

METHODOLOGICAL NOTES

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Superluminal electromagnetic solitons in nonequilibrium media

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Contents

Abstract. The possibility of stable faster-than-light propagation of ultimately short (without high-frequency carrier) electromagnetic solitons, breathers, and nonresonant envelope solitons is discussed based on the simple model of two-component nonequilibrium media undergoing two-level quantum transitions with widely differing eigenfrequencies.

1. Introduction

Today in connection with the development of pulsed laser technology that allows creation of highly nonequilibrium media, interest has revived in the issue of the possible existence of pulses in such media that travel faster than light in vacuum [1]. The superluminal (tachyon) modes exist necessarily in media that are unstable with respect to transition to the equilibrium state [2]. The possibility of faster-than-light propagation in amplifying (active) media of optical monochromatic pulses (MP) was earlier noted in the series of works [3 – 5]. The term 'monochromatic' as applied to a pulse is conventional and can only be used when the spectral width of the pulse $\delta\omega \sim 1/\tau_p$ (where τ_p is the time length of the pulse) is much less than its carrier frequency ω. In other words, the condition under which the pulse may be regarded as monochromatic can be written as $\omega\tau_p \gg 1$. It should be observed that we are talking about the group velocity $v > c$. Such pulses do not carry information in the process, because the propagation of the maximum of the wave packet is associated not with the transfer of energy from one point to another, but with the amplification of the pulse by virtue of the nonequilibrium state of the medium. Associated with this circumstance is also the impossibility of spatial localization of the superluminal object [2]. The difficulty consists in the instability of the stationary light pulses that return the nonequilibrium medium into the original inverted state [1]. On the other hand, the stability of superluminal signals leads to violation of the causality principle [6].

The advances of laser physics over the past decade have made it possible to produce optical pulses about one period of electromagnetic wave long — ultimately short pulses (USP) [7 – 9]. The interaction of such pulses with matter (including nonequilibrium media) exhibits special features not found in the interaction involving MP with a well-defined carrier frequency [10, 11].

In the simplest case, when the medium is assumed to consist of two-level atoms, the dynamics of broadband USP is described by the three-dimensional sine-Gordon equation [10, 11]. We use the term 'broadband' here in the sense that the pulse spectrum $\delta\omega \sim \tau_p^{-1}$ overlaps the frequency ω_0 of quantum transitions in the two-level atoms (that is, $\omega_0/\delta\omega \sim \omega_0\tau_p \ll 1$; see below). In the nonequilibrium case, when prior to the arrival of the pulse in the medium it is mostly the excited states that are occupied, the solution of the sine-Gordon equation formally appears as localized superluminal pulses ($v > c$). Such solutions, however, are found to be unstable (for example, with respect to self-focusing) [12, 13]. The same can be said about the nonresonant envelope solitons [12].

In connection with the above, it would be interesting to study the issue of the feasibility to create such a state of the medium in which it will carry stable USP and nonresonant envelope solitons with superluminal group velocities.

2. Linear analysis

The linear optical dispersion is quite adequately described by the classical Lorentz theory [14], in which the interaction of the optical electron with atomic core is represented by the quasi-elastic restoring force. As a result, the complex index of refraction, whose imaginary part $\varkappa$ is proportional to the coefficient of light damping, and the real part N describes the refractive properties of the medium, is expressed in terms of the parameters of the medium and the frequency ω of light by the Selmeier formula [14]. Figure 1 shows the curves $N(\omega)$ and $\varkappa(\omega)$ relevant to this formula [14]. Observe that the Selmeier formula corresponds to the quantum-mechanical model of a

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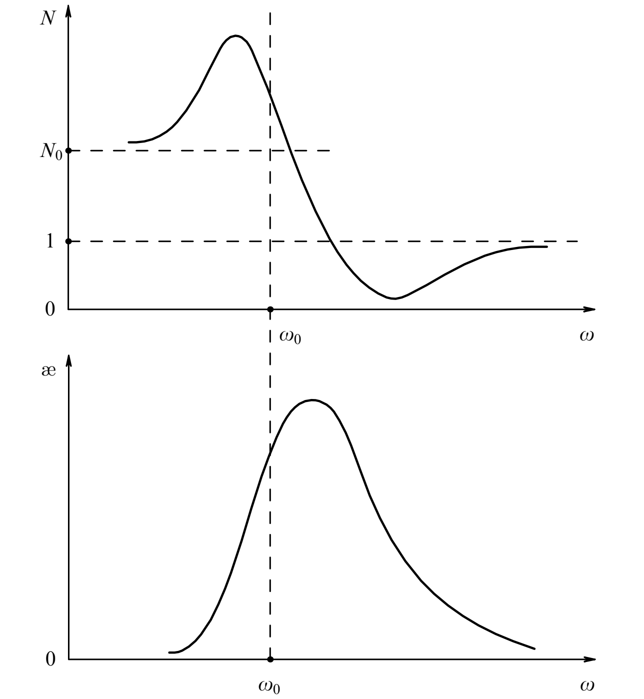

Figure 1. Real refractive index N and absorption coefficient $\varkappa$ as functions of light frequency ω, based on the classical Selmeier formula (ω_0 is the resonant medium frequency, N_0 is the refractive index at zero frequency). In the neighborhood of ω_0, where $N(\omega)$ falls off rapidly due to anomalous dispersion, we formally have $v > c$. However, owing to the strong absorption in this frequency range, the parameter v loses its meaning of the velocity of the wave.

two-level medium, where the frequency of transition between the levels is ω_0.

By definition, the group velocity is $v = \mathrm{d}\omega/\mathrm{d}k$, where k is the wave number corresponding to the frequency ω and expressed through the phase velocity $v_{\mathrm{ph}} = c/N$ as $k = \omega/v_{\mathrm{ph}} = \omega N/c$. Then we have

$$\frac{1}{v} = \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{1}{c}\,\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega N) = \frac{1}{c}\left(N + \omega\frac{\mathrm{d}N}{\mathrm{d}\omega}\right). \tag{1}$$

In the neighborhood of the frequency ω_0 of resonant absorption, owing to the abrupt decrease in N, the inequality $N + \omega\,\mathrm{d}N/\mathrm{d}\omega < 1$ may be met and, as a consequence [see Eqn (1)], $v > c$. However, the damping of light here is so strong that the group velocity loses the meaning of the energy transfer velocity [15].

The curves shown in Fig. 1 correlate with the equilibrium medium. For a medium with the inverse population of quantum levels, the coefficient $\varkappa$ is negative and the regions of normal dispersion ($\mathrm{d}N/\mathrm{d}\omega > 0$) in the equilibrium medium become the regions of anomalous dispersion ($\mathrm{d}N/\mathrm{d}\omega < 0$) in the inverted medium, and vice versa. In the regions where N decreases rapidly with increasing ω, the amplification of waves is now strong, which leads to the development of their instability.

The situation may be somewhat different if we consider a two-component medium with the resonant frequencies ω_1 and ω_2, respectively, when one of the components is in the equilibrium state, whereas the other is inverted. Let in this case the characteristic frequency ω of the electromagnetic wave satisfy the dual inequality

$$\omega_1^2 \ll \omega^2 \ll \omega_2^2\,. \tag{2}$$

The condition $\omega_1^2 \ll \omega^2$ can be satisfied by the eigenfrequencies ω_1 that lie in the IR range. The inequality $\omega_2^2 \gg \omega^2$ holds true for frequencies ω_2 falling within the optical range, corresponding to the electronic-optical transitions. Further on we shall refer to the quantum transitions with the frequency $\omega_1(\omega_2)$ as the 1(2) transitions or 1(2) components. Both types of transitions may pertain to the same molecular structural units (then their respective concentrations n_1 and n_2 are equal) or to different ones (then, possibly, $n_1 \neq n_2$).

The set of constitutive and wave equations for the medium comprised of two types of two-level atoms is given by

$$\frac{\partial^2 U_j}{\partial t^2} = -\omega_j^2 U_j - \omega_j \frac{2d_j E}{\hbar} W_j\,, \quad \frac{\partial W_j}{\partial t} = \frac{2d_j E}{\hbar\omega_j}\,\frac{\partial U_j}{\partial t}\,, \tag{3}$$

$$\Delta E - \frac{1}{c^2}\,\frac{\partial^2 E}{\partial t^2} - \frac{\gamma}{c^2}\,\frac{\partial E}{\partial t} = \frac{8\pi}{c^2}\,\frac{\partial^2}{\partial t^2}\sum_{j=1}^{2} d_j n_j U_j\,. \tag{4}$$

Here, the subscript $j = 1, 2$ denotes the number of the respective component, E is the electric field strength of the pulse, d_j is the transition dipole moment of the jth component, U_j and W_j are the dimensionless dipole moment and the population inversion of atoms of the jth component ($-1/2 \leqslant W_j \leqslant 1/2$), and γ is the radiation loss factor in the medium per unit time. Further on we assume that the duration τ_{p} of the light pulse is much shorter than the time of phase and energy relaxation for both transitions, which allows us to disregard the relaxation terms in Eqn (3).

Linearization of equations (3), (4) implies fixing the values of inversion for both components. In accordance with this, we assume that $W_j = W_{j\infty}$ (where $W_{j\infty}$ is the initial inversion of the jth component). Assuming then that $P, E \sim \exp(\mathrm{i}\omega t)$, where $P = 2\sum_{j=1}^{2} d_j n_j U_j$ is the medium polarization, we find the expressions for susceptibility $\chi = P/E$ and the square of the refractive index ($\gamma = 0$):

$$N^2 = 1 + 4\pi\chi = 1 - \frac{16\pi}{\hbar}\sum_{j=1}^{2}\frac{d_j^2\omega_j n_j}{\omega_j^2 - \omega^2}\,W_{j\infty}\,. \tag{5}$$

By virtue of Eqn (2), in the denominator of the first term ($j = 1$) under the summation sign in expression (5) we may drop ω_1^2, and also drop the frequency squared ω^2 of the external field in the denominator of the second term ($j = 2$). Then from Eqns (1) and (5) we find the condition under which we have $v > c$:

$$\omega_*^2 W_{1\infty} + \omega^2 W_{2\infty} > 0\,, \tag{6}$$

where we have introduced the parameter ω_* (with dimensions of frequency), which is defined as $\omega_* = (d_1/d_2 N_{20}) \times \sqrt{n_1\omega_1\omega_2/n_2}$, $N_{20} = \sqrt{1 - 2\eta W_{2\infty}}$ and has the meaning of the refractive index of the 2-component in the range of frequencies that satisfy condition (2): $\eta = 8\pi d_2^2 n_2/(\hbar\omega_2)$.

As follows from the Kramers – Kronig relations [16], the dispersion (the dependence of χ on the frequency) inevitably leads to absorption of the light field in the medium. Observe also that the absorption coefficient $\varkappa$ has sharp peaks in the regions of anomalous dispersion ($\partial\chi/\partial\omega \sim \partial N^2/\partial\omega < 0$). This anomalous dispersion pattern is always associated with absorption [14]. From the same Kramers – Kronig relations it follows that the change of sign of the absorption coefficient alters the nature of dispersion: the dispersion becomes normal in the neighborhoods of the resonant frequencies, and

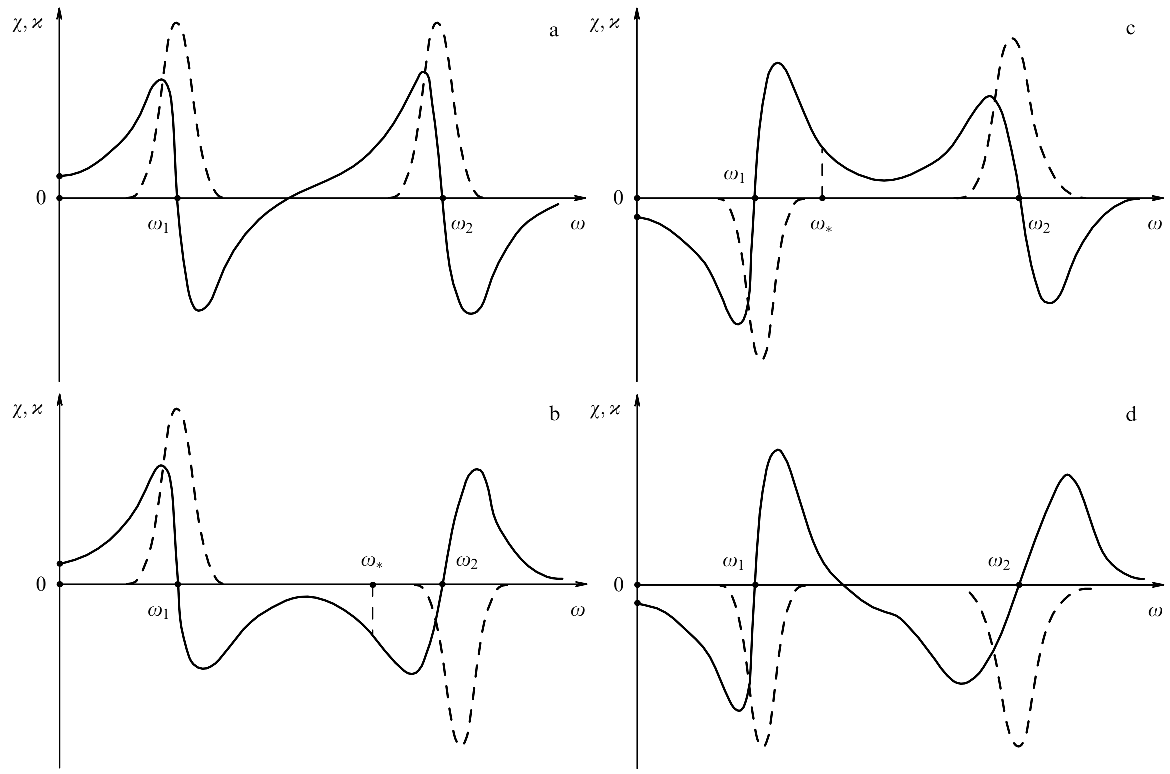

Figure 2. Curves $\chi(\omega)$ (solid lines) and $\varkappa(\omega)$ (dashed lines) for different states of the two-component medium with resonant frequencies ω_1 and ω_2: (a) absorbing medium: $v < c$ at $\omega_1^2 \ll \omega^2 \ll \omega_2^2$; (b) absorbing – amplifying medium: $v > c$ at $\omega_*^2 < \omega^2 \ll \omega_2^2$; (c) amplifying – absorbing medium: $v > c$ at $\omega_1^2 \ll \omega^2 < \omega_*^2$; (d) amplifying medium: $v > c$ at $\omega_1^2 \ll \omega^2 \ll \omega_2^2$.\

anomalous elsewhere. Figure 2 depicts the curves $\chi(\omega)$ and $\varkappa(\omega)$ for the four different states of the two-component medium. As follows from Eqn (6), in the case of an absorbing medium (AbM, $W_{1\infty} = W_{2\infty} = -1/2$), the group velocity is always lower than c. At the same time, in an absorbing – amplifying medium (AbAmM, $W_{1\infty} = -W_{2\infty} = -1/2$), and in an amplifying – absorbing medium (AmAbM, $W_{1\infty} = -W_{2\infty} = 1/2$), we have $v > c$ at $\omega > \omega_*$ and $\omega < \omega_*$, respectively. The amplifying medium (AmM, $W_{1\infty} = W_{2\infty} = 1/2$), according to Eqn (6), always corresponds to superluminal group velocities in the frequency range between ω_1 and ω_2. In all four cases we must pay attention to inequalities (2), which identify the frequency range where absorption and amplification are relatively weak.

Condition (6) does not imply the possibility of transmission of information faster than the speed of light, because plane monochromatic waves cannot carry information. Linear wave packets formed by the wave groups falling within the frequency range in question, will rapidly spread because of the dispersion.

3. Nonlinear wave equation

The ultimately short pulses are so called because their spectrum is so broad that it is not possible to identify the carrier frequency [17, 18]. So as to be able to avail ourselves of the conditions (2) in this case as well, we must set $\omega \sim \tau_p^{-1}$, and then conditions (2) will take on the form used in Refs [10, 11, 17]: $(\omega_1\tau_p)^2 \ll 1$ and $(\omega_2\tau_p)^2 \gg 1$. At that time for the 1-transitions ($j = 1$) in the right-hand side of the first equation in (3), by virtue of inequalities (2), we may drop out the term $\omega_1^2 U_1$, and the solution of Eqn (3) for $j = 1$ will be obviously written in the form [10, 11]

$$W_1 = W_{1\infty}\cos\theta\,, \quad \frac{\partial U_1}{\partial t} = -\omega_1 W_{1\infty}\sin\theta\,, \tag{7}$$

where

$$\theta = \left(\frac{2d_1}{\hbar}\right)\int_{-\infty}^{t} E\,\mathrm{d}t'\,,$$

and $W_{j\infty}$ is the inversion of j-transitions prior to the arrival of USP.

For $j = 2$, the left-hand side of the first equation in set (3) is small [see Eqn (2)]. Neglecting this left-hand side in the zero approximation, we find $U_2 \simeq -2d_2EW_2/(\hbar\omega_2)$. Substituting this expression into the left-hand side of the first equation in (3), we get in the next approximation

$$U_2 = -\frac{2d_2E}{\hbar\omega_2}W_2 + \frac{2d_2W_{2\infty}}{\hbar\omega_2^3}\,\frac{\partial^2 E}{\partial t^2}\,. \tag{8}$$

Here in the second term of the right-hand side we set $W_2 = W_{2\infty}$ because the spectrum of the pulse does not contain Fourier components in resonance with the 2-transitions [see Eqn (2)], and so the latter are excited only slightly. In this way, the variation of W_2 becomes of the same order of magnitude as the small parameter $(\omega_2\tau_p)^{-2}$.

Substitution of $U_2 \simeq -2d_2EW_{2\infty}/(\hbar\omega_2)$ into the second equation in set (3) with subsequent integration leads to the expression

$$W_2 = W_{2\infty}\left[1 - 2\left(\frac{d_2E}{\hbar\omega_2}\right)^2\right]. \tag{9}$$

Using relations (4), (7) – (9), we get

$$\Delta\theta - \frac{N_{20}^2}{c^2}\frac{\partial^2\theta}{\partial t^2} - \frac{\gamma}{c^2}\frac{\partial\theta}{\partial t} = \alpha\sin\theta + \beta\left(\frac{\partial\theta}{\partial t}\right)^2\frac{\partial^2\theta}{\partial t^2} + \mu\frac{\partial^4\theta}{\partial t^4}\,, \tag{10}$$

where

$$\alpha = -\frac{16\pi d_1^2\omega_1 n_1 W_{1\infty}}{\hbar c^2}\,,$$

$$\mu = -\frac{16\pi d_2^2 n_2 W_{2\infty}}{\hbar c^2\omega_2^3}\,,\quad \beta = \frac{3d_2^2\mu}{2d_1^2}\,. \tag{11}$$

The parameter μ accounts for the dispersion, and β for the cubic nonlinearity of the 2-component. At the same time, the dispersion and nonlinearity caused by the 1-transitions enter the right-hand side of Eqn (10) in a nonadditive way as the term $\alpha\sin\theta$.

Further analysis in this paper is based on the particular solutions of equation (10) in the form of solitary running pulses.

4. Dissipative soliton

Consider a nonequilibrium dissipative medium, in which radiation loss plays a vital part ($\gamma \neq 0$). We pursue the one-dimensional solution to Eqn (10) in the form of a steady pulse travelling along the z-axis with the velocity v — that is, $\theta = \theta(t - z/v)$. Obviously, such a solution is possible when the medium has an energy content sufficient for compensating the loss. Let the 1-component before the arrival of USP resides in the inverted state $W_{1\infty} = 1/2$. Then $\alpha = -|\alpha| < 0$. The travelling USP, taking in the energy from the 1-transitions, transfers them to the ground state, after which the energy dissipates in the medium because of the loss. As a result of such a 'balance', a steady pulse with an 'area'

$$\theta_\infty \equiv \left(\frac{2d_1}{\hbar}\right)\int_{-\infty}^{+\infty} E\,\mathrm{d}t' = \pi\,,$$

that is, a π-pulse, can form in the medium. In accordance with this, we select the ansatz [19, 20]

$$\dot\theta = \frac{1}{\tau_\mathrm{p}}\sin\theta\,, \tag{12}$$

where the dot above θ denotes differentiation with respect to $t - z/v$.

Substitution of relation (12) into (10) at $\Delta = \partial^2/\partial z^2$, and subsequent bringing the coefficient of $\sin 4\theta$ to zero, establishes the linkage between the dipole moments of the two transitions for which the ansatz (12) holds true: $d_2 = 2d_1$. This restriction is very artificial. However, it allows the exact solution of equation (10) to be written in the form of a running pulse, which in turn permits the basic features of propagation of solitary electromagnetic waves in nonequilibrium media with dissipation to be traced.

Equating further the coefficients of $\sin\theta$ and $\sin 2\theta$ in both sides of Eqn (10), and making use of Eqn (11), we get

$$\frac{1}{\tau_\mathrm{p}} = \frac{8\pi d_1^2 n_1\omega_1}{\hbar\gamma}\,,$$

$$\frac{1}{v} = \frac{1}{c}\left\{1 - 2\eta W_{2\infty}\left[1 - \frac{1}{(\omega_2\tau_\mathrm{p})^2}\right]\right\}^{1/2}\,. \tag{13}$$

Integration of formula (12) leads to the expression for the electric field profile of the travelling pulse:

$$E = \frac{\hbar}{2d_1\tau_\mathrm{p}}\,\mathrm{sech}\left(\frac{t - z/v}{\tau_\mathrm{p}}\right)\,. \tag{14}$$

From expressions (7) and (9) we get for the inverse population of both components:

$$W_1 = -|W_{1\infty}|\tanh\left(\frac{t - z/v}{\tau_\mathrm{p}}\right)\,,$$

$$W_2 = W_{2\infty}\left[1 - \frac{1}{2(\omega_2\tau_p)^2}\,\mathrm{sech}^2\left(\frac{t - z/v}{\tau_\mathrm{p}}\right)\right]\,. \tag{15}$$

Hence it follows that after cessation of the pulse (14), the state of the 2-component coincides with its original state. Observe that if prior to the arrival of USP this component was inverted ($W_{2\infty} > 0$), then the speed of the pulse is $v > c$ [see Eqn (13), condition $(\omega_2\tau_\mathrm{p})^2 \gg 1$, and Fig. 3a]. Since $(\omega_2\tau_\mathrm{p})^2 \gg 1$, then, as follows from Eqn (13), one finds $v \approx c/N_{20}$. This is true until the value η becomes equal to one. As $\eta \to 1$, we have $N_{20} \to 0$ and, according to Eqn (13), the velocity tends to $v \to (\omega_2\tau_\mathrm{p})c$ rather than to infinity, as would follow from the approximate expression $v \approx c/N_{20}$. Given the smallness of the parameter $(\omega_2\tau_\mathrm{p})^{-2}$, the condition of feasibility to form pulse (14) can be written as $\eta < 1$, because otherwise the speed v becomes imaginary. And if $W_{2\infty} < 0$ (AmAbM), then $v < c$ (Fig. 3b).

The state of the 1-component, as follows from Eqn (15), is changed to the opposite after cessation of USP in the form (14): the initial inversion $W_{1\infty} = 1/2$ is replaced by the final ground state ($W_1 = -1/2$ as $t \to +\infty$). In this way the overall state of the two-component medium is changed after cessation of USP in the medium. Therefore, before passing the second, third, and so forth superluminal solitons of the form (14), (13) through this medium, it must be wholly brought back to the state with $W_{1\infty} > 0$, $W_{2\infty} > 0$, which takes energy. As a result, the information that the pulse is applied to the input gate of the medium becomes known at the exit of the medium before the occurrence of an event. Accordingly, the question of faster-than-light transmission of information using pulses (14), (13) has no meaning. Besides, the parameters of the pulse itself (length, velocity and amplitude), as follows from Eqns (13), (14), are rigidly fixed by the parameters of the medium and its initial state (the values of $W_{1\infty}$ and $W_{2\infty}$). It follows that there are no free parameters in the solution (14). This means that as the pulse travels in the medium and acquires its asymptotic shape (14), it loses information about its own parameters at the point of entry into the medium in question. This information dissipates as radiation loss together with the energy stored in the medium. Accordingly, the pulse of the form (14) does not carry information either at $v < c$ ($W_{2\infty} < 0$) or at $v > c$ ($W_{2\infty} > 0$). The conditions (2) at $\omega \sim \tau_\mathrm{p}^{-1}$, according to Eqn (13), impose restrictions on the parameters of the medium that need to be complied with to provide for the formation of the pulse (14): $(\gamma/\Omega_1)^2 \ll 1 \ll (\gamma/\omega_1)^2$, where $\Omega_1 = 8\pi d_1^2 n_1/\hbar$.

Steady pulses with the properties described above are often referred to as dissipative solitons or autosolitons [21, 22]. From expressions (14), (15) and from Fig. 3 we see that the solitons describing the motion of the electric field of the USP and the inversion of the 2-component are 'running

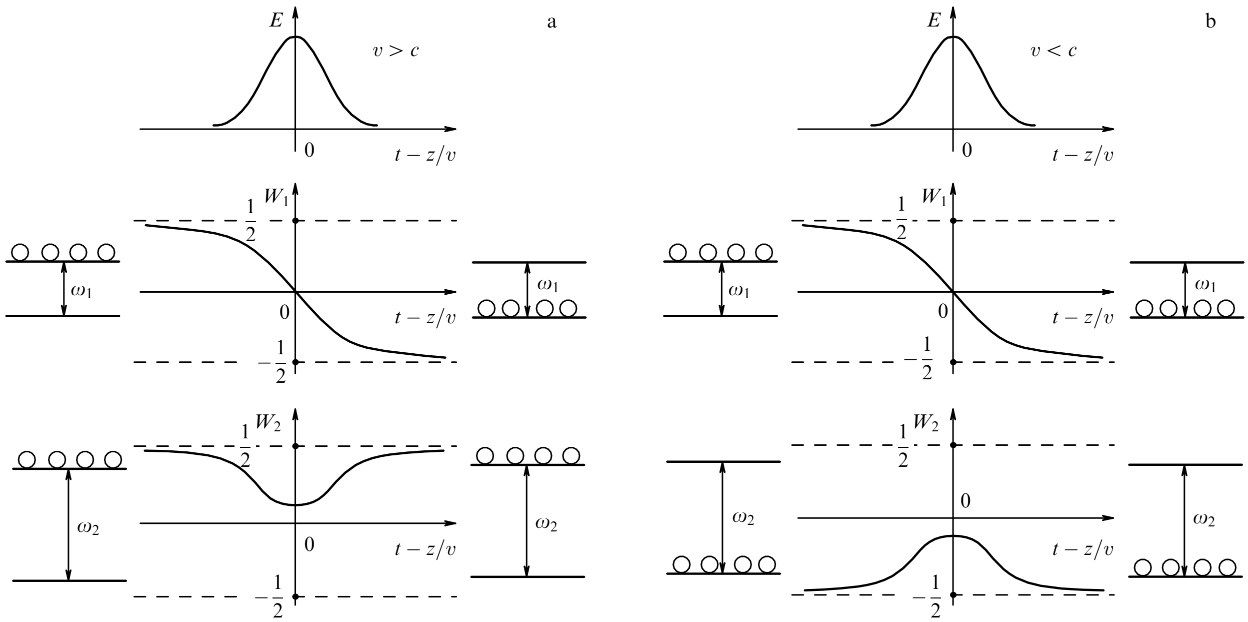

Figure 3. Electric field profiles of the ultimately short autosoliton and the corresponding inversions of the 1- and 2-component in the comoving frame of reference in a medium with radiation loss. The state of the 1-component changes after cessation of the autosoliton, whereas the 2-component returns to the initial state: (a) amplifying medium: $v > c$; (b) amplifying–absorbing medium: $v < c$.

pulse' type autosolitons [21, 22]. At the same time, the inversion of the 1-component propagates in the regime of a 'running front' [21, 22], which ensures the irreversible character of the change of state in the dissipative medium. In a sense, autosolitons are similar to self-oscillations. The asymptotic form of the latter also does not depend on the way they had been excited (that is, on the initial conditions), but is wholly determined by the parameters of the system in which they occur. It is only necessary that the initial conditions should not leave the region of attraction to the limit cycle corresponding to self-oscillations. In this way, self-oscillations in nonequilibrium nonlinear systems, like autosolitons, 'forget' information about the initial circumstances of their excitation.

The region of attraction of the input signal to autosoliton (14) is determined by the condition that its total 'area' θ_∞, as follows from Eqn (12), should fall within the range $0 < \theta_\infty < 2\pi$. This ensures the stability of the autosoliton discussed in this section.

5. Superluminal solitons and breathers in conservative medium

In this section we shall consider a supersonic soliton in a loss-free medium ($\gamma = 0$). Then at $\beta = 3\mu/2$, equation (10) admits an exact one-dimensional solution in the form of a running soliton-like pulse [23]. In our case, the condition $\beta = 3\mu/2$, as follows from Eqn (11), corresponds to equality between the transition dipole moments for both components: $d_2 = d_1$. Here we find such solution from a more general standpoint corresponding to the three-dimensional case. With this purpose we use the method of averaged Ritz–Witham type variational principle [12, 24]. Such an approach will allow us to study the stability of such a pulse. For this we note that equation (10) at $\gamma = 0$ can be written out as the Euler–Lagrange equation, using the density of the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\nabla\theta)^2 - \frac{N_{20}^2}{2c^2}\left(\frac{\partial\theta}{\partial t}\right)^2 + \alpha(1 - \cos\theta) + \frac{\mu}{8}\left(\frac{\partial\theta}{\partial t}\right)^4 - \frac{\mu}{2}\left(\frac{\partial^2\theta}{\partial t^2}\right)^2 . \tag{16}$$

The state of the loss-free medium remains the same after cessation of the steady soliton in it. Below we take the soliton in the extended sense, without assuming its elastic interaction with other solitons similar to it. Then its total 'area', according to Eqn (7), equals $\theta_\infty = 2\pi$. Following Ref. [12], we use the trial solution in the form

$$\theta = 4\arctan\left\{\exp\left(\rho(\mathbf{r})\left[t - \frac{\Phi(\mathbf{r})}{c}\right]\right)\right\}, \tag{17}$$

where $\Phi(\mathbf{r})$ and $\rho(\mathbf{r})$ are the 'fast' and the 'slow' functions of the coordinates, respectively.

Drawing an analogy with the plane monochromatic wave [14], we call $\Phi(\mathbf{r})$ the soliton eikonal. Substituting formula (17) into (16), disregarding the derivatives of $\rho(\mathbf{r})$ [12], and integrating the resulting expression with respect to t, we get the averaged Lagrangian

$$L = \frac{1}{2}\int_{-\infty}^{+\infty}\mathcal{L}\,\mathrm{d}t = \left(\frac{\nabla\Phi}{c}\right)^2\rho - \left(\frac{N_{20}}{c}\right)^2\rho + \frac{\alpha}{\rho} + \frac{\mu}{3}\rho^3 . \tag{18}$$

Using L to write out the Euler–Lagrange equations in Φ and ρ, we find

$$(\nabla\Phi)^2 = N_{20}^2 + c^2\left(\frac{\alpha}{\rho^2} - \mu\rho^2\right), \tag{19}$$

$$\nabla(\rho\nabla\Phi) = 0 . \tag{20}$$

The set of equations (19), (20) can be regarded as the equations of geometrical optics for ultimately short solitons. It is clear from their derivation that a validity condition for this approach is the smallness of the change of soliton's amplitude $\sim \rho$ over its length. Equation (19) can be called the equation of a soliton eikonal. It defines the velocity v of propagation of the soliton wavefront in the direction normal to each point. Indeed, differentiating the equation of propagation of the soliton front $\rho(t - \Phi/c) = \text{const}$ and neglecting the change of the 'slow' variable ρ, we arrive at $\mathrm{d}t = \mathrm{d}\Phi/c = |\nabla\Phi|\,\mathrm{d}s/c$, where $\mathrm{d}s$ is the displacement of the soliton front in the direction of the normal. Hence it follows that $v = \mathrm{d}s/\mathrm{d}t = c/|\nabla\Phi|$. In principle, it is pertinent to develop further the Huygens type constructions [14], which can be used for viewing the dynamics of all parts of the soliton front at any subsequent time. Such a stepwise procedure corresponds to the numerical solution of the set of equations (19), (20).

In the one-dimensional case, when ρ and Φ are functions of z, the set (19), (20) is easily integrated: $\rho = \tau_\mathrm{p}^{-1} = \text{const}$, $\Phi = cz/v$. The slow function ρ here has the meaning of the inverse duration of the soliton, in terms of which its group velocity is directly expressed:

$$\frac{1}{v} = \sqrt{\left(\frac{N_{20}}{c}\right)^2 + \alpha\tau_\mathrm{p}^2 - \frac{\mu}{\tau_\mathrm{p}^2}}\,. \tag{21}$$

The wavefronts of such solitons are the planes perpendicular to the z-axis. According to solution (17), the electric field profile of USP is then

$$E = \frac{\hbar}{2d_1}\frac{\partial\theta}{\partial t} = \frac{\hbar}{d_1\tau_\mathrm{p}}\operatorname{sech}\left(\frac{t - z/v}{\tau_\mathrm{p}}\right). \tag{22}$$

It should be emphasized that expressions (21), (22) give the exact solution of equation (10) at $\Delta = \partial^2/\partial z^2$, $\gamma = 0$, and $\beta = 3\mu/2$ in the form of a one-dimensional ultimately short soliton (that is, a soliton without the high-frequency carrier), which can also be obtained using the ansatz

$$\dot{\theta} = \frac{2}{\tau_\mathrm{p}}\sin\frac{\theta}{2}\,, \tag{23}$$

where, like in Eqn (12), the dot over θ denotes differentiation with respect to $t - z/v$. The difference between solutions (12) and (23) is due to the fact that the 'areas' θ_∞ of the autosoliton and soliton in the conservative medium are not the same.

From formulas (17), (7) and (9) we find the appropriate laws of variation of the population for both components:

$$W_1 = W_{1\infty}\left[1 - 2\operatorname{sech}^2\left(\frac{t - z/v}{\tau_\mathrm{p}}\right)\right],$$
$$W_2 = W_{2\infty}\left[1 - \frac{2}{(\omega_2\tau_\mathrm{p})^2}\operatorname{sech}^2\left(\frac{t - z/v}{\tau_\mathrm{p}}\right)\right]. \tag{24}$$

Hence it follows that both components of the medium regain their initial state after cessation of the pulse, unlike the lossy medium. The center of the soliton profile (22) corresponds to the change in the sign of inversion of the 1-component, and a slight change [see Eqn (2)] of the 2-component state, in accordance with arguments developed in Section 2 (Fig. 4).

In addition, solution (22) involves a free parameter, for which we have selected the length τ_p of the soliton. The value of the free parameter determines the speed and the amplitude of the signal. This is another important distinction between the soliton passing through a conservative medium and the autosoliton.

Now let us analyze the stability of the soliton (22). With this purpose we note that the set of equations (19), (20) can be rewritten as the Bernoulli integral and the continuity equation for steady vortex-free flow of ideal liquid:

$$\frac{\mathbf{V}^2}{2} + \int\frac{\mathrm{d}p}{\rho} = \text{const}\,, \qquad \nabla(\rho\mathbf{V}) = 0\,, \tag{25}$$

where the 'velocity' $\mathbf{V}$ is defined as $\mathbf{V} = \nabla\Phi/c$, and the linkage between 'pressure' p and 'density' ρ is expressed by the equation

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\alpha}{\rho^2} + \mu\rho^2\,. \tag{26}$$

Hence in a straightforward way follows the condition of stability of soliton (22) as the steady flow criterion of an ideal liquid of the type of equations (25), (26): $\mathrm{d}p/\mathrm{d}\rho > 0$. Making the replacement $\rho = \tau_\mathrm{p}^{-1}$ in the expression for $\mathrm{d}p/\mathrm{d}\rho$, we

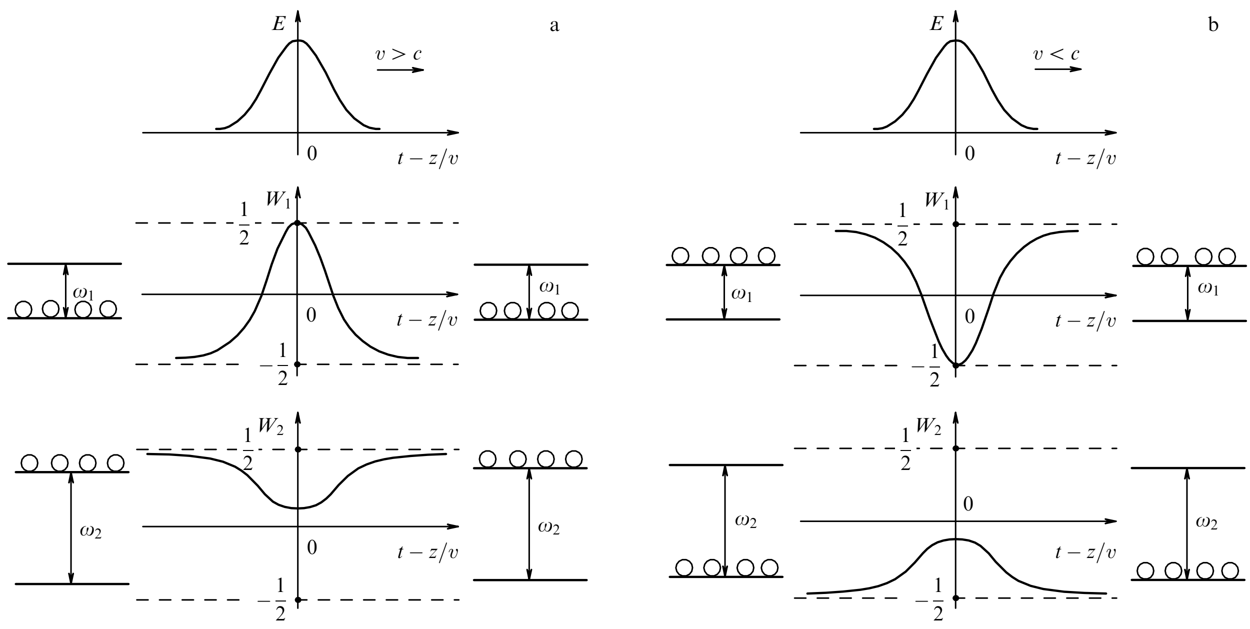

Figure 4. Stable running electric field profiles of the ultimately short autosoliton and the corresponding inversions of the 1- and 2-component in a loss-free medium: (a) absorbing – amplifying medium: $v > c$; (b) amplifying – absorbing medium: $v < c$.

obtain

$$\alpha\tau_p^2 + \mu\tau_p^{-2} > 0\,. \tag{27}$$

Condition (27) can be readily interpreted as follows. From Eqn (21) it appears that, as long as inequality (27) is satisfied, the velocity v is a steadily increasing function of the parameter τ_p^{-1} {the soliton amplitude [see also Eqn (22)]}. Therefore, the portions of the soliton with a larger amplitude, corresponding to the center of the USP cross section, overtake during its propagation the peripheral portions whose amplitude is smaller. If condition (27) is not satisfied, we come to the opposite pattern which eventually leads to self-focusing of the soliton. We write the condition at which the soliton velocity is $v > c$ [see Eqn (21)] in the form

$$\alpha\tau_p^2 + \mu\tau_p^{-2} < \frac{1 - N_{20}^2}{c^2}\,. \tag{28}$$

Straightforward analysis of Eqn (28) indicates that $v > c$ if either $\alpha < 0$ (1-component inverted) or $\mu < 0$ (2-component inverted). As follows from Eqn (27), soliton (22) is unstable in the case of an inverse population of both components ($\alpha < 0$ and $\mu < 0$). For possible observation of the superluminal soliton, its length must simultaneously satisfy conditions (27) and (28).

Let us consider in greater detail the case of $\alpha > 0$, $\mu < 0$ ($W_{1\infty} = -1/2$, $W_{2\infty} = 1/2$, $N_{20} < 1$). From inequalities (27), (28) and formulas (11) we find that the length τ_p of a stable superluminal soliton must fall within the interval (see Fig. 5)

$$\tau_m < \tau_p < \tau_c\,, \tag{29}$$

where

$$\tau_m = \frac{1}{\omega_2}\sqrt{\frac{2}{q}}\,, \quad \tau_c = \frac{\sqrt{2}}{\omega_2}\left(1 - \sqrt{1 - q^2}\right)^{-1/2},$$

$$q^2 = 4\frac{n_1\omega_1}{n_2\omega_2} < 1\,. \tag{30}$$

To have condition $(\omega_2\tau_p)^2 \gg 1$ satisfied at the edges of the interval (29), we must require that $q \ll 1$. Then τ_c will be approximated as $\tau_c = 2/(\omega_2 q)$. Observe that at the maximum admissible value $q = 1$, the width of the interval (29) goes to zero and rapidly increases with decreasing q. In other words, the smaller q, the bigger the chance that condition (29) will be satisfied. As follows from Eqn (21), τ_m corresponds to the soliton length for which its velocity is the highest:

$$v_{max} = c\left[1 - \eta(1 - q)\right]^{-1/2}. \tag{31}$$

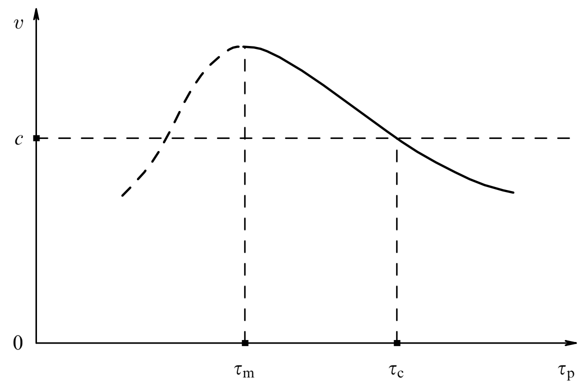

Figure 5. Velocity v of a stable ultimately short soliton in two-component absorbing – amplifying loss-free medium as function of its length τ_p; in the interval $\tau_m < \tau_p < \tau_c$, we have $v > c$.

Assuming that $q \ll 1$, we have $v_{max} = c/N_{20}$. This value practically coincides with the velocity of the autosoliton in a nonequilibrium medium with energy dissipation [see Eqn (1)]. This conclusion is true only if the parameter η is much less than unity. When $\eta \to 1$, then $N_{20} \to 0$ and the velocity v_{max} tends to $c/\sqrt{q}$ rather than to infinity, as would follow from the expression $v_{max} \approx c/N_{20}$. A similar situation is encountered in the case of a superluminal autosoliton in a dissipative medium (see Section 4). Since the parameter q can theoretically be arbitrarily small, there is formally no upper limit imposed on the velocity of the soliton. At $\eta(1 - q) > 1$, the value of v_{max} becomes imaginary, and a soliton of the corresponding length τ_p is no longer feasible.

The necessary condition $q < 1$, at which $v > c$, together with the condition of stability of the soliton, $\tau_p > \tau_m$, can be written as

$$\frac{4\omega_1}{\omega_2} < \frac{n_2}{n_1} < (\omega_1\tau_p)(\omega_2\tau_p)^3\,. \tag{32}$$

Taking, for example, $\omega_1\tau_p = 0.1$, $\omega_2\tau_p = 10$ [see Eqn (2)], we get $\omega_1/\omega_2 = 0.01$ and, as follows from Eqn (32), $0.04 < n_2/n_1 < 100$. Setting $n_2 = 4n_1$, one finds $q = 0.1$, $\omega_2\tau_m \approx 4.5$, $\omega_2\tau_c \approx 20$. Then $c < v < v_{max}$. Setting $\eta = 0.2$, we get $v_{max} \approx 1.1c$. Provided that $\eta = 1$, then $v_{max} \approx 3.2c$. Recall that v_{max} is achieved at $\tau_p = \tau_m$. Thus, the highest velocities of solitons are possible when $\eta \to 1$ ($N_{20} \to 0$). In cases like this, the soliton may travel at velocities reaching several times the speed of light. If the medium is not so dense and $\eta \ll 1$, then v is greater than c by just a few percent.

An analysis similar to what has just been done indicates that in the opposite case (the 1-component inverted: $\alpha < 0$, $\mu > 0$), conditions (2), (27) and (28) are incompatible. Therefore, in such a case the stable propagation of only slower-than-light solitons is possible (Fig. 4b). Notice also that in strictly one-component two-level systems conditions (2), (27) and (28) are not compatible with one another as well. Indeed, if $\mu = 0$ ($N_{20} = 1$), from inequalities (27) and (28) we get two opposing conditions: $\alpha > 0$ and $\alpha < 0$, respectively. This implies that the superluminal ($\alpha < 0$) soliton of the sine-Gordon equation is not stable in the inverted two-level medium [see Eqn (10) with $\gamma = \beta = \mu = 0$]. This agrees with the earlier studies of monochromatic envelope solitons in a two-level resonance medium [1]. However, in the absence of 1-component ($\alpha = 0$) at $\mu > 0$ (the ground-state 2-component), conditions (27) and (28) do not contradict one another, but Eqn (28) is incompatible with (2). In this case $c/N_{20} < v < c$ [11]. If the 2-component is inverted ($\mu < 0$) at $\alpha = 0$, the soliton is unstable because condition (27) is not satisfied.

The presence of the free parameter τ_p in the solutions (21), (22) emphasizes the dependence of the characteristics of the soliton formed in the medium on the pulse parameters at the point of entry into the medium.

In the general case, the analytical study of the linkage between the parameters of the input signals and the resulting solitons is by no means simple. This is largely because equation (10) is nonintegrable. Observe, however, that by virtue of formula (1) each term in the right-hand side of Eqn (10) at $\gamma = 0$ is infinitesimal of a higher order with respect to small parameters $\varepsilon \sim (\omega_1\tau_p)^2 \sim (\omega_2\tau_p)^{-2} \ll 1$ than the terms in the left-hand side. Therefore, one can apply to the approximation of unidirectional soliton propagation along the z-axis with the velocity close to c/N_{20} [25]. With this purpose we introduce the local time $\tau = t - N_{20}z/c$ and the 'slow coordinate' $\zeta = \varepsilon z$. Going over to these new independent variables, we get $\partial/\partial t = \partial/\partial\tau$, $\partial/\partial z = -(N_{20}/c)\partial/\partial\tau + \varepsilon\partial/\partial\zeta$. Hence, neglecting ε^2, we find $\partial^2/\partial z^2 \approx (N_{20}/c)^2\partial^2/\partial\tau^2 - 2\varepsilon(N_{20}/c)\partial^2/\partial\tau\,\partial\zeta$. Going back to variable z, from formula (10) at $\gamma = 0$, $\beta = 3\mu/2$ ($d_1 = d_2$) we come to an equation of the type

$$\frac{\partial^2\theta}{\partial z\,\partial\tau} + a\sin\theta - \frac{3}{2}b\left(\frac{\partial\theta}{\partial\tau}\right)^2\frac{\partial^2\theta}{\partial\tau^2} - b\frac{\partial^4\theta}{\partial\tau^4} = \frac{c}{2N_{20}}\Delta_\perp\theta\,, \quad (33)$$

where $a = c\alpha/(2N_{20})$, $b = c\mu/(2N_{20})$, and $\Delta_\perp$ is the transverse Laplacian.

It should be recorded that Eqn (33) can be derived from the density of the Lagrangian

$$\mathcal{L} = \frac{1}{2}\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial\tau} - a(1-\cos\theta) - \frac{b}{8}\left(\frac{\partial\theta}{\partial\tau}\right)^4 + \frac{b}{2}\left(\frac{\partial^2\theta}{\partial\tau^2}\right)^2 - \frac{c}{4N_{20}}(\nabla_\perp\theta)^2\,. \quad (34)$$

Substituting here the trial solution (17) and taking advantage of the slowness of function ρ, we find the averaged Lagrangian

$$L \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\mathcal{L}\,\mathrm{d}\tau = -\rho\frac{\partial\Phi}{\partial z} - \frac{a}{\rho} - \frac{b}{3}\rho^3 - \frac{c}{2N_{20}}\rho(\nabla_\perp\theta)^2\,. \quad (35)$$

The corresponding Euler – Lagrange equations then become

$$\frac{\mathbf{V}_\perp^2}{2} + \frac{\partial\phi}{\partial z} + \int\frac{\mathrm{d}p}{\rho} = 0\,, \qquad \frac{\partial\rho}{\partial z} + \nabla_\perp(\rho\mathbf{V}_\perp) = 0\,, \quad (36)$$

where

$$\phi = \left(\frac{c}{N_{20}}\right)\Phi\,, \qquad \mathbf{V}_\perp = \nabla_\perp\phi\,,$$

$$\mathrm{d}p/\mathrm{d}\rho = \left(\frac{2c}{N_{20}}\right)\left(\frac{a}{\rho^2} + b\rho^2\right).$$

The set of equations (36) is comprised of the dynamic equations for the potential flow of an ideal liquid, in which the role of time is played by the coordinate z, and the potential of the field of velocities $\mathbf{V}_\perp$ is proportional to the soliton eikonal. The first equation has the meaning of the Cauchy integral for the 'nonstationary' flow, and the second is the continuity equation. The linkage between the 'pressure' p and 'density' ρ up to a constant coefficient coincides with Eqn (26). Because of this, the criterion of stability of the soliton answering to equation (33) does not differ from that in Eqn (27). Using, for example, ansatz (23), it is easy to see that the one-soliton solution of equation (33) at $\Delta_\perp\theta = 0$ in the laboratory system of coordinates has the form (22), and the soliton's velocity as a function of τ_p is given by

$$\frac{1}{v} = \frac{N_{20}}{c} + a\tau_p^2 - \frac{b}{\tau_p^2}\,. \quad (37)$$

This relation can be derived from formula (21) by expanding the latter in a power series of a small parameter $(c/N_{20})^2(\alpha\tau_p^2 - \mu/\tau_p^2)$, which amounts to assuming that the velocity of the soliton v is close to c/N_{20}. Observe that the solitons of Eqn (33) are true solitons at $\Delta_\perp\theta = 0$ because they possess the property of elastic interactions with their like, since this equation is integrated using the inverse scattering problem technique [26]. The disadvantage of equation (33) as compared with (10) is that it does not provide a framework for considering the head-on interaction of pulses. It would be worthwhile to consider analytically what the profiles of input signals are capable of producing superluminal solitons in the medium in question. Let us illustrate the above arguments for the case when the input signal decomposes in the absorbing – amplifying medium into two solitons. The two-soliton solution of Eqn (33) then assumes the form [26]

$$E(z,t) = \frac{\hbar}{2d}\frac{\partial\theta}{\partial t} = \frac{2\hbar}{d}\frac{\partial}{\partial t}\arctan\left[\frac{\exp S_1 + \exp S_2}{1-(\tau_{p1}-\tau_{p2})^2(\tau_{p1}+\tau_{p2})^{-2}\exp(S_1+S_2)}\right], \quad (38)$$

where $S_j = [t - (z/v_j) + t_j]/\tau_{pj}$, $1/v_j = N_{20}/c + a\tau_{pj}^2 - b/\tau_{pj}^2$ ($j = 1, 2$), τ_{p1} and τ_{p2} are free parameters that depend on the profile of the input signal $E(0,t)$, and t_j are the constants that define the time interval $\Delta t_{12} = |t_2 - t_1|$ between the local maxima of the pulse profile $E(0,t)$ applied to the input ($z = 0$) of the medium (Fig. 6). Obviously, τ_{pj} and v_j have the meaning of the length and velocity of jth soliton, respectively, when the solitons are far enough from each other. Setting $z = 0$ in Eqn (38), we find the class of initial profiles that give rise to two different solitons in the medium. If the shape of the profile $E(0,t)$ is such that both the parameters τ_{pj} satisfy inequality (29), then the velocities of the two emerging solitons are superluminal. The faster soliton will be the one for which the parameter τ_p is smaller. If, for example, τ_{p1} satisfies constraints (29), whereas $\tau_{p2} > \tau_c$, τ_m, then the speed of the first soliton will be greater than c, while the second soliton will be subluminal. Assume now that $\tau_{p2} < \tau_c$, τ_m, while τ_{p1} still satisfies inequality (29). In this case at least one of the solitons (for which $\tau_{p2} < \tau_m$) will be unstable. The situation here depends considerably on the ratio between the rates of development of instability and the decomposition of the input signal into separate solitons. In any case, the soliton creation from the initial signal takes some time. Because of this, it is not obvious that the time from the moment the signal is fed to the input of a nonequilirbium medium to the moment it is detected at the exit will be less than L/c, where L is the length of the medium sample in the direction of the soliton

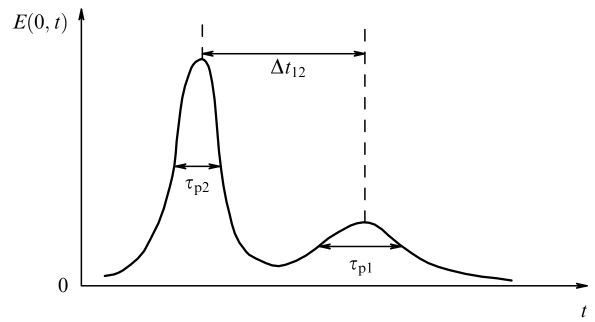

Figure 6. Input pulse $E(0,t)$ corresponding to the two-soliton solution of equation (33). If τ_{p1} and τ_{p2} satisfy condition (29), both the solitons appeared are superluminal. The stability of this solution is ensured by the condition $\tau_{p1}, \tau_{p2} \ll \Delta t_{12}$.

propagation. This issue calls for further investigation. Obviously, the best option is to apply such a signal to the medium input whose shape is close to profile (22) and whose length τ_p satisfies condition (29). Then, according to the arguments developed above, the speed of the soliton in the quasi-nonequilibrium medium will exceed the speed of light in vacuum. Here it is important to add that our considerations with respect to the stability of the two-soliton solution to equation (33) hold good when $\Delta t_{12} \gg \tau_{p1}, \tau_{p2}$. As a matter of fact, criterion (27) has been obtained from the condition of small perturbation of the exact solution (22) [see Eqn (17)]. To be able to talk about the stability of solution (34), we have to perform the procedure of obtaining the averaged Lagrangian for the small perturbation of this two-soliton solution, which meets with mathematical difficulties. If, however, in the initial profile we have $\Delta t_{12} = |t_2 - t_1| \gg \tau_{p1}, \tau_{p2}$, and the peak of the smaller amplitude follows the higher peak (see Fig. 6), then the above conclusions concerning the stability of the one-soliton solutions will also apply to solution (34). If, however, in the input profile $E(0, t)$ the shorter peak comes before the taller one, then the two peaks will originally tend to come closer to each other, which will eventually lead to the violation of the assumption that they are distant from each other, and hence the conclusions regarding the stability of the two-soliton solution, based on applying criterion (27) separately to τ_{p1} and τ_{p2}, are doubtful. Obviously, the analysis of the stability of many-soliton solutions meets with immense mathematical difficulties.

Somewhat different is the situation with the breather solutions of Eqn (33), which can be obtained from Eqn (38) if we assume that τ_{p1} and τ_{p2} are complex conjugate. Taking $\tau_{p1,2} = \tau_p/(1 \pm i\omega\tau_p)$, $t_1 = t_2 = 0$, we find from Eqn (38) the following solution

$$E = \frac{\hbar}{2d_1}\frac{\partial\theta}{\partial t} = \frac{2\hbar}{d_1}\frac{\partial}{\partial t}\arctan\left\{\frac{1}{\omega\tau_p}\operatorname{sech}\left(\frac{t - z/v}{\tau_p}\right) \times \sin\left[\omega\left(t - \frac{z}{v_{ph}}\right)\right]\right\}, \quad (39)$$

where v and v_{ph} are the group and the phase velocities of the breather, respectively, which are expressed in terms of its frequency ω and length τ_p as follows:

$$\frac{1}{v} = \frac{N_{20}}{c} + \frac{a}{\omega^2 + \tau_p^{-2}} + b(3\omega^2 - \tau_p^{-2}), \quad (40)$$

$$\frac{1}{v_{ph}} = \frac{N_{ph}}{c} = \frac{N_{20}}{c} - \frac{a}{\omega^2 + \tau_p^{-2}} + b(\omega^2 - 3\tau_p^{-2}). \quad (41)$$

Here we have introduced the phase refractive index N_{ph}. From Eqn (39) we see that the area of the breather is $\theta_\infty = 0$. Then as follows from Eqns (7) and (9), after the passage of the breather the medium returns to its initial state. At $\omega\tau_p < 1$, solution (39) describes an ultimately short pulse type breather, whose length is just about one period of electromagnetic oscillations. If, on the other hand, $\omega\tau_p \gg 1$, then, as follows from Eqns (39)–(41), the breather becomes an envelope soliton

$$E = \frac{2\hbar}{d_1\tau_p}\operatorname{sech}\left(\frac{t - z/v}{\tau_p}\right)\cos\left[\omega\left(t - \frac{z}{v_{ph}}\right)\right], \quad (42)$$

$$\frac{1}{v} = \frac{N_{20}}{c} + \frac{a}{\omega^2} + 3b\omega^2, \quad \frac{1}{v_{ph}} = \frac{N_{ph}}{c} = \frac{N_{20}}{c} - \frac{a}{\omega^2} + b\omega^2. \quad (43)$$

This solution can be obtained by applying the approximation of a slowly changing envelope directly to Eqn (33). In accordance with this, we represent the field of the pulse as follows

$$E = \frac{1}{2}\mathcal{E}(z, \mathbf{r}_\perp, \tau)\exp\left[i(\omega\tau - qz)\right] + \text{c.c.}, \quad (44)$$

where $\mathcal{E}(z, \mathbf{r}_\perp, \tau)$ is the slowly changing envelope, ω is the carrier frequency of the pulse, and q is the wave number in the comoving frame of reference, linked with the wave number k in the laboratory system of coordinates by the relation $k = q + \omega N_{20}/c$. According to the approximation of a slowly changing envelope, we have

$$\left|\frac{\partial\mathcal{E}}{\partial\tau}\right| \ll \omega|\mathcal{E}|, \quad \left|\frac{\partial\mathcal{E}}{\partial\tau}\right| \ll q|\mathcal{E}|. \quad (45)$$

Using multiple integration by parts, we arrive at

$$\theta = \frac{2d_1}{\hbar}\int_{-\infty}^{\tau} E\,d\tau' = \frac{2d_1}{\hbar}\left(\frac{\mathcal{E}}{i\omega} + \frac{1}{\omega^2}\frac{\partial\mathcal{E}}{\partial\tau} + \frac{i}{\omega^3}\frac{\partial^2\mathcal{E}}{\partial\tau^2} + \cdots\right) + \text{c.c.} \quad (46)$$

The role of the characteristic time scale of the pulse is played here by the inverse frequency ω^{-1}. Then the nonresonance condition (2) holds true. Since the spectral width $\delta\omega$ of the envelope pulse is much less than ω, its spectrum, according to Eqn (42), does not contain resonant Fourier components. Because of this, the excitation of each of the medium components by the pulse field is negligible. Consequently, one obtains $\sin\theta \approx \theta - \theta^3/6$. Also substituting field strength (44) into equation (33) and using Eqns (45), (46), we arrive at the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial\mathcal{E}}{\partial z} + \frac{k_2}{2}\frac{\partial^2\mathcal{E}}{\partial T^2} - \frac{\omega}{c}N_2|\mathcal{E}|^2\mathcal{E} = \frac{c}{2N_{20}\omega}\Delta_\perp\mathcal{E}, \quad (47)$$

where $T = \tau - (a/\omega^2 + 3b\omega^2)z$, $k_2 \equiv \partial^2 k/\partial\omega^2 = 2(3b\omega - a/\omega^3)$ is the parameter of group dispersion, $N_2 = -(1/4)(c/\omega)\times(d_1/\hbar)^2 k_2$ is the nonlinear index of refraction of the medium, found from the expression

$$N_{ph} = N + N_2|\mathcal{E}|^2, \quad (48)$$

where N is the linear part of the phase index of refraction (48) [see Eqn (1)]. As we see, the condition of feasibility to form envelope solitons, $k_2 N_2 < 0$, is automatically satisfied here [27]. The one-soliton solution of Eqn (47) at $\Delta_\perp\mathcal{E} = 0$ with due account for expression (44) exactly coincides with formulas (42), (43). In this way, the envelope soliton of the nonlinear Schrödinger equation (47) is the limiting case of the breather of equation (33) at $\omega\tau_p \gg 1$. Earlier a similar conclusion for other equations was made in Refs [27, 28]. NLS solitons are stable with respect to self-focusing if $N_2 < 0$ (a defocusing medium) [29]. The same conclusion is implied by the method of the averaged Lagrangian applied to the NLS [12]. Then, using the explicit expression for N_2, we may write out the following condition

$$\frac{a}{\omega^4} - 3b < 0. \quad (49)$$

In the general case, the phase velocity of the breather is given by expression (41). As the amplitude of the pulse field E_m increases, the rate of induced transitions becomes higher. Because of this, the duration of solitons and breathers becomes shorter. Accordingly, the amplitude is a steadily increasing function of the variable $\rho = \tau_p^{-1}$. Expressions (22), (39) and (42) confirm this general property of the soliton-like solutions.

Let us expand formula (41) in a Taylor series in ρ^2:

$$\frac{1}{v_{\rm ph}} = \frac{1}{v_{\rm ph}(0)} + n_{2\rho}\rho^2 + \ldots ,$$

where $1/v_{\rm ph}(0) = N_{20}/c - a/\omega^2 + g\omega^2$ is an expression coinciding with that for the envelope soliton [see Eqn (43)], and $n_{2\rho} \equiv [\partial(1/v_{\rm ph})/\partial\rho^2]_{\rho=0} = a/\omega^4 - 3g \sim N_2$. In line with the conclusion about the monotonic increase of $E_{\rm m}(\rho)$, the quantities $n_{2\rho}$ and N_2 have the same sign. Then the condition of stability of the envelope soliton $N_2 < 0$ can also be written as $n_{2\rho} < 0$, which coincides with inequality (49).

In order to extend condition (49) to the case of breathers with arbitrary values of $\omega\tau_p$, we use the following considerations. If the index of refraction $N_{\rm ph} \equiv c/v_{\rm ph}$ increases with increasing $E_{\rm m}$ (or ρ), then the core of the breather in its cross section, where $E_{\rm m}$ is the highest, is travelling slower than its peripheral parts. This leads to self-focusing of the breather. If $N_{\rm ph}$ decreases with increasing ρ, then the breather is stable with respect to transverse perturbations. Thus, one finds

$$\frac{\partial}{\partial\rho^2}\left(\frac{1}{v_{\rm ph}}\right) < 0 \,. \tag{50}$$

From this and from equation (41) we find the condition of stability of the breather (39):

$$\frac{a}{(\omega^2 + \tau_p^{-2})^2} - 3b < 0 \,, \tag{51}$$

which becomes inequality (49) when $(\omega\tau_p)^2 \gg 1$. The criterion (51) together with the conditions

$$\frac{a}{\omega^2 + \tau_p^{-2}} + b(3\omega^2 - \tau_p^{-2}) < \frac{1 - N_{20}}{c} \tag{52}$$

[see Eqn (40)] and

$$\omega_1^2 \ll \omega^2 + \tau_p^{-2} \ll \omega_2^2 \tag{53}$$

defines the feasibility of stable propagation in our two-component medium of superluminal breathers, whose limiting cases (at $\omega\tau_p \gg 1$) are nonresonant envelope solitons. Condition (53) is a natural extension of above condition (2), as well as of condition $\omega_1^2 \ll \tau_p^{-2} \ll \omega_2^2$ used for pulses without high-frequency carrier.

The analysis of inequalities (51)–(53) indicates that the existence of stable superluminal breathers is possible in AmAbM and AmM. In the first case ($W_{1\infty} = 1/2$, $W_{2\infty} = -1/2$ or $a < 0$, $b > 0$), the breather is absolutely stable, since at $a < 0$ and $b > 0$ inequality (51) is satisfied automatically. Given condition (53), it is easy to find from formula (52) the realizability criterion for $v > c$:

$$\omega^2 + \tau_p^{-2} < \frac{3\omega_c^4}{\omega_2^2} = \left(\frac{n_1}{n_2}\right)\omega_1\omega_2 \,, \tag{54}$$

where the frequency parameter ω_c is given by the expression

$$\omega_c^4 = \frac{n_1}{2n_2}\,\omega_1\omega_2^3 \,. \tag{55}$$

Since, according to condition (53), we have $\omega^2 + \tau_p^{-2} \gg \omega_1^2$, then $n_1/n_2 \gg \omega_1/\omega_2$. The last inequality is easily satisfied, for example, at $n_1 = n_2$. From criterion (54) at $\omega\tau_p \gg 1$ we find the condition that the group velocity of the envelope soliton is $v > c$: $\omega_1 \ll \omega \ll \sqrt{3}\omega_c^2/\omega_2$.

In the case of AmM ($W_{1\infty} = W_{2\infty} = 1/2$ or $a < 0, b < 0$), the range of stability of the breather (39), as follows from conditions (51) and (53), is defined by the dual inequality

$$\omega_1^2 \ll \omega^2 + \tau_p^{-2} < \omega_c^2 \,. \tag{56}$$

Then the group velocity of the breather, expressed by formula (40), is always greater than the speed of light in vacuum. The condition $\omega_c^2 \gg \omega_1^2$ required for the formation of the breather in a two-component nonresonant medium is rewritten in the form [see Eqn (55)] $3n_2/n_1 \ll (\omega_2/\omega_1)^3$. This inequality is easily satisfied, for example, at $n_2 \simeq n_1$, because $\omega_2 \gg \omega_1$. Observe that when only the second component is present ($n_1 = 0$), there are no stable envelope breathers or nonresonant solitons in the amplifying medium. Condition (56) for envelope nonresonant solitons can be written as $\omega_1^2 \ll \omega^2 < \omega_c^2$.

The issue about information transfer by electromagnetic solitons in conservative media, like in the case of dissipative media, calls for special treatment. In a nonequilibrium lossy medium, as noted in the previous section, the superluminal solitons whose formation depends on the presence of energy dissipation (autosolitons) do not carry information about the input pulse, because the corresponding solutions do not contain free parameters. At the same time, the soliton type solutions in conservative medium involve free parameters (length τ_p for solitons without high-frequency carrier or ω and τ_p for breathers), whose values depend on the profile of the input pulse. It would seem therefore that the ultimately short subluminal solitons in nonequilibrium lossless medium carry information about the profiles of input pulses which initially gave rise to such solitons.

A more comprehensive analysis indicates, however, that this is not the case. Let us use the example of USP (22) for analyzing the mechanisms of their propagation in absorbing ($v < c$) and amplifying ($v > c$) media. In the former case, the leading edge of the incoming pulse 'is eaten away' by the absorbing atoms (Fig. 7a). This shapes the front portion of the pulse profile, and the atoms start to move to the excited states. The photons from the middle portion of the pulse are absorbed to a lesser extent than the leading photons, because on their way forward they meet fewer atoms in the ground state (Fig. 4b). The trailing photons of the pulse, which meet with little resistance in the medium, start 'pushing on' the photons travelling in front (thus causing self-compression of the pulse and increasing its peak intensity as compared with the intensity of the input signal) and cause induced radiation from the excited atoms. As the atoms drop back to the ground state, the probability of induced radiation decreases and in this way the smooth tail of the soliton is formed. The important circumstance here is that the leading and the trailing edges are formed by the pulse itself through absorption and further reemission of its energy. The greater the amplitude of the input signal, the more intensive the processes of absorption and reemission, and the steeper the edges of the soliton formed in the medium. Thus, in the absorbing medium ($v < c$) the solitons formed carry information about the profiles of input pulses.

The situation with a nonequilibrium medium is different. Consider first the case when a nearly rectangular pulse is applied to the point of entry of a nonequilibrium medium (Fig. 7b). The spectrum of the leading edge of this pulse is dominated by the high frequencies satisfying the conditions $\omega \gg \omega_1, \omega_2$. Accordingly, for both components of the medium we have solutions similar to formulas (7). Then the

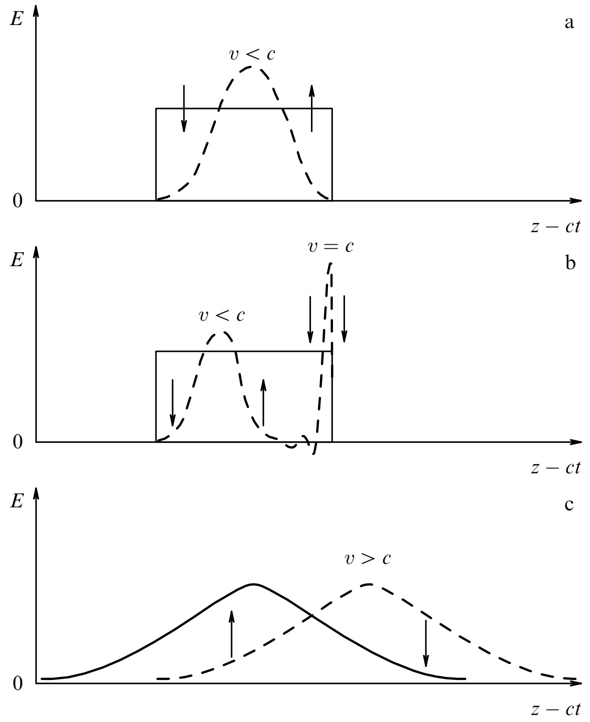

Figure 7. Mechanisms of soliton formation in equilibrium (a) and nonequilibrium (b, c) media. In case (a) the initial squared pulse (solid line) gives rise to a subluminal ultimately short soliton (dashed line) which carries information about the parameters of the input pulse. A squared pulse applied to the nonequilibrium medium [case (b)] does not give rise to superluminal solitons. Instead, an enhancing leading edge is created, whose velocity equals c (precursor), followed by one or several subluminal solitons (dashed line) which carry information about the parameters of the input pulse profile. Case (c) corresponds to superluminal solitons in nonequilibrium medium. Their formation and stable propagation require that the initial profile should have a descending leading edge defined over the entire length of the medium. Therefore, no information is transmitted in this case. The up and down arrows correspond to absorption or emission of the pulse energy.

wave equation for this part of the input pulse coincides with the sine-Gordon equation ($\gamma = 0$, $d_1 = d_2 = d$)

$$\Delta\theta - \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} = \alpha_{12}\sin\theta\,, \tag{57}$$

where $\alpha_{12} = -(16\pi d^2/\hbar c^2)(\omega_1 n_1 W_{1\infty} + \omega_2 n_2 W_{2\infty})$. In the case of an AbAmM ($W_{1\infty} < 0$, $W_{2\infty} > 0$) and $n_1 \simeq n_2$, since $\omega_1 \ll \omega_2$, we have $\alpha_{12} < 0$.

To analyze the solution (57) in this case we introduce, following Ref. [10], the self-simulated variable $\xi = z^2 - c^2t^2$, after which relation (57) at $\Delta = \partial^2/\partial z^2$ and $\alpha_{12} < 0$ assumes the form [10]

$$\xi\theta'' + \theta' = -\frac{|\alpha_{12}|}{4}\sin\theta\,, \tag{58}$$

where the prime at θ denotes differentiation with respect to ξ.

Numerical calculations indicate that the solution of equation (58) for E, decreasing over the interval $[-\infty, +\infty]$, is a sign-alternating function of variable ξ with a total area equal to π (a π-pulse) [30]. This function differs from zero mainly near $\xi = 0$ [which allows us to write $\xi = (z + ct)(z - ct) \approx 2z(z - ct)$], increasing during propagation along the z-axis in amplitude with simultaneous self-compression. This circumstance allows us to carry out an approximate analytical study of the solution of equation (58). Taking advantage of the fact that near $\xi = 0$ — that is, near the maximum of E — the function θ is smooth, we disregard the first term in the left-hand side of Eqn (58). After integration of the resulting equation, we find

$$\mathrm{tg}\,\frac{\theta}{2} = \exp\left[-\frac{|\alpha_{12}|}{2}z(z - ct)\right],$$

$$E = \frac{\hbar|\alpha_{12}|}{4d}\,z\,\mathrm{sech}\left[\frac{|\alpha_{12}|}{2}z(z - ct)\right]. \tag{59}$$

Substituting Eqn (59) into (7), we get

$$W_{1,2} = W_{1,2\infty}\,\tanh\left[\frac{|\alpha_{12}|}{2}z(z - ct)\right]. \tag{60}$$

From formulas (59) we see that the amplitude and the inverse width of the π-pulse travelling in the AbAmM with velocity c both grow in proportion to z [10]. As follows from Eqn (60), the amplifying π-pulse switches the 2-component to the ground state ($W_{2\infty} > 0$) and, at the same time, excites the 1-component ($W_{1\infty} < 0$).

Away from $\xi = 0$, equation (58) can be linearized by writing $\theta = \pi + Y$, $|Y| \ll 1$. Then we arrive at

$$\xi Y'' + Y' - \left(\frac{|\alpha_{12}|}{4}\right)Y = 0\,. \tag{61}$$

The solution of this last equation is expressed in terms of the zero-order (for Y) and first-order (for E) Bessel functions:

$$Y \sim J_0\left(\sqrt{|\alpha_{12}|(c^2t^2 - z^2)}\right),$$

$$E \sim z\,\frac{J_1\left(\sqrt{|\alpha_{12}|(c^2t^2 - z^2)}\right)}{\sqrt{|\alpha_{12}|(c^2t^2 - z^2)}}\,. \tag{62}$$

Expressions (62) hold true when $ct > |z|$ (oscillating tail of the π-pulse, see Fig. 7b). If $|z| > ct$, we get the solution of equation (61) as the MacDonald function, which for E in the asymptotic limit $|\alpha_{12}|\xi \gg 1$ assumes the form

$$E \sim z\left[|\alpha_{12}|(z^2 - c^2t^2)\right]^{-3/4}\exp\left\{-\sqrt{|\alpha_{12}|(z^2 - c^2t^2)}\right\}. \tag{63}$$

According to expressions (60) and (63), as the π-pulse travels on, its leading edge becomes steeper and steeper, leaving in its wake the weak quasi-periodical pulsations of the electric field strength, accompanied by small oscillations of inverse population in the vicinities of $W_1 = -W_{1\infty} = 1/2$ and $W_2 = -W_{2\infty} = -1/2$ [see Eqns (62) and (7) at $\theta = \pi + Y$].

In this way, in the wake of the π-pulse an amplifying–absorbing medium forms (for which $\alpha_{12} > 0$), which in our case is not capable of hosting the superluminal solitons. Accordingly, subluminal solitons are formed from the remaining part of the rectangular input pulse, whose mechanism of formation has been discussed above. Information about the profile of the input pulse in this case is carried across the nonequilibrium medium by the π-pulse travelling at the speed of light, which plays the role of a precursor, and by the solitons that follow. The trailing front of the rectangular input pulse, like the leading front, contains frequencies $\omega \ll \omega_1, \omega_2$ in its spectrum. However, the corresponding Fourier components are not strong enough to excite the atoms of the 2-component and to form the 2π-soliton of equation (57) by the trailing front, where $\alpha_{12} > 0$ (because

$W_{1\infty} > 0, W_{2\infty} < 0$). For the same reason, the leading and the trailing edges of the rectangular pulse, fed to the absorbing medium, are not capable of each separately giving rise to solitons.

The mechanism of faster-than-light propagation of the pulse in a nonequilibrium medium has been described in sufficient detail in Ref. [1]. The descending portion of the pulse running before its core causes induced radiation from the excited atoms (2-transitions), so that some of them drop back to the ground state. As a result, the descending portion is replaced by the new core (Fig. 7b). At the site of its local disposition, the atoms occur in the ground state after emitting radiation (Fig. 4a). The former core is absorbed by the 1-component and by some of the 2-transitions that have fallen back to the ground state. As a result, what used to be the core becomes the exponentially falling off tail. In this way, by virtue of local amplification in a nonequilibrium medium, the pulse profile travels faster than the photons themselves. In papers [5, 31] this mechanism of superluminal propagation was called 'reshaping'.

It should be noted that the pulse energy in this case is not transferred from one point to another, as was the case in the absorbing medium, but is instead taken from the inverted medium by the exponentially falling off portion travelling far ahead of the core. The slow abatement of the front portion ($\tau_p > \tau_c$) results in the slow proceeding of the induced processes. As a result, the rate of the profile reshaping can be slower than c. When the front edge becomes steeper ($\tau_p < \tau_c$), the amplitude of the profile increases, which speeds up the induced processes of emission and reabsorption. As a consequence, the speed of the pulse profile propagation may be faster than the speed of light. If in some way we cut off the leading portion of the pulse, this will lead to a gradual steepening of its front edge (that is, to its deformation) [1]. This steepening facilitates the broadening of the pulse spectrum towards the higher frequencies $\omega \gg \omega_1, \omega_2$. In this case, the speed of the pulse front portion becomes close to c. The trailing edge of the pulse, whose velocity is $v > c$, pushes on the leading edge from behind, which leads to strong compression of the pulse and amplification of its peak. Eventually such a pulse is transformed into the π-pulse discussed above, which travels in the unbounded medium in the regime of amplification with the group velocity $v = c$.

Thus, the steady propagation of superluminal ultimately short solitons, breathers, and envelope solitons in a nonequilibrium medium requires their smooth exponentially localized profiles to be set from the outset for $0 < z < +\infty$. Therefore, the information about the initial parameters of the pulses with the profiles like those defined in Eqns (22), (39) or (42), for which $v > c$, is *known over the entire expanse of the medium even before they start to traverse it*.

Consequently, there is no reason to speak about the transmission of information in a nonequilibrium medium at superluminal speed by electromagnetic solitons, and there is no reason to question the fundamental causality principle.

The envelope solitons discussed here are nonresonant with respect to quantum transitions of both the components. Therefore, their carrier frequencies lie far from the absorption (amplification) bands (see Fig. 2). The case of AmM (Fig. 2d) in this respect is similar to the situation considered in Ref. [32]. The authors of monograph [32] observed a superluminal pulse in nonequilibrium atomic vapor of cesium excited by two waves of continuous Raman pumping at the frequencies ω_1 and ω_2, respectively. Then the frequency ω of the superluminal pulse lay between ω_1 and ω_2, and its Rabi frequency was much smaller than the detuning $\omega_2 - \omega_1$. This last circumstance prevented the pulse from containing Fourier components in resonance with the pumping waves.

From the standpoint of quantum transitions, the λ-scheme was realized in a three-level system with two close lower levels and a remote third level. The nonequilibrium state was mainly produced by the initial population of the middle level. In this case, the situation is formally similar to that shown in Fig. 2d, where the role of eigenfrequencies ω_1 and ω_2 is played by the frequencies of the pumping waves. Since the carrier frequency of a superluminal pulse lies outside of the lines of resonant Raman amplification, the authors of Ref. [32] concluded that in such a case the mechanism of pulse propagation cannot be explained by the process of reshaping, described above in this paper and in a series of earlier works [1, 3, 5, 31].

Instead of reshaping, they suggested a mechanism based on the wave properties of light: the interference in different Fourier components of the wave packet in a medium with an anomalous dispersion.

In this respect it should be noted that anomalous dispersion in nonresonant region between the two lines of resonance amplification is such only because the medium is nonequilibrium (see Section 2 above). The presence of nonequilibrium state implies that the medium has stored a certain amount of energy. The induced release of this energy at the site of location of the pulse front edge, and its return by the tail portion, provide for superluminal propagation of the wave packet maximum.

In this way, as we see it, in both resonant and nonresonant cases the group velocity of the pulse may exceed the speed of light in vacuum through the mechanism of reshaping. The only difference is that in the resonant case the nonequilibrium medium releases a much greater amount of stored energy, and the populations of quantum levels change considerably; in the nonresonant case, however, the deformation of initial populations is small because the interaction between the field and the medium is not strong.

Thus, the process of reshaping involves only a small portion of the energy stored in the medium. In any case, the electric field of the pulse produces dipole moments in the atoms (brings them into quantum superposition states), which, according to the second equation in (3), brings about a change in the population inversion. Because of this, the reshaping is still the most likely mechanism of superluminal propagation of the pulse, observed in Ref. [32].

Since for the envelope solitons we have $\theta \ll 1$, then one finds $\cos\theta \approx 1 - \theta^2/2$. Accordingly, from Eqns (7), (9), (39), and (42) we get

$$W_1 = W_{1\infty}\left[1 - \left(\frac{2}{\omega\tau_p}\right)^2 \operatorname{sech}^2\left(\frac{t - z/v}{\tau_p}\right)\right],$$
$$W_2 = W_{2\infty}\left[1 - \left(\frac{2}{\omega\tau_p\omega_2\tau_p}\right)^2 \operatorname{sech}^2\left(\frac{t - z/v}{\tau_p}\right)\right]. \qquad (64)$$

Hence, as well as from the condition $(\omega_2\tau_p)^2 \gg 1$, it follows that the level populations actually do not change much in the medium hosting nonresonant envelope solitons ($\omega\tau_p \gg 1$), including superluminal solitons.

The stability of solitons with respect to dispersion spreading is ensured exclusively by nonlinearity. The inversion of the population, being the energy characteristic of the medium, exhibits a nonlinear dependence on the field of the pulse. Therefore, its change (whether large or small) is caused by nonlinearity. We see that the stable propagation of the wave packet in the dispersion frequency range, nonlinearity, the change in the population inversion, and the mechanism of reshaping (at $v > c$) are all very closely connected with one another.

To end this section, let us note that the application to the two-component model is very important. In all cases considered above, one component of the medium provides for the superluminal propagation of the pulse, while the other ensures the stability of this process.

6. Conclusions

The model of the two-component medium, proposed in this paper, has allowed us to trace the qualitative differences in the superluminal propagation of solitons in dissipative and conservative nonequilibrium (quasi-nonequilibrium) media.

The solitons considered here in lossy and loss-free media are stable only as a matter of convention, because the media that host these solitons are unstable themselves. The lifetime T_R of the excited states undergoing transitions in the optical spectrum (2-transitions) is $\sim 10^{-8}$ s to an order of magnitude, which is much less than the corresponding time for the IR range, into which the 1-transitions fall. Therefore, it is far from easy to create in an experiment the conditions when $v > c$. The propagation time τ_{prop} of a soliton in a quasi-nonequilibrium medium must satisfy the condition $\tau_{prop} \sim L/c \ll T_R$. Taking $T_R \sim 10^{-8}$ s, we find that $L < 1$ m. Assuming also that $\omega_1 \sim 10^{13}$ s^{-1}, $\omega_2 \sim 10^{15}$ s^{-1}, we find that conditions (2) can be satisfied at $\tau_p \sim 10^{-14}$ s.

Along with the electromagnetic pulse, the superluminal propagation involves the polarization $P = 2(d_1 n_1 U_1 + d_2 n_2 U_2)$ of the nonequilibrium medium, which ought to be accompanied by superluminal Cherenkov–Vavilov radiation [1, 33, 34]. Related to the dynamic polarization are the elementary excitations of the medium (quasi-particles), known as polaritons. Accordingly, one may say that expression (40) defines the group velocity of polaritons with the energy $\hbar\omega$ and lifetime τ_p in the state with the given energy. Indeed, the finiteness of the pulse length τ_p leads to broadening of its spectrum or the spectrum of elementary excitations ($\delta\omega \sim \tau_p^{-1}$). Accordingly, τ_p may be interpreted as the lifetime of such excitations. Setting in formula (40) $\omega = 0$, we get the polariton condensate centered at zero frequency with a bandwidth of the order of τ_p^{-1}. This bandwidth results from the interaction between polaritons as a result of nonlinearity. In this case, expression (40) goes over into (37) and defines the group velocity of an ultimately short soliton. Therefore, soliton (22) may be regarded as a bunch of polaritons whose lifetime in the state with zero energy equals τ_p. Since in this case $\tau_p^{-1} > \omega = 0$, this implies that polaritons (as quasi-particles) lose their individuality. Interaction between quasi-particles turns out to be very strong. The situation is different with polaritons corresponding to the envelope solitons (42), for which $\omega \gg \tau_p^{-1}$. Here we have the condensate of polaritons at a frequency ω with a small spectral width, which implies that the interaction between quasi-particles is weak. Because of this, the individual characteristics of polaritons in this case are emphasized.

On the other hand, the superluminal solitons of equations (33) and (47) exhibit the 'particle-like' property of elastic interaction with their like. Therefore, the solitons themselves, which are bunches of polaritons, may be regarded as quasi-particles (nonlinear excitations) in a nonequilibrium medium. However, one must remember that solitons in the strict meaning of this term (solitary pulses that retain their shape after interaction with one another) are only the spatially one-dimensional solutions of equation (47), because at $\Delta_\perp \mathcal{E} \neq 0$ this equation is nonintegrable [12, 13, 29]. As far as equation (33) is concerned, this question has not yet been investigated. It is easy to see, however, that Eqn (33) includes as special cases the sine-Gordon equation for θ ($b = \Delta_\perp \theta = 0$), and the modified Korteweg–de Vries equation for $E \sim \partial\theta/\partial\tau$ ($a = \Delta_\perp \theta = 0$). As is known, these two equations are nonintegrable in the spatially inhomogeneous case [13, 29]. Therefore, one may assume that equation (33) is also nonintegrable at $\Delta_\perp \theta \neq 0$, and therefore its solutions in the form of superluminal pulses in all three dimensions do not exhibit the soliton (particle-like) properties of elastic interaction with their like.

On the strength of the arguments developed above, one may speak of superluminal group velocities in the system of elementary and nonlinear excitations of a nonequilibrium medium, which, like polaritons, have a finite lifetime T_R. It is clear that one can ascribe a lifetime τ_p to the polaritons in a certain state only if $\tau_p \ll T_R$. When $\tau_p \sim T_R$, it is not possible to separate the lifetime of the polaritons from the lifetime of the medium in the nonequilibrium state. Here the constitutive equations (3) must be supplemented with the relaxation terms, which will make the mathematics much more complicated.

The Kramers–Kronig type dispersion relations, which introduce asymmetry between the past and the future on the microscopic level [35], express the principle of microscopic causality [36, 37]. On the other hand, as we know, the second law of thermodynamics is based on the irreversibility of relaxation processes which over the time bring different objects into the state of thermodynamic equilibrium with respect to each other. As a result of such processes, the entropy of the system tends to its maximum. In this way the 'arrow of time' [38] (the absence of symmetry between the past and the future) stands out on the macroscopic level. The causality related to the second law of thermodynamics can be called macroscopic causality [36].

The treatment used in this work relates to 'ideal' media [39], for which $T_R = \infty$. Formally this leads to a situation when the atomic excited state becomes in a certain sense stable. In this case it is possible to distinguish clearly two subsystems within the medium: the excited subsystem, and the subsystem thermalized according to the Boltzmann statistics. Such a sharp distinction between the two subsystems implies that the medium does not occur in the state with maximum entropy, and is therefore far from equilibrium. The assumption that the lifetime of such a state is $T_R = \infty$, in some sense contradicts the second law of thermodynamics. This circumstance ought to lead to violation of the macroscopic causality, which seems to open the way for superluminal propagation of a signal (that is, the transmission of information). One must remember, however, that in our case the superluminal objects (electromagnetic solitons) are extended (in the macroscopic sense) rather than point-like objects. Because of this, the violation of causality principle is fictitious — it occurs through the mechanism of reshaping and is only observed on the macroscopic level. This fictitious effect is manifested when we look at the wave packet crossing the boundary between the equilibrium and nonequilibrium media: under certain conditions the wave packet may emerge from the nonequilibrium medium sooner than the maximum of the input pulse reaches the medium of interest [32]. In reality, however, because of reshaping the descending front edge of the pulse is replaced by another maximum, and it is this maximum that is registered at the exit. All this time the original maximum is to be found close to the interface. This effect can be manifested especially clear when pulses are fed to nonequilibrium media prepared in the form of thin films [40]. At the same time, a superluminal pulse will propagate in a homogeneous nonequilibrium medium (without interfaces) virtually without changing its shape [1, 39].

'Violated' macroscopic causality does not affect the fundamental principle of microscopic causality, therefore each photon travels from one atom to another at the velocity c. One can speak about violation of the macroscopic causality principle, associated with the dynamics of a large number of photons interacting with the electric dipole transitions, only when $\tau_{\rm prop} < T_{\rm R}$.

Equations (10), (33) and (47) do not possess the property of Lorentz invariance (there is asymmetry between the space and time derivatives) because the constitutive equations (3), (4), unlike the Maxwell equations, are nonrelativistic. In this case there is no need for the strictly relativistic approach, because the characteristic velocities of electrons in atoms subjected to interaction with electromagnetic pulses are much less than the speed of light. This once again confirms the fact that the superluminal solitons are to a large extent the creature of the nonequilibrium medium, and not only the consequence of the relativistic nature of the electromagnetic field. The mechanism of reshaping once again puts everything in place.

Assuming that the irreversibility of physical phenomena is a fundamental law [41], there is no real equality between time and space, as stated by the special theory of relativity, because of the existence of the 'time arrow' [38]. The problem of the linkage between irreversibility and the feasibility of superluminal group velocities [38], leaving unshakeable the fundamental principle of microscopic causality, is still awaiting solution.

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References

1. Oraevsky A N *Usp. Fiz. Nauk* **168** 1311 (1998) [*Phys. Usp.* **41** 1199 (1998)]
2. Andreev A Yu, Kirzhnits D A *Usp. Fiz. Nauk* **166** 1135 (1996) [*Phys. Usp.* **39** 1071 (1996)]
3. Basov N G et al. *Zh. Eksp. Teor. Fiz.* **50** 23 (1966) [*Sov. Phys. JETP* **23** 14 (1966)]
4. Kirzhnits D A, Sazonov V N, in *Éĭnshteĭnovskiĭ Sbornik 1973* (Ed. V L Ginzburg) (Moscow: Nauka, 1974) p. 84
5. Chiao R Y, Kozhekin A E, Kurizki G *Phys. Rev. Lett.* **77** 1254 (1996)
6. Bludman S A, Ruderman M A *Phys. Rev. D* **1** 3243 (1970) [Translated into Russian, in *Éĭnshteĭnovskiĭ Sbornik 1973* (Ed. V L Ginzburg) (Moscow: Nauka, 1974) p. 190]
7. Becker P C et al. *Phys. Rev. Lett.* **63** 505 (1989)
8. Chernikov S V et al. *Opt. Lett.* **18** 476 (1993)
9. Tamura K, Nakazawa M *Opt. Lett.* **21** 68 (1996)
10. Belenov E M et al. *Pis'ma Zh. Eksp. Teor. Fiz.* **47** 442 (1988) [*JETP Lett.* **47** 523 (1988)]
11. Belenov E M, Nazarkin A V, Ushchapovskiĭ V A *Zh. Eksp. Teor. Fiz.* **100** 762 (1991) [*Sov. Phys. JETP* **73** 422 (1991)]
12. Zhdanov S K, Trubnikov B A *Kvazigazovye Neustoĭchivye Sredy* (Quasi-gas Unstable Media) Ch. 6 (Moscow: Nauka, 1991)
13. Ablowitz M J, Segur H *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM, 1981) Ch. 4 [Translated into Russian (Moscow: Mir, 1987)]
14. Sivukhin D V *Obshchiĭ Kurs Fiziki* T. 4 *Optika* (General Course in Physics, Vol. 4, Optics) (Moscow: Nauka, 1985) p. 520
15. Mandelshtam L I *Lektsii po Optike, Teorii Otnositel'nosti i Kvantovoĭ Mekhanike* (Lectures in Optics, Theory of Relativity and Quantum Mechanics) (Moscow: Nauka, 1972)
16. Loudon R *The Quantum Theory of Light* (Oxford: Clarendon Press, 1973) Ch. 4 [Translated into Russian (Moscow: Mir, 1976)]
17. Kozlov S A, Sazonov S V *Zh. Eksp. Teor. Fiz.* **111** 404 (1997) [*JETP* **84** 221 (1997)]
18. Maĭmistov A I *Kvantovaya Elektron.* **30** 287 (2000) [*Quantum Electron.* **30** 287 (2000)]
19. Sazonov S V *J. Phys.: Condens. Matter* **7** 175 (1995)
20. Parkhomenko A Yu, Sazonov S V *Zh. Eksp. Teor. Fiz.* **114** 1595 (1998) [*JETP* **87** 864 (1998)]
21. Vasil'ev V A, Romanovskiĭ Yu M, Yakhno V G *Avtovolnovye Protsessy* (Autowave Processes) Ch. 1 (Moscow: Nauka, 1987)
22. Kerner B S, Osipov V V *Avtosolitony — Lokalizovannye Sil'noneravnovesnye Oblasti v Odnorodnykh Dissipativnykh Sistemakh* (Autosolitons —Localized Highly Nonequilibrium Regions in Homogeneous Dissipative Systems) Ch. 1 (Moscow: Nauka, 1991)
23. Kosevich A M, Kovalev A S *Solid State Commun.* **12** 763 (1973)
24. Anderson D *Phys. Rev. A* **27** 3135 (1983)
25. Vinogradova M B, Rudenko O V, Sukhorukov A P *Teoriya Voln* (Wave Theory) 2nd ed. (Moscow: Nauka, 1990) p. 161
26. Konno K, Kameyama W, Sanuki H *J. Phys. Soc. Jpn.* **37** 171 (1974)
27. Vederko A V et al. *Vestnik Mosk. Univ., Ser. 3: Fiz. Astron.* **33** (3) 4 (1992)
28. Dodd R K et al. *Solitons and Nonlinear Wave Equations* (London: Academic Press, 1982, 1984) [Translated into Russian (Moscow: Mir, 1988)]
29. L'vov V S *Nelineĭnye Spinovye Volny* (Nonlinear Spin Waves) Ch. 2 (Moscow: Nauka, 1987)
30. Lamb G L, Jr. *Elements of Soliton Theory* (New York: John Wiley & Sons, 1980) [Translated into Russian (Moscow: Mir, 1983)]
31. Kuritski G et al. *Opt. Spektrosk.* **87** 551 (1999)
32. Wang L J, Kuzmich A, Dogariu A *Nature* **406** 277 (2000)
33. Faĭngol'd M I, in *Éĭnshteĭnovskiĭ Sbornik 1974* (Eds V L Ginzburg, G I Naan) (Moscow: Nauka, 1976) p. 276
34. Ginzburg V L *Teoreticheskaya Fizika i Astrofizika* (Theoretical Physics and Astrophysics) Ch. 8 (Moscow: Nauka, 1975) [Translated into English (Oxford: Pergamon Press, 1979)]
35. Bogolyubov N N, Shirkov D V *Vvedenie v Teoriyu Kvantovannykh Poleĭ* (Introduction to the Theory of Quantized Fields) 4th ed. (Moscow: Nauka, 1984) [Translated into English (New York: John Wiley, 1980)]
36. Bilaniuk O M, Sudarshan E C G *Phys. Today* **22** (5) 43 (1969) [Translated into Russian, in *Éĭnshteĭnovskiĭ Sbornik 1973* (Ed. V L Ginzburg) (Moscow: Nauka, 1974) p. 112]
37. Blokhintsev D I *Prostranstvo i Vremya v Mikromire* (Space and Time in the Microworld) Ch. 5 (Moscow: Nauka, 1970) [Translated into English (Dordrecht: Reidel, 1973)]
38. Kadomtsev B B *Dinamika i Informatsiya* (Dynamics and Information) Ch. 5 (Moscow: Red. Zh. 'Usp. Fiz. Nauk', 1999)
39. Oraevskiĭ A N *Sorosovskiĭ Obrazovatel'nyĭ Zh.* (10) 75 (1999)
40. Blaauboer M et al. *Phys. Rev. A* **57** 4905 (1998)
41. Prigogine I *From Being to Becoming* (San Francisco: W.H. Freeman and Co., 1980) [Translated into Russian (Moscow: Nauka, 1985)]