

PACS numbers: **71.30.+h**, 73.23.Hk, **74.50.+r**, **74.76.-w**

Chernogolovka 2000: Mesoscopic and strongly correlated electron systems

5. Superconductor–metal–insulator transitions

Quantum superconductor–metal transition in a proximity array

M V Feĭgel'man, A I Larkin, M A Skvortsov

Abstract. A theory of the zero-temperature superconductor-metal transition is developed for an array of superconductive islands (of size d) coupled via a disordered two-dimensional conductor with the dimensionless conductance $g = \hbar/e^2R_\square \gg 1$. At $T = 0$ macroscopically superconductive state of the array with lattice spacing $b \gg d$ is destroyed at $g < g_c \approx 0.1\ln^2(b/d)$. At high temperatures the normal-state resistance between neighboring islands at $b = b_c$ is much smaller than $R_Q = h/4e^2$.

In two-dimensional (2D) systems, qualitative arguments based on duality between Cooper pairs and vortices lead to the prediction [1] that the superconductor–insulator transition happens at the universal quantum value $R_Q \equiv h/4e^2$ of the resistance per square $R_\square$. Although a number of experiments (cf. [2] and references therein) seem to be in agreement with this prediction, other data demonstrate strong deviations from it [3–6]. Phenomenological picture of duality is not able to predict the system's parameters (e.g., the value of the normal-state resistance) leading to the quantum critical point: a microscopic theory is needed to find it. Competition between Josephson coupling E_J and charging energy E_C is known [7, 8] to be the driving mechanism of zero-temperature phase transitions between the superconductive and insulating states in artificial arrays [3, 9], films [10, 11] and bulk materials [12]. In such systems there are no free electrons at very low temperatures due to Cooper pairing, but pairs may become localized due to Coulomb repulsion. This is the 'bosonic' mechanism of superconductivity suppression.

Homogeneously disordered superconductive films [4–6, 13] present another group of systems where quantum fluctuations lead to destruction of superconductivity. The theory of T_c suppression in such films was developed in Ref. [14]. The qualitative idea behind this theory is that disorder-enhanced Coulomb repulsion leads to the decrease of Cooper attraction and thus to the decrease of T_c. The superconductive transition temperature vanishes [14, 15] when the dimensionless film conductance $g = \hbar/e^2R_\square$ decreases down to $g_{Fin} = (2\pi)^{-2}\ln^2(1/T_{c0}\tau_{tr})$, where T_{c0} is the BCS transition temperature and τ_{tr} is the elastic scattering time. This second ('fermionic') mechanism of superconductivity suppression is clearly different from the first one [7], since its basic feature lies in disappearance of Cooper pairs altogether. Experimental data supporting the fermionic mechanism are reviewed in Ref. [15]. A drawback of this theory is that it neglects quantum fluctuations of the bosonic field (i.e., it can be considered as a kind of the BCS theory with the renormalized attraction constant). For a phenomenological comparison of the bosonic and fermionic mechanisms, see Ref. [16].

In this paper we study a model for quantum breakdown of superconductivity, which lies in between the two limiting cases discussed above. We consider an array of small superconductive (SC) islands (of radius d each) in contact with a thin film of dirty normal (N) conductor with the dimensionless conductance $g \gg 1$. The distance between neighboring islands is $b \gg d$ (more precisely, b^{-2} will be the concentration of islands). Resistance R_T of the interface between each island and the film is low: $G_T = \hbar/e^2R_T \gg 1$. Islands are thick enough, to prevent suppression of superconductivity inside them. The corresponding condition for the superconductive gap reads $\Delta_{SC} \gg G_T/\nu V_i$, where V_i is the island's volume and ν is the density of states. The above inequality ensures that the lifetime of electrons in the SC island (which is finite due to the tunneling across the interface) is much longer than $\hbar/\Delta_{SC}$. We assume also that $G_T^2 \gg 4\pi g$; the meaning of this condition will be explained below. We will show that macroscopic superconductivity in such a system at $T = 0$ becomes unstable with respect to quantum fluctuations at g less than

$$g_c = \mathcal{G}_c\left(\frac{1}{\pi}\ln\frac{b}{\tilde{d}}\right)^2, \qquad (1)$$

where $\mathcal{G}_c \sim 1$ will be determined below, and $\tilde{d} \sim d$. Equation (1) presents our main result (obtained within logarithmic accuracy), which shows that the critical sheet resistance $R_{\square c} = \hbar/e^2g_c$ is much less than the quantum resistance R_Q, provided $\ln(b/d) \geq 3$. Moreover, the same is valid for the normal-state resistance between neighboring islands

$$R_n = \frac{R_{\square c}}{\pi}\ln\frac{b}{d} \sim 3\frac{R_Q}{\ln(b/d)}.$$

This result is at odds with usual arguments based on the model of resistively shunted Josephson junctions [17]: in that model superconductive behavior of a single junction is preserved at $T = 0$ as long as it has resistance $R < R_Q$. The physical reason for this discrepancy is that we account for the

M V Feĭgel'man, A I Larkin, M A Skvortsov L D Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, 117940 Moscow, Russian Federation

A I Larkin Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA

discrete nature of charge transport between the SC islands, which was neglected in the Caldeira – Leggett model adopted in Ref. [17].

We will follow an idea presented in Ref. [18], where a simplified version of the considered model was analyzed (cf. also [19]). Namely, we make use of the long-range nature of the Josephson coupling J_{ij} between SC islands due to the proximity effect in the film, which scales as $J_{ij} \propto r_{ij}^{-2} \exp[-c(r_{ij}/L_T)]$, where $L_T = (\hbar D/T)^{1/2}$ is the thermal coherence length (cf. Refs [18, 20]), $c \sim 1$, and D is the diffusion constant in the film. At low temperatures the interaction radius L_T diverges indicating that the position of the quantum phase transition can be found in the mean-field approximation (MFA) analogous to the one developed in Ref. [21]. Within the MFA, macroscopic superconductive coherence sets in at

$$\frac{1}{2\hbar}\mathcal{J}(T)\mathcal{C}(T) \geq 1\,, \tag{2}$$

where

$$\mathcal{J}(T) = \sum_j J_{ij}, \quad \mathcal{C}(T) = \int_0^{1/T} \mathrm{d}\tau C_0(\tau)\,, \tag{3}$$

and $C_0(t) = \langle \cos[\theta(0) - \theta(t)]\rangle$ is the single-island autocorrelation function of the order-parameter phase. Thus, to find the superconductor – metal transition line we need to determine the functions $\mathcal{J}(T)$ and $\mathcal{C}(T)$ at low temperatures. We will discuss the $T = 0$ limit mainly, and present the results for low $T > 0$ at the very end of the paper.

Calculation of $\mathcal{J}(0)$ is rather straightforward. The result depends on the relation between the dimensionless Cooper repulsion constant λ_n defined at the energy scale $\omega_d = D/d^2$ and the ratio $G_T/4\pi g$ which is a measure of the interface transparency. At low interface transparencies, $G_T/4\pi g \ll \lambda_n$, it is possible to consider the lowest order process over G_T, and one obtains [18]

$$\mathcal{J}(0) = \frac{G_T^2}{16 v \lambda_n^2} \frac{1}{b^2 \ln(b/d)}\,. \tag{4}$$

This result is similar to the one obtained in Ref. [20] for a one-dimensional S-N-S geometry: in both cases $\mathcal{J}(0)$ formally diverges as the Cooper repulsion constant goes to zero. However, the decrease of λ_n at a fixed G_T drives the system out of the low-transparency limit. At $\lambda_n \leq G_T/4\pi g$ full nonlinear treatment of the proximity problem is necessary, accounting multiple Andreev reflections from SC islands. At the same time, the effect of λ_n upon $\mathcal{J}(0)$ can be neglected in this limit (for 2D geometry studied here). To find the coupling energy $J(r)$ between two islands situated at the distance $r \gg d$, one has to solve the Usadel equation [22] in the film. Actually, it is more convenient to calculate the maximum superconducting current between the islands, $I_c(r) = (2e/\hbar)J(r)$. The value of the supercurrent can be determined as the integral over the current density, taken along the straight line which lies in the middle between two islands, and is perpendicular to the vector $\mathbf{r}$ connecting them. If the distance between the islands is very large, $\ln(r/d) \gg 1$, anomalous part of the Green function is small along the above-mentioned line, that allows† to linearize the Usadel equation at relevant energies $E \sim \hbar D/r^2$ and find $I_c(r)$. Thereby one obtains $J(r) = (\pi^3/4) g D/r^2 \ln^2(r/d)$ and, finally,

$$\mathcal{J}(0) = \frac{\pi^4}{2} \frac{gD}{b^2 \ln(b/d)}\,. \tag{5}$$

The key point in the discussion of the $T = 0$ transition is to determine $\mathcal{C}(T \to 0)$. We will see that $\mathcal{C}(0)$ depends exponentially on the film conductance g. If islands do not have ohmic contacts with the film (coupling via capacitance C_j only) then $\mathcal{C}(0) = \hbar/E_C = \hbar C_j/2e^2$. In our case $\hbar/\mathcal{C}(0)$ plays the role of an effective charging energy E_C^* of an island that survives in spite of good conductance around. To make ideas transparent, we first discuss a simplified model [18] with sufficiently strong Cooper-channel repulsion in the film, $\lambda_n \gg G_T/4\pi g$. Then dynamics of the phase $\theta(t)$ of a single SC island can be described by a simple imaginary-time action,

$$S_0[\theta] = -\frac{G_A}{8\pi} \int\!\!\int_0^{1/T} \mathrm{d}t\,\mathrm{d}t' \frac{\cos[\theta(t) - \theta(t')]}{(t - t')^2}\,. \tag{6}$$

Here $G_A = G_T^2/4\pi g \lambda_n$ is the Andreev subgap conductance (normalised to $e^2/\hbar$) in the limit of weak proximity effect, valid under the condition $\lambda_n \gg G_T/4\pi g$ [18, 23]‡. Expression (6) is valid at low energies, $\omega \leq \omega_d \exp(-1/\lambda_n)$, while at higher energies, $\omega_d \exp(-1/\lambda_n) \leq \omega \leq \omega_d$, one has $G_A(\omega) = (G_T^2/4\pi g)\ln(\omega_d/\omega)$. Thus, the Andreev conductance is large compared to $4e^2/h$ for all frequencies below ω_d if the condition $G_T^2 \gg 4\pi g$ is fulfilled. At lower values of G_T, N-S transport across the interface is suppressed by the usual Coulomb blockade effect governed by the junction's charging energy $2e^2/C_j$. We are not interested in this trivial effect, and will assume the condition $G_T^2 \gg 4\pi g$ to be valid.

For large G_A one can start from the Gaussian approximation for $S_0[\theta(t)]$. Then the Fourier-transformed correlator of phase fluctuations $\langle|\theta_\omega|^2\rangle_0 = 4/|\omega| G_A$, and, hence,

$$C_0(t) = \exp\left\{-\frac{1}{2}\langle[\theta(t) - \theta(0)]^2\rangle_0\right\} \propto t^{-4/\pi G_A}\,.$$

At $G_A > 4/\pi$, $\mathcal{C}(T \to 0)$ diverges that seems to indicate that at large G_A superconductivity is always stable at $T = 0$, in agreement with Ref. [17]. The crucial point is to note that the employed Gaussian approximation breaks down at a finite time scale t^*, due to downscale renormalisation of G_A. This renormalisation is caused by the periodicity of the action $S_0[\theta]$ as a functional of $\theta(t)$, that is, in physical terms, by the charge quantization. This problem is analogous to the one studied by Kosterlitz [24]. Translating his results to the present case, one gets the renormalization group (RG) equation $\mathrm{d}G_A(\zeta)/\mathrm{d}\zeta = -4/\pi$, with $\zeta = \ln \omega_d t$. This equation is to be solved with the initial condition $G_A(0) = G_A$. As a result, at the time scale $t^* \sim \omega_d^{-1} \exp(\pi G_A/4)$ the renormalized Andreev conductance $G_A(t^*)$ decays down to the value of order of unity [18]. At longer time scales $C_0(t)$ decays approximately as t^{-2}, so the integral

$$\mathcal{C}(0) \sim t^* \sim \omega_d^{-1} \exp\frac{\pi G_A}{4}$$

Taking into account that $\mathcal{J} \sim b^{-2}$, and using Eqn (2), one obtains [18] the critical distance between islands $b_c \sim d \exp(\pi G_A/8)$.

† Similar calculation can be found in Appendix C of Ref. [22].

‡ The result for G_A given in Ref. [18] is overestimated by the factor of 2 due to a numerical mistake.

However, this result is valid under the condition $\lambda_n \gg G_T/4\pi g$ which is difficult to realize simultaneously with the inequality $G_T^2 \gg 4\pi g$ needed to avoid trivial Coulomb blockade effects. Indeed, at energies $E \ll \hbar/\tau_{tr}$, Cooper interaction constant $\lambda(E)$ is determined by the RG equation [14] which we present in a simplified form [23] valid for $\ln(\hbar/E\tau_{tr}) \ll g$ when renormalization of g can be neglected:

$$\frac{d\lambda}{d\zeta} = -\lambda^2 + \lambda_g^2, \quad \lambda_g = \frac{1}{2\pi\sqrt{g}}, \tag{7}$$

and $\lambda(\zeta = 0) = \lambda_n$. The fixed point solution of Eqn (7), $\lambda = \lambda_g$, is too small to fulfil both the above inequalities together. Therefore, typically the approximation of single-parameter RG for G_A is not valid, and we should reconsider the problem of the subgap N-S conductance in the presence of three different effects acting simultaneously:

(i) disorder-enhanced multiple Andreev reflections [25] which increase $G_A(\omega)$ when the coherence length $\sqrt{D/\omega}$ grows;

(ii) Cooper-channel repulsion λ which reduces G_A [18, 26, 27];

(iii) quantum fluctuations of the phase $\theta(\tau)$ which destroy coherence between Andreev reflections and suppress $G_A(\zeta)$ at long time scales.

To treat all these effects together, we employ the functional RG method for the proximity-effect action in the Keldysh form [26].

Like in the simplified model [18] discussed above, the constant $\mathcal{C}(0)$ is determined (with exponential accuracy) by the value of time t^* when $G_A(\zeta = \ln \omega_d t^*)$ becomes of the order of 1, since at longer times $C_0(t)$ decays fast. However, the equation for $G_A(\zeta)$ is much more complicated now as it includes an infinite set of parameters. To derive the corresponding RG equations, we start from the Keldysh action for a SC island in contact with a disordered metal, derived in Ref. [23]. It can be represented as a sum $S = S_{bulk} + S_{bound}$ of the bulk and boundary (the last term in Eqn (8)) contributions:

$$S = \frac{i\pi\nu}{4}\mathrm{Tr}\Big[D(\nabla Q)^2 + 4i\big(i\tau_z\partial_t + \overleftrightarrow{\varphi} + \overleftrightarrow{\Delta}\big)Q\Big] + \mathrm{Tr}\,\boldsymbol{\varphi}^T V^{-1}\boldsymbol{\varphi} + \frac{2\nu}{\lambda}\mathrm{Tr}\,\boldsymbol{\Delta}^+\sigma_x\boldsymbol{\Delta} - \frac{i\pi G_T}{4}\mathrm{Tr}\,Q_S Q\,. \tag{8}$$

The bulk action, S_{bulk}, is a functional of three fluctuating fields: the matter field $Q(\mathbf{r}, t, t')$ in the film (its average value gives the time-domain representation of the electron Green function $G(\mathbf{r}, \mathbf{r}')$ at $\mathbf{r} = \mathbf{r}'$), the electromagnetic potential $\boldsymbol{\varphi}(\mathbf{r}, t)$, and the order-parameter field $\boldsymbol{\Delta}(\mathbf{r}, t)$ used to decouple the quartic interaction vertex in the Cooper channel. $Q(\mathbf{r}, t, t')$ is a matrix in the time domain, and in the direct 4×4-dimensional product $K \otimes N$ of the Keldysh and Nambu–Gor'kov spaces. Pauli matrices in the K and N spaces are denoted by σ_i and τ_i, respectively. The field Q satisfies a nonlinear constraint $Q^2 = 1$ and can be parametrized as $Q = e^{-W/2}\Lambda e^{W/2}$ with $\{W, \Lambda\} = 0$, where $\Lambda = \Lambda_0\tau_z$ is the metallic saddle point and

$$\Lambda_0(\epsilon) = \begin{pmatrix} 1 & 2F(\epsilon) \\ 0 & -1 \end{pmatrix}_K. \tag{9}$$

The matrix $F(\epsilon) = \tau_0 f(\epsilon) + \tau_z f_1(\epsilon)$ has the meaning of a generalized distribution function. The object $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^T$ is a vector in the Keldysh space, with φ_1, φ_2 being the classical and quantum components of the φ-field. $\overleftrightarrow{\varphi}$ is a shorthand notation for the matrix $\overleftrightarrow{\varphi} = \varphi_1\sigma_0 + \varphi_2\sigma_x$. Similarly, $\boldsymbol{\Delta} = (\Delta_1, \Delta_2)^T$, and $\overleftrightarrow{\Delta}$ stands for a 4×4 matrix

$$\overleftrightarrow{\Delta} = [\tau_+\Delta_1 - \tau_-\Delta_1^*]\sigma_0 + [\tau_+\Delta_2 - \tau_-\Delta_2^*]\sigma_x\,,$$

where $\tau_\pm \equiv (\tau_x \pm i\tau_y)/2$. In terms of the σ-model action (8), diffuson and Cooperon collective modes of the electron system are described as slow fluctuations of the Q-matrix over the manifold $Q^2 = 1$. The last (boundary) term in Eqn (8) describes an elementary tunneling process between the SC island and the N metal. The matrix Q_S describes the state of the SC island. At the low-energy scales $\epsilon \ll |\Delta_{SC}|$ it is expressed via the phase $\theta(t)$:

$$Q_S = -i\tau_+ \exp(i\overleftrightarrow{\theta}) + i\tau_- \exp(-i\overleftrightarrow{\theta})\,, \tag{10}$$

where $\overleftrightarrow{\theta} = \theta_1\sigma_0 + \theta_2\sigma_x$.

The action (8) contains a fluctuating scalar potential field $\boldsymbol{\varphi}$ accounting for the direct Coulomb interaction in the density–density channel. Major effects of this interaction are: (i) local electroneutrality of electron liquid at low frequencies, and (ii) zero-bias anomaly in the tunneling DOS [28]. Both effects can be taken care of by means of a special gauge transformation [29]:

$$Q_{tt'} \to \exp[i\overleftrightarrow{K}(t)\tau_z]Q_{tt'}\exp[-i\overleftrightarrow{K}(t')\tau_z]\,,$$

and $\boldsymbol{\varphi}(t) \to \boldsymbol{\varphi}(t) + \partial_t\mathbf{K}(t)$. The 'Coulomb phase' $\mathbf{K}(t)$ is a linear functional of $\boldsymbol{\varphi}(t)$ chosen in a way to cancel all terms which are linear both in $\boldsymbol{\varphi}(t)$ and in the matrix field $W(\mathbf{r}, t, t')$. Then the effect (i) is contained in the tree level of the transformed effective action, whereas (ii) comes from the simplest loop correction [23, 29]. After the above gauge transformation the phase $\boldsymbol{\theta}_j(t)$ of the j-th island enters the action in the combination $\boldsymbol{\theta}_j(t) - 2\mathbf{K}(t, \mathbf{r}_j)$ only. Now the important point of our discussion comes about: the phases $\boldsymbol{\theta}_j(t)$ of each island are not fixed by any external source, and should be integrated out. Thus the shift of integration variable $\boldsymbol{\theta}_j(t) \to \boldsymbol{\theta}_j(t) - 2\mathbf{K}(t, \mathbf{r}_j)$ eliminates $\mathbf{K}(t, \mathbf{r}_j)$ from the action, together with both effects (i) and (ii). In other terms, the present problem of unconstrained phase $\boldsymbol{\theta}(t)$ fluctuations can be treated as if it would be no Coulomb interaction, since Gaussian terms in the action containing electric field are decoupled from the redefined $\boldsymbol{\theta}(t)$ variable. It is thus legitimate to neglect electroneutrality and calculate frequency-dependent subgap conductance $G_A(\omega)$ as if the outer normal contact would be placed at the distance $R_\omega = \sqrt{D/\omega}$ from the SC island. We emphasize that the same would be wrong for a usual problem of N-S conductance between contacts with fixed voltages, where the full size of the N film, $L \gg R_\omega$, does enter the result, adding the term $(R_\square/2\pi)\ln(L/R_\omega)$ into the resistance, cf. Section VI of Ref. [26].

Next we use the RG method to integrate consecutively over fast degrees of freedom in the action (8), which is defined with ω_d being a high-energy cutoff. At each step of the RG procedure one has to eliminate fast modes $W_{\epsilon_1\epsilon_2}$ in the N film, with either $\Omega^* > \max(Dq^2, \epsilon_1 - \epsilon_2) > \Omega$ (for diffusons) or $\Omega^* > \max(Dq^2, \epsilon_1 + \epsilon_2) > \Omega$ (for Cooperons), and fast (with $\Omega^* > \omega > \Omega$) fluctuations of the order-parameter phase $\boldsymbol{\theta}$ on the SC island (where Ω is the running infrared RG cutoff). The above integration results in a correction to the action of slow variables proportional to the increment of the 'logarithmic time' $\Delta\zeta = \ln(\Omega^*/\Omega)$. The

structure of the boundary term in the action (8) is not reproduced under the RG [23], instead higher-order terms $\mathrm{Tr}(Q_S Q)^n$ are generated, which are all relevant in the case of strong proximity effect. The full boundary action can be written in the form [26]

$$S_{\mathrm{bound}} = \sum_{n=1}^{\infty} S_n = -\mathrm{i}\pi^2 g \sum_{n=1}^{\infty} \gamma_n(\zeta)\, \mathrm{Tr}(Q_S Q)^n \,. \tag{11}$$

At the energy scale ω_d, the multicharge action (11) reduces to the last term in Eqn (8), that is, $\gamma_1(0) = G_T/4\pi g$ and $\gamma_{n\geq 2}(0) = 0$. Integration over all fast modes down to the energy scale $\omega = \omega_d \exp(-\zeta)$ defines the action (11) with the set of parameters $\gamma_n(\zeta)$.

In the model [18] of large λ_n discussed above, separation of scales was possible: at relatively short time scales the term $S_2 \propto \gamma_2 \,\mathrm{Tr}\,(Q_S Q)^2$ was generated under the action of the RG [23], that lead to a constant value of $G_A \gg 1$ (other $\gamma_{n\geq 3}$ were still small). At longer time scales fluctuations of θ became important, being determined by the action (6). In the full problem considered now, all parameters γ_n are important, and all types of fluctuations should be considered simultaneously. The corresponding RG equations for the 'charges' $\gamma_n(\zeta)$ had been derived in Ref. [26] for the case when the SC island is connected to an external circuit and its phase is fixed by an external bias V through the Josephson relation $\mathrm{d}\theta/\mathrm{d}t = 2eV$. In the absence of an external contact, all phases do fluctuate and these fluctuations lead to an additional logarithmic renormalization of the action.

To calculate the corresponding correction one has to average the action (11) over high-frequency fluctuations of the phase $\boldsymbol{\theta}$. Its propagator is determined by the Gaussian part of the same action and has the form

$$\langle \theta_i(\omega)\theta_j(-\omega)\rangle = \frac{2\Pi_\omega^{ij}}{\omega G_A(\zeta_\omega)} \,, \tag{12}$$

where $G_A(\zeta_\omega) = 16\pi g \sum_{n=1}^{\infty}(-1)^n n^2 \gamma_{2n}(\zeta_\omega)$ is the scale-dependent Andreev conductance, $\zeta_\omega = \ln(\omega_d/\omega)$, and

$$\Pi_\omega = \begin{pmatrix} 2\coth(\omega/2T) & 1 \\ -1 & 0 \end{pmatrix} . \tag{13}$$

Let us consider the n-th term of the action (11), $S_n = -\mathrm{i}\pi^2 g \gamma_n \,\mathrm{Tr}(Q_S Q)^n$, and find the one-loop correction from it, $\Delta^{(n)}\gamma_k$, to the parameters γ_k due to phase fluctuations with frequencies ω in the range $\Omega < |\omega| < \Omega_*$. This correction will be proportional to $\Delta\zeta = \ln(\Omega_*/\Omega)$. To find it, one has to separate $\overleftrightarrow{\boldsymbol{\theta}}$ into the sum of the slow part $\overleftrightarrow{\boldsymbol{\theta}}''$ (with frequencies $|\omega| < \Omega$), and the fast part $\overleftrightarrow{\boldsymbol{\theta}}'$ (with $\Omega < |\omega| < \Omega_*$), expand S_n to the second order in $\overleftrightarrow{\boldsymbol{\theta}}'$ and average over it with the correlator (12), taking into account all possible pairings. There are two types of pairing: two fast variables $\overleftrightarrow{\boldsymbol{\theta}}'$ can belong to the same matrix Q_S (cf. Eqn (10)), or to two different Q_S. The first case is simple: on averaging over $\overleftrightarrow{\boldsymbol{\theta}}'$ each of the terms S_n reproduces its structure, and the whole effect of fluctuations is to modify γ_n:

$$\Delta^{(n)}\gamma_n = -2\frac{n\gamma_n \Delta\zeta}{\pi G_A(\zeta)} \,.$$

The pairing of the second type involves averaging over the fast variables $\overleftrightarrow{\boldsymbol{\theta}}'(t_1)$ and $\overleftrightarrow{\boldsymbol{\theta}}'(t_2)$ which belong to different Q_S. Therefore these two Q_S subdivide the whole cyclic product $(Q_S Q)^n$ under the trace into two parts:

$$\mathrm{Tr}(Q_S Q)^n = \mathrm{Tr}\, Q_S(t_1) A(t_1,t_2) Q_S(t_2) B(t_2,t_1) \,,$$

where $A = Q(Q_S Q)^p$, $B = Q(Q_S Q)^{n-p-2}$, and p is an integer number which distinguishes different pairings. An example with $n = 8$ and $p = 5$ is shown in Fig. 1. For certainty we will assume that the fast energy ω runs over the matrix A (the opposite situation can be treated analogously by replacing $p \leftrightarrow n - p - 2$). Within the RG precision, fast bulk fields Q entering the product A can be substituted by $\Lambda(\omega) = \Lambda_0(\omega)\tau_z$ (we will explain this below) while slow matrices Q_S entering A can be taken at coinciding times $t \approx t_1 \approx t_2$.

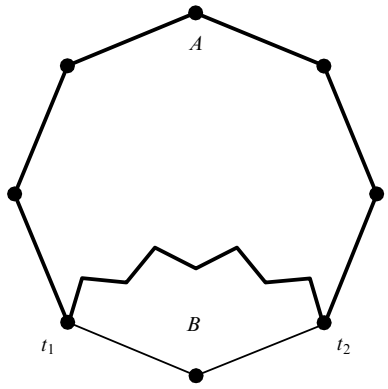

Figure 1. An example of averaging $\langle S_n\rangle$ over phase fluctuations: $n = 8$, $p = 5$ (see text for details). Dots denote $Q_S(t)$, straight lines denote $Q(t,t')$, and the zigzag line stands for the correlator (12) of fast phases taken from $Q_S(t_1)$ and $Q_S(t_2)$. The fast energy ω runs over the zigzag line and the part A of the diagram, the corresponding elements being shown by thick lines.

Averaging is performed with the help of Eqn (10) and the following algebraic relations (which can be proved by direct calculation):

$$\Big\langle \overleftrightarrow{\boldsymbol{\theta}}' \exp(\pm\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\Lambda_0 \Big[\exp(\mp\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\Lambda_0 \exp(\pm\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\Lambda_0\Big]^k \times \exp(\mp\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\overleftrightarrow{\boldsymbol{\theta}}'\Big\rangle = 0 \,, \tag{14}$$

$$\Big\langle \overleftrightarrow{\boldsymbol{\theta}}' \Big[\exp(\pm\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\Lambda_0 \exp(\mp\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\Lambda_0\Big]^k \times \exp(\pm\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'')\overleftrightarrow{\boldsymbol{\theta}}'\Big\rangle = \frac{4\Delta\zeta}{\pi G_A}\exp(\pm\mathrm{i}\overleftrightarrow{\boldsymbol{\theta}}'') \,. \tag{15}$$

According to these equations, only pairings with odd $p = 2k - 1$ produce a nonzero contribution. For them we have

$$\mathcal{N}_k \equiv \langle Q_S \Lambda (Q_S \Lambda)^{2k-1} Q_S\rangle = (-1)^{k+1}\frac{4\Delta\zeta}{\pi G_A} Q_S \,, \tag{16}$$

where fast $\overleftrightarrow{\boldsymbol{\theta}}'$ are taken from the first and the last Q_S under the correlator, and the sign $(-1)^k$ results from commutation of Pauli matrices in the Nambu space. Then, collecting all combinatorial factors, one gets

$$\Delta S = (-1)^k \mathrm{i} n\gamma_n \frac{4\pi g \Delta\zeta}{G_A}\mathrm{Tr}(Q_S Q)^{n-2k} \,.$$

We see that the paring with $p = 2k - 1$ modifies S_{n-2k} with $\Delta^{(n)}\gamma_{n-2k} = (-1)^{k+1} 4n\gamma_n \Delta\zeta/\pi G_A(\zeta)$. Thus, under the action

of the RG, the term S_n generates the descending series of terms: S_n, S_{n-2}, S_{n-4}, ... The RG equation for the coefficient γ_n reads

$$\Delta\gamma_n = -\frac{2\Delta\zeta}{\pi G_{\rm A}(\zeta)}\left[n\gamma_n + 2\sum_{k=1}^{\infty}(-1)^k(n+2k)\gamma_{n+2k}\right]. \quad (17)$$

Finally, we have to explain the validity of the substitution $Q \mapsto \Lambda$ in the 'fast' product A. Such reduction implies that no slow diffusons or Cooperons could be connected to the part A of the diagram on further steps of the RG. This statement is evident for Cooperons, but not for diffusons. Indeed, Cooperons depend on the fast frequency $\epsilon_1 + \epsilon_2 = 2\omega$, whereas diffusons depend on $\epsilon_1 - \epsilon_2$ which can be slow, and thus might contribute to the RG equation on further steps. Actually, however, they do not. The situation here is very similar to the one described during the derivation of the λ term in the RG, cf. Section V of Ref. [26]: each diagram containing a diffuson in the 'fast' product A produces several nonzero contributions, but their sum vanishes.

To prove this cancellation property, it is sufficient to consider the case when one Q from the part A is connected to some other object O by a diffuson, cf. Fig. 2. Since the pairing between S_n and the object O is of diffusonic character, the latter ought to have the form $O \propto {\rm Tr}(Q_{\rm S}\Lambda)^{2m}W$ with even number of $Q_{\rm S}$. Pairing is performed with the help of the contraction rules from Ref. [26] and yields two terms:

$${\rm Tr}\, Q_{\rm S}(t_1)BQ_{\rm S}(t_2)\left[(\Lambda Q_{\rm S})^{2k+2m-1} - (\Lambda Q_{\rm S})^{2k-2m-1}\right]\Lambda. \quad (18)$$

To obtain the final result, the expression (18) should be averaged over phase fluctuations. Note that Eqn (16) is valid for any integer k. Therefore, averaging (18) with the help of Eqn (16) yields $\mathcal{N}_{k+m} - \mathcal{N}_{k-m} = 0$, q. e. d.

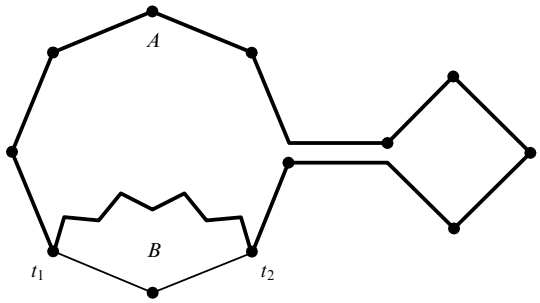

Figure 2. An example of diffuson pairing of a matrix Q from the fast part A of the diagram shown in Fig. 1 to the object ${\rm Tr}(Q_{\rm S}\Lambda)^4 W$. The double line denotes the diffuson propagator.

To simplify the RG equation, we introduce, following Ref. [26], a function of an auxiliary continuous variable x according to the definition $u(x,\zeta) = \sum_{n=1}^{\infty} n\gamma_n(\zeta)\sin nx$. Then the full RG equation for the function $u(x,\zeta)$ reads:

$$u_\zeta + uu_x + \lambda(\zeta)\, u\left(\frac{\pi}{2},\zeta\right)\sin x = -\frac{2}{\pi G_{\rm A}(\zeta)}\mathcal{F}[u(x,\zeta)], \quad (19)$$

where $\mathcal{F}[u(x,\zeta)] \equiv [u(x,\zeta)\tan x - u(\pi/2,\zeta)\sec x]_x$, and the initial condition is $u(x,0) = (G_{\rm T}/4\pi g)\sin x$. The functional RG equation (19) with zero R.H.S. was derived in Ref. [26]. In the present problem the R.H.S. describes the effect of phase fluctuations; it is obtained as the Fourier-transform of Eqn (17). The scale-dependent subgap conductance $G_{\rm A}(\zeta)$ is determined by the solution of Eqn (19) as $G_{\rm A}(\zeta) = 4\pi g u_x(\pi/2,\zeta)$.

To find the parameter $\mathcal{C}(0)$ with exponential accuracy, we integrate Eqn (19) together with Eqn (7) for $\lambda(\zeta)$. Written in the rescaled variables $s = \zeta/2\pi\sqrt{g}$, $w(x,s) = 2\pi\sqrt{g}u(x,\zeta)$, and $\tilde{\lambda} = \lambda/\lambda_g$, Eqn (19) acquires the form

$$w_s + ww_x = -2\frac{\mathcal{F}[w(x,\zeta)]}{w_x(\pi/2,s)} - \tilde{\lambda}(s)w\left(\frac{\pi}{2},s\right)\sin x, \quad (20)$$

with the initial condition $w(x,0) = A\sin x$, where $A \equiv G_{\rm T}/2\sqrt{g} \gg 1$. The solution of Eqn (20) weakly depends on the ratio $\tilde{\lambda}(0) = \lambda_n/\lambda_g$ which is assumed to be not very large. At $s \ll 1$, the function $w(x,s)$ is close to the solution of Eqn (20) with zero R.H.S., which, at $s \geq A^{-1}$, is given by $w(x,s) \approx x/s$ for $x \in (0,\pi)$. As s grows, the R.H.S. terms become increasingly important, and eventually reduce $G_{\rm A}(\zeta) = 2\sqrt{g}w_x(\pi/2,s)$ down to the value of the order of 1 at the critical value of $\zeta^* = 2\pi\sqrt{g}s_{\rm c}$. The value of $s_{\rm c} \sim 1$ was determined, for several values of $\tilde{\lambda}(0)$, via numerical solution of Eqn (20) in the limit $A \to \infty$:

$$s_{\rm c} = \begin{cases} 1.25, & \text{for } \tilde{\lambda}(0) = 0, \\ 1.17, & \text{for } \tilde{\lambda}(0) = 1, \\ 1.13, & \text{for } \tilde{\lambda}(0) = 2. \end{cases} \quad (21)$$

As it was explained above, the value of $\zeta^* = 2\pi\sqrt{g}s_{\rm c}$ determines $\mathcal{C}(0)$ with exponential accuracy:

$$\mathcal{C}(0) = \frac{\mathcal{B}}{\omega_d}\exp(2\pi\sqrt{g}s_{\rm c}), \quad (22)$$

where pre-exponential factor $\mathcal{B}$ is left undetermined (to find it, the two-loop RG calculation would be necessary). Combining Eqns (2), (5) and (22), we find that macroscopic superconductivity in the array is destroyed when the distance b between neighboring islands becomes larger than the critical value $b_{\rm c}(g)$ defined as

$$\frac{b_{\rm c}(g)}{d} \approx \frac{\pi^2}{2}\left(\frac{\mathcal{B}\sqrt{g}}{\pi s_{\rm c}}\right)^{1/2}\exp(\pi\sqrt{g}s_{\rm c}). \quad (23)$$

Finally, inverting relation (23) one finds the result (1) for the critical film conductance $g_{\rm c}(b)$, with

$$\mathcal{G}_{\rm c} = \frac{1}{s_{\rm c}^2} = \begin{cases} 0.64, & \text{for } \tilde{\lambda}(0) = 0, \\ 0.73, & \text{for } \tilde{\lambda}(0) = 1, \\ 0.79, & \text{for } \tilde{\lambda}(0) = 2. \end{cases} \quad (24)$$

nd $\tilde{d} = d\pi^{3/2}\mathcal{B}^{1/2}g^{1/4}/2s_{\rm c}^{1/2}$.

At $g > g_{\rm c}(b)$ macroscopic superconductive transition occurs at $T > 0$. Close to the critical point (1), at $b \leq b_{\rm c}(g)$, the transition temperature is primarily determined by the temperature dependence of $\mathcal{J}(T)$ which is reduced compared to $\mathcal{J}(0)$ since couplings $J(r)$ are strongly suppressed at $r \geq L_{\rm T}$. At low temperatures

$$\mathcal{J}(T) = \mathcal{J}(0)\frac{\ln(L_{\rm T}/b)}{\ln(L_{\rm T}/d)}. \quad (25)$$

The same expression, with L_T replaced by $l_H = \sqrt{\pi\hbar c/eH}$, determines $\mathcal{J}(0, H)$ in the presence of a weak transverse magnetic field H. To find the critical temperature $T_c(b)$ and the $T = 0$ value of the upper critical field H_{c2}, one uses Eqn (2) together with Eqns (22), (25). The result is that both $T_c(g)$ and $H_{c2}(g)$ scale in the same way, and drop fast at $b \to b_c(g)$:

$$\ln\frac{T^*}{T_c} \approx \ln\frac{\Phi_0}{H_{c2}b^2} \approx \frac{2\ln(b/d)}{b_c^2(g)/b^2 - 1}\,, \qquad (26)$$

where $T^* = \hbar D/b^2$, and $\Phi_0 = hc/2e$ is the flux quantum. The form of Eqn (26) suggests that the behavior of the array near the quantum critical point $b = b_c(g)$ can be understood in terms of the standard BCS-type theory with a *macroscopic* effective Cooper attraction λ_{eff}, which vanishes at $b \to b_c(g)$. Equation (26) is valid for $b/b_c(g) \geq [2\ln(b/d)]^{-1/2}$. This inequality ensures that T_c is small compared both to T^* (under this condition the proximity coupling is long-range) and to $\hbar/\mathcal{C}(0)$. The latter condition allows one to approximate $\mathcal{C}(T)$ by $\mathcal{C}(0)$ while deriving Eqn (26). At shorter $b \ll b_c(g)\sqrt{\lambda_g}$, the transition occurs at $T_c \sim T^*$. Here $L_{T_c} \sim b$, the MFA is not applicable and the transition is governed by thermal fluctuations. Similarly, the characteristic scale of magnetic field which affects superconductive state in such an array is just Φ_0/b^2: at higher fields formation of a superconductive glass state is expected. Additional limitation for our results from the higher-T side is due to the neglect of quasi-particles inside SC islands, the corresponding temperature scale being $T_{\text{parity}} = \Delta_{SC}/\ln(\nu V_i \Delta_{SC})$. On the other hand, the conclusion about metallic nature of the system's state at $b > b_c(g)$ is limited to the temperature scale $T \geq T_{\text{loc}} \sim \hbar\omega_d \exp(-\pi^2 g)$, since we neglected weak localization effects (the value of g entering this estimate is defined at the length scale of the order of d). Determination of weak-localization corrections to the conductance of an array with $b > b_c(g)$ would need the account of dephasing effects due to fluctuations of island's phases. This is an interesting problem which we left for the future studies.

Finally, we comment briefly on the similar problem of small superconductive grains of radius d immersed into a 3D metal with bulk resistivity ρ. In this case the simple method of Ref. [18] can be used. The characteristic Coulomb energy $\hbar/\mathcal{C}(0) \approx \hbar\omega_d \exp(-\pi\hbar/4e^2 R_A)$ in the tunneling limit $R_T \gg R_N$; here $R_A = R_T^2/R_N$ and $R_N = \rho/4\pi d$. Proximity coupling energy is estimated as $J(r) \sim (\hbar/e^2 R_T)^2/r^3\nu$. The point of the SC-M transition is then given by $\hbar\rho/16e^2 d R_T^2 \cong 3\ln b/d$.

In conclusion, we developed a theory of quantum superconductive-metal transition in a 2D proximity-coupled array. The critical resistance $R_{\Box c}$ is non-universal and small compared to the quantum resistance. Near the quantum critical point the system behaves as a BCS-like superconductor with the effective Cooper attraction constant vanishing at $R_\Box \to R_{\Box c}$.

We are grateful to A Kamenev and Yu V Nazarov for useful discussions. This research was supported by the NSF grant DMR-9812340 (A I L), RFBR grant 98-02-16252, NWO – Russia collaboration grant, Swiss NSF – Russia collaboration grant 7SUPJ062253.00 and by the Russian Ministry of Industry, Science and Technology via the project "Mesoscopic electron systems for quantum computing" (M V F and M A S).

References

1. Fisher M P A *Phys. Rev. Lett.* **65** 923 (1990)
2. Goldman A M, Markovic N *Phys. Today* **51** (11) 39 (1998)
3. van der Zant H S J et al. *Phys. Rev. Lett.* **69** 2971 (1992); *Phys. Rev. B* **54** 10081 (1996)
4. Gantmakher V F et al. *Pis'ma Zh. Eksp. Teor. Fiz.* **71** 231 (2000) [*JETP Lett.* **71** 160 (2000)]
5. Chervenak J A, Valles J M, Jr. *Phys. Rev. B* **61** R9245 (2000)
6. Mason N, Kapitulnik A, cond-mat/0006138
7. Efetov K B *Zh. Eksp. Teor. Fiz.* **78** 2017 (1980) [*Sov. Phys. JETP* **51** 1015 (1980)]
8. Fazio R, Schön G *Phys. Rev. B* **43** 5307 (1991)
9. Delsing P et al. *Phys. Rev. B* **50** 3959 (1994)
10. Jaeger H M et al. *Phys. Rev. B* **40** 182 (1989)
11. Haviland D B, Liu Y, Goldman A M *Phys. Rev. Lett.* **62** 2180 (1989)
12. Gantmakher V F et al. *Pis'ma Zh. Eksp. Teor. Fiz.* **64** 713 (1996) [*JETP Lett.* **64** 767 (1996)]
13. Hebard A F, Paalanen M A *Phys. Rev. Lett.* **65** 927 (1990)
14. Finkel'stein A M *Pis'ma Zh. Eksp. Teor. Fiz.* **45** 37 (1987) [*JETP Lett.* **45** 46 (1987)]; *Sov. Sci. Reviews Sect. A Phys. Rev.* (Ed. I M Khalatnikov) **14** (Pt. 2) 3 (1990)
15. Finkel'stein A M *Physica B* **197** 636 (1994)
16. Larkin A I *Ann. Phys.* (Leipzig) **8** 507 (1999)
17. Schmid A *Phys. Rev. Lett.* **51** 1506 (1983); Bulgadaev S A *Pis'ma Zh. Eksp. Teor. Fiz.* **39** 264 (1984) [*JETP Lett.* **39** 315 (1984)]
18. Feĭgel'man M V, Larkin A I *Chem. Phys.* **235** 107 (1998)
19. Spivak B Z, Zyuzin A, Hruska M, cond-mat/0004058
20. Aslamazov L A, Larkin A I, Ovchinnikov Yu N *Zh. Eksp. Teor. Fiz.* **55** 323 (1968) [*Sov. Phys. JETP* **28** 171 (1969)]
21. Vaks V G, Larkin A I, Pikin S A *Zh. Eksp. Teor. Fiz.* **53** 313 (1967) [*Sov. Phys. JETP* **26** 188 (1968)]
22. Usadel K *Phys. Rev. Lett.* **25** 507 (1970); Nazarov Yu V *Phys. Rev. Lett.* **73** 1420 (1994)
23. Feĭgel'man M V, Larkin A I, Skvortsov M A *Phys. Rev. B* **61** 12361 (2000)
24. Kosterlitz J M *Phys. Rev. Lett.* **37** 1577 (1976)
25. Hekking F W J, Nazarov Yu V *Phys. Rev. B* **49** 6847 (1994)
26. Skvortsov M A, Larkin A I, Feĭgel'man M V *Phys. Rev. B* **63** 134507 (2001)
27. Zhou F et al. *Phys. Rev. B* **52** 4467 (1995); Stoof T H, Nazarov Yu V *Phys. Rev. B* **53** 14496 (1996)
28 Al'tshuler B L, Aronov A G, Lee P A *Phys. Rev. Lett.* **44** 1288 (1980)
29. Kamenev A, Andreev A F *Phys. Rev. B* **60** 2218 (1999)

Fluctuation effects in high sheet resistance superconducting films

J M Valles Jr., J A Chervenak, S-Y Hsu, T Kouh

Abstract. As the normal-state sheet resistance, R_n, of a thin-film superconductor increases, its superconducting properties degrade. For $R_n \simeq h/4e^2$ superconductivity disappears and a transition to a nonsuperconducting state occurs. We present electron tunneling and transport measurements on ultrathin, homogeneously disordered superconducting films in the vicinity of this transition. The data provide strong evidence that fluctuations in the amplitude of the superconducting order parameter dominate the tunneling density of states and the resistive transitions in this regime. We briefly discuss possible sources of these amplitude fluctuation effects. We also describe

J M Valles Jr., J A Chervenak, T Kouh Department of Physics, Brown University, Providence, RI USA
S-Y Hsu Department of Electrophysics, National Chiao Tung University, Hsinchu 300, Taiwan