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Dynamic chaos interference in Hamiltonian systems: experiment and potential radiophysics applications

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Abstract. **The sign correlation of quasiperiodic oscillations with close incommensurable frequencies forms a dynamic chaos, which interferes like noise with a single interference peak and is controlled by the delay of its constituent oscillations. This property of oscillations with incommensurable frequencies can be employed in multichannel information transfer systems to form radar reception patterns and obtain uninterrupted coherent key streams in symmetric cryptographic systems. The review of known results on the generation and properties of quasiperiodic oscillations is complemented by a description of new experiments.**

1. Introduction

Dynamic chaos as a deterministic irregular motion has attracted the attention of experts in different scientific fields (see, for instance, Refs [1 – 3]). Particularly rapid is the development of chaotic dynamics in radiophysics — the realm best suited to the pursuit of experimental investigations.

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Relatively recently, an investigation was made into the possibility of using electronic circuits with strange attractors for information transfer protection from unauthorized access [4]. The identity of parameters of the transmitting and receiving chaotic modules of these circuits ensures their strong coupling, making it possible to perform consistent reception of desired signals concealed in chaos. A start was made on the development of such circuits once the feasibility of controlling dynamic chaos came to light. This determined the main line of employment of strange attractors as information signals in information transfer and storage systems [5].

The aim of our paper is different. The objective is to show the possibility of using the dynamic chaos of mappings representing quasiperiodic oscillations, i.e. oscillations with incommensurable (irrationally related) frequencies, for communication systems. Oscillations of this kind are generated by parametric circuits with regular attractors [6].

During the intensive investigations of parametric circuits in the 1960s, dynamic chaos was not yet known to radiophysicists, and therefore the chaotic properties of parametric circuits were associated with noise and did not arouse practical interest. That is why several promising applications related to the chaotic properties of parametric circuits have been given no consideration. We now believe that this gap should be compensated, the more so as quasiperiodic oscillations have long come to the attention of radiophysicists [7, 8].

At present, the study of dynamic chaos in experimental radiophysics and electronics is primarily limited to the investigation of strange attractors, whereas systems with regular attractors remain mainly a source of mathematical problems and an object of computer simulations.

In experiments with oscillations having incommensurable frequencies, oscillations of a two-circuit parametric oscillator are used [8]. Under certain conditions, such an oscillator can be considered a conventionally conservative Hamiltonian chaotic system with a constant phase volume [3]. The chaotic dynamics of such systems is determined by uniform quasiperiodic motion irrationally winding a torus — a regular attractor of the system formed by phase trajectories.

The dynamics of Hamiltonian systems is the subject of three modern special theories: the egrodic theory [9], which studies the instability of chaotic motion; the KAM theory, which considers its stability; and the stochastic theory, which unites both extreme theories on the basis of nonlinear resonance [10] (the acronym KAM is associated with the names of Kolmogorov, Arnol'd, and Moser [11 – 13]).

The remarkable property of Hamiltonian systems to generate irregularity in the mutual mappings of regular stable motions was noted even by Poincare [14]. Present-day numerical investigations showed that 'this irregularity can be indistinguishable from randomness in the case of irrationally related frequencies' [15].

This can be observed experimentally in the mutual mappings of two rectangular waves with close incommensurable periods, with one wave defining a periodic sequence of zeroes and unities and the other defining regular samplings of these elements with a repetition rate incommensurable with the frequency of the sequence. The absence of resonances common to rectangular waves with incommensurable periods makes the results of samplings virtually random, as in the case of tossing up an ideal coin. Therefore, a random (Bernoulli's) sequence of zeroes and unities forms — a physical model of dynamic chaos, in which binary elements and their combinations are evenly mixed.

The frequency proximity of quasiperiodic oscillations is fundamentally significant in their interference, because in this case the oscillation frequency incommensurability alone proves not to suffice to fully mix their phases and obtain a continuous spectrum, which characterizes the chaotic properties of the resultant oscillation.

In practice, one quite often restricts oneself to so-called Poincare mappings, which relate quasiperiodic oscillations stroboscopically (in short-term samplings), at time intervals multiple to one of the periods of quasiperiodic oscillations. In this case, the proximity of the frequencies of quasiperiodic oscillations may no longer be of fundamental significance for obtaining a resultant oscillation with a continuous spectrum.

The result of the interaction of quasiperiodic oscillations is termed chaotic, or quasirandom. Here, the randomness is caused not only by the incommensurability of the oscillation frequencies, but also by the impossibility of precisely prescribing the initial conditions of motion, which stems from the continuity of the phase space of a classical system. That is why the phase trajectories of oscillations with different initial conditions diverge with time; this occurs more noticeably, the less frequent are the samplings for the Poincare mappings. Moreover, random transitions from one phase trajectory to another, referred to as Arnol'd diffusion [1], are possible under external perturbations, which makes motion even less predictable.

The manifestations of order and chaos are particularly pronounced and diverse in the mappings of quasiperiodic oscillations. Since these mappings are determined by the dynamics of a Hamiltonian system, of particular interest is their evolution from regular trajectories of the degenerate mode to local resonances of the nondegenerate mode and further (via intermittency) to dynamic chaos. The evolution of mappings is observable in the tuning of partial frequencies of a two-circuit generator.

Depending on the tuning of the resonators in a two-circuit parametric oscillator, a periodic pump voltage can excite either rationally or irrationally related oscillations, whose respective mathematical models are resonant and nonresonant tori.

In a degenerate mode, the multiplicity of frequencies results in self-synchronization of parametric oscillations and pump oscillations. As the circuits are tuned to pass into a nondegenerate mode, the self-synchronization energy lowers and the resonant tori begin to collapse. These collapses, or bifurcations, are observed as diffusion and jump-overs of the phase of parametric oscillations relative to the pump phase. On the resonant tori thus collapsed, isles of local resonances form, which next give way to nonresonant tori that are free from mutual synchronization and prove to be more stable (firmer) than the resonant tori.

Since chaos is inherent in the incommensurability of the frequencies of quasiperiodic oscillations, narrow-band filtration, frequency division, and amplitude limitation can be applied to them. These rational transformations permit simplifying the problems related to dynamic chaos production and control over the interference of the corresponding chaotic binary sequences. These operations can be accomplished with the aid of frequency separation (filtration) of quasiperiodic oscillations and control over their delay prior to sign correlation. In this case, all electronic circuits, with the exception of parametric oscillators, can be made of digital integrated elements.

The frequency ratio in a nondegenerate oscillator is, generally speaking, a hidden parameter of regular attractors, because the oscillation frequency ratio is hard to judge from oscillograms. That is why the frequency incommensurability of quasiperiodic oscillations can be experimentally determined from the form of their mutual mapping and its autocorrelation function.

Chaotic models possess the properties of dynamic chaos [16] (a continuous spectrum and a decaying autocorrelation function), and are a time-unlimited analog of algorithmic noise-type signals, which are used extensively in communication systems [17]. The possibility of obtaining binary chaotic sequences from quasiperiodic oscillations allows their autocorrelation function to be controlled with the aid of oscillation delay. With a sufficiently high stability of their incommensurable frequencies, this property can find extensive use in the solution of several topical radiophysical problems.

In particular, two-channel four-beam interference of quasiperiodic oscillations, which are used as radar signals, can be employed to form sharp radar reception patterns and suppress specular reflections (multipath propagation).

Producing dynamic chaos from quasiperiodic oscillations received by spatially diversed receivers in symmetric cryptographic communication systems makes it possible to accomplish the hidden 'generation' of uninterrupted coherent key streams, which can be used to cipher and decipher the transmitted information. In this case, special preparation of the key-stream formation system affords the optimal conveyance of a secret communication channel. The problems of preparation of suchlike statistical systems with minimal relaxation times were considered in Ref. [9].

In writing this article we allocated a separate section to the principal properties of radio oscillations with incommensurable frequencies. The reason is that they have not been systematized in the literature, while this information may be beneficial where the radiophysical applications of quasiperiodic oscillations under consideration are involved. Moreover, to do this has taken additional experimental investigation. The findings of these investigations are used in subsequent sections, and they are discussed in Section 7.

2. Quasiperiodic oscillations

In the literature, oscillations with incommensurable frequencies are commonly referred to as quasiperiodic oscillations. Commensurability and incommensurability are the relative properties of uniform quantities which do or do not have a common measure. If one such quantity is assumed to be unity (taken as the reference quantity), then the quantities commensurable with it are expressed by rational numbers and those incommensurable with it by irrational ones. This pertains also to the frequencies of oscillations which do or do not have common resonances. Quasiperiodic oscillations are therefore oscillations with irrationally related frequencies (ω_1 and ω_2), and their frequency ratio is expressed by an irrational number, for instance, by the number $m_1 = (\omega_2/\omega_1) = (2^{1/2} - 1) = 0.414\ldots$

In Hamiltonian systems, oscillations with irrational frequency ratios may have no common resonances for an indefinitely long time. This is indicative of the high stability of the irrational relationship of their frequencies. In the above example we adopt the frequency $\omega_2 = 1$ as the reference. The other frequency is then $\omega_1 = 0.414\ldots$ In this case, irrespective of which of the two irrationally related frequencies is taken to be unity, their difference $\Delta\omega = |\omega_1 - \omega_2|$ is also an irrational number and is expressed by a unique infinite continued fraction [18]. In our example, $\Delta\omega = |0.414\ldots - 1| = 0.585\ldots$ Of course, this irrationality is of no physical interest, and measuring instruments always round off such quantities to rational values.

Experimentally, the situation may be different. One continuous rectangular wave defines a regular sequence of zeroes and unities with a frequency ω_1, and a similar reference wave defines a regular sequence of short samplings of these elements with a frequency ω_2. In this case, each sampling is a quantized phase value (0 or π) of the signal wave (with the frequency ω_1) relative to the change of sign of the reference wave (with the frequency ω_2). If the wave periods are commensurable, the sequence of samplings contains resonances and is a result of repeated measurements of the difference frequency $\Delta\omega$ in the form of a rational, periodically repeated binary number.

If the wave periods are incommensurable, the sequence of samples is void of resonances and represents the difference frequency $\Delta\omega$ by an irrational binary number of the form ...0101101... Such a binary sequence may be infinite, but it cannot be periodic owing to the absence of common resonances in the frequencies of the waves under comparison. Because of this, it has a decaying autocorrelation function.

The properties outlined correspond to those of a deterministic dynamic chaos, which is also highly sensitive to the initial conditions of excitation of quasiperiodic oscillations. It manifests itself in the breaking of correlation between binary sequences obtained from quasiperiodic oscillations with different initial phases.

Since irrational numbers form a continuum on the number axis, their measure is assumed to be unity and the measure of rational numbers to be zero. Similar measures also apply to the frequencies of real radio signals, in which natural beats appear under a nonsynchronous interaction. The interaction of nondegenerate parametric oscillations, despite the fact that they are irrationally related owing to the regular pump voltage, is also attended by beats.

In the general case, the interaction of two oscillations whose frequencies are disproportionate (not close) produces an aperiodic oscillation with a discrete spectrum consisting of two spectral lines, which, under certain conditions, is also characteristic of quasiperiodic oscillations. That is why the mathematical result of superposition of two periodic functions with incommensurable frequencies is commonly termed a quasiperiodic function by physicists and an almost periodic function by mathematicians. Such a function retains the property of being almost periodic under any rational transformation. In particular, its autocorrelation function is also an almost periodic function [19, 20].

Experiment shows that the statistical properties of incommensurable observables are most pronounced when their scales are proportionate (with respect to frequencies, this manifests itself in the overlap of resonances) and that a simple mixture of oscillations with incommensurable but not close (disproportionate) frequencies is not chaos. If the frequencies are scale-proportionate (the resonances overlap), they can be approximately related as $\Delta\omega \leqslant 0.5(\omega_1 + \omega_2)$.

To make efficient use of quasiperiodic oscillations, it is expedient to resort to strong irrational numbers most remote from their rational approximations [18]. A strong irrational number is the 'golden section' equal to $0.5\,(5^{1/2} - 1)$. It is defined as the ratio of the smaller portion of a linear segment divided in the mean and extreme ratios to the larger portion, equal to the ratio of this larger portion to the entire segment [21].

Rational approximations of the 'golden section' are the fractions 2/3, 3/5, 5/8, ..., whose denominators make up the Fibonacci number sequence [22]. The irrational frequency ratio equal to the 'golden section' can be obtained in a nondegenerate two-circuit parametric oscillator (see Section 3.1).

According to Ref. [19], any rational transformations can be applied to quasiperiodic oscillations. Among them are narrow-band filtration (for frequency separation) and oscillation amplitude limitation (for obtaining binary mappings and applying digital devices). With the aid of frequency division it is possible to draw apart the frequencies in the spectrum and accomplish frequency matching to make them scale-proportionate (with the overlapping of the resonances), and also to thin out the sequence of their mutual mappings with a given cadence frequency Ω_s. The frequency of thinning out may be either regular or irregular, but irregular thinning hinders the observation of local resonances.

Due to the transformations specified above, in the formation of chaos it is possible to obtain samplings with a nearly maximum cadence frequency controllable over a wide range, from a chaotic binary sequence prepared in a mixer of a phase comparator type. Such a specially prepared signal is free of local resonances, which can be checked by its continuous spectrum at the mixer output.

The range of incommensurable frequencies can be broadened by thinning out the Poincare mappings. The thinning results in the decorrelation of the signal phases [23] and is, under certain conditions, equivalent to frequency matching. This is possible when the thinning frequency is so selected that all local resonances (or intrinsic phase correlations of the signal that fall within the intervals between samplings) are automatically rejected.

In the case of interference of quasiperiodic oscillations with close incommensurable frequencies, the difference frequency $\Delta\omega$ exhibits instability, which leads to beats. This instability results from the relative phase mixing of the quasiperiodic oscillations, whereas the frequencies under measurement themselves may be stable (see Section 3.4) and the difference calculated from the measured frequencies may be stable. In the interference of oscillations with close incommensurable frequencies, a continuous spectrum is observed. Having no common resonances, they interfere like noise with a single interference peak at zero.

When considering the properties of oscillations with incommensurable frequencies, noteworthy is the fact the autocorrelation functions of purely periodic oscillations that have incommensurable frequencies are also purely periodic oscillations with the same incommensurable frequencies. Therefore, the intensity of interference of two quasiperiodic oscillations produced by purely periodic oscillations with incommensurable frequencies is made up of the independent results of interference of each coherent pair of purely periodic oscillations.

Naturally, the results outlined above depend on the phase difference between the interfering coherent oscillations. Because of this, two signals with incommensurable frequencies can be used to form an infinite number of quasiperiodic signals with a degree of mutual correlation sensitive to the initial conditions. The role of initial conditions is played by two parameters — the phase difference between coherent oscillations of one frequency and the phase difference between coherent oscillations with a different, incommensurable frequency.

In experiments, the sign correlation of two rectangular waves with close incommensurable periods T_1 and T_2 (Fig. 1a) generates a discrete Poisson stream (a cascade, in the mathematical terminology) of pulses of random length (Fig. 1b), which occur at random points in time, as in band-clipped noise with zero average value [24]. Here, the randomness is a consequence of the noncoincidence and relative mixing of the zeroes of rectangular waves with close incommensurable periods.

The statistical transformation properties manifest themselves in the simple mathematical model shown in Fig. 1c. It is of the form of two infinite arithmetical progressions of the forms $\{i\} = (0, \tau_1, 2\tau_1, 3\tau_1, \ldots)$ and $\{k\} = (0, \tau_2, 2\tau_2, 3\tau_2, \ldots)$ whose terms are real numbers having no common multiple, which recur at intervals $\tau_1 = T_1/2$ and $\tau_2 = T_2/2$, where T_1 and T_2 are the incommensurable periods of rectangular waves (this representation was borrowed from Ref. [19]).

If the sequences $\{i\}$ and $\{k\}$ are superimposed, it is possible to bring into coincidence only two numbers, for instance, zeroes or any pair of values $i\tau_1$ and $k\tau_2$, where i and k are the respective serial numbers of the terms of the first and second sequences. Shifting the zeroes by uniform intervals $\varepsilon > 0$ shifts the coincidence of the i and k values by different 'random' distances. In this case, even for $\varepsilon \to 0$ the point of coincidence may turn out to be at infinity. At the same time, any relative displacement of the zeroes brings about a coincidence of some other values i and k. These values may be formally taken as the new initial conditions and the new coincidence of the zeroes, after which the mutual position of the sequences is completely repeated. These repetitions are infinite in number.

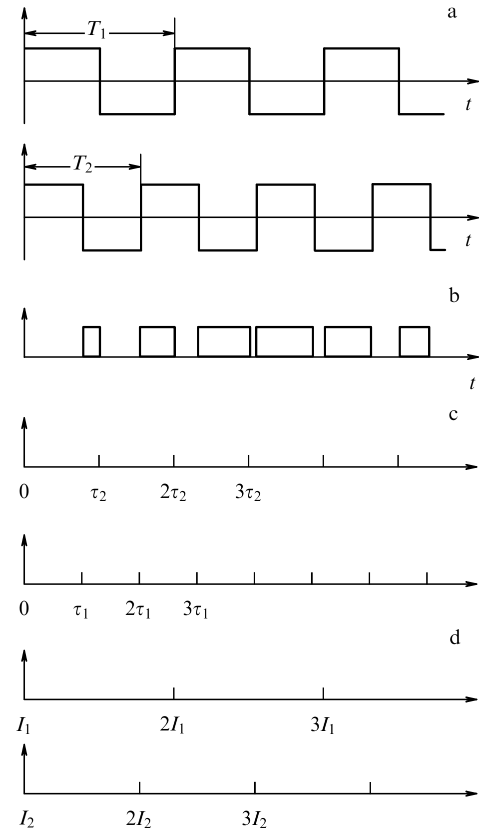

Figure 1. (a) Regular rectangular waves with incommensurable periods, (b) random pulse stream, (c) infinite arithmetical progressions, and (d) interference maxima of rectangular waves with incommensurable periods.

As the sequence $\{i\}$ is rolled up in a circle that contains two equidistant 'unit' intervals $[0 - \tau_1]$ and $[\tau_1 - 2\tau_1]$ equivalent to the intervals $[0 - \pi]$ and $[\pi - 2\pi]$, the sequence $\{k\}$, infinitely 'wound' on this circle, never closes up. Its even and odd serial numbers $k = 0, \tau_2, 2\tau_2, 3\tau_2, \ldots$ are mixed between the 'unit' intervals $[0 - \tau_1]$ and $[\tau_1 - 2\tau_1]$ and are distributed along the circle with a uniform density.

The properties of dynamic chaos of mappings and the possibility to control its interference clearly manifest themselves in the model presented above. Here, the selection and mixing of alternative states (0 and 1) are present, as is the case with Bernoulli systems, along with the mixing of coincidences of all values of i and k belonging to the sequences $\{i\}$ and $\{k\}$. This entails the mixing of an infinite set of other 'random' intervals between any pairs of values of i and k not belonging to the same progression. It follows from the incommensurability of the progression periods that, in an unrolled (statistical) model, equal 'random' intervals cannot be present among the infinite number of 'random' intervals in one period of displacement of the zeroes.

In the model under consideration, irrationality and the sensitivity to displacement result in ergodicity and randomness. Nevertheless, the model dynamics is deterministic and periodic, because the arithmetical progressions are periodic, and the coincidence of progression terms is equivalent to the repetition of the same initial conditions (coincidence of zeroes), which can generate an infinite number of repetitions of random mappings. These properties, which demonstrate the Poincare recurrence theorem, significantly distinguish this model from the classical torus model, which represents Hamiltonian systems, and also from real physical systems.

A radiophysics analog of the model described above is, for instance, a parametric quantizer of the phase of radio oscillations whose frequency is incommensurable with the pump frequency (see Section 6.1). For physical systems, however, the initial conditions are irreproducible and their parameters are inherently unstable, with the result that they cannot be fully deterministic.

"The classical model of quasiperiodic oscillations produced in nature is an attractive invariant torus on the surface of which a quasiperiodic motion without self-intersection of trajectories is possible" [25]. The system of mappings formed from oscillations with proportionate incommensurable frequencies is close in properties to a conservative system in which there exists a continuum of tori surrounding a periodic orbit, and a random transition from one trajectory of motion to another one is possible under perturbations (Arnol'd diffusion [1, 26]).

In the binary version (with a limitation on the oscillation amplitude), the continuum of tori is likely to degenerate into a continuum of phase trajectories that have no self-intersections and regular mutual intersections and are located on the surface of two tori enclosed inside one another. In the formation of a bit sequence, this results in the mixing of individual elements and their combination, which explains its ergodicity and removes the apparent contradiction to the recurrence theorem. A model of this type leaves room for a phase transition from one trajectory to another under the action of perturbations and admits delays between the oscillation phases.

Practical applications of quasiperiodic oscillations require a stable irrational frequency ratio. It can be provided in a parametric system of weakly coupled oscillators between which the pump energy is distributed in proportion to its scattering by the diffusion law (since, naturally, a real system also possesses dissipative properties). An algorithmic approach to the theory of Hamiltonian systems shows that quasiperiodic motion is such a system can be not only irregular (owing to the 'uncoupling' of phase correlations and phase mixing), but stable as well [12].

The limitation of oscillation amplitudes required to employ digital devices and sign correlation leads to Bernoullis-type mappings and broadens the spectrum of a quasiperiodic or almost periodic function $f(x)$. The lengths of the intervals $\tau(\varepsilon)$ that mark off the boundaries of almost-periods of the function $f(x)$ on the number axis, as well as their distribution density, depend on the chosen accuracy of coincidence of repetitive values of the function: $\varepsilon \geqslant |f(x+\tau) - f(x)|$.

As the limitation strengthens, the range of possible values of the function $f(x)$ narrows ($\varepsilon \to 0$), the almost-periods lengthen, and their distribution density decreases indefinitely [19]. Therefore, a full limitation transfers an almost periodic function into the class of aperiodic functions. Under the limitation, the frequency ratio is conserved, and, if they are proportionate (overlapping of the resonances), the almost-period length and the Poincare recurrence point tend to infinity.

It was shown in Ref. [9] that the discrete-to-continuous spectrum transformation in a conservative system takes place only if the system relaxation results from phase mixing (the ergodicity alone proves to be insufficient for the system relaxation). This is also observable in a conventionally conservative dynamic system of weakly coupled oscillators with incommensurable frequencies.

However, as theory [10] and experiment show, the frequency incommensurability alone is also insufficient for a uniform phase mixing in such a system and for a passage to a continuous spectrum: the frequency scales should be proportionate, i.e. the resonances should overlap [10], resulting in phase mixing and an almost complete 'uncoupling' of phase correlations.

The rate of phase mixing is determined by the rate of their relative motion, i.e., the magnitude of the frequency difference of quasiperiodic oscillations, $\Delta\omega = |\omega_1 - \omega_2|$. Accordingly, the relaxation time τ_r is determined by the inverse quantity $2\pi/\Delta\omega$. Therefore, the difference frequency of oscillations with incommensurable frequencies determines the minimal time interval between the samplings and, hence, sets an upper limit on the sampling thinning frequency Ω_s in the production of chaos. At the same time, the difference frequency $\Delta\omega$ determines the character of the autocorrelation function of the formed stream of samplings.

For a sign correlation, the scale disproportion of quasiperiodic oscillations with irrationally related frequencies generates local resonances — sign-variable chains of samplings which expand the interference peak. Strongly separated frequencies ($\omega_1 \ll \omega_0$ or $\omega_0 \ll \omega_1$) lead to quasiperiodic oscillations, to which there correspond almost periodic autocorrelation functions with a discrete spectrum. However, such frequencies can be matched by means of their division. In this case, the study of mapping dynamics of parametric oscillations can begin with retuning the oscillator from the degenerate mode, whereby the fitted frequencies can be made equal and $\Delta\omega = 0$ (see Section 4.2).

In the oscillator retuning, the frequency ratio $m_1 = \omega_1/\omega_2$ runs a sequence of rational and irrational values. In the Poincare mappings local resonances are observed whose duration and regularity are related to a partial phase mixing and depend on the difference frequency $\Delta\omega$ and the sampling frequency. As a result, the transition from the degenerate mode to a nondegenerate one proves to be most complicated for the analysis of dynamics of a Hamiltonian system, although insignificant for the radiophysical applications considered below.

We note that for relatively close irrationally related frequencies a complete 'uncoupling' of phase correlations occurs. The overlapping of incommensurable frequencies in the mixer band does not signify the occurrence of their common resonances. As in the case of phase mixing in the overlapping of nonlinear resonances, considered in Ref. [10], it brings about an increase in the weight of the fractional (irrational) part of the frequency ratio, reflected by the difference frequency $\Delta\omega$, and, accordingly, an enhancement in the effect of randomness.

In the event of strongly separated frequencies ($\omega_1 \ll \omega_2$ or $\omega_1 \gg \omega_2$), the 'uncoupling' of phase correlations is limited. This weakens the effect of randomness and results in local resonances. Therefore, the chaotic properties of the interaction of oscillations with incommensurable frequencies manifest themselves in full measure only provided they are scale-proportionate. In other words, the scale proportion of the frequencies (or overlapping of the resonances) is a condition that enhances disorder in the interaction of oscillations with irrationally related frequencies.

As the difference frequency $\Delta\omega$ increases while the frequencies still remain close, the mode of local resonances is replaced with intermittency. In this mode, the beat frequency is still low, because it is related to a slow variation of the phase difference and to a weak phase mixing. However, for a sign correlation, this now gives rise to shorter binary sequences of random duration. Such streams of samplings with a relatively infrequent random phase change-over are similar to the intermittency observed in chaotic systems on the threshold of synchronization [3].

If the frequencies are more remote but still relatively close, the average beat frequency is already rather high. This is a consequence of the rapid chaotic variation of the phase difference in the intense uniform phase mixing. For a sign correlation, it gives rise to virtually δ-correlated binary streams of samplings with a continuous spectrum. Therefore, by changing the difference frequency of the signals, it is possible to vary the degree of correlation of binary elements in the stream of samplings and the shape of the autocorrelation function from a time-broadened Poisson distribution to a δ-function-type distribution.

Two problems ensue from the above consideration. The first involves preparing a stationary statistical system with a continuous spectrum and frequency matching, i.e. ensuring their scale proportion. In such a system (without thinning out the samplings), the phase of a quantized oscillation should not have more than one inversion in an interval of phase quantization (interval between samplings). In this case, obtaining a δ-correlated sampling stream affords the minimal time of 'uncoupling' of phase correlations of quasiperiodic oscillations. As this takes place, the sampling frequency is bounded from above only by the system relaxation time, i.e., by the difference oscillation frequency $\Delta\omega$. Hence we obtain an approximate operating range of irrational ratios between proportionate incommensurable frequencies (or oscillation periods):

$$\left[(2^{1/2}-1)=0.41\ldots,\ \ldots,(3^{1/2}-1)=0.73\ldots\right],$$

in which, amidst the rational numbers, there exists an infinite number of relatively strong irrational numbers. We note that, if a parametric phase quantizer (PPQ) is used as the sign correlator, the problem is solved by matching the frequency of the quantized signal and the band of the cold circuit of the PPQ [27].

The second problem is related to the appearance of a constant component for a nonideal amplitude limitation, which introduces asymmetry in the distribution of oscillation phases and manifests itself in the inequality of probabilities of binary elements in the stream of large-volume samplings. In this case, the symmetry of distribution of oscillation phases needs to be restored upon amplitude limitation. To eliminate the constant signal component upon the limitation, it is expedient to use frequency division, which transforms the signal into a meander, i.e., ensures equidistant positions of the quantization intervals $[0-\pi]$ and $[\pi-2\pi]$.

We note that equidistant positions of the cells of a classical macrosystem are a necessary condition to prepare it for the corresponding statistical measurements [9]. We also note that this condition is automatically fulfilled in a PPQ, which behaves as a perfect limiter. Because of this, a PPQ can be used as a sign correlator in a circuit for the phase quantization of a strong signal whose frequency is incommensurable with that of the pump signal (see Section 6.1).

On performing the above matchings, the thinning of a Poisson sampling stream with a frequency Ω_s, which does not exceed the difference frequency of the signals $\Delta\omega$, yields a δ-correlated binary model of dynamic chaos. These models are practically unlimited in number and, for a synchronous sampling thinning, they can be made coherent at any points of a two-channel reception with the aid of phase self-tuning of coherent oscillations.

As noted above, the Poisson streams of binary samplings of quasiperiodic oscillations interfere like noise with a single interference peak, which can be controlled owing to the periodicity of each of the oscillations. That is why a controllable interference of such sampling streams can be practically realized only in a four-beam interference scheme with a frequency separation of oscillations that have incommensurable frequencies. When compensating for the delay between coherent oscillations in such a scheme, the maximum of the total intensity, which corresponds to the interference of independent sampling streams, is equal to the sum of the maximum intensities of two-beam interference of each coherent pair of constituent oscillations. It is evident that, with synchronous samplings, the coherence of resultant Poisson streams corresponds to tuning the four-beam interferometer to the peak of total intensity.

3. Generation of quasiperiodic oscillations

To produce dynamic chaos from quasiperiodic oscillations, it is necessary to generate oscillations with stable irrationally related frequencies. In classical systems (owing to the continuity of the parameter phase space), the frequencies of the generated oscillations are inherently irrational, because the fraction of their rational values has a zero Lebesgue measure, while the fraction of the irrational ones has the power of continuum. The natural instability of parameters of uncoupled oscillators cannot afford a stable irrational frequency ratio of their oscillations.

Irrational frequency ratios can be obtained in nonlinear parametric systems. Experiment shows that a stable form of self-organization of the chaotic mode is possible in such systems. Oscillations with irrationally related frequencies can be generated, in particular, in a nondegenerate two-circuit capacitive parametric oscillator operating with a continuous pump voltage.

3.1 Nondegenerate parametric oscillator

A nondegenerate parametric oscillator is a chain of two oscillators weakly coupled through a nonlinear semiconductor capacitor with a capacitance modulated by the voltage produced by a separate pump oscillator. The theory of parametric oscillators and frequency dividers was elaborated in the 1930s [28], and practical applications of these devices in the radio and microwave ranges were investigated in the 1960s; see, for instance, Refs [8, 27, 29] and references therein. Parametric oscillators of different types and designs are described in these papers, including three-circuit ones with a resonance pump chain as well as multi-circuit and distributed schemes.

In our experiments, we used balanced-circuit oscillators (Fig. 2), with partial compensation for the pump voltage at a bridge assembly (D) of D202A-type silicon varicaps operating

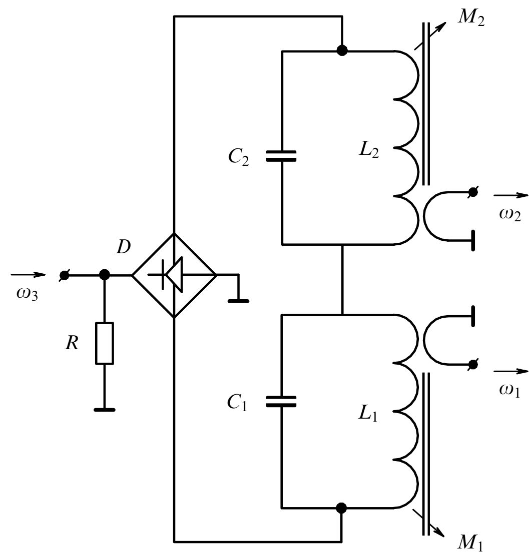

Figure 2. Schematic diagram of a two-circuit parametric oscillator.

with an autobias. A pump with an effective voltage $U_{\mathrm{eff}} = 3.5$ V and a fundamental frequency $\omega_3/2\pi = 3$ MHz was employed to obtain oscillation. The frequencies of the degenerate mode were $\omega_1/2\pi = 1$ MHz and $\omega_2/2\pi = 2$ MHz. The retuning of LC circuits with capacitances $C_1 \approx C_2$ was performed with magnetic cores $M_1 \approx M_2$ of inductance coils $L_1 > L_2$.

According to the Manley–Rowe theorem (see, for instance, Ref. [8]), the sum of partial frequencies $\omega_1 + \omega_2$ in a two-circuit parametric oscillator is equal to the pump frequency ω_3, while the phases of the normal oscillations are related as $\varphi_1(t) + \varphi_2(t) = \varphi_3(t)$ and are, relative to the pump phase, which can be set equal to zero ($\varphi_3 \equiv 0$), anticorrelated (dichotomous): $\varphi_1 = -\varphi_2$. In the degenerate mode, the oscillation frequencies ω_1 and ω_2 and the pump frequency ω_3 are rationally related; the conditions for their self-synchronization are fulfilled in this case, which makes the frequency difference and the phase difference constant.

Retuning the oscillator to the nondegenerate mode implies bringing the eigenfrequencies of the circuits closer together to maintain their coupling with the Manley–Rowe relation retained. In the nondegenerate mode, the conditions for self-synchronization between the oscillations of the resonators are not fulfilled, and the oscillation frequency ratios are in general irrational. In this case, the oscillation phases φ_1 and φ_2 are free relative to the pump phase φ_3, and the phase difference becomes an aperiodic function of time: $\varphi_1 - \varphi_2 = \Delta\varphi(t)$. The pump voltage maintains forced oscillations in each resonator, giving up energy to them synchronously, in pulses and regularly, in general, with a frequency irrationally related to the frequencies of the normal oscillations of the resonators.

Their phases can be imagined to move away from the pump pulses like billiard balls, i.e., specularly symmetric relative to the direction of the incident ball (see Ref. [8] and Fig. 12 below). As a result, the synchronism of the phases φ_1 and φ_2 relative to the pump phase φ_3 persists and the irrational frequency ratio ω_1/ω_2 remains stable. (Such motion is termed stochastic phase synchronism [30].)

Therefore, a complex behavior of the voltage across the nonlinear capacitor of the oscillator arises in the nondegenerate mode. The spectrum of the voltage contains, as in the degenerate mode, two normal frequencies ω_1 and ω_2, and also the pump frequency ω_3. The superposition of these three oscillations with nonoverlapping frequencies is observed as an aperiodic or almost periodic oscillation with a ternary discrete spectrum.

3.2 Regular attractors

As already noted, the aim of our paper is to show potential practical applications of oscillations with irrationally related frequencies. Naturally, this is possible on the basis of the present-day notions of the dynamic chaos in Hamiltonian systems, which is not a result of conventional fluctuations and is stable enough for the control of its interference.

In dissipative systems, according to the Ruelle–Takens theory, temporal instability appears for three or more incommensurable frequencies (turbulence related to the transformation of tori into strange attractors). In contrast, stable states in integrable Hamiltonian systems are, according to the KAM theorem, possible even if the number of irrationally related frequencies is $n > 2$. This is one of the reasons to classify such systems as conventionally conservative, although this is related primarily to the fact that the pump oscillator permanently compensates for the energy losses [3].

At the same time, multi-circuit schemes are also known, both with series and parallel connections of $n > 2$ nonlinear resonators weakly coupled to the common pump voltage generator (see, for instance, Ref. [8] and Section 6.4). Lastly, irrational tori are produced in Hamiltonian systems and by the harmonics of nonlinear combination oscillations. The practical problem to be solved first upon the excitation of a nondegenerate mode of a parametric oscillator is therefore the extraction (filtration) of normal oscillations.

Without considering higher combination constituents, the mathematical model of motion in the three-dimensional phase space of a system with frequencies ω_1, ω_2, and ω_3 shows up as a double-layer torus or as KAM tori (nonresonant tori) enclosed inside one another. According to the KAM theorem, as the perturbation increases, the last to collapse in a multifrequency Hamiltonian system (following resonant and nonresonant tori) are regular KAM tori with the strongest irrational frequency ratio, like, for instance, the 'golden section' [3].

In a regular attractor formed by three limiting cycles of stationary oscillations of a two-circuit parametric oscillator and a pump oscillator, the trajectories of motion are located in different, possibly incommensurable, orbits. The periods of the torus and of the winding of each phase-trajectory layer are related as the oscillation periods that correspond to the pump and oscillation frequencies. Because of this, the trajectory loops on each layer are closed for a rational frequency ratio and opened for an irrational one.

We note that the experimental techniques of observing chaotic attractors of radio circuits are described in Ref. [31]. Smooth attractors can be obtained upon filtration of the normal oscillations of a parametric oscillator.

Figure 3 shows a portion of a double-layer torus which is a model of a Hamiltonian system with frequencies ω_1, ω_2, and ω_3 (which prescribes the period of the double-layer torus). The phase trajectories of oscillations with the frequency ω_1 make up the outer torus K and those of oscillations with the frequency ω_2 the inner torus L. Also shown here is the Poincare plane P, which transversally intersects the tori, as

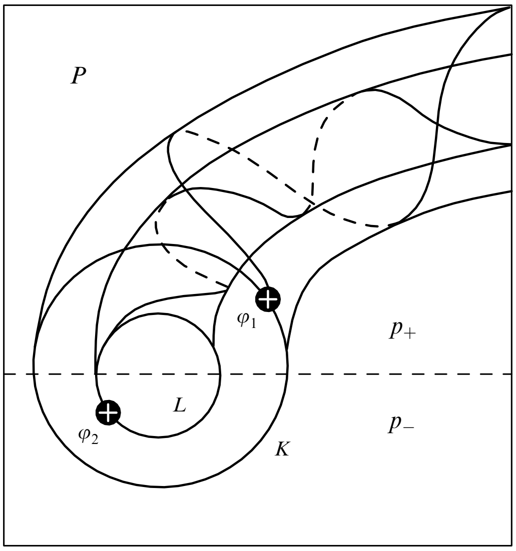

Figure 3. Section of regular attractors by the Poincare plane.

the phase trajectories are strobed with the pump frequency ω_3, and the points of intersection of the phase trajectories φ_1 and φ_2 with the plane P.

The sequences of intersection points $\{\varphi_{1i}\}$ and $\{\varphi_{2k}\}$, where i and k assume the values 1, 2, 3 ..., are Poincare mappings that relate the dynamic states of the system (phase and energy) stroboscopically, i.e., at a time interval T. The Poincare mappings form a discrete binary basis for the production of dynamic chaos, i.e., a complete (base) chaotic model. In the formation of different random sequences, these mappings are commonly thinned out with a frequency $\Omega_s = \omega_3/d$ multiple to the pump frequency, where d is the so-called Poincare number of rotations.

If the frequencies are incommensurable, the phase trajectories cover the entire torus with a uniform density. This property referred to as the transitivity on a torus is, according to the Birkhoff theorem, equivalent to ergodicity [31]. By the Poincare recurrence theorem, the trajectory of motion on any torus returns to a given neighborhood of some point an infinite number of times. On the one hand, this leads to ergodicity, but on the other this is precisely the reason why ergodicity is not a sufficient property for the production of chaos. In the case of a torus, however, this process may be accompanied with an even stronger property — mixing. Mixing is one of the central concepts of modern ergodic theory since it is related to probability, i.e., to the generation of statistically independent events [9].

The motion on a torus is defined by the Arnol'd mapping [3, 26], which possesses the property of mixing for irrationally related frequencies, i.e., it distorts any surface element so strongly that the element behaves as if it spread over the entire surface with time. This accounts for the virtually uniform mixing of the phase sequences $\{\varphi_{1i}\}$ and $\{\varphi_{2k}\}$ $(i, k = 1, 2, 3, \ldots)$ in the K and L contours (intersections of the tori by the plane P). Also mixed in this case are the hits of phase on the half-planes p_+ and p_-, which conventionally denote equidistant intervals of their quantization. Experimental scans of the thinned-out Poincare mappings of resonant and nonresonant tori are given in Fig. 6 (see below).

It was shown in Ref. [10] that for a uniform phase mixing the spectrum of quasiperiodic oscillations becomes continuous and "... the lower bound for the return frequency of the trajectory of motion to a given neighborhood of some point proves to be zero, despite the fact that the return is bound to occur an infinite number of times in accordance with the Poincare theory. Therefore, motion is inherently fluctuant, and the statement that statistical notions are inappropriate on time intervals exceeding the Poincare cycle prove to be erroneous." In the case of quantization of the phase of quasiperiodic oscillations in the production of dynamic chaos, both individual elements of the bit sequence and any of their combinations are mixed, which does not, naturally, contradict the recurrence theorem.

For a uniform mixing of oscillation phases φ_1 and φ_2, their distributions over the contours K and L are of the form $W(\varphi_{1,2}) = 1/2\pi$ and their hits on the half-planes p_+ and p_- are not correlated. This determines the distribution of the corresponding signed samplings in the formation of chaos. We note that, again, for a binary phase quantization of a strong signal, which is incommensurable with the pump of a PPQ and has a uniform phase distribution, the probabilities of the corresponding values 0 and π (or 0 and 1) are [27] $P(0) = 1/2$, $P(1) = 1 - P(0) = 1/2$.

As noted in Section 1, theory [10] and experiment show that a uniform phase mixing is possible only with the overlapping of resonances, i.e., if the frequencies are scale-proportionate. If the oscillation frequencies in a real oscillator are disproportionate, they should therefore be pre-matched to be used for chaos production. In this case, the chaotic model will differ from a real oscillation system not only by the thinned-out dynamics mapping and overlapping of the resonances, but by the continuous spectrum as well.

Figure 4a shows the development of several periods of a nonresonant torus T and the phase trajectory with a period $T_1 \gg T$ in the case of unmatched incommensurable frequencies ω and ω_1. Figure 4b depicts the corresponding sequence of 'oppositely signed' Poincare mappings (bright and dark). This is a weak-mixing case, whereby a fraction of the phase samplings (shaded in Fig. 4b and marked in Fig. 4a) do not regularly participate in the process. Limited sequences of stable phase correlations make up local resonances, which leads to a discrete spectrum and an almost periodic auto-

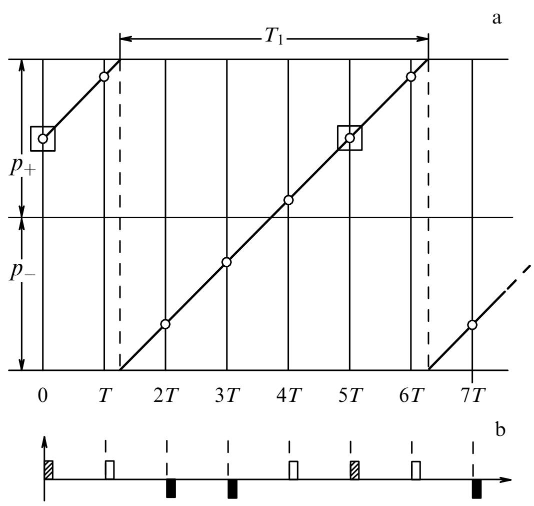

Figure 4. Development of a torus (arbitrary scale).

correlation function of the chaotic model. In this case, only the oppositely signed neighboring mappings — bright and dark — are in fact mixed up.

3.3 Hidden parameters of regular attractors

Along with normal frequencies ω_1 and ω_2, the spectrum of a two-circuit oscillator contains an infinite number of higher harmonics and combination frequencies, which descend in amplitude and are of the form $n_1\omega_1 + n_2\omega_2$, where n_1 and n_2 are integers. These constituents, which are produced by the capacitance nonlinearity of the varicaps, partially fall into the resonance bands of the oscillators. If the amplitudes are high enough, they are responsible for either the self-synchronization of normal oscillations (a degenerate mode) or for self-modulation and period doubling — effects occurring on the threshold of synchronization and fractal in character [3, 6].

In particular, in the retuning of the frequencies of the resonators and the pump, a two-circuit parametric oscillator exhibits a fractal structure of alternating degenerate and nondegenerate modes, which was considered in Ref. [8] prior to the discovery of dynamic chaos in such systems. Naturally, it was associated with the loss of synchronization and attributed to the effect of fluctuations. The domains of excitation of subharmonics of integer multiplicity $1/n = 1/2$, $1/3, \ldots$ are separated by narrower domains of excitation of subharmonics of fractional multiplicity k/n, which, in turn, are separated by the states of instability, wherein the ratios of the normal frequencies are irrational (k and n are natural, mutually prime numbers: $n > k \geqslant 1$).

If the ratio of the lower oscillation frequency to the pump frequency is equal, for instance, to the irrational number $m_1 = \omega_1/\omega_3 = (2^{1/2} - 1) = 0.414\ldots$ lying between the first two subharmonics of integer multiplicity, it is surrounded by the infinite sequence of rational numbers

$$\frac{1}{3} < \frac{2}{5} < \ldots < \frac{239}{577} < \ldots < m_1 < \ldots < \frac{169}{408} < \ldots < \frac{3}{7} < \frac{1}{2}\,,$$

whose denominator determines the order of the subharmonic oscillation. Related to these numbers are the domains of high-order subharmonic generation separated by domains of the nondegenerate mode. Therefore, there exists an infinite cascade of period doubling, which is responsible for the occurrence of subharmonics with frequencies $2^{-n}\omega_0$, where ω_0 is the normal frequency. Experimental observation of subharmonics of order $n = 500$ was reported in Ref. [8].

Not nearly all oscillations with rational frequency ratios are provided with a reliable self-synchronization and are realized in the form of high-order subharmonics, and therefore the nondegenerate mode is observed in a relatively broad pump frequency band (about 1%). To put it differently, the set of local resonances (multiple frequencies) is hidden in the nondegenerate mode owing to the absence of self-synchronization and superficial evidence for it. In this case, the irrationality of, for instance, the K torus relative to the pump is practically possible if the L torus is rational (see Fig. 3), which, of course, also expands the nondegenerate-mode band.

In experiment, hidden (local) resonances occupy about 50% of the nondegenerate-mode domain and manifest themselves as almost periodic autocorrelation functions with a specific envelope whose aperiodic decay stems from the limited coherence time of the pump phase.

3.4 Frequency stability of quasiperiodic oscillations

Upon narrow-band filtration, the normal oscillations of a two-circuit parametric oscillator can be viewed from a practical standpoint as purely periodic functions. In our experimental setup, a Ch6-31 frequency synthesizer with a frequency instability within 10^{-8} was used as the pump oscillator. However, according to Ref. [33], the actual stability of parametric oscillations can be made much higher than the pump stability, this being so not only due to its frequency division.

Akhmanov et al. [33] derived a condition for the constancy of oscillation frequencies under slow correlated deviations of intrinsic frequencies possible for different temperature instabilities of the oscillator circuit parameters. This condition implies that the oscillation frequencies remain invariable if the increments of the intrinsic frequencies under temperature variations are related as normal oscillation bands. Since this condition can be satisfied in our experiments, the frequency instability can be assumed to be of the order of 10^{-9}.

Therefore, it is possible to ensure a fairly high stability of incommensurable frequencies in the nondegenerate mode. This property of radio signals of this type can be very useful for specific radiophysics applications: when controlling the interference of deterministic chaos, the stability of incommensurable frequencies (or the phase coherence time of each quasiperiodic oscillation) determines the maximum permissible delay that retains the coherence of chaotic models.

4. Production of dynamic chaos

To impart a simple quantized form, convenient for radio transmission and delay control, to complex parametric oscillations and, in doing so, retain the periodicity and the incommensurability of the normal frequencies, match (bring closer together) the frequencies, and ensure equal distances between the points where the phase passes through zero, some requisite transformations were used, viz., the filtration of the normal oscillations (separation of the frequencies ω_1 and ω_2), limitation of their amplitudes to logical levels of 0 and 1, and frequency division by an even integer. This allowed observing the phase correlation of different oscillations, including those of different parametric oscillators with a common pump. With the aid of frequency dividers, all signals were matched with the sixth pump subharmonic $\Omega_3 = \omega_3/6$; thus, they had the symmetric appearance of meanders with periods T_1, T_2, T_3 and T_1', T_2' and close frequencies of the order of 0.5 MHz:

$$\Omega_1 = \frac{\omega_1}{2} \approx \Omega_2 = \frac{\omega_2}{4} \approx \Omega_1' = \frac{\omega_1'}{2} \approx \Omega_2' = \frac{\omega_2'}{4} \approx \Omega_3 = \frac{\omega_3}{6}\,. \tag{1}$$

In the matching of different frequencies ω_1 and ω_2 of one oscillator, divisions by different integers destroys the mirror symmetry of their phases. For this reason, in studies of the dichotomous properties of the phases of parametric oscillations, we used signals with close frequencies (ω_1 and ω_1', ω_2 and ω_2') produced by different two-circuit nondegenerate oscillators with a common pump at a frequency ω_3.

4.1 Experimental setup

A schematic structural diagram of the experimental setup is shown in Fig. 5. Two identical two-circuit parametric

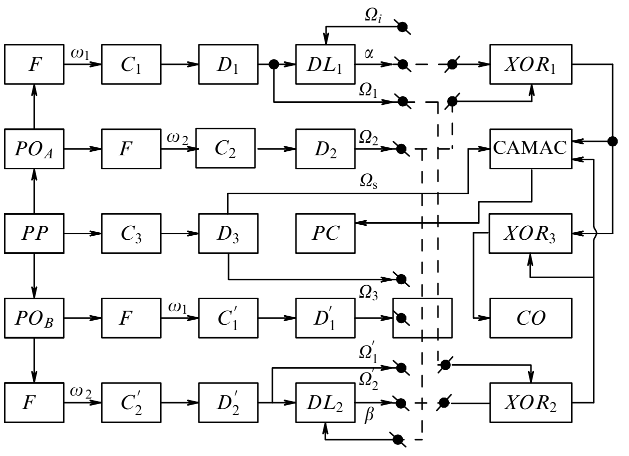

Figure 5. Schematic structural diagram of the facility for mapping quasiperiodic oscillations and observing their interference.

oscillators PO_A and PO_B operate with a common generator of continuous pump PP. Filters F separate out the frequencies of normal oscillations. Comparators C limit the amplitudes to logical levels of 0 and 1. Frequency dividers D provide frequency matching and the formation of meanders. The oscillation phases are discretely varied by delay lines DL. The sign correlation is performed by phase comparators XOR — modulus-two adders.

Employing a frequency synthesizer as a pump oscillator and digital delay lines furnished a good reproducibility of experimental results. All devices, with the exception of the parametric oscillators and filters F, were made of digital components [34].

Different ways of connecting the signals with frequencies (1) to phase comparators were realized experimentally; the dashed line in Fig. 5 shows the connection used to study the interference of signals with frequencies Ω_1 and Ω_2. The setup also involved power supply units, a dual-beam oscilloscope, and a spectrum analyzer, which are not shown in Fig. 5.

We note some operational features of the facility. The sign correlation of signals in the XOR comparators yields a binary Poisson stream — the basic model of dynamic chaos. Thinning out the basic model in CAMAC permits us to obtain a virtually unlimited number of different chaotic models corresponding to different sampling frequencies Ω_s, which could be set either by a separate oscillator, or by means of an additional division of any one of the frequencies (1).

As already noted, the maximum sampling frequency Ω_s is limited from above by the difference frequency of incommensurable oscillations: $\Omega_s \leqslant \Delta\Omega$. This condition defines the boundary mode between intermittency and dynamic chaos. However, for a CAMAC transmission capacity of no higher than 30 kHz, the frequency transformation (1) was quite sufficient to verify this condition.

The operations of sign correlation and sampling are as follows. If the signs of meanders (for instance, T_1 and T_2) at the inputs of the phase comparator XOR coincide, the level of the signal voltage at the comparator output is minimal, and a value of 0 is assigned to it; if the signs are opposite, this level is maximal, and a value of 1 is assigned to it. For an uninterrupted succession of incommensurable meanders at the inputs of XOR, an aperiodic variation of logical voltage levels is observed at its output. They are fed to the CAMAC input, where the stream is thinned with a sampling frequency Ω_s. The resultant bit stream (the final result of chaos formation) is brought out to a display screen via CAMAC for visual diagnostics and recording. The autocorrelation function of the stream was calculated with a personal computer PC. Spectrum observations were made at the output of the oscillators, filters, and XOR comparators.

4.2 Experimental conditions and results

The indication of mappings in the form of a bit stream on a display is convenient in that it permits observing the evolution of dynamic modes when retuning the circuits of a parametric oscillator. In the raster of samplings (the so-called dump), which reflects motion in the system, characteristic manifestations of this evolution are observed: hidden (local) resonances with regular controllable sequences of zeroes and unities, intermittancy with long (also controllable) random sequences of these symbols, and chaos, in whose kaleidoscopic dynamics regular correlations and possibilities for control cannot be found. Among the visual effects of the chaotic raster is the apparent insensitivity of the system to external perturbations, although in reality it is in chaotic dynamics that its highest sensitivity is manifested.

Dynamic chaos was observed in a 'deeply nondegenerate' mode, wherein neither self-synchronization nor attendant threshold effects can be found: there are no period-doubling bifurcations. The absence of such bifurcations is an indication that the system is stable [35]. The stability is also manifested in the abruptness of transitions from one dynamic state to another (see Section 4.3) and in the stable repeatability of these effects without no hysteretic effects.

Nevertheless, the nondegenerate mode commences with bifurcations like oscillation phase jumps related to the collapse of resonant tori with a decrease in the self-synchronization energy. The traces of these collapses are local resonances hidden to an oscilloscopic inspection but readily observable on the display screen.

Figure 6 shows typical fragments of bit streams (top) and their autocorrelation functions (bottom), which correspond to (a) local resonances, (b) intermittancy, and (c) dynamic chaos. To enhance contrast, the zeroes and unities in dumps are represented by spaces and rectangles. These experimental results were obtained for a sign correlation of matched signals of the PO_A oscillator with close frequencies $\Omega_1 = \omega_1/2$ and $\Omega_2 = \omega_2/4$ at a sampling frequency of 10^4 Hz not synchronized with the frequency of the pump oscillator PP. Similar results are also observed for a sign correlation of other matched frequencies (1).

We note that thinning out samplings with the frequency of a 'free' oscillator, uncoupled to the pump oscillator, does not hinder observing chaos. However, in the observation of hidden resonances this introduces (floating) stroboscopic distortions, which show up in raster dynamics as motion turbulent in time. If the frequency of the 'free' oscillator were irrationally related to the frequency of the oscillations under study, this would lead to chaos in the thinning of the mappings of hidden resonances. However, as one could expect, no 'masking' of hidden resonances by chaos is observed here.

To put it differently, uncoupled oscillators do not furnish stable irrationally related frequencies. A relation of this sort is an intrinsic property of a Hamiltonian system [35]. This can also be regarded as a manifestation of self-organization of the chaotic mode in a conventionally conservative dynamic system possessing an extremely high stability and the maximum entropy.

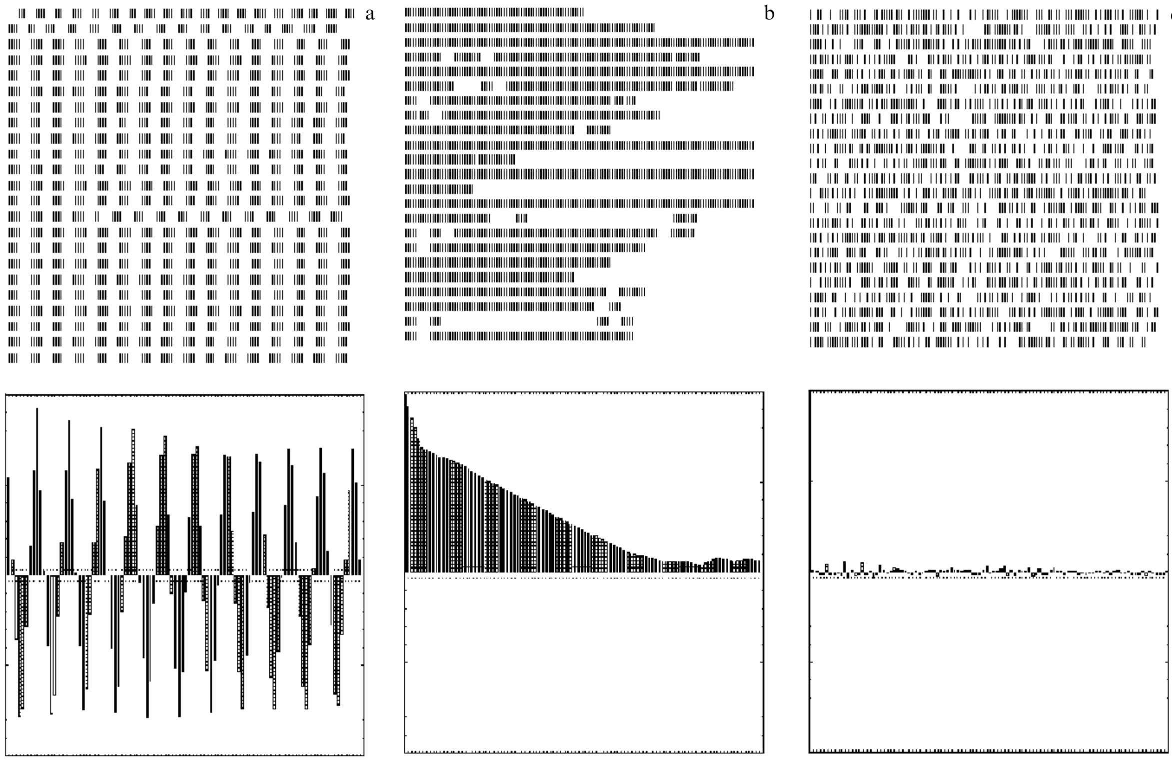

Figure 6. Dumps of Poincare mappings (top) and their autocorrelation functions (bottom): (a) local resonances, (b) intermittancy, (c) dynamic chaos.

Such properties are also realized in more complex systems. In particular, it is possible to observe experimentally that the stochastic phase synchronism involves oscillations of several nondegenerate oscillators coupled only by a common pump (see Section 6.4).

The nondegenerate mode originates as the self-synchronization of parametric oscillations is lost; the self-synchronization mechanism was comprehensively studied in Ref. [8]. When the pump frequency is retuned, self-synchronization fails earlier than one of the *K* or *L* tori (see Fig. 3) becoming irrational, i.e., the irrationality of at least one of the tori occurs in the nondegenerate mode some distance away from the self-synchronization threshold. This is one of the reasons for the relative broadening of the band of the nondegenerate mode with a variation of the pump frequency. Half this band (corresponding to 1% of the pump frequency) can be used for a crude practical estimate of stability of the chaotic mode. We note that the band in which all frequencies are irrationally related proves to be substantially narrower, of the order of 1% of the pump frequency.

A parametric oscillator as a physical system is, naturally, subject to technical fluctuations, the temperature instability of its circuit elements, and the flicker noise of the quartz pump oscillator. The phase diffusion of parametric oscillations arising from such perturbations can be observed in the mappings of hidden resonances and in their autocorrelation functions.

According to the Manley – Rowe theorem, slow fluctuations should be dichotomous relative to the pump phase. This is likely to account for a virtually perfect coherence of two chaotic models synchronously obtained for a sign correlation of the oscillations of two pairs of close frequencies from two different parametric oscillators operated with a common pump. Another version follows from the KAM theorem that irrational tori are stable against small perturbations, which distort the trajectories of motion only slightly, but do not eliminate their irrational relation.

A two-circuit parametric oscillator as a coupled system is characterized by a common capacitance and a common inductance [8]. That is why the retuning of one circuit results in a partial retuning of the frequency of the second one, which approaches the frequency of the first circuit. In the degenerate mode with a frequency ratio $1:2:3$, the matched frequencies are equal: $\Omega_1 = \Omega_2 = \Omega_3$. In the oscillator retuning to a nondegenerate mode for a constant voltage and a fixed pump frequency ω_3, the frequency $\Omega_3 = \omega_3/6$ matched with it remains constant, and the oscillation frequencies come closer together.

As the partial frequencies approach each other, $\omega_1 \to \omega_1''$ and $\omega_2 \to \omega_2''$ (Fig. 7a), the matched frequencies Ω_1'' and Ω_2''

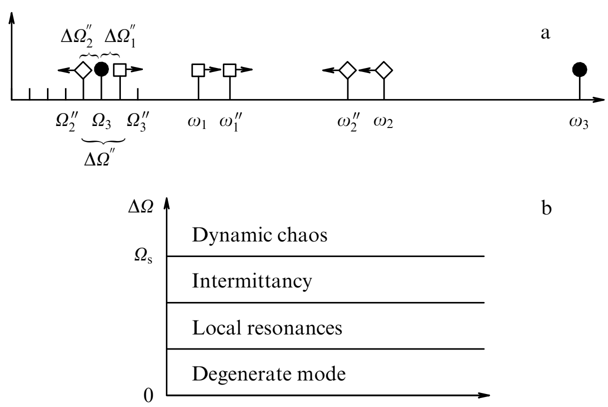

Figure 7. (a) Retuning of partial and matched frequencies of a nondegenerate oscillator with a fixed pump frequency and (b) diagram of the domains of dynamic chaos evolution.

diverge, and their difference frequency $\Delta\Omega'' = \Omega_1'' - \Omega_2''$ increases. The difference frequencies between the matched partial frequencies and the pump frequency also increase: $\Delta\Omega_1'' = \Omega_3 - \Omega_1''$ and $\Delta\Omega_2'' = \Omega_3 - \Omega_2''$. In experiment, the matched frequencies were measured with the frequency meter CO at the outputs of the frequency dividers D (see Fig. 5).

Of special interest for chaos production are the KAM tori with the strongest irrational frequency ratio and high difference frequencies, for instance $\omega_1/\omega_2 = (5^{1/2} - 1)/2 = 0.618\ldots$, known as the 'golden section'. It is shown in an arbitrary scale in the following diagram, where the pump frequency ω_3 is taken as the unit frequency:

$\omega_3 \equiv 1$

$\omega_2 = 0.618\ldots$ | $\omega_1 = 0.382\ldots$

Strong irrational ratios ensure a dependable stability of nonresonant tori and the highest difference frequency $\Delta\Omega$ of the matched frequencies needed to attain a high sampling frequency Ω_s in the production of chaos.

The 'golden section' of the pump frequency $\omega_3/2\pi = 3$ MHz gives frequencies (from here on they are expressed in kilohertz) $\omega_1/2\pi = 1146\ldots$, $\omega_2/2\pi = 1854\ldots$, strong irrational ratios $\omega_2/\omega_3 = 0.618\ldots \approx \omega_1/\omega_2 = 0.618\ldots$ (the 'golden section'), $\omega_1/\omega_3 = 0.382\ldots$, and the highest difference frequency equal to

$$\frac{|\Delta\omega_3|}{2\pi} = \frac{|\omega_3 - \omega_1|}{2\pi} = 1854\ldots$$

For comparison we note a strong irrational ratio $(2^{1/2} - 1)$. The corresponding section is shown in the diagram

$\omega_3 \equiv 1$

$\omega_2 = 0.586\ldots$ | $\omega_1 = 0.414\ldots$

For a pump frequency $\omega_3/2\pi = 3$ MHz, this section corresponds to the frequencies $\omega_1/2\pi = 1243\ldots$, $\omega_2/2\pi = 1757\ldots$, close irrational ratios $\omega_1/\omega_2 = 0.707\ldots$, $\omega_1/\omega_3 = 0.414\ldots$, $\omega_2/\omega_3 = 0.586\ldots$, and the highest difference frequency equal to

$$\frac{\Delta\omega_3}{2\pi} = \frac{|\omega_3 - \omega_1|}{2\pi} = 1757\ldots$$

Since the matched frequencies of the degenerate mode are

$$\frac{\Omega_1}{2\pi} = \frac{\omega_1}{4\pi} = \frac{\Omega_2}{2\pi} = \frac{\omega_2}{8\pi} = \frac{\Omega_3}{2\pi} = \frac{\omega_3}{12\pi} = 500\ \text{kHz}\,,$$

for a matched pump frequency $\Omega_3/2\pi = 500$ kHz, the 'golden section' 0.618... gives the maximum difference frequency equal to

$$\frac{|\Delta\Omega_1|}{2\pi} = \frac{|\Omega_1 - \Omega_2|}{2\pi} = 573\ldots - 212\ldots = 361\ldots,$$

and the maximum difference frequency given by the section 0.414... is two times lower:

$$\frac{|\Delta\Omega_1|}{2\pi} = \frac{|\Omega_1 - \Omega_2|}{2\pi} = 621\ldots - 439\ldots = 182\ldots$$

Therefore, the selection of a 'section' and the corresponding frequencies with the highest difference is of significance in the production of chaos. At the same time, apart from the choice of the 'section', it is important that the difference frequencies increase under the transformation of frequencies by division in accordance with expression (1) (Fig. 7a), allowing the solution of both problems: to ensure a stable irrationality and a fast response (supposedly up to 10% of the pump frequency).

A physical experiment does not allow an observation of the fine fractal structure of the dynamics evolution in the mappings of regular attractors. This shows up in that the sizes of the domains of local resonances, intermittancy, and chaos do not correspond to the Lebesgue measures of rational and irrational numbers. This could be related to the fact that a limited resonator Q-factor results in an overlap of resonance and nonresonance states with the oscillator retuning, and the frequency ratios do not resolve the fine structure of the continuous series of resonant and nonresonant tori.

In the dynamics of mappings obtained from matched signals (1), it is possible to isolate only a limited series of three stable domains comprising local resonances, intermittancy, and dynamic chaos. All the mapping domains with different forms of motion are determined and distinguished by the rate of phase mixing, which is governed by the difference of matched frequencies $\Delta\Omega = \Omega_1 - \Omega_2$ and is measured relative to the sampling frequency.

As the difference frequency $\Delta\Omega$ is varied, the sampling frequency Ω_s should remain constant, because it characterizes the mapping rate: low difference frequencies ($\Delta\Omega < \Omega_s$) correspond to a slow phase mixing and high frequencies ($\Delta\Omega > \Omega_s$) to a fast one. Therefore, by varying the ratio of frequencies $\Delta\Omega$ and Ω_s it is possible to control the distribution of elements in chaotic streams.

4.3 Evolution of the chaos in Hamiltonian systems

As the oscillation frequencies are retuned for a constant pump frequency, a specific evolution of the mapping dynamics is observed, in which chaos borders on intermittancy. This corresponds to one of the known typical transitions to chaos [3]. At the same time, it is also possible to point out other states of the nondegenerate mode, also related to the oscillation-to-pump frequency ratios.

The evolution of chaos commences with local resonances, which are observed in the mappings formed from matched oscillation frequencies (1). As one of the circuits is retuned, the self-synchronization breaks down and the oscillator passes to the nondegenerate mode with non-synchronized regular attractors, in which the oscillation phases are relatively free [8]. In the sequence of samplings this shows up in the form of local resonances within limited time intervals, which are clearly visible in phase trajectories and have periodic autocorrelation functions with slowly decaying envelopes (Fig. 6a).

In the passage to this state, the breakdown of self-synchronization is due to the reduction of the energy of the combination constituents that fall within the transparency bands of the resonators, but no stable irrational relation between the oscillations still takes place here. There is no full mixing of the phases of local resonances in their relative diffusion, and therefore the signal spectrum at the output of XOR_1 is discrete. Local resonances were also observed with phase correlations of disproportionate irrationally related frequencies for their individual matching unrelated to expressions (1). The domain of these resonances is located above the chaos domain shown in Fig. 7b.

Intermittancy is observed for low difference frequencies (of the order of 10^3 Hz) and a slow phase mixing ($\Omega_s = 10^4$ Hz). In this state, the ratio of the frequency of the retuned circuit to the pump frequency and, hence, to the frequency of the second circuit becomes irrational, and the phase difference is a random function of time. In this case, local resonances pass into intermittancy — long random sequences of zeroes and unities, corresponding to irregular changes of the phase difference and having a Poisson-type autocorrelation function (Fig. 6b).

The transition to chaos from intermittancy for a constant sampling frequency $\Omega_s = 10^4$ Hz was observed at close difference frequencies $\Delta\Omega$ of the order of 10^4 Hz. The ratio between the frequency of the retuned circuit and the pump frequency remains irrational in this state. However, as the difference frequency increases, the relaxation time shortens, and the phase mixing quickens. For a constant mapping rate, this is attended with the transformation of the autocorrelation function into a δ-shaped function (Fig. 6c). In the production of chaos from matched frequencies (1), the upper limit of the thinning frequency supposedly approaches a frequency of the order of 10^5 Hz.

At the same time, as shown by experiment, for a stable irrational ratio between the frequencies of the first circuit and the pump, the ratio of the frequency of the second circuit to the pump frequency may be close to a rational number. In this case, the phase difference of the matched frequencies is a slowly varying, almost periodic function, and its autocorrelation function has an aperiodic envelope similar to that depicted in Fig. 6a.

The common feature of transitions from local resonances to intermittancy and from intermittancy to chaos is that in mappings they are abrupt and bifurcation-like. Figure 7b shows an experimental diagram of the sequence of the domains of different dynamic modes in relation to the difference between the matched frequencies $\Delta\Omega$. The boundary between the domains of intermittancy and dynamic chaos is marked by the corresponding value of the sampling frequency Ω_s in Fig. 7b.

The dynamics of mappings obtained from the matched signals of the pump and the second oscillator circuit evolves similarly with the frequency retuning of this circuit. When all the frequencies in the system are irrationally related, dynamic chaos is observed in the correlation of any pair of matched frequencies.

5. Dynamic chaos interference

The above-considered principle of dynamic chaos production in Hamiltonian systems admits long-range transmission of its two quasiperiodic constituents, for instance, in the form of narrow-band signals (prior to frequency matching) or in the form of envelopes of different phase-manipulated carriers (for matched frequencies). In either of these representations, the quasiperiodic oscillations can be received by separate receivers with separated incommensurable frequencies Ω_1 and Ω_2, and the interference of a coherent pair of oscillations with the frequency Ω_1 can be considered separately from the interference of a coherent pair of oscillations with the frequency Ω_2. The intensities I_1 and I_2 in the case of interference of coherent oscillation pairs with equal quantized amplitudes A are [35]

$$I_1 = 2A^2(1 + \cos\alpha)\,, \quad I_2 = 2A^2(1 + \cos\beta)\,, \tag{2}$$

where $\alpha = 2\pi t/T_1$ and $\beta = 2\pi t/T_2$ are the phase differences between the coherent oscillations with the frequencies Ω_1 and Ω_2, respectively.

The interference peaks of rectangular waves with periods T_1 and T_2 produced from the corresponding quasiperiodic oscillations can be conventionally represented as shown in the diagrams given in Fig. 1d. Since the T_1/T_2 ratio is given by an irrational number, the intensity peaks I_1 and I_2 will recur at irrationally related periods T_1 and T_2. To put it otherwise, both sequences of peaks located at points $I_1, 2I_1, 3I_2, \ldots$ and $I_2, 2I_2, 3I_2, \ldots$ are incommensurable, as are two arithmetical progressions whose terms I_1 and I_2 have no common multiple. It is therefore evident that, by superposing the diagrams, it is possible to bring into coincidence only two peaks: either at zero, or at any other (single) point.

In the case of the propagation of two adequate signals in time, such alignment of peaks is possible only in one circumference in a plane or in one surface in space. This can be done either by shifting the upper row of peaks (in Fig. 1d) with the aid of a common delay of the interfering signals with the frequency Ω_1 relative to the signals with the frequency Ω_2 within the limits of the period T_1, or by shifting the lower row of peaks with the aid of a delay of the interfering pair of signals with the frequency Ω_2 relative to the signal pair with the frequency Ω_1 within the limits of the period T_2.

On the superposition of interference patterns, for instance, employing a common correlator, the visibility of the common interference will be independent of which of the two peaks are brought into coincidence, and there will always be one (and only one) point of a common interference peak. Moreover, the periodicity of signals with the frequencies Ω_1 and Ω_2 makes it possible to bring into coincidence the intensity peaks I_1 and I_2 at any (single) point by compensating for the delay α between the received signals with the frequency Ω_1 and by compensating for the delay β between the received signals with the frequency Ω_2, and thereby to ensure the production of a common interference peak.

Therefore, bringing into coincidence the intensity peaks I_1 and I_2 obtained in the two-beam interference with a frequency separation of the signals with frequencies Ω_1 and Ω_2 is equivalent to a four-beam interference, whereby the total intensity I_L is determined by the sum of intensities I_1 and I_2:

$$I_\Sigma = I_1 + I_2 = 2A^2(2 + \cos\alpha + \cos\beta)\,. \tag{3}$$

One can see from relation (3) that the energy of the signals doubles for a four-beam interference with a frequency separation; the average value of the total intensity I_Σ, which corresponds to the absence of correlation for $\alpha = \beta = \pm\pi/2$, also doubles and is equal to $4A^2$. The maximum value of the total intensity, which corresponds to a total correlation of chaotic sequences for $\alpha = \beta = 0$, is equal to $8A^2$. The minimal value of I_Σ, which corresponds to a total anticorrelation of the sequences, is zero. There also appear new intensity values of mixed states ($\alpha = 0, \beta = \pm\pi/2$) and ($\alpha = \pm\pi/2, \beta = 0$) equal to $6A^2$.

Relation (3) implies that it is possible to separately control the interference of oscillations with irrationally related frequencies by varying the parameters α and β in different frequency channels with the use of the delay between the oscillations with equal frequencies within the limits of their periods. Therefore, for in-phase interfering oscillations (for $\alpha = \beta = \pm n\pi$, where $n = 0, 1, 2, \ldots$) it is possible to obtain coherent chaotic bit sequences in correlators connected to spatially diversed receivers. In a four-beam interferometer of this type, synchronization can be obtained with the aid of two diversed circuits of phase self-tuning and synchronous samplings determined by a separate synchronization system.

The experimental setup diagrammed in Fig. 5 allows the realization of the four-beam interferometer with a frequency separation. In experiments the correlations of different combinations of parametric oscillations of oscillators PO_A and PO_B were considered. This figure shows one of the possible variants of investigation of the interference of oscillations with matched frequencies Ω_1 and Ω_2 of the oscillator PO_A. The sign correlations were analyzed with comparators XOR_1 and XOR_2, and the results were compared in comparator XOR_3.

The degree of correlation M of two chaotic bit sequences was measured with the counter CO at the output of the XOR_3 phase comparator. To introduce a delay α between the phases of the oscillations with the frequency Ω_1 and a delay β between the phases of the oscillations with the frequency Ω_2, we used delay lines DL_1 and DL_2 based on type-K155LN1 microcircuit elements. Since the frequencies are close, the delay increments in the α and β angles were approximately equal, about 0.08 rad.

Figure 8 shows the experimental dependence of the degree of correlation M between two chaotic bit sequences on the phase difference α between the signals with the frequency Ω_1 for $\beta = 0$. The dependence on the phase difference β between the signals with the frequency Ω_2 for $\alpha = 0$ is similar. In each channel, it was possible to employ either different comparators and different frequency dividers or the same pair of comparators and frequency dividers common to the two sequences compared. This allowed us to estimate the likelihood of errors introduced by the comparators into the correlation of sequences: it was found to be 0.05.

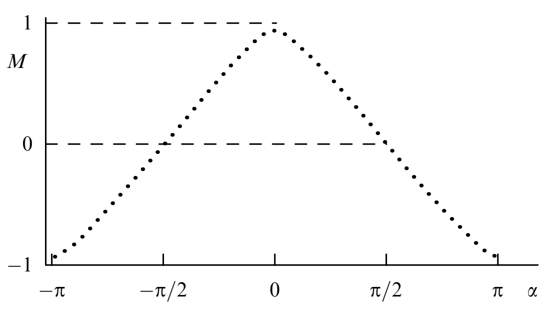

Figure 8. Degree of correlation M of two chaotic sequences as a function of the phase difference α between the signals with the frequency Ω_1.

The results given in Fig. 8 show how the random and periodic properties of the interaction of quasiperiodic radio signals manifest themselves in the four-beam interference. Unlike the interference of repetitive noise-type signals, here, naturally, only one interference peak is observed at the zeroes ($\alpha = \beta = 0$), if the coherent oscillations are inphase at the reception sites. In this case, the synchronous samplings of their sign correlation, i.e., the binary sequences of dynamic chaos, are also coherent.

The coherence of chaotic sequences is determined by the phase coherence of the constituent oscillations and depends not only on the phase delay, but also on the general time delay between them, and the possible bounds on the latter are limited by the frequency stability of the oscillations. This signifies that a narrow-band noise signal can be employed as the pump of parametric oscillators for small delays of the constituent signals. Under certain conditions, these circumstances can be used in specific radiophysics applications, for instance, in cryptographic communication systems to improve their operating immunity.

6. Radiophysics applications

We now consider random number generation, the application of irrationally related frequencies in radar and cryptography, and also the modeling of quantum correlations (see Section 6.4). So far quasiperiodic radio signals have not been applied to location and cryptography. However, experimental investigations have shown that they can be used in multichannel communication systems with frequency separation to produce N-sequences — aperiodic noise-type signals (NTSs) having single-lobe autocorrelation functions. It appears reasonable to compare the signals of this kind with their recurrent algorithmic analogs, such as Barker codes and sequences of maximum length (or M-sequences).

Unlike periodic M-sequences, any N-sequence (or unlimited-length sequence) has no intrinsic period and is an irrational binary number. However, any N-sequence has a continuous spectrum and a δ-like autocorrelation function, which is defined by two incommensurable periods of the constituent quasiperiodic oscillations and is controlled by their delay. That is why, in principle, coordinated filtration is impossible for N-sequences. However, a separate coordinated reception of quasiperiodic oscillations and a coherent accumulation of samplings of sign correlation are possible.

The main operations to produce N-sequences are not more complex than the corresponding operations employed to produce algorithmic NTSs [17]. Practical schemes for obtaining algorithmic NTSs on the basis of shift registers and logical elements were considered, for instance, in Ref. [37].

Unlike their analogs, N-sequences have the property of uniqueness, are void of correlation between the constituent radio signals, and (with their frequency separation) possess an uncorrelated dependence of the autocorrelation function on instrumental delay parameters. These new properties complement the well-known properties of algorithmic NTSs [17] and their nearest analogs based on strange attractors — broadband chaotic signals (BCSs) considered in Ref. [38].

Noise-type signals in the form of different codes have been elaborated specifically for radar and communication, and their use has proved to be quite fruitful. Since the length of algorithmic sequences (codes) is limited, they are inherently periodic. That is why algorithmic NTSs afford coordinated reception and the δ-like shape of their autocorrelation function ensures the formation of a directivity pattern and a delay resolution of the radar required to resolve targets and to suppress the interference from specularly reflected beams.

However, in other applications of communication systems, such as encoding messages or possible recurrence of the code (key), severely deteriorates the protection of information. Naturally this fact is irremovable, despite the existing possibilities for producing algorithmic NTSs of virtually unlimited length [17].

According to Ref. [39], a continuous ergodic δ-correlated key stream represented, for instance, by the binary form of an irrational number is optimal for enciphering messages. This may be an N-sequence formed from quasiperiodic signals, which is adequate in its cryptographic immunity to a singly used Vernam key (see, for instance, Ref. [44]).

The possibility of coordinated filtration is an important property of NTSs. However, only Barker signals are energetically equivalent to harmonic signals with synchronous accumulation, since, in their convolution, the amplitudes of discrete signals add up. In the convolution of an M-sequence, the signal energies are added up, i.e., only half the signal energy is in fact used to improve the signal-to-noise energy ratio, like in the case of a normal noise.

In the reception of quasiperiodic radio signals with a duration T and a total energy equal to $2E$, one half of this energy can be used for synchronous accumulation in a time interval of $0.5T$ and the other half for the formation of N-sequences and the calculation of their mutual correlation functions. In energy efficiency, this is equivalent to the reception of an M-sequence with a duration $2T$.

Therefore, with quasiperiodic signals one can expect an increase in the detection range and an improvement in the resolution of ranging. In this case, detection is ensured by narrow-band filtration of echo signals, while the resolution of targets is ensured by producing N-sequences with a continuous spectrum from the signals and by a single-lobe autocorrelation function. The frequency band of such an 'NTS' is equal to the difference frequency of the matched quasiperiodic signals: $\Delta\omega = |\omega_1 - \omega_2/2|$, where ω_1 and ω_2 are the frequencies of the reflected signals ($\omega_2 > \omega_1$), while $\omega_2/2$ and ω_1 are the proportionate signal frequencies after their matching (from a practical point of view, $\Delta\omega$ can be of the order of $0.1\omega_1$).

6.1 Random number generation

The properties of quasiperiodic oscillations generated by a nondegenerate two-circuit parametric oscillator and the properties of a one-circuit parametric oscillator as a perfect limiter permit employing a combination of these devices as an optimal generator of random binary numbers (including technical applications). This is afforded not only by the probabilistic character of the results of the sign correlation of phases, but also by the absence of any algorithm of the process.

Another important property of the random-number generator is the possibility of synchronization with a similar generator; the synchronization principle was partly considered in Section 6.4 and comprehensively outlined in Ref. [40].

Figure 9 shows a block diagram of a random-number generator. The pump oscillator PP with a frequency ω_3 continuously excites the two-circuit oscillator OP_A with frequencies $\omega_{1,2}$. In the degenerate mode, the frequency ratio is $1:2:3$ ($\omega_1 + \omega_2 = \omega_3$), and in the nondegenerate mode the frequency ratio is irrational. The signal of the upper oscillation frequency ω_2 of the OP_A oscillator is voltage modulated by the modulator M with a frequency $\Omega = \omega_3/n$ produced by the frequency divider D_1, where n is a large integer. The modulated signal serves as a pump for the periodic excitation of the one-circuit oscillator OP_B. The signal of the upper frequency ω_1 of the OP_A oscillator defines the initial excitation condition for the OP_B oscillator with

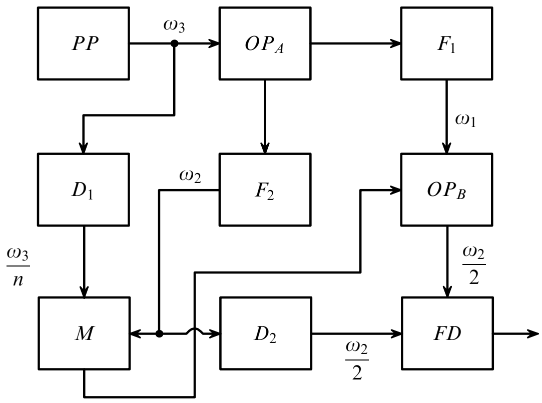

Figure 9. Block diagram of a random number generator.

phases 0 or π. The phase detector FD transforms the quantized phase samplings 0 and π at the OP_B oscillator output into amplitude samplings with logical levels 0 and 1.

The sequence of zeroes and unities in a tuned setup is an irrational binary number. The reference signal of the phase detector FD is produced by the frequency divider D_2. Filters F provide the frequency separation of the signals ω_1 and ω_2. To increase the cadence frequency, a phase manipulation of the voltage of the OP_B pump oscillator can be taken advantage of [27].

6.2 Potential radar applications

The interference properties of quasiperiodic radio signals can be employed to form sharp directivity patterns for radar reception, improve the resolution in target discrimination, and combat specular noise.

Figure 10 shows a diagram of the location of a low-flying target C with a two-frequency interferometric-type locator. The transmitter S emits two quasiperiodic oscillations with incommensurable frequencies Ω_1 and Ω_2. Upon the reflection from the target, the signals are received by the two-channel locator receivers A and B. Sign comparators produce synchronous N-sequences $\{N_A\}$ and $\{N_B\}$ from the signals received. The result of their correlation $M = \langle\{N_A\}\{N_B\}\rangle$ is the output signal of the interferometer. We emphasize that each beam in Fig. 10 contains two oscillations whose periods T_1 and T_2 are incommensurable, and therefore their two amplitude peaks (or zeroes) can coincide only at one point on the time axis. The two-channel narrow-band receivers A and B are equally distant from the transmitter S.

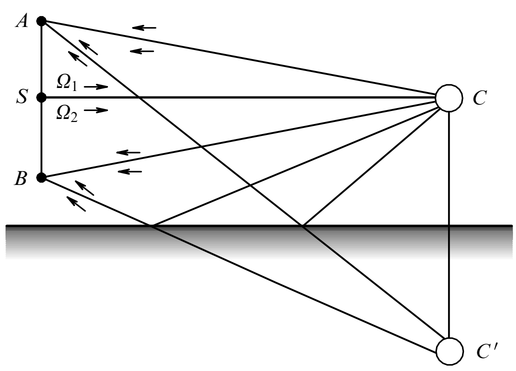

Figure 10. Two-frequency radar scheme for a low-flying target.

One can see from Fig. 10 that the paths of the echo signals (direct and specular) are always different. Therefore, the locator sees only the target C located in the equisignal direction, whereby the coherent signals are in phase, and does not see the specular image of the target C', because both signals reflected, for instance, from a water surface, arrive at the receivers with a delay relative to the 'direct' echo signals.

For a small angle of elevation, this delay is small, and the exact directions of a target and its specular image will be indistinguishable when using algorithmic-type NTSs, i.e., the average direction pointing at the target will be deviated towards the specular image. For quasiperiodic radio signals with incommensurable frequencies, one can expect a higher resolution affording the separation of the target and its specular image.

This can be achieved by employing a periodic modulation (signal interruption) of continuously generated probing signals (accordingly, the delay and interruption of the echo signal reception) and synchronous accumulation of uniform realizations of chaotic N-sequences, i.e., the coherent summation of the principal maxima and the incoherent summation of the randomly distributed side lobes.

For active location with a delay by a time τ, the interference formula involves the correlation functions $R_1(\tau)$ and $R_2(\tau)$ of oscillations with average frequencies $\Omega_{1\text{av}}$ and $\Omega_{2\text{av}}$:

$$I_{\Sigma} = 4A^2\left[1 + R_1(\tau)\cos(\Omega_{1\text{av}}\tau_1) + R_2(\tau)\cos(\Omega_{2\text{av}}\tau_2)\right], \quad (4)$$

where $R_{1,2}(\tau) = 1$ for $\tau = 0$ and fall off to zero as τ tends to infinity.

The dependence of the interference peak on $R_{1,2}(\tau)$ shows that the coherence of chaotic models for a relative delay τ should be ensured by the stability of frequencies Ω_1 and Ω_2. This limits the radar range L to half the coherence length equal to $c\tau_0$, where c is the velocity of light and τ_0 is the coherence time, whereby $R_{1,2}(\tau_0) = 0.5$ and which can be assumed to be equal for the frequencies Ω_1 and Ω_2. For an oscillation frequency instability of the order of 10^{-9} and $\tau_0 \approx 0.5$ (these are the parameters of our experimental setup) with the neglect of other limitations, L can be taken equal to 10^8 km.

6.3 Application to cryptography

Quasiperiodic oscillations open up the possibility of 'generation' of uninterrupted key streams in symmetric, spatially separated cryptographic devices invoked to encipher and decipher messages with the purpose of protecting them from unauthorized access. As noted in Ref. [39], 'the development of continuous generators of coherent key streams is the best we can hope for in the progress of classical cryptography'. Key streams enciphered with secret keys are capable of providing the highest practical cryptographic resistance. The problem of the formation of coherent key streams in symmetric cryptographic systems is solved by employing the four-beam interference of quasiperiodic oscillations.

The 'generation' principle of continuous key streams is explained in Fig. 11. Two subscribers, A and B, of a symmetric cryptographic system receive two continuous nonoverlapping narrow-band signals with incommensurable frequencies ω_1 and ω_2 from a transmitter S aboard an artificial Earth satellite (of the universal time service). Subscribers A and B have a common system for synchronizing samplings [41] and two circuits for the phase self-tuning of the signals received.

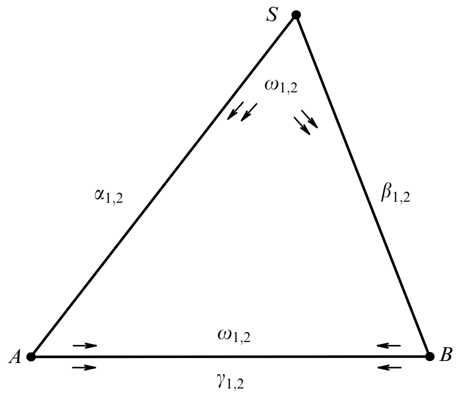

Figure 11. Principle of key stream generation in a symmetric cryptographic system.

The transmission and reception system with a frequency separation of quasiperiodic oscillations depicted in Fig. 11 is a two-channel or a four-beam interferometer. In the general case it has a variable base (the distance between the subscribers A and B) and an indefinite deviation from the equisignal direction of the transmitter S. Because of this, the signals received by the subscribers have different delays: subscriber A receives signals proportional to $\cos(\omega_1 t + \alpha_1)$ and $\cos(\omega_2 t + \alpha_2)$, and subscriber B signals proportional to $\cos(\omega_1 t + \beta_1)$ and $\cos(\omega_2 t + \beta_2)$. The delays of the signals with frequencies ω_1 and ω_2 in the phase self-tuning circuits of subscribers A and B are the same, equal to $2\gamma_1$ and $2\gamma_2$, respectively. Since the phase self-tuning eliminates the $\alpha_{1,2}$ and $\beta_{1,2}$ delays, subscribers A and β obtain in-phase signals proportional to $\cos(\omega_1 t + 2\gamma_1)$ and $\cos(\omega_2 t + 2\gamma_2)$.

The sign correlations of the signals and their synchronous samplings yield random coherent bit sequences, which can be used, upon enciphering with secret keys, as working keys for the bulk enciphering of transmitted messages. The delays $2\gamma_1$ and $2\gamma_2$ are hidden parameters of the cryptographic system. This property, as shown in Section 6.4, can be used to protect the secret keys.

Another version of system adjustment is the optimal selection of the delay between the oscillations of each pair of coherent signals in the interferometer receivers A and B according to the maximum correlation between the received cryptogram and the local one, out of the enciphered key stream. This can be done using an open channel with, for instance, a secret algorithmic key.

6.4 Simulation of quantum correlations

The use of dichotomous properties of the oscillations of a nondegenerate parametric oscillator was considered in Refs [42, 43] in the context of the simulation of EPR-type quantum correlations in the Bohm version (the acronym EPR owes its origin to the names of Einstein, Podolsky, and Rozen). Numerous optical EPR experiments are known to have pioneered quantum cryptography [44].

Since the signals of a nondegenerate oscillator are a classical analog to quantum dichotomous (two-valued) signals applied in quantum cryptography, the experimental setup outlined in Ref. [45] can be regarded as the nearest classical analog to the quantum cryptographic system considered in Ref. [44].

The scheme elaborated in Ref. [45] is a prototype of a symmetric wave cryptographic system in which the principal problems of cryptography — the transfer and protection of a key — are solved by the synchronous production of continuous key streams for subscribers connected by open communication channels. In a cryptographic system of this kind, the subscribers generate, on their own, radio signals with incommensurable frequencies and partially exchange them to obtain phase correlations, whose coherence is determined by the dichotomous properties of the signal phases and by the employment of a common synchronization signal (transmitted via an open communication channel) to synchronize the pump and the sign correlation samplings.

To put it differently, the subscribers receive coherent key streams simultaneously and secretly as they exchange regular continuous signals at incommensurable frequencies. Taken alone, the harmonic signals transmitted via open communication channels carry no information, but their synchronous sign correlation gives rise to dynamic chaos simultaneously at widely separated points in space. One the one hand, this is adequate to the production of continuous streams of information whose amount is infinite and, on the other hand, this principle of information transfer is a classical analog of the quantum teleportation described in Ref. [46].

Of course, there is nothing like the mysterious teleportation here. What does take place is the result of synchronous phase correlations, whose coherence is due to the phase dichotomy of interfering oscillations and whose randomness is caused by the incommensurability of their frequencies. A system of this kind is also an interferometer in which four irrationally related oscillations interact. Controlling correlation in the system was considered comprehensively in Refs [42, 45].

The interference peak of the system is a cosine function of the sum of two parameters, α and β. Each of these parameters is determined by the delay of the oscillation received by the corresponding subscriber, relative to its reference pair generated by the receiving subscriber. This allows the delay of the transmitted oscillations to be compensated, thereby controlling the coherence of key streams by any subscriber, A or B.

The operating principle of the cryptographic system is explained in Fig. 12, which conventionally depicts the drifts of the phases $\varphi_{1,2}$ and $\varphi'_{1,2}$ of the incommensurable frequencies $\omega_{1,2}$ and $\omega'_{1,2}$ of the parametric oscillators PO_A and PO_B, respectively, in dimensionless time $\tau = \omega_3 t$ (relative to the phase of the common pump). The open circles indicate the phase trajectories of the hidden reference signals, arrows indicate the phase trajectories of the signals which subscribers exchange via open communication channels, and the solid squares show the phase trajectories of the signals φ_1^* and $\varphi_2'^*$ received by the subscribers with delays α and β.

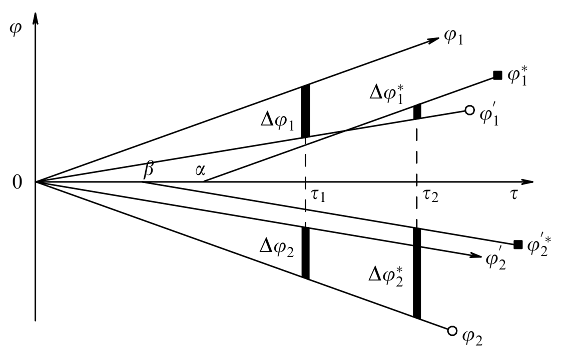

Figure 12. Phase drifts of the oscillations of parametric oscillators relative to the phase of the common pump.

For an in-phase pumping, the phase differences of close frequencies (ω_i and ω'_i, $i = 1, 2$) of different oscillators form dichotomous random time functions: $\Delta\varphi_1(t) = (\varphi_1 - \varphi'_1) = -\Delta\varphi_2(t) = (\varphi_2 - \varphi'_2)$. For this reason, synchronous voltage samplings of the sign correlation between the oscillations at these frequencies give two coherent bit streams possessing δ-shaped autocorrelation functions. In Figure 12 this case (in which the signals are not delayed and $\alpha = \beta = 0$) corresponds to synchronous samplings equal to $\Delta\varphi_1(\tau_1)$ and $\Delta\varphi_2(\tau_1)$.

Figure 12 also shows the result of synchronous samplings of the correlation between the reference oscillation phases and the phases φ_1^* and $\varphi_2'^*$ of the received oscillations, which are shifted by angles α and β owing to the delay in the transmission from one subscriber to the other. To make the result descriptive, the instant of sampling τ_2 is shifted relative to the instant of sampling τ_1. One can see from Fig. 12 that $\Delta\varphi_1^*(\tau) = \varphi_1^* - \varphi'_1 \neq -\Delta\varphi_2^*(\tau) = \varphi_2 - \varphi_2'^*$, i.e., the oscillation delay destroys the coherence of the key streams. Therefore, in order for the correlation to be controllable, an automatic compensation for the phase shifts α and β should be provided for.

Naturally, a conventional phase self-tuning system is not appropriate for incommensurable frequencies. However, the coherence of streams can be authenticated employing an open communication channel and standard exchange protocols, for instance, parity verification [44]. The secrecy of controlling the key stream coherence can be afforded by algorithmic ciphers or by additional key streams produced in parallel with an additional synchronous delay and therefore uncorrelated with the main key streams.

Therefore, in the exchange of signals of different frequencies, for synchronous samplings of the sign correlation between the received and reference oscillations, subscribers A and B can simultaneously receive coherent bit streams to encipher and decipher messages. We note that there is no way of comparing non-close frequencies in this scheme, since their matching may necessitate an asymmetric frequency division, which will result in the violation of signal phase dichotomy.

The cryptographic immunity of a classical cryptographic system of this sort is afforded by hidden parameters: each subscriber transmits only one of two of his signals to the other subscriber, and uses the other signal as a hidden reference parameter for the comparison with the phase of the signal received from the other subscriber. The hidden parameters may also be the frequency and phase of the pump of parametric oscillators, the delay of signals in their transmission, the cadence frequency, and, finally, the secret keys which can be used to additionally improve the cryptographic resistance. However, in any of the possible versions of the scheme with two or four incommensurable frequencies (see Section 6.3) the principal hidden parameters are the delays between the received and reference signals.

The development of any cryptographic system implies the presence of a third 'subscriber' C referred to as the cryptoanalyst. To determine the hidden frequencies, advantage can be taken of regenerative and nonregenerative parametric amplifiers operating with a known pump, which should be synchronized in phase with the pump of the parametric amplifiers of subscribers A and B (additional phase corrections are required in this case). Moreover, cryptoanalyst C should also know the remaining hidden parameters of the cryptographic system. Despite the task's complexity, which determines the cryptographic immunity of the system, this system cannot be regarded as absolutely secure, like any other classical or even quantum system.

Clearly, classical cryptographic systems cannot exhibit a response to interception similar to that inherent in quantum cryptographic systems. However, the existence of hidden parameters may endow them with a like property in the case of the active participation of 'subscriber' C in the phase self-tuning of the system. This is related to the fact that any system is majorant in the selection of a signal for phase self-tuning [41]. When comparing the results of transfer via the secret and open channels, subscribers A and B of the cryptographic system shown in Fig. 11 are therefore able to discover the loss of direct synchronization with each other if the phase of subscriber C is enforced on them and thereby establish the fact of information interception.

7. Discussion of experimental results

1. Owing to the investigations of Hamiltonian systems, the problem of magnetic plasma confinement was solved in the 1960s. A wealth of other possibilities are also known for using the chaotic dynamics of Hamiltonian systems [1 – 3], which have clearly not been exhausted.

Experiments show that two irrationally related oscillations are sufficient to model motion with phase mixing; these may be, for instance, the quasiperiodic oscillations of a nondegenerate two-circuit parametric oscillator, which exhibit unique properties. The nature and properties of these oscillations are the result of a nonlinear resonance and the reality of the continuum of irrational numbers. Irrationally related oscillations can possess a stable frequency incommensurability and a mirror symmetry of phases relative to the pump phase. In this case, they are highly stable, spectrally resolvable, and can be made proportionate to ensure a uniform phase mixing and the obtainment of a continuous spectrum.

The above properties allow the four-beam interference of quasiperiodic signals to be used in radio communication and the exploit of the fact that in the overlapping of resonances their interaction generates chaotic oscillations interfering with one interference peak, which, unlike noise, can be controlled. This is possible due to the fact that the autocorrelation functions of purely periodic oscillations with incommensurable frequencies remain purely periodic functions of the incommensurable frequencies and of the instrumental parameter — delay.

2. In the stationary oscillation mode with continuous pumping, the Q factor of a parametric system is so high that its sensitivity to an external signal is several orders of magnitude lower than to the initial excitation conditions, which determine the trajectories of motion. For a narrowband filtration and a deep limitation of oscillations, part of the information on the system dynamics is clearly deleted, but the ratio and stability of the normal frequencies persist. This makes it possible to check the frequency incommensurability by the form of the autocorrelation function of mappings, control the interference of chaotic models and their coherence, and observe the local resonances of regular attractors.

3. Irrationally related oscillations interfere as absolutely uncorrelated processes, otherwise their frequencies are either commensurable, or disproportionate. The result of their interference is a controlled random process with a multiple stochastic attractor represented by a set of determinate phase trajectories (of the type shown in Fig. 6c) extremely sensitive to the delay between the interfering oscillations. Changes in this delay result in determinate transitions from one attractor to another, which are possible due to Arnol'd diffusion.

4. In the production of chaos, the relative delay of quasiperiodic oscillations is equivalent to a change in the initial conditions of the system dynamics and is manifest in a change of the correlation of chaotic models. In the case of natural dynamic chaos and continuity of phase space, with the prescription of similar initial conditions prohibited, the trajectories of motion are unique. In the case of chaos simulations at widely diversed sites of reception of the constituent radio signals, the full correlation of models is a reality caused by the four-beam interference of quasiperiodic oscillations.

Moreover, the correlation of the models possesses a long-term stability, provided the interfering signals and their sign correlation samplings are synchronous. Under these conditions, the result of synchronous mappings of similar processes is observed, in which the phase diffusion is also synchronous and the trajectories of motion do not have time to diverge.

5. One could expect an effect of fluctuations of reference voltages under limitation. Random noise of this kind does really exist and is responsible for the equalization of phase coincidence and noncoincidence probabilities in phase comparators. In the experimental setup shown in Fig. 5, the ensuing model-correlation deviation from 100 % was about 5% (see Fig. 8). If parametric phase quantizers are employed as phase correlators, the errors amount to fractions of 1%. In the absence of perturbations, the likelihood of errors should tend to zero, since any trajectory is determined by a unique pair of initial parameters from the infinite phase space.

6. In our experiments, uninterrupted observations of stable dynamic chaos were limited to sequences of the order of 10^8 bits for a fixed sampling frequency varied between 10^{-1} and 3×10^4 Hz. The observation duration could be lengthened, but for times exceeding the pump phase coherence time this nevertheless does not provide an answer to the question, posed in Ref. [3], of how 'strong' this deterministic chaos is. However, from the standpoint of interference, the answer is simple: in as much as the irrational frequency relation is stable.

For Hamiltonian systems with nonresonant tori, the long-term stability of motion has been proved by the KAM theory. In experiments, the stable generation of quasiperiodic oscillations is afforded at the level of limiting cycles of the parametric oscillator, where the intrinsic frequency instability does not exceed 10^{-9} (see Section 3.4) and is observed over a relatively broad band of pump frequencies (about 1%).

7. Within some range of pump frequency retuning (of the order of 1%), a nondegenerate two-circuit parametric oscillator exhibits the properties of a dissipative system with two irrational and one rational torus, which is consistent with the Ruelle – Takens theory, and exhibits the properties of a conservative system with three stable irrational tori only in a narrow range of pump frequency retuning (of the order of 0.1%). These properties are an experimental justification for classifying such a system as conventionally conservative [3]. At the same time, two irrationally related frequencies are sufficient to implement the above radiophysics applications.

8. From the general effect of wave interference it follows that controllable N-sequences can be realized in all classical systems operating in acoustic, radio, and microwave ranges, i.e., where possibilities exist not only for generating oscilla-

tions with incommensurable frequencies, but also for their production with the aid of frequency conversion. The applications considered here formally reduce to a doubling of the signal channels of the known radio systems.

8. Conclusions

The interference of oscillations with incommensurable frequencies is a special case of the general effect of wave interference. Its properties were implicitly implied in the papers of Harald Bohr (1887 – 1951), who published a lecture course on the theory of almost periodic functions. The Russian edition of this small monograph [19] gave an example in which the possibility of controlling the interference of purely periodic functions with incommensurable frequencies was predetermined. It is therefore appropriate to term the scheme of two-channel four-beam interference of oscillations with incommensurable frequencies the 'Bohr interferometer.

The papers listed as references make up only a small fraction of the publications on the problem involved. Many of them, for instance, Refs [1 – 3, 10, 26], are reviews in character, i.e., encompass all the main results of research on the chaotic dynamics of Hamiltonian systems and provide a vast bibliography on the adjacent issues and strange attractors [47]. Among the outcomes of this prodigious work of mathematicians and physicists is the recognition that quasiperiodic signals hold promise as a basis for developing interference multichannel communication systems. It is not improbable that quasiperiodic signals are used by Nature in biological systems. However, verifying this supposition is a nontrivial task.

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