Acknowledgements. This research was supported in part by INTAS (Grant 99-1070).

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Hidden SU(4) symmetry in bilayer quantum well at integer filling factors

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Abstract. Phase diagram of a bilayer quantum well at integer filling factors is established using the hidden symmetry method. Three phases: ferromagnetic, canted antiferromagnetic (CAP) and spin-singlet, have been found. We confirm early results of Das Sarma et al. Each phase violates the SU(4) hidden symmetry and is stabilized by the anisotropy interactions.

1. Introduction

Integer filling factors of a 2D electron gas (2DEG) confined to a quantum well in an external magnetic field are special ones because a huge degeneracy of the ground state is gone here. It justifies the Hartree–Fock approximation with the accuracy limited only by normally a small parameter: $V^{\rm int}/\hbar\omega_0$, where $V^{\rm int}$ is the energy of the Coulomb interaction and ω_0 is the frequency of the cyclotron resonance. Such an approach predicts the ground state of a single-layer 2DEG at $\nu = 1$ to be a ferromagnet with the degenerate total spin orientation. The elementary excitations of 2DEG are electron–hole pairs or excitons, and in the limit of vanishing momentum they transform into the elementary excitations of a ferromagnet — spin waves. The latter are gapless [1] and do not interact with each other [2] if Zeeman energy is neglected — the two consequences of the Goldstone theorem. In the limit of large momentum the electron and the hole of an exciton are well separated and they become the elementary charged excitations.

The case of a bilayer 2DEG turned out to be a more rich one, where both spin and pseudo-spin (layer) dynamics become entangled. The Hartree–Fock approximation does not apply here except for two limiting cases. The first one is the case of well separated layers which is a common setup in the experiment [3, 4] and where, theoretically, one starts from the two single-layer ferromagnets in the balanced case of filling factor $\nu = 2$ and makes the perturbation expansion in powers of interlayer interactions [5]. And the second one is the symmetric case defined in such a way that one can freely rotate an electron spinor in both layer and spin spaces. The latter requires to approximate the Coulomb interaction by its symmetric part and to neglect all symmetry-breaking fields like Zeeman energy. The first attempts in this direction dealt with the case of filling factor $\nu = 1$ and relied heavily on the assumption of a saturated spin polarization of electrons [6, 7]. This symmetric approximation turned out to be useful to determine the exciton energy in bilayer [7]. Recent works [8, 9] specialize to the bilayer heterostructure case $\nu = 2$, employ the Hartree–Fock approximation and predict a phase diagram that features three phases: the ferromagnetic, the canted antiferromagnetic and a special spin-singlet phase. In this paper we reproduce the phase diagram of Refs [8, 9] isolating the symmetric and the symmetry-breaking parts of the Hamiltonian in a consistent way. Our approach reveals the Hartree–Fock phase diagram to be indeed exact in the limit $V^{\rm anis}/V^{\rm sym} \to 0$, where $V^{\rm sym}$ is the SU(4)-symmetric part of the bilayer Hamiltonian whereas $V^{\rm anis}$ is anisotropy interactions that reduce the bilayer Hamiltonian symmetry to $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$. We prove the stability of all phases with respect to long-range spatial perturbations. We find that low-energy excitations over the bilayer ground state are governed by the $\mathrm{U}(4)/\mathrm{U}(\nu) \otimes \mathrm{U}(4-\nu)$ coset in nonlinear sigma model.

2. Hamiltonian of 2DEG bilayer

The electronic Hamiltonian of a 2DEG in a confining potential $V(\boldsymbol{\rho})$ and in an external magnetic field H perpendicular to the layer consists of a one-particle part as well as a Coulomb interaction part:

$$H = \int \psi^+_\alpha(\boldsymbol{\rho}) \times \left\{ \frac{1}{2m}[-\mathrm{i}\boldsymbol{\nabla} + \mathbf{A}(\boldsymbol{\rho})]^2 + V(\boldsymbol{\rho}) - |g|\mu_{\rm B} H \sigma^z_{\alpha\beta} \right\} \psi_\beta(\boldsymbol{\rho})\, \mathrm{d}^3\boldsymbol{\rho} + \frac{1}{2}\int\int \frac{e^2}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|} \psi^+_\alpha(\boldsymbol{\rho})\psi^+_\beta(\boldsymbol{\rho}')\psi_\beta(\boldsymbol{\rho}')\psi_\alpha(\boldsymbol{\rho})\, \mathrm{d}^3\boldsymbol{\rho}\, \mathrm{d}^3\boldsymbol{\rho}', \quad (1)$$

where $\alpha, \beta = \pm$ are spin indices and thereafter a sum over repeated indices is implied. We use such units that $\hbar = 1, e = c$ and $H = B = 1$. All distances can be expressed in terms of the so-called magnetic length: $l_H = \sqrt{c\hbar/eH} = 1$. We split three coordinates $\boldsymbol{\rho}$ into a perpendicular to the layer coordinate ξ and two in-plane coordinates $\mathbf{r} = (x, y) = (z, \bar{z})$. We assume that the confining potential is uniform over the plane: $V(\boldsymbol{\rho}) = V(\xi)$, and represents a double-well structure in the transverse direction as shown in Fig. 1, with the two wells being separated by the distance d. We use only two eigenfunctions: the lowest energy symmetric $\chi_{\rm S}(\xi)$ and antisymmetric

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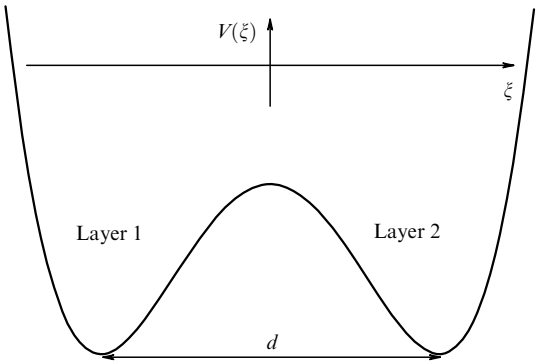

Figure 1. Schematic view of a confining potential $V(\xi)$ in a typical bilayer setup.

$\chi_A(\xi)$, from a set of one-electron eigenfunctions in the confining potential $V(\xi)$ and we expand an electron second-quantized operator in terms of these two eigenfunctions:

$$\psi_\alpha(\boldsymbol{\rho}) = \sum_{\tau,n,p} \chi_\tau(\xi)\phi_{n,p}(\mathbf{r})c_{n\alpha\tau p}\,, \tag{2}$$

where $c^+_{\alpha\tau p}$ and $c_{\alpha\tau p}$ are electron creation and annihilation operators, $\phi_{n,p}(z\bar{z})$ is an electron wave function number p in the Landau gauge in the n's Landau level, the index $\tau = 1, 2$ being the layer index and the layer wave functions read:

$$\chi_{1,2}(\xi) = \frac{\chi_S(\xi) \pm \chi_A(\xi)}{\sqrt{2}}\,. \tag{3}$$

We restrict our model to the case of a sufficiently strong magnetic field, such that the cyclotron energy $\hbar\omega_0$ dominates over the Coulomb, Zeeman and the level splitting, $E_A - E_S$, energies. Thus, we specialize to the lowest Landau level and retain only the term $n = 0$ in (2).

Plugging the wave functions (2) into Eqn (1) we find a 2DEG Hamiltonian as:

$$\begin{aligned} H = {} & \frac{1}{2m} c^+_{\alpha\tau p} c_{\alpha\tau p} - c^+_{\alpha\tau_1 p}\left(t\tau^x_{\tau_1\tau_2} + \mu^z\tau^z_{\tau_1\tau_2}\right)c_{\alpha\tau_2 p} \\ & - |g|\mu_B H\, c^+_{\alpha\tau p}\sigma^z_{\alpha\beta}c_{\beta\tau p} \\ & + \frac{1}{2}\sum_{p_1,\ldots,p_4}\int\!\!\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}^2\mathbf{r}'\; V^{\tau_1\tau_4}_{\tau_2\tau_3}(\mathbf{r}-\mathbf{r}') \\ & \times \phi^*_{p_1}(\mathbf{r})\phi^*_{p_2}(\mathbf{r}')\phi_{p_3}(\mathbf{r}')\phi_{p_4}(\mathbf{r}) e c^+_{\alpha\tau_1 p_1} c^+_{\beta\tau_2 p_2} c_{\beta\tau_3 p_3} c_{\alpha\tau_4 p_4}\,, \end{aligned} \tag{4}$$

where we have defined a hopping constant

$$t = \frac{1}{2}\int\!\!\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}\xi\; \phi^*_p(\mathbf{r})\chi_{\tau_1}(\xi)\tau^x_{\tau_1\tau_2}V(\xi)\chi_{\tau_2}(\xi)\phi_p(\mathbf{r}) \tag{5}$$

and an external electrostatic potential created by an asymmetric gate charge,

$$\mu^z = \frac{1}{2}\int\!\!\int \mathrm{d}^2\mathbf{r}\,\mathrm{d}\xi\; \phi^*_p(\mathbf{r})\chi_{\tau_1}(\xi)\tau^z_{\tau_1\tau_2}V(\xi)\chi_{\tau_2}(\xi)\phi_p(\mathbf{r})\,, \tag{6}$$

whereas the Coulomb interaction matrix reads:

$$V^{\tau_1\tau_4}_{\tau_2\tau_3}(\mathbf{r}-\mathbf{r}') = \int\!\!\int \frac{\chi_{\tau_1}(\xi)\chi_{\tau_2}(\xi')\chi_{\tau_3}(\xi')\chi_{\tau_4}(\xi)}{\sqrt{(\xi-\xi')^2 + (\mathbf{r}-\mathbf{r}')^2}}\,\mathrm{d}\xi\,\mathrm{d}\xi'\,. \tag{7}$$

We use notations τ^x, τ^y and τ^z for the Pauli matrices in the layer space whereas we use notations σ^x, σ^y and σ^z for the Pauli matrices in the spin space. The hopping constant can be related to the splitting of the symmetric and antisymmetric levels: $t = E_A - E_S$. The electrostatic potential μ^z, which can be viewed as a difference between the chemical potentials in the two layers, breaks down the symmetry between the two wells of potential $V(\xi)$. This term appears naturally when a single gate is fabricated to control the electron density in the bilayer. In the limit $d \to 0$, μ^z vanishes too, whereas in the limit of large layer separation, $d \to \infty$, $\mu^z \to \infty$ and electrons reside only on the layer adjacent to the gate. We assume that the energy of a capacitor formed by the two layers is much lower than the characteristic Coulomb energy $e^2/\kappa l_H^3$, per area, where κ is the dielectric constant. We note the invariance of the Coulomb energy (7) under the following transformations: $\tau_1 \leftrightarrow \tau_4$, $\tau_2 \leftrightarrow \tau_3$ as well as $(\tau_1\tau_4) \leftrightarrow (\tau_2\tau_3)$. To fully exploit these symmetries we cast the Eqn (7) into a more suitable representation:

$$V^{\tau_1\tau_4}_{\tau_2\tau_3}(\mathbf{r}-\mathbf{r}') = V^{\mu\nu}(\mathbf{r}-\mathbf{r}')\tau^\mu_{\tau_1\tau_4}\tau^\nu_{\tau_2\tau_3}\,, \tag{8}$$

where τ^0 is the unit matrix, $V^{\mu\nu}$ is a 3×3 symmetric interaction matrix with indices μ, ν running over a set $(0, z, x)$. If there is a symmetry of the Coulomb interaction under an exchange of layers: $(\xi\xi') \leftrightarrow (-\xi - \xi')$ and $1 \leftrightarrow 2$ then it restricts further values of the interaction matrix: $V^{0z} = 0$ and $V^{zx} = 0$. But in the presence of a gate asymmetry we shall keep the matrix element: V^{0z}. Therefore, the Coulomb interaction matrix for symmetric bilayer 2DEG depends on four parameters: $V^{00} > 0$, V^{0x}, $V^{xx} > 0$, $V^{zz} > 0$. We note also that $V^{0x} \sim \chi$, $V^{xx} \sim \chi^2$, whereas $V^{zz} \sim d^2/|z|^3$ as $|z| \to \infty$. In the following we shall neglect small V^{xx} matrix element.

Next, we split the total bilayer Hamiltonian (4) into two parts: the first one contains a dominant Coulomb energy term:

$$\begin{aligned} H^{\mathrm{sym}} = {} & \frac{1}{2m} c^+_{\alpha\tau p} c_{\alpha\tau p} \\ & + \frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} V^{00}(\mathbf{q}) \exp\left(-\frac{\mathbf{q}^2}{2}\right) N(\mathbf{q})N(-\mathbf{q})\,, \end{aligned} \tag{9}$$

where $V^{\mu\nu}(\mathbf{q})$ is the Fourier transform of $V^{\mu\nu}(\mathbf{r})$ and the electron density operator reads:

$$N(\mathbf{q}) = \sum_p c^+_{\alpha\tau p} c_{\alpha\tau p - q_y} \exp(-\mathrm{i}q_x)\left(p - \frac{q_y}{2}\right). \tag{10}$$

This part of the Hamiltonian is invariant under uniform rotations from the SU(4) Lee group in the combined spin and layer space. Every of its eigenenergies is hugely degenerate. Given any eigenstate $|\Psi\rangle_0$, a set of related eigenstates $|\Psi\rangle$ can be generated by applying uniform rotations: $U \in \mathrm{SU}(4)$. For Landau level filling factor $\nu = 1$, $\nu = 2$ and $\nu = 3$ we assume that the bilayer ground state is uniform over p-orbitals:

$$\Psi = \prod_{i=1}^{\nu}\prod_p c^+_{\alpha_i\tau_i p}\,|\mathrm{empty}\rangle\,, \tag{11}$$

and we prove in the next Section that this state is stable with respect to long-range spatial perturbations. One can easily check by inspection that any such wave function (11)

represents an eigenfunction of the $H^{\rm sym}$ (9). The wave function (11) represents the so-called Halperin (1,1,1) multi-component wave function [10] that describes an incompressible quantum Hall effect state.The remaining few terms in the Hamiltonian (4) are treated like perturbations:

$$H^{\rm anis} = -c^{+}_{\alpha\tau_1 p}\left(t\tau^{x}_{\tau_1\tau_2} + \mu^{z}\tau^{z}_{\tau_1\tau_2}\right)c_{\alpha\tau_2 p} - |g|\mu_B H\, c^{+}_{\alpha\tau p}\sigma^{z}_{\alpha\beta}c_{\beta\tau p} + \frac{1}{2}\int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\,V^{\mu\nu}(\mathbf{q})\exp\left(-\frac{\mathbf{q}^2}{2}\right)T^{\mu}(\mathbf{q})T^{\nu}(-\mathbf{q})\,, \quad (12)$$

where (see, e.g. [7])

$$T^{\mu}(\mathbf{q}) = \sum_{p}\; c^{+}_{\alpha\tau_1 p}\tau^{\mu}_{\tau_1\tau_2}c_{\alpha\tau_2 p-q_y}\exp(-\mathrm{i}q_x)\left(p-\frac{q_y}{2}\right) \quad (13)$$

with $(\mu\nu)\neq(00)$. The Hamiltonian (12) breaks down the SU(4) symmetry but it is still invariant under separate rotations in the spin and layer space: $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$. We shall call this part of the Hamiltonian the anisotropy Hamiltonian. It lifts the degeneracy of eigenstates of the SU(4)-symmetric Hamiltonian (9). An important point to note here is that a splitting of energy levels is determined by matrix elements of weak-anisotropy Hamiltonian (12) truncated to a linear space of the symmetric Hamiltonian (9) level degeneracy. There are no Fermi-liquid type renormalizations of the constants of the anisotropy Hamiltonian (12) due to the SU(4)-symmetric Hamiltonian (9). In other words, the mean-field Hartree–Fock approach is perfect for the $\nu=1$, $\nu=2$ and $\nu=3$ cases.

Our guiding analogy in treating the total bilayer Hamiltonian (9), (12) lies in the theory of magnetism. We will see below that there exists a local order parameter, Q, very much like to magnetization. And we aim to express the total bilayer Hamiltonian (9), (12) in terms of this order parameter Q. The exchange-like Hamiltonian (9) has to be expanded in powers of spatial variations of order parameter $Q(\mathbf{r})$ with the second power of gradients being an important contribution, whereas only locally homogeneous Q has to be retained in the anisotropy Hamiltonian (12).

3. SU(4)-symmetric case

According to our plan we first specialize to the SU(4)-symmetric part of the bilayer 2DEG Hamiltonian (9) which is invariant under the global rotations of a four-component electron spinor by the 4×4 matrix U from the SU(4) Lee group. An inhomogeneous state of 2DEG described by an electron field is related to the reference state (11) by a 4×4 unitary matrix $U(\mathbf{r})$. For ground state (11) one can define an occupation number for the electronic states:

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (14)$$

where blocks are 1×1 and 3×3 in the case of $\nu=1,3$, and 2×2 in the case of $\nu=2$.The fifteen generators of SU(4) Lee group, $\{\Sigma^{l}\}$, with $l=1,\ldots,15$, can be subdivided into two complementary sets: the first one includes those generators that do commute with the occupation number matrix (14), and we shall call it an *even* set, whereas the second one includes the remaining generators, and we shall call it the *odd* set. Generators of the *even* set constitute an algebra itself. A Lee group built around the *even* set of generators is called a stabilizer subgroup S of SU(4) Lee group. The *odd* set always contains an even number of generators. Specifically, eight in the case of $\nu=2$ and six in the case of $\nu=1,3$.

A non-homogeneous order parameter matrix Q is then defined as follows:

$$Q(\mathbf{r}) = U(\mathbf{r})NU^{+}(\mathbf{r})\,. \quad (15)$$

This electronic order parameter can be used to average any operator A:

$$\langle A\rangle = \mathrm{tr}(AQ)\,. \quad (16)$$

It is evident that any rotation from the denominator subgroup S leaves the order parameter intact. Thus, rotations in Eqn (15) can be restricted to a coset or, in other words, a physical space of the bilayer 2DEG,

$$\frac{U(4)}{U(\nu)\otimes U(4-\nu)}\,. \quad (17)$$

We refer the reader to Ref. [11] for details of effective Hamiltonian calculation. This method uses gradient expansion and in terms of the order parameter matrix the result reads:

$$H = \frac{E_1}{4}\int\mathrm{tr}(\boldsymbol{\nabla}Q\boldsymbol{\nabla}Q)\,\frac{\mathrm{d}^2\mathbf{r}}{2\pi} + \mathrm{sgn}(B^{z})\frac{E_0}{2}\int\epsilon_{\mu\nu}\,\mathrm{tr}\left(Q\,\partial_{\mu}Q\,\partial_{\nu}Q\right)\frac{\mathrm{d}^2\mathbf{r}}{2\pi}\,, \quad (18)$$

where Coulomb interaction constants

$$E_0 = 2E_1 = \sqrt{\frac{\pi}{2}}\,\frac{e^2}{\kappa l_{\rm H}}\,. \quad (19)$$

In the order parameter matrix representation, the topological index appears as an index of a map of the order parameter coset space into a 2D plane. The Hamiltonian (18) must be invariant under the time reversal symmetry. The time reversal operator can be chosen as a complex conjugate operator: $U\to U^{*}$. To ensure the time reversal, $\mathrm{sgn}(B^{z})$ has been added to the topological index term. The appearance of topological excitations — skyrmions — in the model (18) is a consequence of a well known homotopy group identity:

$$\mathcal{Q} = \pi_2\left[\frac{U(4)}{U(\nu)\otimes U(4-\nu)}\right] = Z\,. \quad (20)$$

4. Anisotropic part of Coulomb energy. Phase diagram

In this Section, we cast the anisotropic part of the bilayer Hamiltonian (12) in terms of the order parameter matrix Q. It can be conveniently done by the following Hartree–Fock average of c-operator product in Eqn (12):

$$\tau^{\mu}_{\tau_1\tau_4}\tau^{\nu}_{\tau_2\tau_3}\langle c^{+}_{\alpha\tau_1 p_1}c^{+}_{\beta\tau_2 p_2}c_{\beta\tau_3 p_3}c_{\alpha\tau_4 p_4}\rangle = \delta_{p_1p_4}\delta_{p_2p_3}\,\mathrm{tr}(Q\tau^{\mu})\,\mathrm{tr}(Q\tau^{\nu}) - \delta_{p_1p_3}\delta_{p_2p_4}\,\mathrm{tr}(Q\tau^{\mu}Q\tau^{\nu})\,, \quad (21)$$

where τ^{μ} acts on four-spinor as $\tau^{\mu}\otimes\sigma^{0}$. Next, we define the following Coulomb anisotropy constants:

$$E^{ab} = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi l_{\mathrm{H}}^2} V^{ab}(|z|) \exp\frac{-|z|^2}{2l_{\mathrm{H}}^2} = (1-\nu^{ab}) \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi l_{\mathrm{H}}^2} V^{ab}(|z|)\,, \tag{22}$$

where $\nu^{ab} \approx 0$ for $(ab) \neq (00)$ in the limit $d \ll l_{\mathrm{H}}$. And, finally, we rewrite the anisotropy Hamiltonian (12) in terms of order parameter matrix Q:

$$\frac{H^{\mathrm{anis}}}{\mathcal{N}} = -\left[t + (\nu-1)E^{0x}\right] \mathrm{tr}(Q\tau^x) - \left[\mu^z + (\nu-1)E^{0z}\right] \mathrm{tr}(Q\tau^z) - |g|\mu_{\mathrm{B}} H\,\mathrm{tr}(Q\sigma^z) + \frac{1}{2}E^{zz}[\mathrm{tr}(Q\tau^z)\,\mathrm{tr}(Q\tau^z) - \mathrm{tr}(Q\tau^z Q\tau^z)]\,, \tag{23}$$

where $\mathcal{N}$ is the number of degeneracy of the Landau level. Equations (18), (23) define the effective long-range Hamiltonian of a bilayer at integer filling factors. At non-zero temperatures thermal fluctuations of the order parameter soften the anisotropy constants in the Hamiltonian (23). The relevant calculation can be found, e.g., in Ref. [12] and the result reads:

$$\left[t + (\nu-1)E^{0x},\ \mu^z + (\nu-1)E^{0z},\ |g|\mu_{\mathrm{B}}H\right]_R = \left[t + (\nu-1)E^{0x},\ \mu^z + (\nu-1)E^{0z},\ |g|\mu_{\mathrm{B}}H\right]\left(\frac{l_{\mathrm{H}}}{R^*}\right)^{8T/E_1},$$
$$E_R^{zz} = E^{zz}\left(\frac{l_{\mathrm{H}}}{R^*}\right)^{24T/E_1}, \tag{24}$$

where the spatial scale $R_*^2 = l_{\mathrm{H}}^2 E_1/\max(t, \mu^z, |g|\mu_{\mathrm{B}}H, E^{zz})$ indicates the excitation wavelength where its anisotropy energy starts to compete with its exchange energy. Note that the three first constants renormalize as an external field whereas E^{zz} constant renormalizes as an easy-axis anisotropy. Although the Coulomb energy $E_1 \sim 100K \gg T \sim 1K$ in most experiments, the specific number: $24 = 3 \times 8$, which is related to the order of anisotropy and to the eight degrees of freedom for thermal fluctuations in the case of SU(4) symmetry, makes the renormalization of the constant E^{zz} noticeable.

As we have seen in previous Section, the order parameter can be parameterized by six or eight angles in the case of $\nu = 1, 3$ or $\nu = 2$. Actually, not every of those rotations corresponds to a physically distinct eigenstate. We restrict the calculation of the total bilayer energy up to a first order in powers of the anisotropy Hamiltonian, which means that we shall need only its diagonal matrix elements. But, these are real matrix elements of course, despite the fact that in an external magnetic field there is no time reversal symmetry. Hence, if the Hamiltonian is real one, so real has to be its ground state. One generates all real eigenstates from a reference state by rotations from the SO(4) subgroup of the SU(4) group. This group has 6 parameters with two of them falling into the denominator subgroup. Thus, the ground state differs from the reference state by just four rotations. One can view locally the 8D manifold of order parameter as a composition of four unit vectors: magnetization of the first and the second layers and the two hopping-tau vectors which represent the distribution of spin-up and spin-down electron density over the two layers. Now the first two term in the Hamiltonian (12) are external fields acting on these four vectors. On the other hand the Coulomb energy couples pairs of tau vectors via an exchange interaction. This instructive picture allows us to identify only three special global rotations that do change the total bilayer energy. We start with the case $\nu = 2$ and we use a set of trial many-electron wave functions parameterized by the three angles of rotations relevant in our case, $\theta_\pm$ and ϑ:

$$\prod_p U(\vartheta, -\vartheta) R(\theta_+, \theta_-)\ c_{+1p}^+ c_{-2p}^+\ |\mathrm{empty}\rangle\,, \tag{25}$$

where spins in the layer 1,2 are first rotated by $\pm\vartheta$:

$$U_{\beta\tau_2}^{\alpha\tau_1}(\vartheta, -\vartheta) = \left(\frac{\tau^0 + \tau^z}{2}\right)_{\tau_1\tau_2} \exp\left(\mathrm{i}\frac{\vartheta}{2}\sigma^y\right)_{\alpha\beta} + \left(\frac{\tau^0 - \tau^z}{2}\right)_{\tau_1\tau_2} \exp\left(-\mathrm{i}\frac{\vartheta}{2}\sigma^y\right)_{\alpha\beta}, \tag{26}$$

and then wave functions of electrons with spin $= \pm$ spill over the two layers, the process described by two distinct angles, $\theta_\pm$:

$$R_{\beta\tau_2}^{\alpha\tau_1}(\theta_+, \theta_-) = \left(\frac{\sigma^0 + \sigma^z}{2}\right)_{\alpha\beta} \exp\left(\mathrm{i}\frac{\theta_+}{2}\tau^y\right)_{\tau_1\tau_2} + \left(\frac{\sigma^0 - \sigma^z}{2}\right)_{\alpha\beta} \exp\left(-\mathrm{i}\frac{\theta_-}{2}\tau^y\right)_{\tau_1\tau_2}. \tag{27}$$

This set includes the singlet-liquid state at $\theta_\pm = \pi/2$ and $\vartheta = 0$ and the canted antiferromagnetic state at $\theta_\pm = 0$. The order parameter reads:

$$Q = URNR^+U^+\,, \tag{28}$$

with N being the electron density calculated with the reference state of previous Section (see Eqn (14)). Now we substitute Eqn (28) into the anisotropic Hamiltonian (23) to find the total anisotropy bilayer energy as:

$$E^{\mathrm{anis}} = -E^{zz}\cos\theta_+\cos\theta_- - (t + E^{0x})\cos\vartheta(\sin\theta_+ + \sin\theta_-) - (\mu^z + E^{0z})(\cos\theta_+ - \cos\theta_-) - |g|\mu_{\mathrm{B}}H\sin\vartheta(\cos\theta_+ + \cos\theta_-)\,. \tag{29}$$

The minimum of this energy corresponds to three phases: (i) ferromagnetic, $\vartheta = \pi/2$, $\theta_+ = \theta_- = 0$; (ii) spin singlet, $\vartheta = 0$, $\theta_+ = \pi - \theta_- = \theta$; and (iii) canted antiferromagnetic state otherwise, as it is shown in Fig. 2. It is identical to that found in Ref. [9]. A line of continuous phase transitions between the ferromagnetic phase and the canted antiferromagnetic phase is given by the following equation:

$$\left[(E^{zz} + |g|\mu_{\mathrm{B}}H)^2 - \left(\mu^z + E^{0z}\right)^2\right]|g|\mu_{\mathrm{B}}H = \left(t + E^{0x}\right)^2(E^{zz} + |g|\mu_{\mathrm{B}}H)\,. \tag{30}$$

In the spin singlet phase the interlayer mixing phase θ is determined by equation

$$\left(E^{zz}\sin\theta + t + E^{0x}\right)\cos\theta = \left(\mu^z + E^{0z}\right)\sin\theta\,. \tag{31}$$

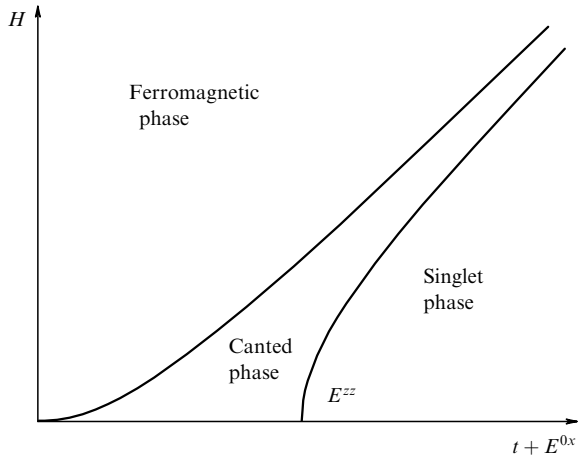

Figure 2. Phase diagram in the gate-symmetric case.

A phase transition line that separates the spin singlet phase from the canted antiferromagnetic phase is given parametrically by equation

$$\left[(t + E^{0x})\sin\theta - E^{zz} + \left(\mu^z + E^{0z}\right)\cos\theta\right]\left(t + E^{0x}\right) = (|g|\mu_B H)^2 \sin\theta\,, \tag{32}$$

where θ is determined from Eqn (31). This phase transition is a continuous one also.

In the case of $\nu = 1$, the Coulomb interaction energy is canceled except for a small v^{ab} correction defined in Eqn (22) and the total bilayer energy reads:

$$E^{\text{anis}} = -t\sin\theta - \mu^z\cos\theta - |g|\mu_B H\cos\vartheta + E^{zz}v^{zz}\cos^2\theta\,. \tag{33}$$

The minimum of this energy is given by electron spin directed along the magnetic field, $\vartheta = 0$, whereas $\theta = \tan^{-1} t/\mu^z$. There is no phase transition in the case of $\nu = 1$ and the only phase can be characterized as ferromagnetic in both the spin and the layer spaces. Actually, there exists a phase transition between an incompressible quantum Hall state and a compressible metallic state [13], but it happens when $V^{\text{sym}} \sim V^{\text{asym}}$ and is beyond our mean field method.

The case of $\nu = 3$ formally reduces to the case of $\nu = 1$ although here the Coulomb interaction energy is not identically zero. We find renormalizations to the one-particle electron Hamiltonian whereas the total energy being similar to the case of $\nu = 1$:

$$E^{\text{anis}} = -(t + 2E^{0x})\sin\theta - (\mu^z + 2E^{0z})\cos\theta - |g|\mu_B H\cos\vartheta\,, \tag{34}$$

There is no phase transition in this case either.

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