. Thus, expanding (33) in powers of a small f_{mn}, one obtains:

$$\begin{aligned} Z &= S_{\rm II} + S_{\rm SQED} + O(\alpha', \Gamma_{\rm N})\,,\\ S_{\rm SQED} &\propto \int {\rm d}^{p+1}x\left\{\frac{1}{2}f_{mn}^2 + \frac{1}{2}|\partial_m\phi^i|^2 + \ldots\right\}. \end{aligned} \tag{42}$$

The dots in the second row stand for the fermionic superpartner terms. The latter could be recovered from the fact that this supersymmetric QED (SQED) is maximally supersymmetric in $(p+1)$ dimensions. In fact, we know from (31), (32) the number of SUSY transformations under which the theory (42) is invariant. This number is 16 — half of 32, which is the total number of components of supercharges in type II string theories.

There is also another way to find the number of SUSY transformations under which (42) is invariant. One could consider ten-dimensional $\mathcal{N} = 1$ (maximally supersymmetric: there are 16 components of supercharges) SQED:

$$L = \frac{1}{2} f_{\mu\nu}^2 + \frac{\mathrm{i}}{2}\, \bar{\Psi} \hat{\partial} \Psi\,, \tag{43}$$

where Ψ are Majorana–Weyl spinors and the super-partners of a_μ. Then one can make a reduction of the theory to $(p+1)$ dimensions. That is when one considers all the fields in the theory to be independent of $(9-p)$ coordinates [12].

This way, changing the notation from a_i to ϕ_i $(i = p+1, \ldots, 9)$, one gets the theory (42) with the proper fermionic content. Furthermore, during this procedure the number of SUSYs is increased with respect to $\mathcal{N} = 1$ in ten dimensions [12]. In fact, the ten-dimensional fermions Ψ are rearranged into representations of the smaller Poincaré group $\mathrm{P}(p+1)$. Hence, from a single ten-dimensional fermion we obtain several lower-dimensional ones.

The low energy action (42) can also be found by another approach [10]. At low energies the strings that terminate on Dp-branes look like massless vector (a_m) and scalar (ϕ_i) excitations — the massless excitations in open string theory [3]. Furthermore, in the limit $g_s \to 0$ the coupling of open strings attached to Dp-branes with closed strings in the bulk is suppressed. At this point one finds that the low energy theory for such excitations is SUSY QED — the only supersymmetric and gauge-invariant action containing the smallest number of powers of the field derivatives.

The last point of view is helpful in understanding the low energy theory describing a bound state of Dp-branes [10]. Let us consider N parallel Dp-branes with the same p. In this situation in addition to the strings which terminate on the same Dp-brane, there are strings stretched between different branes. Furthermore, because the strings are oriented, there can be two types of strings stretched between any two Dp-branes. The strings attached with both ends to the same Dp-brane yield familiar massless vector excitations living on the brane. On the other hand, the stretched strings yield vectors with masses proportional to the distances between corresponding Dp-branes. They are charged with respect to the gauge fields living on the Dp-branes at their ends. Therefore, the latter vector excitations are similar to the $\mathrm{W}^{\pm}$-bosons in gauge theories with spontaneous symmetry breaking. They acquire masses through a kind of Higgs mechanism — splitting of Dp-branes — and become massless when the Dp-branes approach each other.

Hence, the world-volume theory on the bound state of N Dp-branes is nothing but the $\mathrm{U}(N)$ maximally supersymmetric SYM theory [10]:

$$\begin{gathered} S \propto M_p \alpha'^2 \int \mathrm{d}^{p+1}x\, \mathrm{Tr} \left\{ \hat{f}_{mn}^2 + |\mathrm{D}_m \hat{\phi}_i|^2 + \sum_{i>j} [\hat{\phi}_i, \hat{\phi}_j]^2 + \ldots \right\}, \\ \hat{f}_{mn} = \partial_{[m} \hat{a}_{n]} + \mathrm{i}[\hat{a}_m, \hat{a}_n]\,, \qquad \mathrm{D}_m = \partial_m + \mathrm{i}[\hat{a}_m, \quad]\,. \end{gathered} \tag{44}$$

Dots in this action stand for fermionic terms. Furthermore, all possible positions of the Dp-branes, composing this bound state, are given by VEV's of the $\mathrm{U}(N)$ matrix $\hat{\phi}_i$. Note that the potential in the action (44) has flat directions. These flat directions are not lifted by quantum corrections due to the SUSY invariance of the action (44). Thus, the $\mathrm{U}(1)$ factor in the decomposition $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)$ describes the center of mass position of the Dp-brane bound state.

Unfortunately, we do not know any rigorous derivation of (44) from first principles such as the definition of the Dp-branes, though, there is a non-canonical way to formulate the non-Abelian version of (32), (33), and hence of (42) [24], which can be useful for the derivation of (44).

Anyway, to sharpen the reader's understanding we give one more argument in favor of the appearance of SYM on the Dp-branes. When one has a stack of Dp-branes, the strings which terminate on them carry Chan–Paton indices, enumerating these Dp-branes. Hence, one obtains sources like (14) for their massless excitations, where $\hat{a}_i \to \hat{\phi}_i$ $(i = p+1, \ldots, 9)$. This, as we know, leads at large distances to SYM theory and shows that the theory (44) is the reduction of ten-dimensional $\mathcal{N} = 1$ SYM to $(p+1)$ dimensions.

It is worth mentioning at this point that one can also consider BPS bound states of different types of Dp-branes (with different p's) [4]. However this lies outside the scope of our discussion.

5. AdS/CFT-correspondence

We see that the Dp-branes allow two different descriptions depending on the distance from which one looks at them. From far away, the D-branes look like sources for gravity solitons, while at small distances one observes their quantum fluctuations described by SYM theory. It seems that both limits are unrelated to each other; however, this is not so. To understand why, from now on we are going to discuss one of the simplest situations.

We consider a stack of N D3-branes in a ten-dimensional type IIB SUGRA. The D3-branes are on top of each other at $x_4 = \ldots = x_9 = 0$ and occupy $0, \ldots, 3$ directions. The corresponding SUGRA soliton is the self-dual R–$\tilde{\mathrm{R}}$ D3-brane (29), (37), (41) with

$$R_3^4 = 4\pi g_s N \alpha'^2\,, \qquad Q_3 \propto N\,. \tag{45}$$

Note that the *classical* SUGRA description is applicable when $R_3 \gg \sqrt{\alpha'}$, that is when $g_s N \gg 1$ (note that $g_s \to 0$). Otherwise string theory corrections are relevant and deform the soliton (37).

The geometry of the D3-brane soliton is as follows: It has asymptotically flat boundary conditions at spatial infinity, as at $r \gg R_3$ the ratio $(R_3/r)^4$ becomes much smaller than unity. At the same time, near the position of the source $(r = 0)$ there is an infinite throat region of a constant curvature:

$$\mathrm{d}s^2 = \frac{r^2}{R_3^2} (\mathrm{d}x_m\, \mathrm{d}x^m) + \frac{R_3^2}{r^2}\, \mathrm{d}r^2 + R_3^2\, \mathrm{d}\Omega_5^2\,, \quad \exp[-\Phi] = \mathrm{const}\,. \tag{46}$$

By definition the throat is the region where $r \ll R_3$, so that in (41) the unity can be neglected with respect to $(R_3/r)^4$. In this way one obtains (46) from (37)–(41).

As one can directly check, the metric (46) has a constant scalar curvature equal to R_3. The curvature does not diverge and the D3-brane is a non-singular soliton. In fact, the metric (46) has the geometry of $\mathrm{AdS}_5 \times \mathrm{S}_5$, where AdS_5 is a five-dimensional Anti-de-Sitter space and S_5 is a five-sphere — de-Sitter space. Both of these manifolds are known to have constant scalar curvatures: S_5 has a positive while AdS_5 has a negative curvature. They are both solutions to five-dimensional Einstein equations with positive and negative cosmological constants, correspondingly.

We now describe the geometry of AdS_5 space. There are many ways to present AdS_5 space (see, for example, Ref. [8]), but we find the following description convenient. Algebraically AdS_5 space can be represented as the *universal cover* of a sub-manifold in a six-dimensional flat space (W, V, X_q, where $q = 1, \ldots, 4$) with signature $(-,-,+,+,+,+)$. The equation defining this sub-manifold is [8]:

$$W^2 + V^2 - \sum_{q=1}^{4} X_q X_q = R_3^2\,, \tag{47}$$

where R_3 is the radius of the sub-manifold and of AdS_5 space. Thus, AdS_5 admits the natural action of the global SO(4,2), which is its isometry group.

The metric on the ambient flat six-dimensional space is:

$$ds^2 = -dW^2 - dV^2 + \sum_{q=1}^{4} dX_q\, dX_q\,. \tag{48}$$

The metric on the universal cover of the manifold (47) can be found by solving equation (47):

$$\begin{aligned} &V = R_3 r t\,,\\ &W = \frac{1}{2r}\left[1 + r^2\left(R_3^2 + \sum_{q=1}^{3} x_q^2 - t^2\right)\right],\\ &X_4 = \frac{1}{2r}\left[1 - r^2\left(R_3^2 - \sum_{q=1}^{3} x_q^2 + t^2\right)\right],\\ &X_q = R_3 r x_q\,, \quad q = 1, \ldots, 3\,. \end{aligned} \tag{49}$$

Substituting this solution into equation (48) we obtain the metric for AdS_5 space:

$$ds^2 = \frac{r^2}{R_3^2}\left(-dt^2 + \sum_{q=1}^{3} dx_q\, dx_q\right) + \frac{R_3^2}{r^2}\, dr^2\,, \tag{50}$$

which coincides with the metric for the AdS_5 part in (46) if $x_m = (t, x_q)$, where $q = 1, \ldots, 3$.

Now let us define the boundary of AdS_5 space. If W, V, X_q (where $q = 1, \ldots, 4$) tend to infinity, after dividing the coordinates by a positive constant one obtains an equation defining the boundary:

$$W^2 + V^2 - \sum_{q=1}^{4} X_q\, X_q = 0\,. \tag{51}$$

The boundary is a four-dimensional manifold, because (51) is invariant under the scalings $W \to \lambda W$, $V \to \lambda V$, $X_q \to \lambda X_q$ for a real non-zero λ.

Making use of the scaling with a positive λ, one can map (51) into the locus:

$$W^2 + V^2 = \sum_{q=1}^{4} X_q\, X_q = 1\,, \tag{52}$$

which is a copy of $(S^1 \times S^3)/Z_2$. In this space, we must factor over Z_2 because there is a remaining symmetry under $W \to -W$, $V \to -V$, $X_q \to -X_q$ transformations. The universal cover of (47) has the universal cover of (52) as a boundary, which is $R^1 \times S^3$. The latter manifold is a conformal compactification of the four-dimensional Minkowski space $R^{3,1}$. Indeed, for the conformal compactification of $R^{3,1}$ one adds a point at the spacelike infinity.

In terms of metric (50) this could be clarified as follows. There are two parts of the AdS_5 boundary: the first one is at $r \to \infty$, which is a four-dimensional Minkowski space (t, x_q) where $q = 1, \ldots, 3$; the second part of the boundary is the point $r = 0$. These considerations imply that there is a natural action of SO(4,2) on the conformal compactification of the Minkowski space. This group now defines four-dimensional conformal transformations. Note that under a generic conformal transformation the point $r = 0$ is mapped to a point inside $R^{3,1}$. That is the reason why the compactification of $R^{3,1}$ is referred to as conformal.

Note that SUGRA on AdS_5 space is invariant under a global SO(4,2) symmetry. Furthermore, SUGRA on the throat (46) of the D3-brane is invariant under $\mathcal{N} = 8$ SUSY.

Now let us consider the SYM description of the D3-brane. This description is applicable when $g_s \to 0$, and as follows from (44), the description is given by $\mathcal{N} = 4$ four-dimensional SYM:

$$\begin{aligned} S = \frac{1}{4\pi g_s}\int d^4x\ \mathrm{Tr}\,\Bigg\{&\frac{1}{2}\hat{f}_{mn}^2 + \frac{1}{2}\,|D_m\hat{\phi}_i|^2 + \frac{1}{2}\sum_{i>j}^{6}\big[\hat{\phi}_i, \hat{\phi}_j\big]^2\\ &+ \frac{i}{2}\sum_{I=1}^{4}\hat{\bar{\Psi}}^I \hat{D}\,\hat{\Psi}_I - \frac{i}{2}\hat{\Psi}^I\big[\hat{\phi}_{IJ}, \hat{\bar{\Psi}}^J\big] + \mathrm{c.c.}\Bigg\}\,, \end{aligned} \tag{53}$$

where $\hat{\phi}_{IJ} = \hat{\phi}_i \gamma_{IJ}^i$ and γ_i^{IJ} are six-dimensional Dirac matrices.

One can see from this formula that $4\pi g_s = g^2$, and so when $g_s \to 0$ the perturbative expansion of SYM is well defined. The theory (53) has a vanishing β-function because of the perfect cancellation of quantum corrections due to bosons and fermions. Hence, g is just a non-renormalizable constant, which is in accordance with the fact that $g_s = \exp[2\Phi] = \mathrm{const}$. Furthermore, at any value of g the theory is invariant under four-dimensional conformal transformations given by the SO(4,2) group. The conformal symmetry extends the $\mathcal{N} = 4$ SUSY invariance of SYM theory in question to $\mathcal{N} = 8$ SUSY.

This shows that SO(4,2) is naturally realized both on the SYM and SUGRA sides, which is a good sign that $\mathcal{N} = 4$ SYM theory should be related to type IIB SUGRA on the $AdS_5 \times S_5$ space with a self-dual $R-\tilde{R}$ four-form flux [7] [6]. Note that the *classical* type IIB SUGRA description is valid when $R_3/\sqrt{\alpha'} \to \infty$, which corresponds, according to (45), to taking $N \to \infty$ as well as $g_s N \to \infty$ (note that $g_s \to 0$). Hence, *strongly* coupled $\mathcal{N} = 4$ SYM theory in the large N limit is applicable in absolutely the same situation as type IIB SUGRA on an $AdS_5 \times S_5$ background. These naive considerations favor a relation between the two theories which will be given further support below.

5.1 ABC of the AdS/CFT-correspondence

We now wish to present in a formal way the relation which we are going to study below. The relation is between $\mathcal{N} = 4$ four-dimensional SU(N) SYM and type IIB SUGRA in an $AdS_5 \times S_5$ background with an $R-\tilde{R}$ four-form flux [7, 8]. It establishes that as $g_s N \to \infty$, while $g_s \to 0$ and $N \to \infty$:

$$\begin{aligned} &\left\langle \exp\left[-i\sum_j \int d^4x\ J_0^j(x)\,\mathcal{O}^j\right]\right\rangle\\ &\qquad \approx \exp\left\{-iS^{\min}\big[(AdS_5)_N \times (S_5)_N\big]_{J^j|_u = J_0^j}\right\}. \end{aligned} \tag{54}$$

[7] The self-dual $R-\tilde{R}$ four-form flux is present because the $AdS_5 \times S_5$ geometry appears from the D3-brane which is charged with respect to this field.

The average on the LHS is taken in *strongly coupled* large $N\,\mathrm{SU}(N)$, $\mathcal{N}=4$ SYM theory; $\{\mathcal{O}^j\}$ is a complete set of local operators, which respects the symmetries of the problem. On the RHS of (54) $S^{\min}$ is a type IIB SUGRA action in an $\mathrm{AdS}_5\times \mathrm{S}_5$ background with a self-dual $\mathrm{R}-\tilde{\mathrm{R}}$ four-form flux.

The action is minimized on classical solutions for all its fields: note that as $R_3/\sqrt{\alpha'}\propto g_s N\to\infty$ string theory corrections to this SUGRA theory are suppressed. The classical solutions in SUGRA are represented schematically as J_j: for example, j can contain tensor indices. These solutions take values $J^j|_u = J_0^j$ at the four-dimensional hyper-surface $r=u<R_3$ in the AdS_5 space and display some asymptotic behavior as $u\to R_3$ [8]. These values J_0 serve as sources in the LHS.

Thus, we see that type IIB SUGRA in the *bulk* of the AdS_5 space is related to SYM theory living on *four-dimensional hyper-surfaces* ($r=u$ for an arbitrary u) inside the space in question. This is the so called holography phenomenon [21–23] in quantum field theory.

Relations between the different parameters on both sides of (54) are:

$$R_3^4 = 4\pi g_s N\alpha'^2\,,$$
$$g^2 = 4\pi g_s = \mathrm{const}\,,$$
$$M_{\mathrm{UV}} = \frac{R_3}{\alpha'}\,, \tag{55}$$

number of units of $\mathrm{R}-\tilde{\mathrm{R}}$ four-form flux

$=$ rank of the gauge group $= N\propto Q_3\,,$

energy scale in the SYM theory $=\dfrac{u}{\alpha'}\,,$

where M_{UV} is the UV cutoff for SYM. Indeed, the generating functional of the SYM correlation functions [the LHS of (54)] has UV divergences and needs to be regularized. Hence, the SYM generating functional evolves under the renormalization group (RG) flux. This is despite the fact that there are no quantum corrections to the classical action (53) of four-dimensional $\mathcal{N}=4$ SYM theory. AdS_5 SUGRA needs to be regularized as well, as we discuss below, and the natural regularization parameter is again R_3 [7, 8].

Before discussing the meaning of relation (54) let us emphasize that it is similar to relation (12) between string theory and gravity. In this case SUGRA theory appears as an effective theory of SYM. One of the differences from the string theory statement (12) is that now we get '$Z=\exp\left[-\mathrm{i}S(\mathrm{sources})\right]$' because SYM is a *second quantized* theory.

The relation between the two theories in question should be understood as follows: there is a quantum type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}_5$ with a $\mathrm{R}-\tilde{\mathrm{R}}$ background, which is valid at any energies and yet to be found. This string theory is weakly coupled when $g_s\to 0$, so to keep $g_s N$ fixed one should take $N\to\infty$. At energies smaller than R_3/α' the superstring theory in question has two degenerate limits, one of which happens when $g^2 N\propto g_s N\ll 1$. It is described by weakly coupled $\mathcal{N}=4$ SYM at large N, which is a well defined theory. Another case is when $R_3^4/\alpha'^2\propto g_s N\gg 1$. In this limit one must deal with a strongly coupled SYM theory whose definition is not known. Then the proper description for $g_s N\gg 1$ is given by a weakly coupled (*classical*) type IIB SUGRA on the background under consideration.

5.2 Interpretation

Consider now type IIB string theory in an $\mathrm{AdS}_5\times\mathrm{S}_5$ background with $\mathrm{R}-\tilde{\mathrm{R}}$ flux corresponding to a D3-brane. This theory is quantum gravity, therefore, one should average over all metrics with the asymptotically AdS boundary conditions. As a result, the correlation functions in this theory are independent of the choice of the metric. Hence, the correlators are independent of the coordinates of the operators acting in the bulk of AdS_5. Thus, in the theory all correlators for operators placed in the bulk of AdS_5 are trivial. Moreover, because AdS_5 space does not contain any asymptotically flat part, SUGRA in an AdS_5 background is always strongly coupled in the sense that there are no asymptotic states.

Thus, in AdS_5 SUGRA it is natural to consider a quantity that generates correlation functions of operators acting at the boundary of AdS_5 space. This quantity is nothing but a wave-functional in the SUGRA theory. The operators in question should be those that create or annihilate various SUGRA particles at the boundary. The classical limit of such a generating functional is the RHS of (54). It is important that correlations between operators acting at the boundary of AdS_5 are non-trivial. In fact, after fixing the boundary in AdS_5 space, there is a natural [8] choice of metric on the boundary within the conformal class given by the bulk metric [8] (50).

In other words, gravity in AdS_5 is entirely described by an SO(4,2) (conformally) invariant field theory living only on its boundary, or on any four-dimensional hyper-surface with $r=u\leqslant R_3$. The generating functional considered above for AdS_5 gravity theory is equivalent to the generating functional of a four-dimensional conformal field theory [9]. The question to be answered is what kind of conformal theory is living on the four-dimensional hyper-surfaces in the AdS_5 space?

Now that we have established how the correspondence (54) can be understood from the bulk theory point of view, let us clarify how the things are seen from the boundary theory point of view. Defining the classical limit of the gravity generating functional at the boundary, one can find (via SUGRA equations of motion) its value at any hyper-surface $r=u$. On the boundary theory side this is seen as a RG flux from the cutoff R_3/α' to the energy scale u/α'. In fact, the LHS of (54) is nothing but the Wilsonian effective action for the boundary theory, which is defined at the energy scale $r=u$.

At the same time, the asymptotic behavior of sources (coefficient functions) J_0 as $r=u\to R_3$ is given by perturbative β-functions in the boundary theory. Note that coefficient functions of the Wilsonian effective action depend not only on u but also on the coordinates of the four-dimensional space time (x_m). This fact is necessary for Holography to be valid from the point of view of the theory confined to the boundary.

To explain this consider that it is Holography which allows one to find the generating functional in the boundary theory at the energy scale u/α' if one knows the value of this functional at any other energy scale, independently of whether it is bigger or smaller than u/α'. For example, if one knows the generating functional of the boundary theory at the energy scale $u/\alpha'<R_3/\alpha'$, then it is possible to find its value at the cutoff scale R_3/α'.

[8] The boundary metric is obtained by multiplication of Eqn (50) by $1/r^2$ and taking $r\to\infty$.

[9] Compare this statement with (54).

Now we temporarily forget about the AdS/CFT-correspondence and just look at what happens to the boundary theory. In the RG evolution of this theory we integrate over high energy modes. If in this integration one was only keeping information about divergent counter-terms in the limit $R_3/\alpha' \to \infty$, there would be no way to recover the UV theory from the IR one. In fact, there could be many different UV theories which would flow to the same IR one. However, this clearly contradicts the principle of Holography.

To restore Holography one must keep all information about high energy modes in the RG evolution of the theory. This is done by keeping *all* counter-terms, even those which are finite as $R_3/\alpha' \to \infty$. In this way all information about high energy modes is encoded in terms of all sources J_0 provided the latter are only functions of x_m. Specifically, we mean that in the latter case any variation of the fields in the theory can be compensated by a variation of the sources $J_0(x)$. Thus, if one knows the values of all J_0 (i.e. one knows the SYM generating functional) at some $r = u$ it is possible to find them at any other $r = u_1$.

Unfortunately, there is no rigorous derivation of the equality (54) and one can not straightforwardly trace the 'boundary' theory. Hence, the best that can be done now is to present different points of view and to give some self-consistency arguments in favor of the correspondence.

Below we explain why SUGRA on the asymptotic flat space of the whole D3-brane soliton should decouple from relation (54); why AdS_5 SUGRA is related to $\mathrm{SU}(N)$ SYM rather than to $\mathrm{U}(N)$; why the limits $N \to \infty$ and $g_s N \to \infty$ should be taken; why R_3/α' (u/α') plays the role of the UV cut off (energy scale) in SYM theory; what specifies which field in SUGRA is related to which operator in SYM and vice versa.

5.3 Qualitative notes

Let us consider what is going on with an N D3-brane bound state at very low energies as measured by an observer at infinity [6, 16]. According to (42) in this limit the observer sees free (non-interacting) ten-dimensional SUGRA in the bulk: all interactions are suppressed, because Γ_{N} is small with respect to the characteristic scale in the theory. In fact:

$$S \propto \frac{1}{\Gamma_{\mathrm{N}}} \int \mathrm{d}^{10}x \sqrt{-G}\,\mathcal{R} + \ldots$$
$$\propto \int \mathrm{d}^{10}x \left[(\partial h)^2 + \sqrt{\Gamma_{\mathrm{N}}}\,(\partial h)^2 h + \ldots \right]. \qquad (56)$$

Here we have parametrized the metric as $G = \eta + \sqrt{\Gamma_{\mathrm{N}}}\,h$, where η is the flat metric and h represents small fluctuations around it.

Because all interactions are suppressed, free SUGRA decouples from the D3-brane excitations which are described by $\mathrm{SU}(N)$ SYM (53). Of all D3-brane excitations described by $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)$ SYM those that correspond to the $\mathrm{U}(1)$ part are not decoupled from free SUGRA. In fact, they describe the center of mass degrees of freedom and correspond to the source for the corresponding D3-brane soliton. Hence, these excitations are coupled to the bulk SUGRA even in the low energy limit.

That is only one way of looking at things. Another point of view is that according to (36) and (37), free SUGRA seen by the observer at infinity is decoupled from the SUGRA living in the throat region (46) of the $\mathrm{R} - \tilde{\mathrm{R}}$ D3-brane. In fact, the bulk massless particles decouple from the throat region, because their low energy absorption cross section by the D3-branes scales as [13]:

$$\sigma \propto \omega^3 R_3^8\,, \qquad (57)$$

where ω is the energy of an in-going scalar particle as measured by an observer at infinity. The cross section vanishes as ω decreases. This behavior can be understood as follows: in the low energy limit the wavelengths of particles in the bulk become much bigger than the typical gravitational size of the brane R_3. Hence, long wavelength fluctuations do not see regions of size $\sim R_3$.

At the same time, (57) is equivalent to the grey-body factor for the D3-brane soliton. In this language the behavior of the grey-body factor (57) can be understood as follows: As we lower the energy (as measured by a distant observer) of the excitations whose wave-function is centered close to the position of the brane ($r \ll R_3$), these excitations find it harder and harder to climb up the gravitational potential of the D3-brane and escape to the asymptotic region. As a result, the throat region and asymptotic one do not interact with each other in the low energy (as measured by a distant observer) limit.

In conclusion, there are two pictures describing the same phenomenon. In both cases we have two decoupled theories in the low energy limit from the point of view of a distant observer. In both cases one of the decoupled theories is free SUGRA in the ten-dimensional flat space. So, it is natural to identify the other two systems which appear in both descriptions [6]. The latter systems are $\mathcal{N} = 4$ four-dimensional $\mathrm{SU}(N)$ SYM and type IIB SUGRA in an $\mathrm{AdS}_5 \times \mathrm{S}_5$ background with a self-dual $\mathrm{R} - \tilde{\mathrm{R}}$ four-form flux.

What is most important for the whole picture is that the two theories possess finite (*non-zero*) energies [6]. In fact, their energy scales are those which are seen by an observer in the throat (at a fixed r less than R_3) rather than those which are seen by an observer at infinity. Note that the g_{tt} component of the D3-brane metric is not constant. Hence, the energy E_r of an object as measured at a constant position r and the energy E_∞ measured by an observer at infinity are related by the red-shift factor:

$$E_\infty = f_3^{-1/4} E_r\,. \qquad (58)$$

This implies that the same object, having a fixed finite energy, as being brought closer and closer to $r = 0$, will appear to have a smaller and smaller energy to an observer at infinity.

5.4 Additional arguments

In this subsection we present more calculations in favor of the correspondence (54).

1. First, we explain how one finds relations between operators on the LHS and fields on the RHS of (54). For the massless excitations in SUGRA one can use (33) or its non-Abelian generalization [24]. Take for example the dilaton field. It couples to SYM as follows:

$$\Delta_\Phi S \propto \int \mathrm{d}^4 x\, \exp\left[-\Phi(x_m, \phi_i)\right]$$
$$\times \left[f_{mn}^2 + \sum_{i=1}^{6} |\partial_m \phi_i|^2 + \text{fermions} \right]. \qquad (59)$$

Note that the dilaton field depends on the ϕ_i fields in addition to x_m. In other words the dilaton field is a function of all ten coordinates rather than only of four x_m.

We are going to consider small fluctuations of the dilaton field around the background (46). Hence, we expand $\exp[-\Phi]$ in powers of the dilaton field and the field itself in powers of ϕ_i. Then from (59) we obtain:

$$\Delta_\Phi S_n \propto \int \mathrm{d}^4x\,\partial_{i_1}\ldots\partial_{i_n}\,\Phi(\phi_i,x)\Big|_{\phi_i=0} \times \left[\phi_{i_1}\ldots\phi_{i_n}\left(f_{mn}^2+\sum_{i=1}^{6}|\partial_m\phi_i|^2+\text{fermions}\right)\right]. \quad (60)$$

We can see from this that the n-th spherical harmonic of the dilaton field in S_5 (i.e. a KK mode in S_5) couples to the operator

$$\mathcal{O}_n^{\Phi}[\phi_i,a_m]\propto \phi_{i_1}\ldots\phi_{i_n}\left(f_{mn}^2+\sum_{i=1}^{6}|\partial_m\phi_i|^2+\text{fermions}\right).$$

The non-Abelian generalization of this operator is:

$$\mathcal{O}_n^{\Phi}[\hat\phi_i,\hat a_m]\propto \mathrm{Tr}\left[\hat\phi_{(i_1}\ldots\hat\phi_{i_n)}\left(\hat f_{mn}^2+\sum_{i=1}^{6}|\mathrm{D}_m\hat\phi_i|^2 +\frac{1}{2}\sum_{i>j}[\hat\phi_i,\hat\phi_j]^2+\text{fermions}\right)\right]. \quad (61)$$

One can conclude from this that the zero mode of the dilaton field ($n=0$) couples to the SYM action (53). Similarly from (33) one can find that the zero mode of the graviton field $G_{mn}(x,\phi_i=0)$ couples to the SYM energy-momentum tensor.

In general the method of finding relations between SUGRA fields and SYM operators is based on matching their symmetry properties under the group SO(4,2) [16]. Remarkably, it appears that for each SUGRA field in the chiral representation of (the SUSY extension of) SO(4,2) group there is a SYM operator which transforms in the same representation [16] and vice versa.

It is worth mentioning at this point that there are other symmetry arguments in favor of the validity of the AdS/CFT-correspondence [16, 25], though we are not going to discuss them here.

2. Second, bearing the above considerations in mind, let us examine relation (54) in more detail. Following [7, 8], we consider the zero mode of the dilaton. The action for a dilaton field in the AdS_5 background in the linear approximation is [7, 8]:

$$S(\Phi)=\frac{\pi^2R_3^8}{32\Gamma_\mathrm{N}}\int\mathrm{d}^4x\,\mathrm{d}z\,\frac{1}{z^3}\left[(\partial_z\Phi)^2+(\partial_m\Phi)^2\right]+\ldots \quad (62)$$

Here the metric on the AdS_5 space is taken as:

$$\mathrm{d}s^2=\frac{R_3^2}{z^2}(\mathrm{d}z^2+\eta^{mn}\,\mathrm{d}x_m\,\mathrm{d}x_n)\,,\qquad z=\frac{R_3^2}{r}\,. \quad (63)$$

In this metric the boundary of AdS_5 space consists of the Minkowski space in the region $z=0$ plus a point in the region $z\to\infty$.

Action (62) is divergent for those classical solutions which are regular on the boundary and fall off for large z [7, 8]. To regularize this divergence it is naturally to cutoff AdS_5 space at $z=\epsilon\propto\alpha'/R_3$. This is an infrared (IR) regularization of AdS SUGRA. Now any classical solution with $\Phi(z=\epsilon,x)=\Phi_0(x)$ can be expanded in terms that obey $\Phi(z=\epsilon,x)=\exp[\mathrm{i}k_mx^m]$, where k_m is the four-momentum [8]. The unique normalizable [7, 8] solution with the latter boundary condition and which is regular as $z\to\infty$ is [7, 8]:

$$\Phi(x_m,z)=\frac{(kz)^2\mathcal{K}_2(kz)}{(k\epsilon)^2\mathcal{K}_2(k\epsilon)}\exp\left[\mathrm{i}k_m\,x^m\right],\qquad k=|k_m|\,. \quad (64)$$

Here $\mathcal{K}_2$ is the modified Bessel function.

The action for this solution is [7, 8]:

$$S^{\min}(\Phi_0)\propto N^2\int\mathrm{d}^4x\int\mathrm{d}^4y\,\Phi_0(x)\Phi_0(y) \times\left(\epsilon^2+|x_m-y_m|^2\right)^{-4}+O(\epsilon^2)\,, \quad (65)$$

where $\Phi_0(x)=\exp[\mathrm{i}k_mx^m]$. The prefactor N^2 in the integrals (65) emerges because $R_3^2\propto N^2$. There is no contribution to (65), which is of the order of Φ_0^2, from higher order corrections in (62).

At the same time, the generating functional in the SYM picture is:

$$Z(\Phi_0)=\int\mathcal{D}\hat a_m\ldots\exp\left\{-\frac{\mathrm{i}}{g^2}\int\mathrm{d}^4x\,\mathrm{Tr}\left[\hat f_{mn}^2+\ldots\right] +\frac{\mathrm{i}}{g^2}\int\mathrm{d}^4x\,\Phi_0(x)\,\mathrm{Tr}\left[\hat f_{mn}^2+\ldots\right]\right\}. \quad (66)$$

Dots in this equation stand for the superpartners of a_m. Now $\Phi_0(x)$ is the source for the operator that is the SYM classical action. According to (54) and (33) it should be equal to the dilaton's boundary value $\Phi(z=\epsilon,x)$.

Integrating over the SYM fields in (66), we get:

$$Z(\Phi_0)=\text{const}\cdot\exp\left\{-\text{const}\cdot\mathrm{i}\int\mathrm{d}^4x\int\mathrm{d}^4y\,\Phi_0(x)\Phi_0(y) \times\left\langle\mathrm{Tr}\left[\hat f_{lm}^2(x)+\ldots\right]\mathrm{Tr}\left[\hat f_{np}^2(y)+\ldots\right]\right\rangle+\ldots\right\} \quad (67)$$

up to the *quadratic* order in the dilaton. Because of restrictions imposed by $\mathcal{N}=4$ SUSY invariance we know the exact value of the correlator:

$$\left\langle\mathrm{Tr}\left[\hat f_{lm}^2(x)+\ldots\right]\mathrm{Tr}\left[\hat f_{np}^2(y)+\ldots\right]\right\rangle\propto\frac{N^2}{|x_m-y_m|^8}\,. \quad (68)$$

Here N^2 appears as the number of degrees of freedom in SYM theory. In fact, $\mathcal{N}=4$ SYM theory is superconformal and, hence is not confining: the degrees of freedom are the same at *all* scales.

Now in Eqns (67), (68) a UV divergence appears when $x=y$. It can be regularized via point splitting. Concisely, this means that all distances in four-dimensional space-time must be larger than some regularization parameter ϵ'. In this regularization scheme we have:

$$\left\langle\mathrm{Tr}\left[\hat f_{lm}^2(x)+\ldots\right]\mathrm{Tr}\left[\hat f_{np}^2(y)+\ldots\right]\right\rangle \propto\frac{N^2}{\left(\epsilon'^2+|x_m-y_m|^2\right)^4}+\text{contact terms}\,. \quad (69)$$

In conclusion, if we equate $\epsilon'=\epsilon$, we find an agreement between the LHS and RHS of (54). Furthermore, we find that the IR regularization on the SUGRA side is related to the UV one in SYM [7, 8]. Thus, R_3/α' plays the role of a UV regularization on the SYM side. Note that one can vary the SYM UV regularization parameter as well as the position of the boundary of the AdS_5 space by SO(4,2) transformations. In other words, one can place the hyper-surface on which SYM lives at any position $r = u$ inside AdS_5 space using an SO(4,2) transformation.

The check we just performed can also be extended to other SYM operators and SUGRA fields [16].

3. Third, at this point one could ask: What is the meaning of string theory for $\mathcal{N} = 4$ SYM? Normally such a string representation means confinement in the theory [1]. In fact, consider the Wilson loop:

$$W(C) = \mathrm{Tr}\,\mathrm{P}\exp\left[\mathrm{i}\oint_C \mathrm{d}x_m\,\hat{a}_m\right]. \tag{70}$$

In this formula C is some contour inside four-dimensional space-time and the trace is taken in the fundamental representation of the gauge group.

The string representation of YM theory means that the Wilson loop expectation value can be represented as a sum over string world-sheets Σ_C having C as their boundary:

$$\left\langle W(C)\right\rangle = \sum_{\Sigma_C} \exp\left[-\mathrm{i}S(\Sigma_C)\right], \tag{71}$$

for some string theory action $S(\Sigma_C)$. If one takes a large loop C in Euclidean space this becomes:

$$\left\langle W(C)\right\rangle \propto \exp\left[-\mathcal{A}(\Sigma_C^{\min})\right], \tag{72}$$

where $\mathcal{A}(\Sigma_C^{\min})$ is the area of the minimal surface $\Sigma_C^{\min}$ spanned by C. This suggests a linear potential between the sources in the fundamental representation of the gauge group and hence confinement [1].

One can use the AdS/CFT-correspondence in Euclidean space [26 – 28] to find a representation like (72) for the Wilson loop expectation value in $\mathcal{N} = 4$ SYM. The answer for strongly coupled SYM theory is the same as in (72), but now $\mathcal{A}(\Sigma_C^{\min})$ is a regularized [26, 27] area of the minimal surface spanned by the contour C. The latter now lives on the boundary of AdS space. At the same time, the string world-sheet lives inside AdS_5 space.

Note that no confinement is expected for a conformal theory, because in such a theory one has the same degrees of freedom at all scales. Thus, the question appears: why does an answer like (72) for the Wilson loop average in $\mathcal{N} = 4$ SYM theory not lead to confinement? In other words, as the area enclosed by C on the boundary is scaled up, why is the area $\mathcal{A}(\Sigma_C^{\min})$ not scaled up proportionately? It is the AdS geometry that is helpful [28].

In fact, the answer to this question is clear from SO(4,2) invariance: If we rescale C by $x_m \to tx_m$, with a large positive t, then by conformal invariance we can rescale $\Sigma_C^{\min}$, by $x_m \to tx_m$ and $z \to tz$ [see (63)], without changing its area $\mathcal{A}$. Thus the area $\mathcal{A}$ need not be proportional to the area enclosed by C on the boundary. Since, however, in this process we had to scale $z \to tz$ with a very large t, the surface $\Sigma_C^{\min}$ which is bounded by a very large circle C should extend very far away from the boundary of AdS_5 space. This is perfectly consistent with AdS geometry. Direct calculation in Refs [26, 27] shows that these considerations are correct.

There exist other arguments in favor of the validity of the AdS/CFT-correspondence [16], but we shall stop here, since we hope that this is enough to convince the reader that the AdS/CFT-correspondence is justified.

6. Conclusions

Thus we see that the AdS/CFT-correspondence provides the first example of a string theory description of SYM. It is worth mentioning that analogues of the AdS/CFT-correspondence can also be established for SYM theories in other dimensions [29]. Moreover, it can be generalized to conformal YM theories with less SUSY [30, 31]. There are generalizations of the AdS/CFT-correspondence for non-conformal theories [28, 32, 33].

Furthermore, as is usual for such statements, which relate two seemingly unrelated theories, this correspondence is useful for both of its constituents [16]. Besides the fact that the correspondence suggests a string description of SYM, it gives a quantum description of gravity in terms of SYM. We mean that at distances much smaller than the characteristic string scale (when $g^2N \ll 1$) we have a SYM description of quantum gravity: as we mentioned the AdS SUGRA appears as an effective theory for SYM. Also, as we noticed above, the AdS/CFT-correspondence gives an explicit example of the Holography phenomenon, which can be important for understanding quantum gravity.

For integrity we would like to criticise the status of the whole subject. First, we see that it is possible to find a string description of YM theory only in the most simplified situation. In fact, the string description is found when YM theory is maximally supersymmetric, when the large N limit is taken and it is more or less testable only for the strong coupling $g^2N \to \infty$. Second, even in the latter situation the correspondence is not rigorously derived from first principles.

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7. Appendix. BPS states

In this appendix we define BPS solitons for completeness. We present here a standard simple exercise [11] which can, as we hope, help understand why BPS solitons are so special.

Let us consider a two-dimensional scalar SUSY theory:

$$S = \int \mathrm{d}^2\sigma \left[\frac{1}{2}(\partial_a\phi)^2 + \frac{\mathrm{i}}{2}\,\bar{\Psi}\,\hat{\partial}\Psi - \frac{1}{2}\,V^2(\phi) - \frac{1}{2}\,V'(\phi)\,\bar{\Psi}\Psi\right], \tag{73}$$

where Ψ is a Majorana fermion, and $V(\phi)$ is an arbitrary function (it could be $V = -\lambda\left(\phi^2 - \phi_0^2\right)$ or $V(\phi) = -\sin\phi$, for example). The theory is invariant under SUSY transformations with conserved Neuther current:

$$s^a = (\partial_b\phi)\gamma^b\gamma^a\,\Psi + \mathrm{i}V(\phi)\,\gamma^a\,\Psi\,. \tag{74}$$

Working with the chiral components $\Psi^{\pm}$ of the Fermi field, the chiral components $Q^{\pm}$ of the SUSY charge can be written as follows:

$$Q_{\pm} = \int \mathrm{d}\sigma_2 \big|(\partial_1 \phi \pm \partial_2 \phi)\, \Psi_{\pm} \mp V(\phi)\, \Psi_{\mp}\big| \,. \tag{75}$$

In this notation the SUSY algebra is:

$$Q_{+}^2 = p_+ \,, \qquad Q_{-}^2 = p_- \,,$$

$$Q_+ Q_- + Q_- Q_+ = 2 \int \mathrm{d}\sigma_2\, V(\phi)\, \frac{\partial \phi}{\partial \sigma_2} \,, \tag{76}$$

where $p_{\pm} = p_1 \pm p_2$. The RHS of the third equality here is the so called central charge $\mathcal{Z}$ of the SUSY algebra. It is proportional to the topological charge in the theory. In fact, for example, if $V(\phi) = -\sin\phi$, then

$$\mathcal{Z} = \int_{-\infty}^{+\infty} \mathrm{d}\sigma_2\, \frac{\partial}{\partial \sigma_2}\, (2\cos\phi) \,.$$

The latter is non-zero only for (anti-) kink solutions.

From the algebra (76) one finds that:

$$p_+ + p_- = \mathcal{Z} + (Q_+ - Q_-)^2 = -\mathcal{Z} + (Q_+ + Q_-)^2 \,, \tag{77}$$

hence, $p_+ + p_- \geqslant |\mathcal{Z}|$. For a single particle state with mass M at rest this implies

$$p_- = p_+ = M \geqslant \frac{1}{2}\, |\mathcal{Z}| \,. \tag{78}$$

This bound is saturated for the BPS states, when as seen from (77)

$$(Q_+ + Q_-)|\mathrm{BPS}\rangle = 0 \quad \text{ or } \quad (Q_+ - Q_-)|\mathrm{BPS}\rangle = 0 \,.$$

For example, this condition is satisfied for all kink and anti-kink solutions of this theory. Thus, the BPS states compose small representations of the SUSY algebra: some combination of supercharges acts trivially on the state, and hence does not generate superpartners [12].

The last feature of the BPS states is crucial. In fact, if SUSY is not broken (which can be checked from the beginning by calculation of the Witten index for the theory), adiabatic variations of the theory parameters do not change representations of the SUSY algebra. Hence, if Eqn (78) holds at some values of the parameters, it always holds and the BPS states survive quantum corrections. Moreover one can control the renormalization of the mass and charge using Eqn (78), and if there is enough SUSY neither mass nor charge are renormalized at all.

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