current flowing through the sample should decrease the minimum on the capacitance curve. The number of strips decreases with the current, and, ultimately, at very high currents, only one strip (cf. Fig. 5b) will survive in the bulk of the sample, which is equivalent to the pinch discussed above. This is a qualitative description. The quantitative consideration should be based on a self-consistent calculation of the distribution of potentials and electron density around the incompressible current-carrying strip.

We would also like to discuss the case of low currents when the peak is narrow and arises near the center of the minimum (Fig. 3b). The width of the minimum is determined by the dispersion of the electron density in the sample [4] and, hence, by the range of the parameters at which strips of incompressible electron phase exist in a sample, whereas the width of the peak is determined by another range of the parameters at which current flows only through the incompressible strips, i.e., at which percolation over the regions with incompressible phase takes place.

We believe that the peaks observed on the capacitance curve suggest that the current flows through the regions in the bulk of the sample, whose positions change as the gate voltage varies. The peaks can hardly be explained in terms of edge currents.

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References

1. Smith T P et al. *Phys. Rev. B* **32** 2696 (1985)
2. Pikus F G, Efros A L *Phys. Rev. B* **47** 16395 (1993)
3. Jungwirth T, Smrčka L *Phys. Rev. B* **51** 10181 (1995)
4. Dorozhkin S I, Dorokhova M O *Pis'ma Zh. Eksp. Teor. Fiz.* **71** 606 (2000) [*JETP Lett.* **71** 417 (2000)]
5. Gerhardts R R, Gudmundsson V *Phys. Rev. B* **34** 2999 (1986)
6. Dorokhova M O, PhD Thesis (Chernogolovka: Institute of Solid State Physics, 2000)
7. Pudalov V M, Semenchinskiĭ S G *Pis'ma Zh. Eksp. Teor. Fiz.* **44** 526 (1986) [*JETP Lett.* **44** 677 (1986)]
8. Gornik E et al. *Phys. Rev. Lett.* **54** 1820 (1985)
9. Eisenstein J P et al. *Phys. Rev. Lett.* **55** 875 (1985)
10. Dolgopolov V T, Zhitenev N B, Shashkin A A *Zh. Eksp. Teor. Fiz.* **94** 307 (1988) [*Sov. Phys. JETP* **67** 1471 (1988)]
11. Thouless D J *Phys. Rev. Lett.* **71** 1879 (1993)
12. Buttiker M *Phys. Rev. B* **38** 9375 (1988)
13. Chklovskii D B, Shklovskii B I, Glazman L I *Phys. Rev. B* **46** 4026 (1992)
14. Efros A L *Phys. Rev. B* **45** 11354 (1992)
15. Shashkin A A, Dolgopolov V T, Dorozhkin S I *Zh. Eksp. Teor. Fiz.* **91** 1897 (1986) [*Sov. Phys. JETP* **64** 1230 (1986)]
16. Semenchinskiĭ S G *Zh. Eksp. Teor. Fiz.* **91** 1804 (1986) [*Sov. Phys. JETP* **64** 1068 (1986)]
17. Dorozhkin S I et al. *Pis'ma Zh. Eksp. Teor. Fiz.* **63** 67 (1996) [*JETP Lett.* **63** 76 (1996)]
18. Efros A L, Pikus F G, Burnett V G *Phys. Rev. B* **47** 2233 (1993)

The problem of Coulomb interactions in the theory of the quantum Hall effect

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Abstract. We summarize the main ingredients of a unifying theory for abelian quantum Hall states. This theory combines the Finkel'stein approach to localization and interaction effects with the topological concept of an instanton vacuum as well as Chern–Simons gauge theory. We elaborate on the meaning of a new symmetry ($\mathcal{F}$ invariance) for systems with an infinitely ranged interaction potential. We address the renormalization of the theory and present the main results in terms of a scaling diagram of the conductances.

1. Introduction

In this contribution we discuss some of the recent advancements in the theory of the quantum Hall effect [1–3]. In particular, we address some of the main steps in the development of a theory [4] that combines the *instanton vacuum* approach to spin polarized, free electrons [5] with the *Finkel'stein treatment* of the Coulomb interactions [6] in the disordered systems.

The electron gas with an infinite-range interaction potential is, in many ways, very different from what we know about the theory of free electrons. This class of problems belongs to a different universality class of quantum transport phenomena and it is characterized by a typical interaction symmetry which we term $\mathcal{F}$ invariance [1]. $\mathcal{F}$ invariance is intimately related to the electrodynamic $U(1)$ gauge invariance and it has major consequences for the renormalization of the theory [2].

The main physical objective of our theory is to unify the different aspects of (abelian) quantum Hall states originated from different sources have been studied over the years independently. They include the quantum critical behavior of the quantum Hall plateau transitions [7], composite fermion theory or the Chern–Simons mapping between integral and fractional quantum Hall states [8], the Luttinger liquid theory of quantum Hall edge excitations [9], as well as the stability or robustness of the quantization phenomenon due to the disorder [10]. For a detailed exposure we refer the reader to the literature and here, we only present a brief introduction to the subject.

2. Matrices in frequency and replica space

Diffusive modes are encoded in the unitary matrix field variables $Q^{\beta\gamma}_{nm}$ [6]. Here, the superscripts represent the replica indices ($\beta,\gamma = 1, 2, \ldots N_r$ where $N_r \to 0$ at the end of all calculations) and the subscripts denote the Matsubara frequency indices ($n, m = 0, \pm 1, \pm 2, \ldots \pm N'_{\max}$ where the cutoff $N'_{\max}$ is sent to infinity in the end). The matrices Q generally obey the constraints

$$Q^\dagger = Q, \quad Q^2 = 1, \quad \mathrm{tr}\, Q = 0$$

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and they can be represented by $Q = T^\dagger \Lambda T$, where T is a unitary matrix and Λ is a diagonal matrix

$$(\Lambda)^{\beta\gamma}_{nm} = \delta^{\beta\gamma}\delta_{nm}\,\mathrm{sgn}(m)\,.$$

In order to facilitate a discussion of the electrodynamic $U(1)$ gauge invariance of the theory, we generally follow a very specific cut-off procedure in frequency space and a very specific set of algebraic rules for matrix manipulation which we term $\mathcal{F}$ algebra [1]. For example, if we write the matrix Q in the form $Q = \Lambda + \delta Q$ then the δQ matrix is generally taken as a *small* matrix in frequency space, i.e. its matrix elements are non-zero only for $n, m = 0, \pm 1, \pm 2, \ldots \pm N_{\max}$, where $N_{\max} \ll N'_{\max}$. In other words, the unitary rotation T mixes amongst (positive and negative) frequencies with a *small* cutoff ($N_{\max}$) whereas the $U(1)$ gauge transformations generally involve *large* matrices with a *large* cutoff ($N'_{\max}$) in frequency space.

By employing distinctly different cutoffs $N_{\max}$ and $N'_{\max}$ in Matsubara frequency space, both of which are sent to infinity in the end, the problem of electrodynamic gauge invariance simplifies dramatically [1]. Physically, the cut-off procedure is motivated by the vastly different energy scales that generally characterize the elastic scattering processes ($1/\tau_{\mathrm{el}}$) on the one hand, and the bandwidth of the electron gas on the other hand. However, the rules of $\mathcal{F}$ algebra can be shown to have a quite universal significance for disordered electron systems. For example, it successfully describes the dynamics of chiral edge excitations in quantum Hall systems. It has been used as a microscopic basis for deriving, from first principles, the complete Luttinger liquid theory of edge excitations for abelian quantum Hall states [3, 9].

We generally need the introduction of two more (*large*) matrices . Firstly, the diagonal matrix η,

$$(\eta)^{\beta\gamma}_{nm} = \delta^{\beta\gamma} m \delta_{nm}$$

which is the matrix representation of (imaginary) time derivative. Secondly, there are the off-diagonal matrices I^α_n

$$\left(I^\alpha_n\right)^{\beta\gamma}_{kl} = \delta^{\alpha\beta}\delta^{\alpha\gamma}\delta_{n,k-l}$$

which are the generators of the $U(1)$ gauge transformations. For more details on the rules of $\mathcal{F}$ algebra and various algebraic identities, we refer the reader to the original papers [1–3].

3. The σ model action

The action consists of three terms [1]

$$S[Q] = S_\sigma[Q] + S_{\mathrm{F}}[Q] + S_{\mathrm{C}}[Q]\,. \tag{1}$$

Here, the first term

$$S_\sigma[Q] = \frac{\sigma^0_{xx}}{8}\,\mathrm{Tr}\left(\partial_\mu Q\right)^2 - \frac{\sigma^0_{xy}}{8}\varepsilon_{\mu\nu}\mathrm{Tr}Q\,\partial_\mu Q\,\partial_\nu Q \tag{2}$$

represents the standard *nonlinear σ model* [5] for spinless free electrons in two dimensions and in the presence of a static perpendicular magnetic field. The 'Tr' symbol stands for both the spatial integration and the trace 'tr' over all matrix indices (Matsubara as well as replica).

The second term S_{F} is the *singlet interaction* term first introduced by Finkel'stein [6]. It can be written in three equivalent ways [1]:

$$\begin{aligned} S_{\mathrm{F}}[Q] &= -\pi z_0 T \int \mathrm{d}^2x \left[\sum_{\alpha,n_i} Q^{\alpha\alpha}_{n_1n_2} Q^{\alpha\alpha}_{n_3n_4}\delta_{n_1+n_3,n_2+n_4} + 4\,\mathrm{tr}\,\eta Q\right] \\ &\quad + \mathrm{const} \\ &= -\pi z_0 T \int \mathrm{d}^2x \left[\sum_{\alpha,n} \mathrm{tr}\,I^\alpha_n Q\,\mathrm{tr}\,I^\alpha_{-n} Q + 4\,\mathrm{tr}\,\eta Q\right] \\ &\quad + \mathrm{const} \\ &= -\frac{\pi}{2} z_0 T \sum_{\alpha,n} \mathrm{Tr}\left[I^\alpha_n, Q\right]\left[I^\alpha_{-n}, Q\right]. \end{aligned} \tag{3}$$

Here, the quantity z_0 represents the singlet interaction amplitude and T is the temperature. The compact notation of the last line indicates that the expression is invariant under $U(1)$ gauge transformations ($\mathcal{F}$ invariance, see Section 4). This expression generally acts as a single operator under renormalization group transformations [2, 4].

The nonlinear σ model part and S_{F} represent, together, the effective action for a system with infinite-range electron–electron interactions. The Coulomb potential appears explicitly only in higher dimensional (irrelevant) terms (S_{C}) which are usually discarded. However, these higher-dimensional operators turn out to be *dangerously* irrelevant and it is generally important to take also the term S_{C} (the so-called Coulomb term) into account. This part of the action can be written as

$$\begin{aligned} S_{\mathrm{C}}[Q] &= \pi T \int \mathrm{d}^2x\,\mathrm{d}^2x' \\ &\quad \times \sum_{\alpha,n} \mathrm{tr}\,I^\alpha_n Q(\mathbf{x}) U^{-1}(\mathbf{x}-\mathbf{x}')\,\mathrm{tr}\,I^\alpha_{-n} Q(\mathbf{x}')\,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} U^{-1}(\mathbf{p}) &= \int \mathrm{d}^2x U^{-1}(\mathbf{x}) \exp(-\mathrm{i}\mathbf{p}\mathbf{x}) \\ &= \frac{\pi}{2}\,\frac{1}{1+\rho U_0(\mathbf{p})} \approx \Gamma\,|\mathbf{p}|\,. \end{aligned}$$

In this expression $\rho = \mathrm{d}n/\mathrm{d}\mu$ represents the thermodynamic density of states and $U_0(\mathbf{p}) = 2\pi e^2/|\mathbf{p}|$ is the bare Coulomb interaction in two dimensions.

3.1 Renormalization

The theory, as it stands, contains only four quantities for which one needs to compute the quantum corrections, i.e. σ^0_{xx}, σ^0_{xy}, z_0 and Γ. As is well known, the quantity σ^0_{xy}, multiplying the topological charge q,

$$q(Q) = \frac{1}{16\pi\mathrm{i}}\mathrm{Tr}\,\epsilon_{\mu\nu} Q\,\partial_\mu Q\,\partial_\nu Q = \frac{1}{4\pi\mathrm{i}}\oint \mathrm{d}x\,\mathrm{tr}\,T\partial_x T^\dagger \Lambda\,, \tag{5}$$

is not affected by the perturbative quantum theory and will be dealt with at a later stage. The quantity Γ, on the other hand, remains strictly unrenormalized and this statement, as it has turned out from the microscopic derivation of the action, should be imposed as a general constraint on the quantum theory [10] (see also Section 7).

This leaves us with two non-trivial renormalizations in perturbation theory, i.e. the inverse coupling constant σ^0_{xx} and the interaction amplitude z_0. The complete list of renormalization group β and γ functions in $2+2\epsilon$ dimension is given by [2]

$$\begin{aligned}
\beta_t &= \frac{\mathrm{d}t}{\mathrm{d}\ln L} = -2\varepsilon t + 2t^2 + O(t^3)\,, \\
\beta_{xy} &= \frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}\ln L} = 0\,, \\
\gamma &= \frac{\mathrm{d}\ln z}{\mathrm{d}\ln L} = -t - t^2\left(3 + \frac{\pi^2}{6}\right) + O(t^3)\,, \\
\gamma_C &= \frac{\mathrm{d}\ln \Gamma}{\mathrm{d}\ln L} = -1.
\end{aligned}$$

Here, we written $t = 1/\pi\sigma_{xx}$, and L denotes the linear dimension of the system. These results are quite similar to those of the classical Heisenberg ferromagnet and the physics of the electron gas in $2+2\epsilon$ dimension can be obtained following the (in many ways) unique field theoretical methodology of dealing with Goldstone modes.

3.2 Fermi liquids versus non-Fermi liquids

It is important to keep in mind, however, that the physics of interacting systems is very different from that of free electron localization. Free electron formalisms, unlike the Finkel'stein formalism, has Q matrix field variables which usually have two frequencies only, i.e. the advanced and retarded ones [5]. A formal but general way of describing the crossover between the single particle and many body formalisms is obtained by varying the frequency cutoff $N_{\max}$ in the Q matrix fields, from unity to infinity. By varying the value of $N_{\max}$, the theory changes fundamentally. The most dramatic effect of putting $N_{\max} \to \infty$ is that the ultraviolet singularity structure of theory (i.e. the β function) changes completely [4]. The presence of the singlet interaction term S_{F} now implies that the problem generally belongs to a different universality class of quantum transport phenomena. Since there is no Fermi liquid principle for the *disordered* electron system with an infinite-range interaction, it is necessary to reconsider the topological concept of a θ or instanton vacuum which previously was introduced and investigated for the free electron theory alone [5, 15].

Along with the renormalization behavior, also the structure of the operators of the theory change as the cutoff $N_{\max}$ is being sent to infinity [2]. A new, previously unrecognized notion of *interaction symmetries* now becomes an integral part of the problem. These symmetries ($\mathcal{F}$ invariance, Section 4) are intimately related to the electrodynamic $U(1)$ gauge invariance and much is yet to be learned about the behavior of the theory in the strong coupling regime.

Before elaborating on symmetries and gauge invariance, however, we wish first to address some of the general ideas that are associated with the perturbative renormalization group results of the previous Section.

Notice that on the basis of the β function or asymptotic freedom alone, one generally expects that the interacting electron gas, in two spatial dimensions, behaves *quasi-metallic* at short distances but it eventually enters a strong coupling phase with a massgap (an *insulating* phase), as the lengthscale is increased.

The renormalization of the interaction terms S_{F} and S_{C}, i.e. the γ and γ_{c} functions, generally determine the *dynamical* scaling in the problem, i.e. the temperature and frequency dependence of physical observables, and this includes the non-Fermi liquid behavior of quantities like the specific heat [2].

The result of the γ indicates that the singlet interaction term S_{F} plays formally the role of an *order parameter* (i.e. the spontaneous magnetization) in the context of conventional critical phenomena phenomenology. Physically it means that the theory generates a (Coulomb) gap in the density of states that enters in the expression for the specific heat [2]. This result is quite different from what one is used to in the theory of free electron localization, or in the theory with a finite value of $N_{\max}$. For example, such free particle concepts like anomalous or *multifractal* fluctuations in the local density of states are no longer valid in the theory with Coulomb interactions. The physics of the γ functions is generally very different.

In Figure 1, we plotted the results for the anomalous dimensions γ and γ_{c} versus t. We see that in the weak coupling or small t regime, the $\gamma(t)$ dominates the $\gamma_{\mathrm{c}}(t)$ indicating that the Coulomb term S_{C} is irrelevant. However, with the $\gamma_{\mathrm{c}}(t)$ function remaining fixed at the value -1, as mentioned before, there is likely a point on the t axis beyond which the $\gamma_{\mathrm{c}}(t)$ dominates the $\gamma(t)$ function. This means that upon entering the strong coupling regime, the roles of S_{F} and S_{C} get interchanged, and the dynamics of the insulating phase is now entirely determined by the Coulomb term S_{C}.

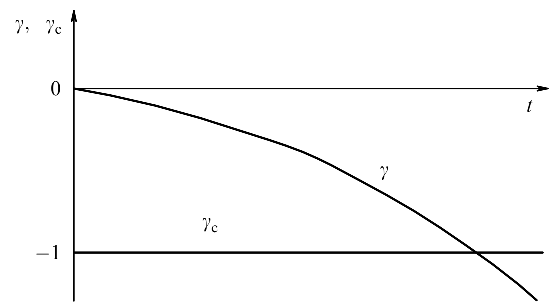

Figure 1. The anomalous dimensions γ and γ_{c} versus t.

This dangerously irrelevant behavior of S_{C} has not been recognized previously but it, obviously, plays a fundamental role in the theory of metals and insulators. This notion cannot be obtained in any heuristic or phenomenological fashion, but it clearly affects the way in which we are going to look upon the complications of the theory in dealing with the quantum Hall effect. For example, if the quantum critical singularities of the quantum Hall plateau transitions [7] are appropriately described by the non-perturbative behavior of the β_{xy} and γ functions of the Finkel'stein theory [4], then this critical behavior is certainly very different from the main expectations of the free particle approximation. Unlike the free particle problem which effectively lives in two spatial dimensions alone, the Finkel'stein theory contains the time variable as an extra non-trivial dimension. This not only destroys any hope of finding an exact (conformal) scheme of critical indices, but also invalidates any explicit or implicit attempt of employing Fermi liquid ideas for quantum Hall systems in the presence of the Coulomb interactions.

4. $\mathcal{F}$ invariance

$\mathcal{F}$ invariance [1] is one of the most important interaction symmetries of disordered systems. It means that the action for systems with an infinite-range interaction potential (like the

Coulomb potential) is invariant under spatially independent $U(1)$ gauge transformations. Such gauge transformations can be represented by a (*large*) unitary matrix W

$$W = \exp\left(\mathrm{i}\sum_{\alpha,n} \phi_n^\alpha I_n^\alpha\right),$$

where the ϕ_n^α are the frequency components of the imaginary time quantity $\phi(\tau)$. The statement of $\mathcal{F}$ invariance can now be written as

$$S[Q] = S[W^{-1}QW].$$

$\mathcal{F}$ invariance is generally violated in systems with a finite-range interaction potential, or free electron systems. One can show that these systems map, under the action of the renormalization group, back onto the free electron theory and, therefore, Fermi liquid ideas can be applied in this case.

The concept of $\mathcal{F}$ invariance implies that only the quantities and correlations that are $\mathcal{F}$-invariant have a simple infrared behavior that generally can be handled with the methodology of the renormalization group. It also implies that the *dynamical scaling* of physical observables like the conductances can only be extracted from the theory if $\mathcal{F}$ invariance is respected by the renormalization scheme that one chooses. For example, the momentum shell or background field methodology generally violates $\mathcal{F}$ invariance and this complicates the computation of e.g. the AC conductances.

On the other hand, the partition function itself, or the response to electromagnetic fields, generally respects statement of $\mathcal{F}$ invariance. It is therefore important to study $\mathcal{F}$-invariant quantities in general and see $\mathcal{F}$ algebra at work.

5. External EM fields

If, for the sake of simplicity, we first consider the theory of weak static magnetic fields, then the various pieces of the action, in the presence of scalar and vector potentials, can be written in a transparent fashion as follows [2]

$$\begin{aligned}
S_\sigma[Q,A] &= \frac{\sigma_{xx}^{(0)}}{8}\,\mathrm{Tr}\left[D_\mu,Q\right]^2 - \frac{\sigma_{xy}^{(0)}}{8}\varepsilon_{\mu\nu}\,\mathrm{Tr}\,Q\left[D_\mu,Q\right]\left[D_\nu,Q\right],\\
S_{\mathrm{F}}[Q,A] &= S_{\mathrm{F}}[Q],\\
S_{\mathrm{C}}[Q,A] &= \pi T\int\frac{\mathrm{d}^2p}{(2\pi)^2}\sum_{\alpha,n}\left[\mathrm{tr}(I_n^\alpha Q) - \frac{1}{\pi T}(A_0)_{-n}^\alpha\right]\\
&\quad\times U^{-1}(\mathbf{p})\left[\mathrm{tr}(I_{-n}^\alpha Q) - \frac{1}{\pi T}(A_0)_n^\alpha\right].
\end{aligned}$$

Here, $D_\mu = \partial_\mu - \mathrm{i}A_\mu$ is the covariant derivative in matrix form with $A_\mu = \sum_{\alpha,n}(A_\mu)_n^\alpha I_n^\alpha$. By the rules of $\mathcal{F}$ algebra, the $U(1)$ gauge invariance can now be formulated by saying that the following set of transformations $Q \to W^{-1}QW$, $(A_\mu)_n^\alpha \to (A_\mu)_n^\alpha + \mathrm{i}\partial_\mu\phi_n^\alpha$, and $(A_0)_n^\alpha \to (A_0)_n^\alpha + \mathrm{i}\omega_n\phi_n^\alpha$ leaves the action invariant.

The theory of strong static magnetic fields B is slightly more complex and has additional terms ($\sigma_{xy}^{\mathrm{II}}$, Section 7) reflecting the B dependence of the electron density.

6. Response at a tree level

As a simple check of the above formulae, we compute the gauge invariant electromagnetic response at a tree level, in the case $\sigma_{xy} = 0$. We obtain

$$S_{\mathrm{eff}}^{\mathrm{tree}}[A] = \frac{1}{T}\sum_{\alpha,n}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\sigma_{xx}^{(0)}}{8}\left[\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 + 4\omega_n U^{-1}(p)/\sigma_{xx}^{(0)}}\right] \times \frac{(E_\mu)_n^\alpha\overline{(E_\nu)_n^\alpha}}{\omega_n}, \tag{6}$$

where $(E_\mu)_n^\alpha$ is the electric field. The charge density can be defined as $n_n^\alpha(p) = -T\delta S_{\mathrm{eff}}[A]/\delta(A_0)_{-n}^\alpha$, then from $S_{\mathrm{eff}}^{\mathrm{tree}}[A]$ one obtains the continuity equation that in ordinary space-time notation can be written as [2]

$$\partial_t n + \boldsymbol{\nabla}\cdot(\mathbf{j}_{\mathrm{C}} + \mathbf{j}_{\mathrm{diff}}) = 0, \tag{7}$$

where $\mathbf{j}_{\mathrm{diff}} = -D_{xx}^{(0)}\boldsymbol{\nabla}n$ with $D_{xx}^{(0)} = \sigma_{xx}^{(0)}/2\pi\rho$ is just diffusive current component and

$$\mathbf{j}_{\mathrm{C}} = \frac{\sigma_{xx}^{(0)}}{2\pi}\mathbf{E}_{\mathrm{tot}},$$

with

$$\mathbf{E}_{\mathrm{tot}} = \mathbf{E}_{\mathrm{ext}} - \boldsymbol{\nabla}\int\mathrm{d}^2x'\,U_0(\mathbf{x}-\mathbf{x}')n(\mathbf{x}')$$

being the electric current density due to the external and internally generated electric fields. The tree level response therefore reproduces the well known results of the theory of metals.

7. Response with quantum corrections

We have computed the complete response to external electromagnetic fields with quantum corrections and checked the gauge invariance at a one loop level. The analysis is rather lengthy and the details will be reported elsewhere [10]. We present, instead, the final result for the continuity equation which can be written in frequency and momentum notation as follows:

$$\omega_\eta\left(n_\eta - \frac{\sigma_{xy}^{(0)}}{2\pi}B^\eta\right) = \mathrm{i}p_\mu\left[\frac{\sigma_{xx}^{\mathrm{I}}}{2\pi}\left(E_\mu^\eta - \mathrm{i}p_\mu U_0 n_\eta\right) - D_{xx}\mathrm{i}q_\mu\left(n_\eta + \mathrm{i}\frac{\sigma_{xy}^{\mathrm{II}}}{2\pi}B^\eta\right)\right]. \tag{8}$$

Here, σ_{xx}^{I} is the *renormalized* longitudinal conductivity which is expressed as $\sigma_{xx}^{(0)}$ plus quantum corrections. Furthermore, $D_{xx} = \sigma_{xx}^{\mathrm{I}}/2\pi\rho$ the renormalized diffusive coefficient and $\sigma_{xy}^{\mathrm{II}}/2\pi = \mathrm{d}n/\mathrm{d}B$.

It is important to remark that the static ($\omega_\eta = 0$) limit of this expression has an important general significance for the renormalization of the theory. Notice that by putting $\omega_\eta = 0$, the expression only contains the thermodynamic quantities such as ρ, the thermodynamic density of states, $\sigma_{xy}^{\mathrm{II}} = \partial n/\partial B$, which is the derivative of the electron density with respect to the static external B, as well as the bare Coulomb interaction U_0. This form of the static response can be shown to be quite generally valid, independently of the effective action of the Q field variables. This means that the thermodynamic quantities ρ, $\sigma_{xy}^{\mathrm{II}}$ as well as U_0 generally do not acquire any quantum corrections and this statement can be imposed as an important general constraint on the quantum theory.

Notice that this constraint does not involve the quantities σ_{xx}, σ_{xy} and the interaction amplitude z which do not appear in the expression for static response. These quantities are therefore the only ones for which quantum corrections are possible (and do occur) in general.

8. Physical observables

The results of the previous Section imply that a general quantum theory of physical observables can be formulated and expressed in terms of $\mathcal{F}$-invariant correlations of the Q field variables. For example, the linear response to an external electric field can be generally expressed in terms of a quantity σ'_{xx} (Kubo formula) as follows:

$$\sigma'_{xx} = -\frac{\sigma^{(0)}_{xx}}{4\eta}\left\langle \mathrm{tr}\left\{\left[I^{\alpha}_{\eta}, Q\right]\left[I^{\alpha}_{-\eta}, Q\right]\right\}\right\rangle + \frac{\sigma^{(0)2}_{xx}}{16\eta D}\int \mathrm{d}^2x'\left\langle \mathrm{tr}\left\{\left[I^{\alpha}_{\eta}, Q\right]\partial_{\mu}Q\right\} \mathrm{tr}\left\{\left[I^{\alpha}_{-\eta}, Q\right]\partial_{\mu}Q\right\}\right\rangle .$$

This result, when evaluated perturbatively [2], is of the general form $\sigma^{(0)}_{xx}$ + quantum corrections. A similar expression exists for the Hall conductance σ'_{xy}. These general results retain their significance also beyond the theory of perturbative quantum corrections. For example, they can be used for non-perturbative (instanton) calculus [4] as well and this has led to the previously unrecognized concept of θ renormalization, or renormalization of the Hall conductance σ_{xy} [5].

For completeness, we mention that the list of effective parameters σ'_{xx} and σ'_{xy} can also be extended to include an effective quantity z' which is associated with the interaction amplitude z_0. The result can be expressed as [2]

$$\sum_{n>0} \omega_n z'(\omega_n) = \frac{\pi}{2} z_0 T \sum_{n>0} \left\langle \mathrm{tr}\left\{\left[I^{\alpha}_{n}, Q\right]\left[I^{\alpha}_{-n}, Q\right]\right\}\right\rangle .$$

9. Instantons

The non-perturbative contributions from a topologically non-trivial sectors of the theory (instantons) have formally the same structure as those obtained in the theory of free electrons [15]. Since the analysis is rather involved [4], we simply present the most important results as illustrated by the renormalization group flow lines in the σ_{xx} and σ_{xy} conductance plane in Fig. 2.

The general consequences of the Coulomb interactions for the quantum critical behavior of the plateau transitions will be reported elsewhere. On the other hand, there is the problem of *robust* quantization of the Hall conductance which is represented in Fig. 2 by the strong coupling fixed points at integer values of σ_{xy}. This aspect of the theory cannot obviously be obtained from any analysis in the weak coupling regime, either perturbative or non-perturbative, and it generally requires a more explicit knowledge of physics of incompressible quantum Hall states.

Recently, a new and general ingredient of the instanton vacuum has been discovered from which the phenomenon of robust quantization can be derived. It has turned out that the edge of the instanton vacuum is generally *massless*, and the theory can be mapped directly onto the more familiar theory of chiral edge bosons in quantum Hall systems [3]. The effective action for massless edge excitations, along with a mass gap for the excitations in the bulk, is *dynamically* generated by the Finkel'stein theory and quantized Hall

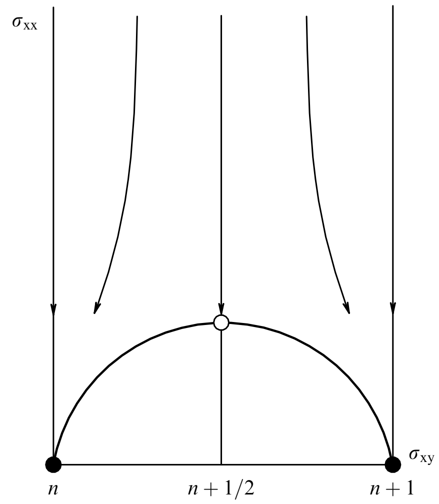

Figure 2. The renormalization gruop flow for the conductances. The arrows indicate the scaling towards the infrared.

conductance now appears as the renormalized quantity σ'_{xy} in the effective action for the edge.

It is important to remark that by extending the instanton vacuum approach to include the effective action for edge excitations in the quantum Hall state [3], the significance of $\mathcal{F}$ invariance has now also been demonstrated in the otherwise forbidden strong coupling regime of the Finkel'stein theory.

10. Chern – Simons statistical gauge fields

The theory of composite fermions [8] is obtained by adding the Chern – Simons statistical gauge fields to the theory. The idea has been discussed at great length and in extensive detail elsewhere. Here, we just mention how the flux attachment transformation by the Chern – Simons gauge fields generates a scaling diagram that includes both the abelian quantum Hall states and the half-integer effect (Fig. 3) [1]. The theory also includes the Luttinger liquid theory for edge excitations [9]. This, then, leads to a unifying theory for both the compressible and incompressible states in the quantum Hall regime.

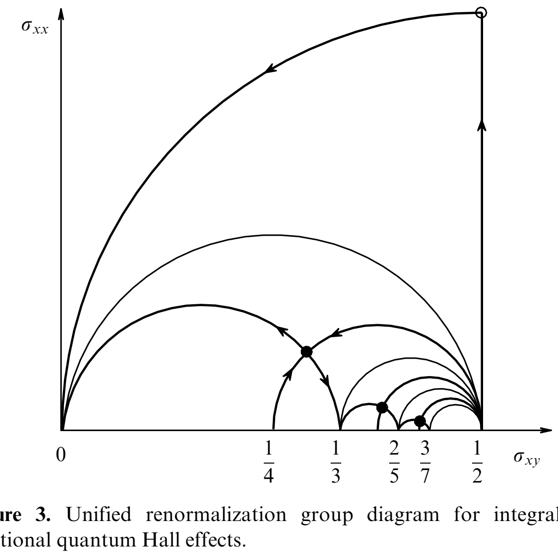

Figure 3. Unified renormalization group diagram for integral and fractional quantum Hall effects.

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References

1. Pruisken A M M, Baranov M A, Škorič B *Phys. Rev. B* **60** 16807 (1999)
2. Baranov M A, Pruisken A M M, Škorič B *Phys. Rev. B* **60** 16821 (1999)
3. Pruisken A M M, Škorič B *Phys. Rev. B* **60** 16838 (1999). The discussion in this paper on the experiments on current scaling is somewhat controversial. See also Ref. [10]
4. Pruisken A M M, Baranov M A *Europhys. Lett.* **31** 543 (1995)
5. Pruisken A M M *Nucl. Phys. B* **235** 277 (1984); Levine H, Libby S, Pruisken A M M *Phys. Rev. Lett.* **51** 20 (1983); *Nucl. Phys. B* **240** [FS12] 30, 49, 71 (1984)
6. Finkel'stein A M *Pis'ma Zh. Eksp. Teor. Fiz.* **37** 436 (1983) [*JETP Lett.* **37** 517 (1983)]; *Zh. Eksp. Teor. Fiz.* **86** 367 (1984) [*Sov. Phys. JETP* **59** 212 (1984)]; *Physica B* **197** 636 (1994)
7. Pruisken A M M *Phys. Rev. Lett.* **61** 1297 (1988); Wei H P, Tsui D C, Paalanen M A, Pruisken A M M *Phys. Rev. Lett.* **61** 1294 (1988); for recent results see van Schaijk R T F et al. *Phys. Rev. Lett.* **84** 1567 (2000)
8. Das Sarma S, Pinczuk A (Eds) *Perspectives in Quantum Hall Effects* (New York: Wiley, 1997); Heinonen O (Ed.) *Composite Fermions* (Singapore: World Scientific, 1998)
9. Škorič B, Pruisken A M M *Nucl. Phys. B* **559** 637 (1999) and references therein.
10. Pruisken A M M, Baranov M A (in preparation)
11. Pruisken A M M *Phys. Rev. B* **31** 416 (1985); Pruisken A M M *Nucl. Phys. B* **285** 719 (1987); *Nucl. Phys. B* **290** 61 (1987)

Hidden SU(4) symmetry in bilayer quantum well at integer filling factors

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Abstract. Phase diagram of a bilayer quantum well at integer filling factors is established using the hidden symmetry method. Three phases: ferromagnetic, canted antiferromagnetic (CAP) and spin-singlet, have been found. We confirm early results of Das Sarma et al. Each phase violates the SU(4) hidden symmetry and is stabilized by the anisotropy interactions.

1. Introduction

Integer filling factors of a 2D electron gas (2DEG) confined to a quantum well in an external magnetic field are special ones because a huge degeneracy of the ground state is gone here. It justifies the Hartree–Fock approximation with the accuracy limited only by normally a small parameter: $V^{\rm int}/\hbar\omega_0$, where $V^{\rm int}$ is the energy of the Coulomb interaction and ω_0 is the frequency of the cyclotron resonance. Such an approach predicts the ground state of a single-layer 2DEG at $\nu = 1$ to be a ferromagnet with the degenerate total spin orientation. The elementary excitations of 2DEG are electron–hole pairs or excitons, and in the limit of vanishing momentum they transform into the elementary excitations of a ferromagnet — spin waves. The latter are gapless [1] and do not interact with each other [2] if Zeeman energy is neglected — the two consequences of the Goldstone theorem. In the limit of large momentum the electron and the hole of an exciton are well separated and they become the elementary charged excitations.

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The case of a bilayer 2DEG turned out to be a more rich one, where both spin and pseudo-spin (layer) dynamics become entangled. The Hartree–Fock approximation does not apply here except for two limiting cases. The first one is the case of well separated layers which is a common setup in the experiment [3, 4] and where, theoretically, one starts from the two single-layer ferromagnets in the balanced case of filling factor $\nu = 2$ and makes the perturbation expansion in powers of interlayer interactions [5]. And the second one is the symmetric case defined in such a way that one can freely rotate an electron spinor in both layer and spin spaces. The latter requires to approximate the Coulomb interaction by its symmetric part and to neglect all symmetry-breaking fields like Zeeman energy. The first attempts in this direction dealt with the case of filling factor $\nu = 1$ and relied heavily on the assumption of a saturated spin polarization of electrons [6, 7]. This symmetric approximation turned out to be useful to determine the exciton energy in bilayer [7]. Recent works [8, 9] specialize to the bilayer heterostructure case $\nu = 2$, employ the Hartree–Fock approximation and predict a phase diagram that features three phases: the ferromagnetic, the canted antiferromagnetic and a special spin-singlet phase. In this paper we reproduce the phase diagram of Refs [8, 9] isolating the symmetric and the symmetry-breaking parts of the Hamiltonian in a consistent way. Our approach reveals the Hartree–Fock phase diagram to be indeed exact in the limit $V^{\rm anis}/V^{\rm sym} \to 0$, where $V^{\rm sym}$ is the SU(4)-symmetric part of the bilayer Hamiltonian whereas $V^{\rm anis}$ is anisotropy interactions that reduce the bilayer Hamiltonian symmetry to $\mathrm{SU(2)} \otimes \mathrm{SU(2)}$. We prove the stability of all phases with respect to long-range spatial perturbations. We find that low-energy excitations over the bilayer ground state are governed by the $\mathrm{U(4)}/\mathrm{U}(\nu) \otimes \mathrm{U}(4-\nu)$ coset in nonlinear sigma model.

2. Hamiltonian of 2DEG bilayer

The electronic Hamiltonian of a 2DEG in a confining potential $V(\boldsymbol{\rho})$ and in an external magnetic field H perpendicular to the layer consists of a one-particle part as well as a Coulomb interaction part:

$$H = \int \psi_\alpha^+(\boldsymbol{\rho}) \times \left\{ \frac{1}{2m} [-\mathrm{i}\nabla + \mathbf{A}(\boldsymbol{\rho})]^2 + V(\boldsymbol{\rho}) - |g|\mu_{\rm B} H \sigma^z_{\alpha\beta} \right\} \psi_\beta(\boldsymbol{\rho})\, \mathrm{d}^3\boldsymbol{\rho} + \frac{1}{2} \int\!\!\int \frac{e^2}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|} \psi_\alpha^+(\boldsymbol{\rho}) \psi_\beta^+(\boldsymbol{\rho}') \psi_\beta(\boldsymbol{\rho}') \psi_\alpha(\boldsymbol{\rho})\, \mathrm{d}^3\boldsymbol{\rho}\, \mathrm{d}^3\boldsymbol{\rho}' , \quad (1)$$

where $\alpha, \beta = \pm$ are spin indices and thereafter a sum over repeated indices is implied. We use such units that $\hbar = 1, e = c$ and $H = B = 1$. All distances can be expressed in terms of the so-called magnetic length: $l_H = \sqrt{c\hbar/eH} = 1$. We split three coordinates $\boldsymbol{\rho}$ into a perpendicular to the layer coordinate ξ and two in-plane coordinates $\mathbf{r} = (x, y) = (z, \bar{z})$. We assume that the confining potential is uniform over the plane: $V(\boldsymbol{\rho}) = V(\xi)$, and represents a double-well structure in the transverse direction as shown in Fig. 1, with the two wells being separated by the distance d. We use only two eigenfunctions: the lowest energy symmetric $\chi_{\rm S}(\xi)$ and antisymmetric