REVIEWS OF TOPICAL PROBLEMS

Gravitational properties of cosmic strings

B É Meĭerovich¶

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<u>Abstract.</u> The gravitational properties of gauge and global relativistic cosmic strings in Abelian Higgs and scalar field models are presented. A complete classification of the strings is given and the ranges of the parameters allowing static configurations are determined. Gravitational properties of cosmic strings in the relevant limiting cases are treated analytically.

1. Introduction

There has been an upsurge of interest in cosmic strings and other topological defects in the last two decades: on the one hand, these defects may play a crucial role in the evolution of the Universe and, on the other, their physical properties differ from those of ordinary matter. The reader interested in learning more about the broad spectrum of the properties of topological defects and about their possible role in the Universe is referred to the monograph by Vilenkin and Shellard [1]. The present review is dedicated to the specific topic of the gravitational properties of relativistic cosmic strings in the Abelian Higgs model.

According to the Standard cosmological model [2], the Universe has been expanding and cooling from a split second after the Big Bang to the present — and remained uniform

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Received 15 January 2001, revised 23 April 2001 Uspekhi Fizicheskikh Nauk **171** (10) 1033–1049 (2001) Translated by E G Strel'chenko; edited by A Radzig and isotropic overall in doing so. In the process of its evolution, the Universe has gone through a chain of phase transitions, including the Grand Unification $(10^{-35} \text{ s after the Big Bang})$, the electroweak phase transition (10^{-11} s) , the formation of neutrons and protons from quarks (10^{-6} s) , recombination $(4 \times 10^5 \text{ s})$, and so forth.

A state above the phase transition temperature usually possesses a higher symmetry than a state below. Upon cooling, a spontaneous breaking of symmetry occurs during the phase transition. Regions with spontaneously broken symmetry, which are more than the correlation length apart, are statistically independent. At the interfaces between these regions, so-called topological defects necessarily arise. The particular types of defects — domain walls, strings, monopoles, or textures — are determined by the topological properties of the vacuum [3]. The domain wall — a transition zone at the interface between two domains in a ferromagnet [4] — is a classical example of a topological defect.

The fundamental role of symmetry breaking in phase transitions was elucidated by Landau [5]. Historically, the notion of a spontaneous breaking of symmetry accompanied by the formation of topological defects arose and has been developed for studying phase transitions in condensed media. In this context, the precursors (and nonrelativistic analogues) of cosmic strings are magnetic vortices in type II superconductors [6] and quantized vortex lines in superfluid ⁴He [7].

Spontaneous breaking of symmetry plays a fundamental role in the modern theory of elementary particles. The symmetry in this context may be 'internal', i.e. not necessarily associated with space-time transformations. The Grand Unification symmetry and the electroweak and isotopic symmetries are examples of such internal symmetries.

A decisive step toward the application of the concept of spontaneous breaking of symmetry to cosmology was taken in 1972 by Kirzhnits [8]. He assumed that, as is the case for solid substances, also in field theory a spontaneously broken symmetry may be restored at a sufficiently high temperature. Thus it is believed that early in the evolution of the Universe, as long as the temperature was sufficiently high, there existed a symmetry between electroweak and strong interactions. As the Universe expanded and cooled, the 'Grand Separation' phase transition took place, with the result that this symmetry was spontaneously broken and the unified interaction broke down into the strong and electroweak interactions.

In 1974, Weinberg [9] suggested that domain walls may have formed at phase transitions early in the evolution of the Universe. The first quantitative analysis of the cosmological consequences of spontaneous breaking of symmetry was given by Zel'dovich, Kobzarev, and Okun' [10]. The increased interest in cosmic strings is due in large part to the works of Zel'dovich [11] and Vilenkin [12] who analyzed the possible role of cosmic strings as sources of the fluctuations that later led to the formation of galaxies.

The theory of the Big Bang raises hopes that the experimental check of new theories may be extended to Planck energies, an energy range totally inaccessible under laboratory conditions. This implies that the classical general relativity must be extrapolated to distances of order 10^{-33} cm, even though the gravitational interaction has been checked experimentally only to fractions of a centimeter. The quantum theory of gravity has not yet been constructed. Under these conditions, of particular significance are exact analytical solutions of Einstein equations; in spite of the complexity of the equations, such solutions prove to be possible for a number of problems. One example is the problem of a relativistic static cosmic string in the Abelian Higgs model.

The physical properties of topological defects differ greatly from those normally seen in ordinary matter. For example, the gravitational mass of a global string is negative, implying that its interaction with matter is repulsive. Cosmic strings have become of major importance in connection with the topological inflation idea [13-15].

The relativistic theory of gravity has stringlike solutions with a Kasner outer metric, which were absent in the nonrelativistic theory. The presence of a singularity in the Kasner metric prompts a temptation to declare these solutions nonphysical [1]. In my view, one should not hasten to do this, however. It is unclear at present precisely which phase transitions caused spontaneous breaking of symmetry during the inflation period. The lack of knowledge about the physical mechanisms of inflation does not allow a unique interpretation of a string-related metric singularity.

A theory based on the Abelian Higgs model is a macroscopic one. In the absence of a microscopic theory, the physical meaning of the order parameter is not specified. This is reminiscent of the situation when the theory of superconductivity was in the making. While in the early 1950s the Ginzburg–Landau macroscopic theory of superconductivity already existed [16], nothing was yet known about Cooper pairing [17], the phenomenon lying at the basis of the microscopic theory of superconductivity.

2. Abelian Higgs model

Among the variety of theories with spontaneous breaking of symmetry, gauge theories are of central importance. Historically, the first example of a gauge theory with a spontaneous breaking of symmetry is the Ginzburg-Landau phenomenological theory of superconductivity [16]. The relativistic extension of this theory to cosmic strings is often referred to in the scientific literature as the Abelian Higgs model.

To take into account the mutual influence of a topological defect and the metric against each other, the total Lagrangian density \mathcal{L}_{tot} of the system is written as the sum

$$\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{grav}} \,. \tag{1}$$

Here,

$$\mathcal{L} = \overline{\mathcal{D}}_{\mu} \overline{\phi} \mathcal{D}^{\mu} \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
⁽²⁾

is the Ginzburg-Landau Lagrangian density written in a general-covariant form, and

$$\mathcal{L}_{\rm grav} = -\frac{1}{16\pi G} R \sqrt{-g}$$

is the Lagrangian density of the gravitational field (where *R* is the scalar spacetime curvature, and *g* the determinant of the metric tensor $g_{\mu\nu}$).

The complex order parameter ϕ in formula (2) is a scalar field also referred to as the Higgs field; $\mathcal{D}_{\mu} = \partial_{\mu} - ieA_{\mu}$ (*e* being the coupling constant); $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the antisymmetric tensor of the vector gauge field A_{μ} , and $V(\phi)$ is a potential allowing a spontaneous breaking of symmetry. The latter is most often taken to be of the 'phi-to-the-fourth' or 'sombrero' ('Mexican-hat potential') form

$$V(\phi) = \frac{1}{4} \lambda (\overline{\phi}\phi - \eta^2)^2 \tag{3}$$

(see Fig. 1). Here, λ is a dimensionless constant, and the quantity η with the dimension of energy specifies the typical energy scale of a spontaneous breaking of symmetry.

 $V(\phi$



Figure 1. 'Sombrero' potential (3).

Varying the Lagrangian (1) with respect to $g_{\mu\nu}$, ϕ and A_{μ} we obtain, respectively, the Einstein equations for the gravitational field $g_{\mu\nu}$, an equation for the wave function ϕ , and an equation for the gauge field A_{μ} . This simple model is invariant under the action of the group U(1) of the local gauge transformations

$$\phi(x) \to \exp\left(i\alpha(x)\right)\phi(x), \quad A_{\mu}(x) \to A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\alpha(x).$$
 (4)

The minimum of the potential $V(\phi)$ is attained at $|\phi| = \eta$. The expected vacuum value of the field ϕ is nonzero at this point, and the symmetry turns out to be spontaneously broken.

By choosing an appropriate gauge $A_{\mu}(x)$, the function ϕ can be made real. Then, for small perturbations near the

minimum:

$$\phi = \eta + \frac{\phi_1}{\sqrt{2}} \,, \qquad \phi_1 \ll \eta \,,$$

the Lagrangian (2) takes the form

$$\mathcal{L} = \frac{1}{2} \left(\hat{o}_{\mu} \phi_{1} \right)^{2} - \frac{1}{2} M_{\rm H}^{2} \phi_{1}^{2} - \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M_{\rm G}^{2} A_{\mu} A^{\mu} \right) + \mathcal{L}_{\rm int} , \qquad (5)$$

where \mathcal{L}_{int} contains cubic and higher-order terms in ϕ_1 and A_u .

The parameters

$$M_{\rm H} = \sqrt{\lambda} \eta , \qquad M_{\rm G} = \sqrt{2} \, e \eta , \qquad (6)$$

we introduced in Eqn (5), can be identified with the masses of the Higgs $(M_{\rm H})$ and gauge $(M_{\rm G})$ particles. The ratio of these masses

$$\beta = \frac{M_{\rm G}}{M_{\rm H}} = \frac{\sqrt{2}\,e}{\sqrt{\lambda}}\tag{7}$$

is a fundamental dimensionless parameter of the theory. In the theory of superconductivity [16], the parameter (7) is identical to the Ginzburg–Landau parameter to within a factor of $\sqrt{2}$: $\kappa = \beta/\sqrt{2}$, and ϕ is the Cooper pair wave function.

The second dimensionless parameter of the theory, viz.

$$\gamma = 8\pi G \eta^2 \,, \tag{8}$$

characterizes the magnitude of the gravitational field. In the natural system of units, where the major fundamental constants are equal to unity:

$$\hbar = c = k_{\rm B} = 1 , \qquad (9)$$

the Newtonian constant of gravitation $G = m_{\text{Pl}}^{-2}$, where $m_{\text{Pl}} = 1.22 \times 10^{19}$ GeV is the Planck mass. Thus, the parameter γ in formula (8) is proportional to the square of the ratio between the spontaneous symmetry breaking energy and the Planck mass.

The properties of a static, infinitely long string can be described by means of functions dependent on a single spatial coordinate x^1 — the 'distance' from the symmetry axis. Neither the time variable x^0 nor the two remaining space coordinates x^2 and x^3 enter the relevant equations.

In the Abelian Higgs model, the cosmic string is described by the vortex solution of the form

$$\phi(x^1, x^2) = \eta f(x^1) \exp(inx^2), \quad A_2(x^1) = -\frac{n}{e} \alpha(x^1).$$
 (10)

If the azimuthal coordinate x^2 varies from zero to 2π , the number *n* of flux quanta should be an integer. The state of the system (10) depends on x^2 , whereas the Lagrangian (2) does not (provided of course that the metric tensor $g_{\mu\nu}$ depends on x^1 alone):

$$\mathcal{L} = \eta^2 \left[g^{11} f^{\prime 2} + n^2 g^{22} (1-\alpha)^2 f^2 \right] - \frac{n^2}{2e^2} g^{11} g^{22} \alpha^{\prime 2} - V.$$
(11)

The string's energy-momentum tensor

$$T_{\mu}^{\nu} = -\delta_{\mu}^{\nu} \mathcal{L} + 2g^{\lambda\nu} \frac{\partial \mathcal{L}}{\partial g^{\mu\lambda}}$$
(12)

is diagonal, so that

$$T_0^0 = T_3^{-3} = -\mathcal{L},$$

$$T_1^{-1} = V - \frac{n^2}{2e^2} g^{11} g^{22} \alpha'^2 + \eta^2 \left[g^{11} f'^2 - n^2 g^{22} (1-\alpha)^2 f^2 \right],$$

$$T_2^{-2} = V - \frac{n^2}{2e^2} g^{11} g^{22} \alpha'^2 - \eta^2 \left[g^{11} f'^2 - n^2 g^{22} (1-\alpha)^2 f^2 \right].$$
(13)

The equality $T_0^0 = T_3^3$ ensures that the system is invariant under Lorentz translations along the string. By analogy with a static macroscopic body, for which $T_0^0 = \varepsilon$, $T_3^3 = -p$ (ε and p are the energy and pressure, respectively), the relation

$$p = -\varepsilon \tag{14}$$

is sometimes treated as the 'equation of state' of a string. Clearly, in the strict sense of the word the equality $T_0^0 = T_3^3$ is a microscopic characteristic of the string and has no relevance to the macroscopic equation of state of a statistical system.

In the presence of a static gravitating string, the spacetime remains invariant under arbitrary transformations of the time variable x^0 as well as under translations x^3 along the string, rotations x^2 about its axis, and Lorentz translations along the string. The metric tensor $g_{\mu\nu}$ defined as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\tag{15}$$

may be considered diagonal, namely

$$g_{\mu\nu} = \text{diag}\left(\exp\left(2F_{0}\right), -\exp\left(2F_{1}\right), -\exp\left(2F_{2}\right), -\exp\left(2F_{3}\right)\right)$$
(16)

The gravitational field is described by the four functions F_0 , F_1 , F_2 , and F_3 .

The general covariance of the equations of general relativity enables an arbitrary choice of the coordinate x^1 . In solving cylindrically symmetric problems, one usually employs the simple cylindrical coordinate system defined by $x^1 = r$. For the metric (15) to be of the form

$$ds^{2} = g_{00} (dx^{0})^{2} - dr^{2} - g_{22} (dx^{2})^{2} - g_{33} (dx^{3})^{2},$$

the coordinate x^1 is chosen in such a way as to satisfy the condition $F_1 = 0$. However, for cylindrically symmetric systems the Einstein equations assume a simpler form if the coordinate x^1 is chosen by imposing the Bronnikov condition¹ [18]

$$F_1 = F_0 + F_2 + F_3 \,. \tag{17}$$

To see which value the coordinate x^1 in this frame of reference takes on the axis of the string and which at infinity,

¹ The coordinate system defined by condition (17) was used in the equilibrium analysis of strong current channels in the general theory of relativity [19].

we go over to the Galilean metric and set $x^0 \equiv t$, $x^2 \equiv \varphi$, $x^3 \equiv z$. Then one finds

$$F_0 = F_3 = 0$$
, $\exp(F_2(x^1)) = r$, $\exp(F_1(x^1)) dx^1 = dr$,
(18)

with $F_1 = F_2$ by virtue of Eqn (17). From this we see that in the Galilean limit the Bronnikov coordinate x^1 and the radius *r* are related by

$$x^{1} - x_{0} = \ln r, \quad \gamma = 0,$$
 (19)

where x_0 is the constant of integration.

Thus, in the coordinate system defined by condition (17) we have $x^1 = -\infty$ on the axis of the string, and $x^1 \to \infty$ as we go in the radial direction away from it. It is convenient to regard $x_0 \equiv \ln r_0$ as the boundary of the core of the string in some sense. Then $x^1 \ge 1$ outside of the string.

In the coordinate system determined by condition (17), we obtain

$$\sqrt{-g} = -g_{11} = \exp\left(2F_1\right)$$

The Ricci tensor takes the simple and transparent form

$$R_{0}^{\ 0} = -g^{11}F_{0}^{\ \prime\prime},$$

$$R_{1}^{\ 1} = -g^{11}[F_{1}^{\ \prime\prime} - 2(F_{2}^{\ \prime}F_{3}^{\ \prime} + F_{3}^{\ \prime}F_{0}^{\ \prime} + F_{0}^{\ \prime}F_{2}^{\ \prime})],$$

$$R_{2}^{\ 2} = -g^{11}F_{2}^{\ \prime\prime},$$

$$R_{3}^{\ 3} = -g^{11}F_{3}^{\ \prime\prime}.$$
(20)

The energy-momentum tensor T_{μ}^{ν} appears in the Einstein equations

$$R_{\mu}^{\nu} = 8\pi G S_{\mu}^{\nu} \tag{21}$$

in the combination

$$S_{\mu}^{\nu} = T_{\mu}^{\nu} - \frac{1}{2} \,\delta_{\mu}^{\nu} T, \qquad T = T_{\mu}^{\ \mu}.$$
(22)

It follows from the Einstein equations (21) subject to the Bianchi identity that the covariant derivative of the energy–momentum tensor (13) vanishes: $T_{\mu;\nu}^{\nu} = 0$. A direct calculation of the covariant derivative gives

$$2g^{11}f'\left[f'' - n^{2}g_{11}g^{22}(1-\alpha)^{2}f + \frac{g_{11}}{2\eta^{2}}\frac{\partial V}{\partial f}\right] \\ + \frac{n^{2}}{e^{2}}g^{11}\alpha'^{2}\left[\eta^{-2}(\alpha'\exp(-2F_{2}))'\right] \\ - 2e^{2}g_{11}g^{22}(1-\alpha)f^{2} = 0.$$
(23)

The expressions in square brackets in Eqn (23) are identical to those in the left-hand sides of the equations for the order parameter and the gauge field:

$$f'' - n^2 g_{11} g^{22} (1 - \alpha)^2 f + \frac{g_{11}}{2\eta^2} \frac{\partial V}{\partial f} = 0, \qquad (24)$$

$$\eta^{-2} \left(\alpha' \exp(-2F_2) \right)' - 2e^2 g_{11} g^{22} (1-\alpha) f^2 = 0.$$
 (25)

Equations (24) and (25) are the Euler-Lagrange equations for the Lagrangian (11). The presence of a linear relation in Eqn (23) means that not all equations in the system (21),

(24), and (25) are independent. The Einstein equations in fact contain the equations of motion in themselves [20]. If in the coupled system containing the Einstein equations one of the equations, say Eqn (24), is regarded as independent, then Eqn (25) will, for $\alpha' \neq 0$, be a consequence of the remaining equations.

Notice here that the symmetries involved in the theory of elementary particles and in astrophysics do not necessarily relate to space – time transformations. They may equally well be internal symmetries, such as the isotopic and electroweak symmetries, the Grand Unification symmetry or even supersymmetry, whose transformations turn bosons and fermions one into another. Internal symmetry transformations do not affect the space – time characteristics of the transformed states and hence are independent of coordinates and time. The topological defects related to the spontaneous breaking of internal symmetries are called global.

From a physical standpoint, gauge and global strings are inherently different objects, and the equations that describe them are also different. The Abelian Higgs model describes gauge strings. Global strings are governed by the Higgs scalar field model, in which there is no gauge field at all. The Lagrangian density in the scalar field model has the form

$$\mathcal{L} = \partial_{\mu}\phi \,\partial^{\mu}\phi - V(\phi) \,. \tag{26}$$

The Abelian Higgs model (2) involves gauge symmetry, and the model (26) global symmetry. Accordingly, the strings in model (2) are called gauge, and in model (26) global.

3. Gauge strings

3.1 Equations

In the absence of ordinary matter the expressions for F_0'' and F_3'' are identical, and we can set ${}^2 F_0 = F_3$. Shifting the origin of coordinates for the functions F_1 and F_2 :

$$F_{1,2} \to F_{1,2} - \ln(e\eta),$$
 (27)

the complete set of equations for a static gauge string can be reduced to the form

$$\left(\alpha' \exp\left(-2F_2\right) \right)' + 2(1-\alpha)f^2 \exp\left(4F_0\right) = 0 ,$$

$$f'' - n^2(1-\alpha)^2 f \exp\left(4F_0\right) - \beta^{-2} \exp\left(2F_1\right) \frac{\partial \tilde{V}}{\partial f} = 0 ,$$

$$F_0'' + \gamma \left[\frac{2\tilde{V}}{\beta^2} \exp\left(2F_1\right) - \frac{n^2}{2} \alpha'^2 \exp\left(-2F_2\right) \right] = 0 ,$$
 (28)

$$F_2'' + \gamma \left[\frac{2\tilde{V}}{\beta^2} \exp\left(2F_1\right) + \frac{n^2}{2} \alpha'^2 \exp\left(-2F_2\right) + 2n^2(1-\alpha)^2 f^2 \exp\left(4F_0\right) \right] = 0 ,$$

$$F_1 = F_2 + 2F_0 .$$

Here \tilde{V} is the dimensionless potential causing spontaneous breaking of symmetry. In the sombrero model (3), the potential is $\tilde{V} = (1/4)(1-f^2)^2$. The string and the gravitational field are described by four functions f, α , F_0 , and F_2 ,

² The regularity condition on the axis implies $F'_0 = F'_3$. The equality $F_0 = F_3$ is achieved by an appropriate choice of the units of measurement for the coordinate x^3 .

and three dimensionless parameters n, β , and γ [see Eqns (7) and (8)].

The shift (27) defines the unit of measurement for the radius r. For a nonrelativistic vortex in a superconductor, $\delta = (e\eta)^{-1}$ is the magnetic penetration depth. In discussing gauge strings below we use $r_0 = (e\eta)^{-1}$ as a measurement unit for the radius r in a cylindrical coordinate system.

3.2 Energy integral

The energy per unit length of the string is equal to

$$E = 2\pi \int \mathrm{d}x^1 \sqrt{-g} \ T_0^{\ 0} \,.$$

Using Eqns (11) and (16), the integrand is found to consist of four positive terms, viz.

$$E = 2\pi\eta^2 \int dx^1 \left[f'^2 + n^2(1-\alpha)^2 f^2 \exp(4F_0) + \frac{n^2}{2} \alpha'^2 \exp(-2F_2) + \frac{2\tilde{V}}{\beta^2} \exp(2F_1) \right].$$
 (29)

The gravitational mass per unit length of the string is given by the Tolman formula³ [20, p. 425]:

$$M = \frac{1}{4\pi G} \iint \mathrm{d}x^1 \, \mathrm{d}x^2 \, \sqrt{-g} \, R_0^{\ 0} = \frac{m_{\mathrm{Pl}}^2}{2} \, F_0'(\infty) \,. \tag{30}$$

3.3 Boundary conditions

The physical requirement that the energy per unit length of the string be finite (which implies that the integral (29) converges), together with the necessary condition for its regularity on the axis determine the boundary conditions

$$F_0'(-\infty) = 0, \quad F_2'(-\infty) = 1, \quad f(-\infty) = 0, \quad \alpha(-\infty) = 0,$$
(31)

$$\left(\widetilde{V}\exp\left(2F_{1}\right)\right)_{x^{1}\to\infty}=0\,,\quad f'(\infty)=0\,,\quad \alpha(\infty)=1\,.$$
(32)

Units for measuring the 'time' coordinate x^0 can be chosen such that

$$F_0(-\infty) = 0. \tag{33}$$

Equations (28) do not explicitly involve the coordinate x^1 ; it enters only through the derivatives. As a result, all the functions depend on x^1 through the combination $x \equiv x^1 - x_0$, where x_0 is the constant of integration. In accordance with Eqn (31) we can set

$$F_2(x^1) = x, \qquad x^1 \to -\infty.$$
(34)

Because F_1 is a growing function of the coordinate x^1 , the conditions $(\widetilde{V} \exp(2F_1))_{x^1 \to \infty} = 0$ and $f'(\infty) = 0$ are both satisfied if $\widetilde{V}(\infty) = 0$. For the sombrero potential (3) we then have

$$f(\infty) = 1. \tag{35}$$

³ Mass per unit length has dimensions of mass per length. In units of Eqn (9), mass has the dimensions of inverse length. As a result, mass/length has the dimensions of mass squared; $m_{\rm Pl}^2 = 1.35 \times 10^{28}$ g cm⁻¹. Note that Tolman's formula for the total mass of an object is derived by assuming that far from the object spacetime is asymptotically flat. This is by no means the case with strings. The integral (30) converges, however, and the constant $F_0'(\infty)$ is a physical characteristic of the string. The relation of this characteristic to the mass per unit length can be established in the Newton limit.

The boundary condition (35) has been in common use thus far. If gravitation is of no significance, this condition is doubtless valid. If gravitation is included, however, the universal boundary condition for $x^1 \rightarrow \infty$ is not relationship (35) but rather the weaker condition

$$f'(\infty) = 0. \tag{36}$$

 $f(\infty)$ itself may be an arbitrary constant, not necessarily unity [21]. It will be shown below that $f(\infty) < 1$ for supermassive gauge strings and also for global strings. Using the boundary condition (35) outside the range of its validity makes it impossible to correctly determine the range of physical parameters for which static string solutions exist.

3.4 General properties of gauge strings

3.4.1 First integrals. The Ricci tensor (20) is a linear function of the second derivatives F_i'' . By introducing a linear combination of the Einstein equations and using the Bronnikov condition (17) it is possible to eliminate the second derivatives and thus to obtain a first integral, namely

$$F_0'(F_0' + 2F_2') = -\gamma \left[\frac{2\widetilde{V}}{\beta^2} \exp(2F_1) - \frac{n^2}{2} \alpha'^2 \exp(-2F_2) + n^2(1-\alpha)^2 f^2 \exp(4F_0) - f'^2 \right].$$
 (37)

The right-hand side of Eqn (37) contains the same terms as in Eqn (29) but with different signs. The boundary conditions (32) require that the right-hand side of the first integral (37) vanishes as $x^1 \rightarrow \infty$. Therefore, one obtains

$$F_0'(\infty) \left[F_0'(\infty) + 2F_2'(\infty) \right] = 0.$$
(38)

Equations (28) yield one more first integral, viz.

$$F_{2}' - F_{0}' = 1 - \gamma n^{2} \left[B + (\alpha - 1)\alpha' \exp(-2F_{2}) \right].$$
(39)

Here the notation was used:

or

$$B = \left(\alpha' \exp\left(-2F_2\right)\right)_{x^1 \to -\infty}.$$
(40)

The physical meaning of the constant *B* is obvious: this is the strength of the gauge field on the axis of the string. In the nonrelativistic case of a superconducting vortex line this is the magnetic field on the axis. The constant *B* is an unknown function of the parameters n, β , and γ , which determines the gravitational field outside the string.

3.4.2 Gravitational field outside the string. Relation (38) shows that gauge strings are of two types, namely

$$F_0'(\infty) = 0, \quad F_0'(\infty) + 2F_2'(\infty) \neq 0$$
 (41)

$$F_0'(\infty) + 2F_2'(\infty) = 0, \quad F_0'(\infty) \neq 0.$$
 (42)

From the boundary condition (32) and relation (39) we have

$$F_2'(\infty) - F_0'(\infty) = 1 - \gamma n^2 B.$$
 (43)

For cases (41) and (42), the metric outside the string is determined by the constant (40).

Both the solutions (41) and (42) degenerate to one and the same solution if the functions F_0' and F_2' vanish as $x^1 \to \infty$. The solutions become identical along the line defined by the condition

$$F_2'(\infty) = 0.$$
 (44)

Conic metric. From relationships (41) and (43) we find

$$F_0'(\infty) = 0, \quad F_1'(\infty) = F_2'(\infty) = 1 - \gamma n^2 B,$$
 (45)

and, hence, the following relation holds for $x^1 \to \infty$:

$$F_1(x^1) = F_2(x^1) = (1 - \gamma n^2 B) x^1 + \text{const}.$$
 (46)

By making the substitution

$$dt = dx^0$$
, $dr = \exp(F_1(x^1)) dx^1$, $d\varphi = dx^2$, $dz = dx^3$,
(47)

the metric outside the string can be reduced to the form

$$ds^{2} = dt^{2} - dr^{2} - (1 - \gamma n^{2}B)^{2}r^{2} d\varphi^{2} - dz^{2}.$$
 (48)

The metric (48) differs from the Galilean metric only by the constant coefficient $(1 - \gamma n^2 B)^2 < 1$ of $r^2 d\phi^2$. The angle φ varies from zero to 2π . In a space with metric (48), the length of a circle with the center on the string axis and of radius *r* is $2\pi r(1 - \gamma n^2 B) < 2\pi r$. The string cuts out a wedge in the plane (r, φ) as shown in Fig. 2a, creating what may be called an 'angle deficiency'

$$\Delta = 2\pi\gamma n^2 B. \tag{49}$$

If the remaining part of the circle is glued along the cutting lines, a cone will form as shown in Fig. 2b. The metric (48) is therefore called 'conic'⁴. The constant *B* depends on the parameters *n*, β , and γ . To find this dependence, it is necessary to solve Eqns (28).



Figure 2. Conic string cuts out a wedge in the (r, φ) plane.

The gravitational mass (30) of a conic string is zero. This is an off-beat example of a macroscopic body with zero mass [23]. At distances larger than the string core radius, such a string has no influence on matter.

In deriving Eqns (47) and (48) it was tacitly assumed that $1 - \gamma n^2 B > 0$. However, as γ increases, so does the angle (49) of the cut out wedge, implying that in analyzing supermassive strings the case $1 - \gamma n^2 B \le 0$ should also be considered. For $1 - \gamma n^2 B < 0$, all the space outside the string turns out to be

⁴ Conic metrics had been studied [22] long before their relevance to cosmic strings was recognized.

cut out, with the result that the radius *r* does not go to infinity but only to a certain limiting value r_{max}^{5} :

$$r = r_{\max} - (\gamma n^2 B - 1)^{-1} \exp\left[-(\gamma n^2 B - 1)x^1 - \text{const}\right],$$

1 - \gamma n^2 B < 0.

The outer metric for $1 - \gamma n^2 B < 0$ takes the form

$$ds^{2} = dt^{2} - dr^{2} - (r_{\max} - r)^{2} (1 - \gamma n^{2} B)^{2} d\varphi^{2} - dz^{2},$$

$$1 - \gamma n^{2} B < 0.$$
(50)

The Kasner metric. For the solutions of the type (42) we obtain

$$F_{0}'(\infty) = -\frac{2}{3}(1 - \gamma n^{2}B),$$

$$F_{1}'(\infty) = -(1 - \gamma n^{2}B),$$

$$F_{2}'(\infty) = \frac{1}{3}(1 - \gamma n^{2}B).$$
(51)

After substituting Eqn (47) we arrive at the outer metric of a Kasner type [24]:

.

$$\mathrm{d}s^{2} = \begin{cases} \left[(1 - \gamma n^{2}B)(r_{\max} - r) \right]^{4/3} (\mathrm{d}t^{2} - \mathrm{d}z^{2}) - \mathrm{d}r^{2} - \\ - \left[(1 - \gamma n^{2}B)(r_{\max} - r) \right]^{-2/3} \mathrm{d}\varphi^{2} \,, \qquad F_{2}'(\infty) > 0 \,, \\ \left[(\gamma n^{2}B - 1)r \right]^{4/3} (\mathrm{d}t^{2} - \mathrm{d}z^{2}) - \mathrm{d}r^{2} - \\ - \left[(\gamma n^{2}B - 1)r \right]^{-2/3} \mathrm{d}\varphi^{2} \,, \qquad F_{2}'(\infty) < 0 \,. \end{cases}$$

$$\tag{52}$$

The fact that the outer metric may be either conic or of a Kasner type was established by Vilenkin [25]. A solution with an outer Kasner metric was found by Laguna and Garfinkle [26].

3.5 Properties of solutions near the degenerate line

For a fixed value of the azimuthal number *n*, the degenerate line (44), along which the solutions (41) and (42) are the same, is defined in the plane of (β, γ) parameters by the equation

$$1 - \gamma n^2 B(n, \beta, \gamma) = 0.$$
⁽⁵³⁾

In the region of $1 - \gamma n^2 B > 0$, solutions with a Kasner outer metric, unlike conic metric solutions, do not extend beyond the limiting radius r_{max} . Conversely, in the region of $1 - \gamma n^2 B < 0$, the Kasner metric is not limited in radius, but the conic one is. This does not happen by chance.

This situation is reminiscent of the electronic term crossing [27, § 79], when any arbitrarily small perturbation breaking the symmetry of the system removes the degeneracy. The electronic terms no longer cross, and the two branches become isolated, with one possessing a limiting radius and the other not.

To illustrate this point, consider the case of a string in the presence of an arbitrarily small amount of ordinary matter with $p \ll \varepsilon$, where p and ε are the pressure and energy of matter, respectively. We shall assume that the distribution of matter in space has the same cylindrical symmetry as the

⁵ The appearance of a limiting radius hints that the usual cylindrical coordinate system is not the most suitable one for the present problem. The coordinate x^1 defined by condition (17) varies from $-\infty$ to ∞ .

string. Now, however, $T_0^0 \neq T_3^3$, and the system as a whole is no longer invariant with respect to the Lorentz translation in the direction of the string. As a result, the Einstein equations with F_0'' and F_3'' are no longer identical:

$$\begin{split} F_0'' &= -\gamma \left[\frac{2\widetilde{V}}{\beta^2} \exp\left(2F_1\right) - \frac{n^2}{2} \,\alpha'^2 \exp\left(-2F_2\right) \right] \\ &+ 4\pi G \varepsilon \exp\left(2F_1\right), \\ F_3'' &= -\gamma \left[\frac{2\widetilde{V}}{\beta^2} \exp\left(2F_1\right) - \frac{n^2}{2} \,\alpha'^2 \exp\left(-2F_2\right) \right] \\ &- 4\pi G \varepsilon \exp\left(2F_1\right), \end{split}$$

so that

$$F_0'(\infty) - F_3'(\infty) = 4G\mathcal{M}\,,\tag{54}$$

where

$$\mathcal{M} = 2\pi \int_{-\infty}^{\infty} \mathrm{d}x^1 \varepsilon \exp\left(2F_1\right)$$

is the mass of matter per unit length.

In the presence of matter, the combination $F_2'F_3' + F_3'F_0' + F_0'F_2'$ does not reduce to a product of two factors, and instead of Eqn (38) we obtain a quadratic equation in $\chi = F_0'(\infty)$:

$$\chi^2 + 2\chi \left[F_2'(\infty) - 2G\mathcal{M} \right] - 4G\mathcal{M} F_2'(\infty) = 0 \,.$$

If $\mathcal{M} \to 0$, then for $F_2'(\infty) > 0$ the solution

$$F_0'(\infty) = 2G\mathcal{M} - F_2'(\infty) + \left[F_2'^2(\infty) + (2G\mathcal{M})^2\right]^{1/2}$$

tends to zero, i.e. it goes over into Eqn (41). For $F_2'(\infty)$ negative, one obtains

$$F_0'(\infty) \to -2F_2'(\infty)$$

in accord with relations (42).

The classification of string solutions in the presence of matter differs from that without matter. The change in the systematics of string solution is illustrated in Fig. 3, which shows that in this case the solutions can no longer be classified into a conic and a Kasner type. Solutions in the presence of



Figure 3. Intersection of solutions with conic and Kasner outer metrics for the example of $F'_0(\infty)$ as a function of $\gamma n^2 B$. In the absence of matter, the conic metric coincides with the *x*-axis, $F'_0(\infty) = 0$, and the Kasner metric is given by $F'_0(\infty) = -(2/3)(1 - \gamma n^2 B)$. In the presence of matter, the two branches split, giving rise to a gap of order $\mathcal{M}/m_{\rm Pl}^2$ between them.

matter should not be classified according to the sign of $F_2'(\infty)$ but rather according to whether or not a limiting radius exists. The conic solution with $F_2'(\infty) > 0$ and the Kasner solution with $F_2'(\infty) < 0$ form one branch, while the conic solution with $F_2'(\infty) < 0$ and the Kasner solution with $F_2'(\infty) > 0$ form the other. In the presence of matter, the two branches split (the upper and lower curves), and between them a gap of order $\mathcal{M}/m_{\rm Pl}^2$ appears. With this systematics, one (lower) branch of the curve possesses a limiting value for the radius $r_{\rm max}$, whereas the other (upper) does not.

As one can see, the presence of an arbitrarily small amount of ordinary matter makes it impossible to classify stringlike solutions into purely conic and purely Kasner solutions. One therefore should not hurry up to conclusions and declare solutions with a Kasner singularity to be nonphysical.

3.6 (β , γ) plane

When numerically integrating Eqns (28) it is important to correctly choose the boundary condition for $x^1 \rightarrow \infty$. For nonrelativistic vortex structures, the boundary condition (35) has traditionally been used. Attempts of Christensen et al. [28] to use the boundary condition $f(\infty) = 1$ for finding the upper bound for the domain of existence of static relativistic strings have been untenable. For the correct determination of the upper bound, the weaker boundary condition (36) should be chosen.

For supermassive conic strings, $F_1(x^1) = (1 - \gamma n^2 B)x^1 +$ const when $x^1 \to \infty$ [see Eqn (46)], with $1 - \gamma n^2 B < 0$. The convergence of the integral of the term $(2\tilde{V}/\beta^2) \exp(2F_1)$ entering into the energy (29) close to the upper bound is ensured by the exponentially decreasing factor $\exp(2F_1)$. The limiting value of the order parameter f should be a constant [otherwise a divergence of the integral due to the term f'^2 would arise in the string energy (29)] — but not necessarily unity. Integrating Eqns (28) subject to the boundary condition (36) yields the bounds for the domains of existence of static solutions [21]. The map of string solutions in the (β, γ) plane is shown in Fig. 4 for n = 1.

The numerical integration in recent work [21] for the degenerate line (53) at $\beta \sim 1$ gave, within the accuracy of the calculations, the dependence

$$\gamma = \beta^{0.535} \,, \tag{55}$$

to which corresponds the curve $\gamma_0(\beta)$ in Fig. 4. The solutions above and below the degenerate line (55) exhibit radically different properties.



Figure 4. 'Map' of string solutions in the (β, γ) plane.

Over the entire region below the line (55), both conic and Kasner solutions exist. For both the conic and Kasner solutions below the line (55), the limiting value of the order parameter $f(\infty) = 1$, and the radius *r* in the usual cylindrical coordinate system varies over the interval $(0, \infty)$. But whereas for the conic solutions we have $F_1'(\infty) > 0$, for the Kasner solutions, on the contrary, $F_1'(\infty) < 0$.

Also above the degenerate line (55) there are regions of static stringlike solutions. Here, for both types of solutions, the radius *r* in the usual cylindrical coordinate system does not go to infinity but rather has a certain finite limiting value r_{max} . For conic solutions above the line (55) we have $F_1'(\infty) < 0$, whereas for Kasner solutions, on the contrary, $F_1'(\infty) > 0$. Above (but close to) the degenerate line, the limiting value of the order parameter $f(\infty)$ decreases with increasing γ for both types of solutions.

Depending on the value of β , there are two possibilities. For conic strings at low $\beta \leq 1$, the limiting value of the order parameter, $f_{\infty} = f(\infty)$, decreases monotonically with γ and at a certain $\gamma = \gamma_{cr}(n,\beta)$ vanishes in such a way that the derivative $|df_{\infty}/d\gamma| \to \infty$. In the region of not small $\beta > \beta_{bif} \ge 1$, the derivative $|df_{\infty}/d\gamma|$ becomes infinite at $\gamma = \gamma_{cr}(n,\beta)$ for finite f_{∞} values. Exactly where f_{∞} approaches zero or $|df_{\infty}/d\gamma|$ becomes infinite (whichever is earlier) determines the upper bound

$$\gamma = \gamma_{\rm cr}(n,\beta) \tag{56}$$

for the domain of existence of static supermassive gauge strings. For conic strings — this is the curve $\gamma_1(\beta)$ in Fig. 4, whereas for Kasner strings — this is the curve $\gamma_2(\beta)$. For $\beta \rightarrow 0$, the upper bounds $\gamma_1(\beta)$ and $\gamma_2(\beta)$ merge at one point: $\gamma_1(0) = \gamma_2(0) = 1.067...$

For conic strings at n = 1, the variation of the limiting values of the order parameter $f_{\infty}(\gamma)$ for $\beta = 1$ and $\beta = 2$ is illustrated in Fig. 5. In the case of $\beta = 1$, f_{∞} vanishes and $|df_{\infty}/d\gamma|$ becomes infinite at the same value of the parameter $\gamma = \gamma_{\rm cr}(n, 1)$. This is an analytically exact result, as discussed below. As to the case of $\beta = 2$, there $|df_{\infty}/d\gamma|$ becomes infinite at a finite value of f_{∞} . Hence, in the region $\beta \ge 1$, a bifurcation point $\beta_{\rm bif}$ must exist above which, as γ increases, the derivative $|df_{\infty}/d\gamma|$ for a conic string becomes infinite earlier than the order parameter f_{∞} vanishes. To this day, the bifurcation point $\beta_{\rm bif}$ has not been found.

3.7 The Bogomol'nyĭ degenerate case

The general property (45) of conic solutions for gauge strings has the implication that the gauge and Higgs fields compen-



Figure 5. Variation of the limiting values of the order parameter $f_{\infty}(\gamma)$ at $\beta = 1$ and $\beta = 2$ for conic strings.

sate each other integrally for any value of the parameter β in the interval $(0, \infty)$. The gauge string problem with potential (3) and a conic outer metric is possessed of internal symmetry associated with the possible interchange of Higgs and gauge particles. A consequence of such a hidden symmetry is that there exist a certain value β , namely

$$\beta = 1 \tag{57}$$

[when the masses (6) of the Higgs and gauge particles are the same], for which the fields compensate each other at any point x^1 , not only integrally. In this degenerate case the system (28) considerably simplifies. For ordinary nonrelativistic vortices [6, 29], Bogomol'nyĭ [30] showed that in the special case (57) the equations for an order parameter and gauge field are actually of first order, rather than of second as in the general case with Eqns (24) and (25).

For the sombrero potential (3), in the Bogomol'nyĭ degenerate case (57) the energy integral (29) reduces to the form

$$E_{\rm B} = 2\pi\eta^2 \int dx^1 \left\{ \left[f' - n(1-\alpha)f\exp(2F_0) \right]^2 + \frac{1}{2} \left[n\alpha'\exp(-F_2) - (1-f^2)\exp(F_1) \right]^2 - n \left[(1-f^2)(1-\alpha) \right]'\exp(2F_0) \right\}.$$
 (58)

It turns out that those solutions of the equations

$$f' - n(1 - \alpha) f \exp(2F_0) = 0,$$
(59)

$$\alpha' \exp(-2F_2) - \frac{1}{n}(1 - f^2) \exp(2F_0) = 0$$

that minimize the energy functional (58) satisfy the system (28). From Eqns (59) and the boundary conditions (31) and (32), the energy (58) is minimum and equals

$$E = 2\pi n\eta^2 \,. \tag{60}$$

In view of Eqns (59), the third equation of the set (28) reduces to $F_0'' = 0$ and allowing for Eqn (33) it is found that for $\beta = 1$ we have

$$F_0'(x^1) = 0$$

for any x^1 in the interval $(-\infty, \infty)$. From the second equation of the set (59) we see that for $\beta = 1$ the constant *B* in Eqn (40) is independent of the parameter γ and is given by

$$B_{\rm B} = \frac{1}{n} \,. \tag{61}$$

3.7.1 Point of double degeneracy. In the (β, γ) plane of Fig. 4, the Bogomol'nyĭ case corresponds to the vertical line $\beta = 1$. It intersects the degenerate line (53) at the point with coordinates

$$\beta = 1 , \quad \gamma = \frac{1}{n} . \tag{62}$$

This is a point of double degeneracy: both in terms of the symmetry of the Higgs and gauge particles and in the sense that solutions with conic and Kasner outer metrics are identical here. The expression for the first integral (39) at $\beta = 1$ can be transformed to the total differential:

$$F_{2}' = 1 - \gamma n + \gamma \left(\ln f - \frac{1}{2} f^{2} \right)'.$$
(63)

Now the function $F_2(x^1)$ can be expressed in terms of the order parameter $f(x^1)$, so that

$$F_2 = (1 - \gamma n)(x^1 - x_0) + \gamma \left(\ln f - \frac{1}{2} f^2 \right) + C, \qquad \beta = 1,$$
(64)

where *C* is the constant of integration. This constant is related to the amplitude of the order parameter *f* for $x^1 \rightarrow -\infty$:

$$f(x^{1}) = \exp\left[n(x^{1} - x_{0}) - \frac{C}{\gamma}\right], \quad x^{1} \to -\infty.$$
(65)

The dependence $C(\gamma, n)$ can be found by solving the equation for f, viz.

$$(\ln f)'' + (1 - f^2) f^{2\gamma} \exp[2(1 - \gamma n)(x^1 - x_0) - \gamma f^2 + 2C] = 0.$$
(66)

The corresponding equation in cylindrical coordinates was derived by Linet [31].

At the point (62) of double degeneracy the explicit dependence on x^1 drops out of Eqn (66), so it can be solved in quadratures. For n = 1 we have

$$2(x^{1} - x_{1}) \exp\left(C - \frac{1}{2}\right) = \int^{f^{2}} \frac{dz}{z \left[1 - z \exp\left(1 - z\right)\right]^{1/2}},$$

$$\beta = 1, \quad \gamma = 1.$$
(67)

Here x_1 is yet another constant of integration. The solution (67) must coincide with Eqn (65) for $x^1 \rightarrow -\infty$. This requirement determines the constant *C* at the point of double degeneracy:

$$C=\frac{1}{2}$$
, $\beta=1$, $\gamma=1$.

3.7.2 Analysis near the upper bound. Above the point of double degeneracy, i.e. for $\gamma n > 1$, the limiting value f_{∞} of the order parameter decreases with increasing γ . In the case of the validity of relation (57), the order parameter f vanishes for $\gamma n = 2$, i.e.

$$\gamma_{\rm cr}(n,1) = \frac{2}{n} \,. \tag{68}$$

Near the upper bound (68), equation (66) can be solved analytically [21].

Let us introduce the function

$$W = \ln f \tag{69}$$

which satisfies the equation

$$W'' + [1 - \exp(2W)] \\ \times \exp[2\gamma W + 2(1 - \gamma n)(x^{1} - x_{0}) - \gamma \exp(2W) + 2C] = 0$$
(70)

together with the boundary conditions

$$W'(-\infty) = n, \qquad W'(\infty) = 0.$$
 (71)

Near the upper bound (68) one has $\exp W \ll 1$, and Eqn (70) can be solved by successive approximations in $\exp W \ll 1$ $(W = W_0 + W_1)$:

$$W_0'' = \exp\left[2\gamma W_0 + 2(1 - \gamma n)(x^1 - x_0) + 2C\right],$$
 (72)

$$W_1'' + 2\gamma W_0'' W_1 = (1+\gamma) W_0'' \exp(2W_0).$$
(73)

The solution to Eqn (72), which goes over to the solution (65) as $x^1 \rightarrow -\infty$, has the form

$$\exp W_0 = f_{\rm m} \exp\left[\left(n - \frac{2}{\gamma}\right)(x^1 - x_0)\right] \\ \times \left[1 + \frac{4}{\gamma} \exp\left(-2(x^1 - x_0)\right)\right]^{-1/\gamma}, \tag{74}$$

where

$$f_{\rm m} = \left(\frac{4}{\gamma}\right)^{1/\gamma} \exp\left(-\frac{C}{\gamma}\right). \tag{75}$$

Relationship (75) clarifies the physical meaning of the constant *C*: this constant relates to the limiting value of the order parameter. A solution (74) to Eqn (72) with an arbitrarily small $f_m \ll 1$ exists for any γ . The energy (29) of such solutions, however, is finite only if $\gamma n < 2$. For $\gamma n > 2$, the order parameter (74) exponentially grows as $x^1 \to \infty$, and the integral governing the energy in Eqn (29) diverges.

The zeroth approximation (74) does not satisfy the boundary condition (71) when $x^1 \to \infty$. However, if we are close to the upper bound (68) then, for $(2/\gamma) - n \le 1$, this 'incorrect' dependence of the zeroth-order solution (74) for $x^1 \to \infty$ can be corrected by including first-order terms. For this it is necessary to solve equation (73) with the boundary conditions

$$W_1(-\infty) = 0, \qquad W_1'(\infty) = \frac{2}{\gamma} - n.$$
 (76)

Equation (73) is a linear, inhomogeneous, second-order differential equation. The corresponding homogeneous equation belongs to the Legendre class:

$$\hat{L}Z + \nu(\nu+1)Z = 0$$
, $\hat{L} = \frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta}$. (77)

The eigenvalues in this case are $v_1 = 1$, $v_2 = -2$, and the two linearly independent solutions of the homogeneous equation are expressed by the Legendre functions

$$P_1(\zeta) = \zeta, \qquad Q_1(\zeta) = \frac{\zeta}{2} \ln \frac{1+\zeta}{1-\zeta} - 1.$$
 (78)

The independent variable ζ is related to the coordinate x^1 in the following way

$$\zeta = \tanh(x^1 - x^*), \quad x^* = x_0 + \frac{1}{2}\ln\frac{4}{\gamma}.$$

The general solution of Eqn (73) takes the form

$$W_{1} = A_{1}\zeta + A_{2}Q_{1}(\zeta) + \frac{f_{m}^{2}\zeta}{2^{n+1}(n+1)} \int_{-1}^{\zeta} dz \, \frac{(1+z)^{n} [nz - (1-z)]}{z^{2}(1-z)} \,.$$
(79)

From the boundary condition (76) for $x^1 \to -\infty$ it follows that $A_1 = A_2 = 0$. The behavior of the solution W_1 as $x^1 \to \infty$ is determined by the third term in formula (79):

$$\frac{f_{\rm m}^2 \zeta}{2^{n+1}(n+1)} \int_{-1}^{\zeta} \mathrm{d}z \, \frac{(1+z)^n \left[nz - (1-z) \right]}{z^2 (1-z)} \\ = -\frac{n f_{\rm m}^2}{n+1} \left(x^1 - x^* \right) + \dots$$
(80)

Comparing formulas (76) and (80) we find

$$f_{\rm m}^{\ 2} = (n+1)\left(1 - \frac{\gamma n}{2}\right), \quad 2 - \gamma n \ll 1.$$
 (81)

Because f_m^2 must be nonnegative, we conclude that the domain of existence of static vortex solutions in the Bogomol'nyĭ case (57) is limited by the inequality $\gamma n < 2$. Close to this bound, the order parameter is found to be

$$f(x^{1}) = \left[(n+1)\left(1 - \frac{\gamma n}{2}\right) \left[1 + 2n \exp\left(-2(x^{1} - x_{0})\right)\right]^{-n} \right]^{1/2},$$

$$\beta = 1, \quad 2 - \gamma n \ll 1.$$
(82)

For the gauge field we obtain

$$\alpha = \left[1 + 2n \exp\left(-2(x^{1} - x_{0})\right)\right]^{-1}, \quad \beta = 1, \quad 2 - \gamma n \ll 1.$$
(83)

The constant C grows logarithmically as the bound is approached:

$$C = \ln (2n) - \frac{1}{n} \ln \left[(n+1) \left(1 - \frac{\gamma n}{2} \right) \right], \quad \beta = 1, \quad 2 - \gamma n \ll 1.$$
(84)

The gravitational field for $2 - \gamma n \ll 1$ is equal to

$$F_1 = F_2 = \frac{1}{2} \ln \left(\frac{n}{2} \left[\cosh \left(x^1 - x_0 - \frac{1}{2} \ln (2n) \right) \right]^{-2} \right).$$

By making the substitution (47) we find the metric in cylindrical coordinates:

$$ds^{2} = dt^{2} - dr^{2} - \frac{1}{4}\sin^{2}(2r) d\varphi^{2} - dz^{2}, \quad 2 - \gamma n \ll 1.$$
 (85)

In the metric (85), the radius *r* varies from zero to the maximum value $r_{\text{max}} = \pi/2$. Close to the string axis $(r \rightarrow 0)$, the metric (85) is Galilean, and for $r \rightarrow r_{\text{max}}$ it goes over into formula (50) with B = 1/n as in Eqn (61) and $\gamma n = 2$.

For Bogomol'nyĭ gauge strings ($\beta = 1$), the dependence $\eta_{\max}(n)$ takes the form

$$\frac{\eta_{\max}(n)}{m_{\rm Pl}} = (4\pi n)^{-1/2} = 0.282 \, n^{-1/2}$$

in accord with formula (68). The numerical computations done by De Laix et al. [15] yielded a different result:

$$\frac{\eta_{\max}(n)}{m_{\text{Pl}}} = a n^p, \quad p = -0.56, \quad a = 0.16.$$

The main reason why the authors of work [15] underestimated $\eta_{\max}(n)$ is that they used the boundary condition f = 1 for $r \to \infty$. This condition is valid only in the region $\gamma n \leq 1$ —so long as the angle deficiency is less than 2π . In reality, the boundary condition valid over the entire range $0 < \gamma < \gamma_{cr}$ is $f'(\infty) = 0$. Curiously, the estimate p = -0.5 which the authors of Ref. [15] call 'naive' actually proves to be exact in the Bogomol'nyı case ($\beta = 1$).

3.8 Solutions with Kasner asymptotics for $\gamma \ll 1$

In the region of small γ , namely for

$$\gamma \ll 1$$
, (86)

the gravitational field of gauge strings with Kasner type outer asymptotics can be determined analytically. The main simplifying factor in the case of inequality (86) is that the regions where the order parameter and the gauge field show the most change are separated in space. Under condition (86), the typical scale of a spontaneous breaking of symmetry is very small, so that the effect of the order parameter on the metric is of no significance. The gauge field, in contrast, plays a dominant role in a Kasner type metric. The constant *B* introduced by expression (40) is large for small $\gamma: B \sim 1/\gamma \ge 1$ [see Eqn (91)]. For this reason, terms of order $\gamma n^2 B$ and $\gamma n^2 B^2$ in equations (28) cannot be neglected even for small γ .

Under condition (86), the gauge field α grows from zero to unity in the region where the order parameter f is still very small. And as long as $f \ll 1$, changes in α and F_2 are described by the equations

$$\alpha' \exp\left(-2F_2\right) = B = \text{const}, \qquad (87)$$

$$F_2'' = -\frac{1}{2} \gamma n^2 B^2 \exp(2F_2).$$
(88)

The solution satisfying the boundary conditions (31) has the form

$$F_2 = -\ln\left[\cosh\left(x^1 - x_0\right)\right] - \frac{1}{2}\ln\left(\frac{1}{2}\gamma n^2 B^2\right),$$
(89)

$$\alpha = \frac{2}{\gamma n^2 B} \left[1 + \tanh(x^1 - x_0) \right].$$
(90)

Employing the boundary condition (32) we may now obtain the constant B:

$$B = \frac{4}{\gamma n^2} , \qquad \gamma \ll 1 . \tag{91}$$

The remaining functions determining the gravitational field are as follows

$$F_{0} = \ln \left[1 + \exp \left(2(x^{1} - x_{0}) \right) \right],$$

$$F_{1} = \ln \left[\exp \left(x^{1} - x_{0} \right) + \exp \left(3(x^{1} - x_{0}) \right) \right] - \frac{1}{2} \ln \frac{2}{\gamma n^{2}}.$$
(92)

The region where the gauge and gravitational fields vary most significantly is $|x^1 - x_0| \sim 1$. For $x^1 - x_0 \ge 1$ we have

$$F_1 = 3(x^1 - x_0) - \frac{1}{2} \ln \frac{2}{\gamma n^2}, \qquad x^1 - x_0 \gg 1.$$
(93)

The order parameter f mainly varies in the region where $exp(2F_1)$ becomes of order unity, i.e.

$$\left|x^1 - x_0 - \frac{1}{6}\ln\frac{2}{\gamma n^2}\right| \sim 1.$$

Since $(1/6) \ln (2/\gamma n^2) \ge 1$, the regions of variation of the functions α and *f* are separated in space.

In region (86), the metric for solutions with Kasner asymptotics (42) may be written in the form

$$ds^{2} = \left[1 + \exp(2x)\right]^{2} \left[(dx^{0})^{2} - (dx^{3})^{2}\right] - \frac{\gamma n^{2}}{2} \left[\exp x + \exp(3x)\right]^{2} (dx^{1})^{2} - \frac{\gamma n^{2}}{8\cosh^{2} x} (dx^{2})^{2}.$$
 (94)

The cylindrical coordinate *r* is related to $x = x^1 - x_0$ by the formula

$$r = \left(\frac{\gamma n^2}{2}\right)^{1/2} \left[\exp x + \frac{1}{3}\exp\left(3x\right)\right].$$
(95)

Close to the axis $(x^1 \rightarrow -\infty)$, metric (94) reduces to the Galilean form:

$$ds^{2} = (dx^{0})^{2} - dr^{2} - r^{2}(dx^{2})^{2} - (dx^{3})^{2}.$$

Outside the string (as $x^1 \to \infty$), relation (94) goes over into the Kasner type metric (52) with $B = 4/\gamma n^2$ defined by Eqn (91).

4. Global strings

4.1 The Goldstone boson

In the simplest scalar field model, the global string is described by the Lagrangian density (26). In the absence of a gauge field, the wave function near the minimum of the potential (3) is complex, namely

$$\phi = \eta + \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \,,$$

and we have, instead of formula (5), the expression

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_1)^2 + \frac{1}{2} (\partial_{\mu} \phi_2)^2 - \frac{1}{2} M_{\rm H}^2 \phi_1^2 + \mathcal{L}_{\rm int}$$
(96)

for the Lagrangian density (26) near the minimum of the potential (3). Now \mathcal{L}_{int} contains cubic and higher-order terms in ϕ_1 and ϕ_2 . As before, the field ϕ_1 in sum (96) represents a Higgs particle of mass $M_{\rm H} = \sqrt{\lambda} \eta$ [see Eqn (6)]. In the case of global symmetry, the field ϕ_2 is massless, and the corresponding scalar particle is called a Goldstone boson.

The fundamental difference between global and gauge strings relates to the Goldstone degree of freedom. The term distinguishing the Goldstone boson in the energy-momentum tensor of the global string decreases very slowly away from the axis [1]. If the curvature of spacetime is neglected, the energy per unit length of an infinite global string diverges. This is a general property of spontaneously broken global symmetry. However, in the general theory of relativity, the greater energy, the more curved is spacetime, with the result that the integration over the cross section of the string yields a finite result.

The gravitational interaction leads to the self-localization of a string. The weaker the gravitational field of the string, the larger the self-localization radius r_L [see below Eqn (111)]. In the limit of the Galilean metric, the self-localization radius becomes infinite. If the global string has a finite length L, then — depending on the relation between the gravitational selflocalization radius r_L and the string length L — two limiting cases are possible.

In the limit of $L \ll r_L$, self-localization is unimportant, and the gravitational properties of the global string may be analyzed using linearized Einstein equations [32]. In the inverse limit, i.e.

$$L \gg r_L \,, \tag{97}$$

the string is localized at distances from the axis much less than the string length. In case (97), the string may be considered one-dimensional, with all its characteristics dependent on a single coordinate — the distance from the axis. In the intermediate case $L \sim r_L$, the string is not one-dimensional, and its gravitational properties cannot be described by linearized Einstein equations.

From the point of view of the gravitational properties of the static global cosmic string, the most interesting case is that given by Eqn (97), and to this we now proceed. For $\gamma > 0$, the order parameter f of the global string is a monotonically decreasing function of γ . If $\gamma \to 0$, the order parameter $f \to 1$, while remaining less than unity.

The vanishing of the order parameter of the global string occurs at

$$\gamma = \gamma_{\rm cr}(n,0)\,,\tag{98}$$

where $\gamma_{\rm cr}(n,\beta)$ is the function (56) which determines the limiting value of γ for the gauge string [36]. Attempts of several authors [33–35] to determine the upper bound on γ using the boundary condition f = 1 could not be successful because the order parameter f corresponding to this upper bound vanishes. The behavior of the gravitational field and order parameter can be determined analytically in the limiting case $\gamma \ll 1$ and also near the upper bound (98).

4.2 Equations for the global string

For the global strings $\alpha = 0$, and the complete set of equations can be written in the form

$$f'' - n^2 f \exp(4F_0) + \frac{1}{2} f(1 - f^2) \exp(2F_1) = 0, \qquad (99)$$

$$F_0'' + \frac{\gamma}{4} (1 - f^2)^2 \exp(2F_1) = 0, \qquad (100)$$

$$F_1'' + \gamma \left[\frac{3}{4}(1-f^2)^2 \exp(2F_1) + 2n^2 f^2 \exp(4F_0)\right] = 0.$$
(101)

Here the notation is the same we have used before for gauge strings. However, instead of the shift (27) we employ

$$F_1 \to F_1 - \ln\left(\sqrt{\lambda\eta}\right). \tag{102}$$

In the Galilean limit $(\gamma \rightarrow 0)$, the coordinate x^1 is related to the radius *r* by the same relation $x^1 - x_0 = \ln r$, but this time, due to the translation (102), the radius *r* is measured in units of the characteristic radius of the global string core, namely

$$r_0 = \left(\sqrt{\lambda}\eta\right)^{-1}.\tag{103}$$

The energy per unit length of the global string is expressed as

$$E = 2\pi\eta^2 \int dx^1 \left[f'^2 + n^2 f^2 \exp(4F_0) + \frac{1}{4}(1 - f^2)^2 \exp(2F_1) \right].$$
 (104)

The gravitational mass is given by the same formula (30) as was used for the gauge string.

From equation (100) and the boundary condition (31) it follows that if $\gamma > 0$ then for a global string we have $F_0'(\infty) < 0$:

$$F_0'(\infty) = -\frac{\gamma}{4} I_1 < 0, \qquad I_1 \equiv \int_{-\infty}^{\infty} dx^1 \left(1 - f^2\right)^2 \exp\left(2F_1\right).$$
(105)

The mass (30) of the global string is negative, and its interaction with ordinary matter is repulsive.

Without a gauge field there is nothing by which to compensate for the Higgs field. Therefore, there is no analogue of conic solutions for the global string, and the only possibility left is a Kasner type solution.

4.3 Self-localization

The term $f^2 \exp(4F_0)$ falls off exponentially as $x^1 \to \infty$, and owing to gravitation the energy integral (29) converges. Without gravitation ($\gamma = 0$) this integral diverges. An arbitrarily weak gravitational interaction eliminates the Goldstone divergence and ensures the self-localization of a global string.

Having regard to $\alpha = 0$ and Eqn (17), the first integral (37) takes the form

$$F_0'(2F_1' - 3F_0') = -\gamma \left[\frac{1}{4} (1 - f^2)^2 \exp(2F_1) + n^2 f^2 \exp(4F_0) - f'^2 \right].$$
(106)

Since for a global string we have $F'_0(\infty) < 0$, from equation (106) it follows that as $x^1 \to \infty$ one arrives at

$$F_1'(\infty) = \frac{3}{2} F_0'(\infty) \,. \tag{107}$$

This means that $F'_1(\infty)$ is also negative, and that the term $\exp(2F_1)$ tends exponentially to zero as $x^1 \to \infty$.

From Eqn (101) and the boundary condition (31) one finds

$$F_1'(\infty) = 1 - \gamma \left(\frac{3}{4}I_1 + 2n^2 I_2\right), \quad I_2 \equiv \int_{-\infty}^{\infty} dx^1 f^2 \exp(4F_1).$$
(108)

Comparing Eqns (105), (107), and (108) we obtain the following relation between the integrals I_1 and I_2 :

$$\frac{3}{8}I_1 + 2n^2 I_2 = \frac{1}{\gamma}.$$
 (109)

For $\gamma \sim 1$, the main contribution to the energy integral (104) comes from the region $|x^1 - x_0| \sim 1$. When $\gamma \ll 1$, the numerical integration of equations (99)–(101) reveals two typical scales in the structure of the string. The order parameter *f* grows from zero to near unity over the range $|x^1 - x_0| \sim 1$, and the main contribution to the energy integral (104) comes from the region $|x^1 - x_0| \sim \gamma^{-1} \ge 1$.

For small γ , the curvilinear coordinate x^1 near the string core is proportional to $\ln r$:

$$x^{1} - x_{0} = \ln \frac{r}{r_{0}}, \quad \gamma \ll 1,$$
 (110)

where r_0 is the string core radius (103). From this the following upper bound is derived for the characteristic self-localization radius:

$$r_L \sim r_0 \exp \frac{1}{\gamma} \,. \tag{111}$$

Relationship (110) is valid in the string core region. At great distances, this linkage between x^1 and r breaks down even for small γ . For this reason, the self-localization radius r_L is actually even smaller: anyway, it cannot exceed the limiting radius r_{max} occurring in expression (129).

The properties of a string may be considered dependent on a single coordinate only if the string structure forms at distances from the axis much shorter than the length L. Therefore, for finite length strings the condition for the applicability of Eqns (99)-(101) is set by the inequality

$$\gamma \ln \frac{L}{r_0} \gg 1.$$
(112)

4.4 Order parameter

If gravitation is taken into account, the boundary condition $f'(\infty) = 0$ ensures the convergence of the energy integral (104). We emphasize once more that $f(x^1)|_{x^1 \to \infty} \equiv f_{\infty}(\gamma)$ is a constant, but not necessarily unity. The order parameter $f_{\infty}(\gamma)$ tends to unity only when $\gamma \to 0$. To impose correctly the boundary condition for $x^1 \to \infty$ is important, in particular, in connection with the recent development of the topological inflation idea [13–15]. Topological inflation may take place if the energy η of spontaneous symmetry breaking exceeds a critical value η_{max} , and then only nonstationary solutions are possible. The sensitivity of topological inflation to η_{max} is indicative of how important it is to have the values of $\eta_{\text{max}}(n)$ exact.

The γ -dependence of the limiting order parameter f_{∞} , found numerically in Ref. [40], is shown in Fig. 6. The order parameter $f_{\infty}(\gamma)$ is a monotonically decreasing function which vanishes at a certain $\gamma = \gamma_{\max}(n)$. For n = 1, the critical value⁶ $\gamma_{\max} = 1.067...$ Accordingly, the maximum energy of spontaneous symmetry breaking is equal to

$$\eta_{\rm max} = \sqrt{\frac{\gamma_{\rm max}}{8\pi}} m_{\rm Pl} = 2.514 \times 10^{18} \,\,{\rm GeV}\,, \qquad n = 1\,. \ (113)$$

⁶ The same value $\gamma_{\text{max}}(n)$ limits the interval of γ values in which static solutions for gauge strings exist in the limit $\beta \rightarrow 0$.



Figure 6. Dependence $f_{\infty}(\gamma)$ for n = 1, 2, 3. The dashed lines represent the analytical asymptotics given by Eqn (141).

As numerical calculations indicate, the critical value $\gamma_{max}(n)$ decreases with increasing *n* (see Table 1).

Table 1.

n	1	2	3	4
$\gamma_{\max}(n)$	1.067	0.456	0.269	0.183

4.5 Analysis for $\gamma \ll 1$

For many cosmological applications, the energy of a spontaneous breaking of symmetry is small compared to the Planck mass, and the gravitational field of a string is weak. The limiting value of the order parameter in this case is very close to unity.

The deviation of the limiting order parameter from unity may have major cosmological implications. This deviation is equivalent to the appearance of a nonzero cosmological constant in the Einstein equations. At the Grand Unification energy scale ($\eta \sim 10^{16}$ GeV) this 'cosmological constant' is very small. However, topological inflation scenarios during (and immediately after) the Planck epoch may prove sensitive to the cosmological constant effect due to the limiting value of the order parameter differing from unity. Therefore, the analytical study of the order parameter and the gravitational field of a weak global string is of keen interest [40].

Notice that for $\gamma \ll 1$ the deviation of the limiting order parameter from unity exhibits an exponentially small effect which cannot be obtained by expanding in powers of γ .

4.5.1 The limit $\gamma = 0$. In the Galilean limit ($\gamma = 0$), the coordinate x^1 is related to the radius *r* by the expression

$$x^1 - x_0 = \ln \frac{r}{r_0}$$

where r_0 is the string core radius (103). For $\gamma = 0$, the order parameter $f = f_0$ satisfies the equation

$$f_0'' - n^2 f_0 + \frac{1}{2} (1 - f_0^2) f_0 \exp(2x) = 0$$
(114)

and the boundary conditions

$$f_0(-\infty) = 0, \quad f_0(\infty) = 1.$$
 (115)

With $\gamma = 0$, integral (105) remains convergent, viz.

$$I_1 = 2n^2, \quad \gamma = 0,$$
 (116)

whereas integral (108) diverges. Outside the string core, for $x \ge 1$, the order parameter is exponentially close to unity:

$$f_0(x) = 1 - n^2 \exp(-2x), \quad x \ge 1, \quad \gamma = 0.$$
 (117)

The structure of the nonrelativistic quantum vortex governed by Eqn (114) has been treated by many authors (see, for instance, Refs [1, 7, 37]).

4.5.2 Gravitational field of a global string for $\gamma \ll 1$. In defining $F_1(x)$ for $x \ge 1$, use has been made of the fact that the order parameter $f_{\infty}(\gamma)$ is exponentially close to unity for $\gamma \ll 1$ [see Eqn (141)]. For $\gamma \ll 1$, in the peripheral region $(x \ge 1)$ equations (100) and (101) reduce to

$$F_0'' = 0$$
, $F_1'' = -2\gamma n^2 f^2 \exp(4F_0)$.

On the strength of Eqns (105) and (107) we have

$$F_0(x) = -\frac{1}{4} \gamma I_1(x - x^*), \qquad (118)$$

$$F_1'(x) = -\frac{3}{8}\gamma I_1 + \frac{2n^2}{I_1} \exp\left(-\gamma I_1(x - x^*)\right), \qquad (119)$$

where the integral I_1 is defined in Eqn (105), and

$$x^* = \frac{1}{2n^2} \int_{-\infty}^{\infty} \mathrm{d}x \, x (1 - f_0^{\ 2})^2 \exp(2x) \tag{120}$$

is a constant of order unity, which can be found numerically. For n = 1, we have $x^* \approx 0.53$.

Equation (119) integrates to the following expression

$$F_1(x) = C - \frac{3}{8} \gamma I_1(x - x^*) - \frac{2n^2}{\gamma I_1^2} \exp\left(-\gamma I_1(x - x^*)\right).$$
(121)

In the region $1 \ll x \ll \gamma^{-1}$, function (121) reduces to

$$F_1(x) = C + x - x^* - \frac{2n^2}{\gamma I_1^2},$$

and we find the constant of integration C:

$$C = x^* + \frac{2n^2}{\gamma I_1^2} \,. \tag{122}$$

Thus, in the region $x \ge 1$, $\gamma \ll 1$, the function sought is given by

$$F_1(x) = \frac{2n^2}{\gamma I_1^2} \left[1 - \exp\left(-\gamma I_1(x - x^*)\right) \right] - \frac{3}{8} \gamma I_1(x - x^*) + x^*.$$
(123)

Formulas (118) and (123) determine the gravitational field of a global string in its major region of variation, where $x \sim \gamma^{-1} \ge 1$.

4.5.3 Maximum radius r_{max} . Let us evaluate r_{max} for a global string at $\gamma \ll 1$. In the region $x \gg 1$, the relation between the coordinate x and the radius r is obtained by substituting

Eqn (123) into Eqn (47) giving

$$dr = \exp\left[\frac{2n^2}{\gamma I_1^2} \left[1 - \exp\left(-\gamma I_1(x - x^*)\right)\right] - \frac{3}{8} \gamma I_1(x - x^*) + x^*\right] dx, \quad x \ge 1.$$

In the region $x \ll \gamma^{-1}$, the linkage between x and r has the form

$$r = \exp x, \qquad x \ll \gamma^{-1}. \tag{124}$$

Let \tilde{x} be any point that satisfies the inequalities

$$1 \ll \widetilde{x} \ll \gamma^{-1} \,. \tag{125}$$

In view of Eqn (124) we have for $x = \tilde{x}$:

$$r = \tilde{r} \equiv \exp \tilde{x} \,. \tag{126}$$

We next use formula (126) as the boundary condition for determining the relation between x and r in the region $x \ge 1$ to obtain

$$r = \exp \widetilde{x} + \int_{\widetilde{x}}^{x} dx \exp \left[\frac{2n^{2}}{\gamma I_{1}^{2}} \left[1 - \exp\left(-\gamma I_{1}(x - x^{*})\right)\right] - \frac{3}{8} \gamma I_{1}(x - x^{*}) + x^{*}\right], \quad x \ge 1.$$
(127)

By virtue of inequalities (125), \tilde{x} drops out from expression (127), and if we make the substitution

$$u = \exp\left[-\frac{3}{8}\gamma I_1(x-x^*)\right],$$

we find that

$$r = \exp x^* - \frac{8}{3\gamma I_1} \exp\left(\frac{1}{2n^2\gamma} + x^*\right)$$
$$\times \int_1^{\exp\left[-\frac{3}{3}\gamma I_1(x-x^*)\right]} du \exp\left(-\frac{u^{8/3}}{2n^2\gamma}\right), \quad x \ge 1.$$
(128)

Letting $x \to \infty$ in this relation, we find that to within a preexponential factor of order unity r_{max} is given by

$$r_{\max} \sim \gamma^{-5/8} \exp \frac{1}{2n^2 \gamma}, \quad \gamma \ll 1, \quad n \sim 1.$$
 (129)

4.5.4 Order parameter in the region $x \sim \gamma^{-1} \ge 1$. The major region of variation of the order parameter f is the core of the string, where $x \sim 1$. In the exterior region $x \sim \gamma^{-1} \ge 1$, the order parameter is very close to unity. To find the deviation of the limiting value of order parameter from unity for $\gamma \ll 1$, it is necessary to find the solution of Eqn (99) for the far peripheral region. We are faced here with the problem of a small parameter in front of the higher derivative⁷. The term f'' entering into Eqn (99) is very small compared to the two other terms. From the equality of these two terms one determines the coordinate dependence of the order parameter in front.

meter *f* on the periphery of the string:

$$f = 1 - n^2 \exp(4F_0 - 2F_1), \quad x \sim \gamma^{-1} \ge 1.$$

With allowance made for Eqns (118) and (123) we obtain

$$f = 1 - n^{2} \exp\left[-\frac{1}{4}\gamma I_{1}(x - x^{*}) - \frac{4n^{2}}{\gamma I_{1}^{2}}\left[1 - \exp\left(-\gamma I_{1}(x - x^{*})\right)\right] - 2x^{*}\right], \quad x \sim \gamma^{-1} \ge 1.$$
(130)

For $\gamma^{-1} \ge x \ge 1$, the order parameter (130) reduces to f_0 defined by Eqn (117), while in the inverse limit $x \ge \gamma^{-1}$ it takes the form

$$f = 1 - n^{2} \exp\left[-\frac{1}{4}\gamma I_{1}(x - x^{*}) - \frac{4n^{2}}{\gamma I_{1}^{2}} - 2x^{*}\right], \qquad x \gg \gamma^{-1}.$$
(131)

4.5.5 The limiting value of the order parameter. Formula (131) would hold as long as $1 - f(x) \ge 1 - f_{\infty}$. In the region where $1 - f(x) \sim 1 - f_{\infty}$, the two last terms in Eqn (99) depend differently on *x*. Considering that

$$1 - f_{\infty} \sim \exp\left(-\frac{4}{3\gamma n^2}\right)$$

[see below Eqn (141)] and comparing

$$n^2 \exp(4F_0) \approx \exp(-2n^2\gamma x)$$

with

$$\frac{1}{2}(1-f^2)\exp(2F_1)\approx \exp\left(-\frac{1}{3\gamma n^2}-\frac{3}{2}n^2\gamma x\right)$$

we see that the two last terms in Eqn (99) may compensate each other only for $x \ll \gamma^{-2}$. (In this estimate it is assumed that $n \sim 1$.) In the outlying region $x \gg \gamma^{-2}$, the last term in Eqn (99) dominates. In the region $x \sim \gamma^{-2}$, the order parameter makes a transition from the exponential behavior, as indicated by Eqn (131), to the limiting value f_{∞} . In order to find f_{∞} , it is necessary to solve Eqn (99) in the far peripheral region, where $x \sim \gamma^{-2}$.

In the region $x \sim \gamma^{-2}$, equation (99) assumes the form

$$f'' + f\left[\frac{1}{2}(1-f^2)\exp\left(\frac{4n^2}{\gamma I_1^2} - \frac{3}{4}\gamma I_1(x-x^*) + 2x^*\right) - n^2\exp\left(-\gamma I_1(x-x^*)\right)\right] = 0.$$

This equation can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\,\frac{\mathrm{d}\psi}{\mathrm{d}t}\right) - \psi = 1 - b^{-1}t^{1/3}\,,\tag{132}$$

by linearizing it in $f_{\infty} - f \ll 1$ and introducing the change of variables:

$$\psi = \frac{f_{\infty} - f}{1 - f_{\infty}},\tag{133}$$

$$t = \left(\frac{4}{3\gamma I_1}\right)^2 \exp\left(\frac{4n^2}{\gamma I_1^2} - \frac{3}{4}\gamma I_1(x - x^*) + 2x^*\right).$$
 (134)

⁷ An analogous situation arises in a dense plasma, when the Debye radius is small compared to the characteristic dimensions of the problem. In this case, the higher derivative can be neglected, and the coordinate dependence of the plasma density is determined by the quasi-neutrality condition.

The region $x \ll \gamma^{-2}$ corresponds to $t \sim \exp(1/\gamma) \gg 1$; $t \sim \exp(-3/2\gamma) \ll 1$ corresponds to the inverse limit $x \gg \gamma^{-2}$. Thus, we are interested in that solution of Eqn (132) which satisfies the boundary conditions

$$\psi \to 0, \qquad t \to 0, \qquad (135)$$

$$\psi \to 1 - b^{-1} t^{1/3}, \quad t \to \infty.$$
 (136)

All the physical parameters of the problem enter into Eqn (132) and the boundary conditions (135) and (136) in the unique combination

$$b \equiv \frac{1 - f_{\infty}}{n^2} \left(\frac{4}{3\gamma I_1}\right)^{2/3} \exp\left[\frac{4}{3} \left(\frac{4n^2}{\gamma I_1^2} + 2x^*\right)\right].$$
 (137)

The general solution of Eqn (132) has the form

$$\begin{aligned} \psi(t) &= C_1 I_0(2\sqrt{t}) + C_2 K_0(2\sqrt{t}) \\ &+ I_0(2\sqrt{t}) \int_0^t \frac{\mathrm{d}t'}{t' I_0^2(2\sqrt{t'})} \int_0^{t'} \mathrm{d}t'' \, I_0(2\sqrt{t''}) \left[1 - b^{-1}(t'')^{1/3}\right], \end{aligned}$$
(138)

where $I_0(x)$ and $K_0(x)$ are the modified Bessel functions. To satisfy the boundary condition (135), it is necessary to set both constants of integration equal to zero: $C_1 = C_2 = 0$. The second boundary condition set by Eqn (136) requires that

$$\int_0^\infty \frac{\mathrm{d}t}{t I_0^2(2\sqrt{t})} \int_0^t \mathrm{d}t' \, I_0(2\sqrt{t'}) \left[1 - b^{-1}(t')^{1/3}\right] = 0 \, dt$$

From this relationship one obtains the constant b entering into Eqn (137).

Considering that

$$\int_0^\infty \frac{\mathrm{d}t}{tI_0^2(2\sqrt{t})} \int_0^t \mathrm{d}t' I_0(2\sqrt{t'}) = 1,$$

we have

$$b = \int_0^\infty \frac{\mathrm{d}t}{t \left[I_0(2\sqrt{t}) \right]^2} \int_0^t \mathrm{d}t' \, (t')^{1/3} I_0(2\sqrt{t'}) = 0.7974 \dots (139)$$

Relations (137) and (139) determine the limiting value f_{∞} of the order parameter:

$$1 - f_{\infty} = bn^2 \left(\frac{3\gamma I_1}{4}\right)^{2/3} \exp\left[-\frac{4}{3}\left(\frac{4n^2}{\gamma I_1^2} + 2x^*\right)\right], \quad \gamma \ll 1.$$
(140)

To correctly determine the *n*-dependence of the preexponential, it is necessary, in the exponent appeared in equation (140), to expand the integral I_1 defined in Eqn (105) up to terms of the first order in γ . Since equation (114) cannot be solved analytically, the common multiplier can only be found numerically. The final result is as follows [40]

$$f_{\infty} = 1 - \mu(n)\gamma^{2/3} \exp\left(-\frac{4}{3\gamma n^2}\right), \quad \gamma \ll 1.$$
 (141)

The numerically determined dependence $\mu(n)$ is presented in Table 2. Within the accuracy of the calculations, this

Table	2.

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	0.594	0.484	0.536	0.607	0.677	0.748	0.817	0.885	0.951	1.015



dependence can be approximated by the formula

$$\mu(n) = \frac{a+bn^2}{n^{4/3}}, \quad a = 0.394, \quad b = 0.215.$$

If one takes a closer look at how the order parameter approaches its limiting value, three — instead of two — key regions of variation of the functions f, F_0 , and F_1 are recognized. The major region of variation of f and F_0 is the core of the string, where $x \sim 1$. Here the order parameter varies from zero to near unity. The negative Tolman mass of the string, governed by Eqn (30), is also localized within the core. The derivative F'_1 varies from unity on the axis to the limiting value (107) outside of the string, in the region $x \sim \gamma^{-1} \ge 1$. In the outlying region $\gamma^{-1} \ll x \ll \gamma^{-2}$, the order parameter approaches unity much more slowly — but still exponentially in accordance with dependence (131). And finally, in the farthest peripheral region $x \sim \gamma^{-2}$, the exponential behavior of f gives way to its flattening out to f_{∞} . The function $\ln(1 - f(x))$ shown in Fig. 7 illustrates the three regions of variation of the order parameter f(x) for n = 1 and $\gamma = 0.1$.

When $\gamma \ll 1$, many properties of global strings can be investigated by means of linearized Einstein equations [32]. Some of these properties, however, do not show up in any order of the expansion of the Einstein-Higgs equations in powers of γ . The exponentially small deviation of f_{∞} from unity, seen from Eqn (141), is a typical example.

The deviation of the limiting value f_{∞} of the order parameter from unity distorts the physical vacuum over the entire space outside the string, effectively leading to the nonzero 'cosmological constant'⁸

$$\Lambda = \frac{\lambda \gamma^2}{8\pi G} (1 - f_\infty)^2 \,. \tag{142}$$

As discussed in monograph [1], an estimate of the fluctuations needed for a galaxy formation process to proceed leads to the constraint $\lambda \leq 10^{-12}$. On the Grand Unification scale $(\eta \sim 10^{16} \text{ GeV})$, the parameter γ introduced by Eqn (8) is of order 10^{-5} , and the 'cosmological constant' Λ is small. However, the 'cosmological constant' of a string nature, governed by Eqn (142), may be of importance in topological

⁸ The possibility of there being a link between the cosmological constant and elementary particles has been pointed out by Zel'dovich [41]. In the present case, the global string plays a similar role. The 'equation of state' of the form $p = -\varepsilon$ is mentioned in Ref. [41].

4.6 Metric for $\gamma \rightarrow \gamma_{max}$

Near the critical value, viz.

$$\gamma_{\max} - \gamma \ll 1 , \tag{143}$$

the metric can be found analytically. Here, the order parameter f is very small, and Eqns (100) and (101) reduce to

$$F_0'' = -\frac{\gamma_{\max}}{4} \exp(2F_1), \qquad (144)$$

$$F_1'' = -\frac{3\gamma_{\max}}{4} \exp(2F_1).$$
 (145)

The solution of Eqn (145) is given by

$$F_1 = \mathcal{B} - \ln \cosh (x^1 - x_0 - x^*),$$
 (146)

where x_0 is the constant of integration, and $\mathcal{B} = -(1/2) \ln (3\gamma_{\text{max}}/4)$ and $x^* = \mathcal{B} + \ln 2$ are chosen in such a way as to satisfy the boundary condition $F_1 = x^1 - x_0$, $x^1 \to -\infty$.

At n = 1 we have

$$\mathcal{B} = 0.11142\ldots, \quad x^* = 0.80457\ldots$$

For F_0 we then obtain

$$F_0 = \frac{1}{3} \left[\mathcal{B} - x^1 + x_0 - \ln \cosh \left(x^1 - x_0 - x^* \right) \right].$$
(147)

The coincidence of the analytical solutions (146) and (147) with the results of the numerical integration of Eqns (99)–(101) is illustrated in Fig. 8, in which $F_0(x)$ and $F_1(x)$ are plotted as functions of $x = x^1 - x_0$ for n = 1 and $\gamma = \gamma_{\text{max}} = 1.067$.



Figure 8. F_0 and F_1 as functions of $x = x^1 - x_0$. Dots are the results of the numerical integration of the set of equations (99)–(101) with $\gamma = 1.067$; solid lines depict the analytical solutions (147) and (146).

After substitution of Eqn (47), the metric assumes the form

$$ds^{2} = g_{00}(r)(dt^{2} - dz^{2}) - dr^{2} - g_{22}(r) d\phi^{2}.$$
 (148)

Here the following notation was used:

$$g_{00}(r) = \cos^{4/3} (r \exp(-x^*)),$$

$$g_{22}(r) = \exp(2x^*) \sin^2 (r \exp(-x^*)) \cos^{-2/3} (r \exp(-x^*)).$$

$$r = \exp x^* \arctan \left[\exp \left(x^1 - x_0 - x^* \right) \right]$$

and varies from zero to its limiting value $r_{\text{max}} = (\pi/2) \exp x^*$, whereas x^1 varies from $-\infty$ to $+\infty$. Near the axis $(r \to 0)$, the metric (148) is Galilean:

$$ds^{2} = dt^{2} - dr^{2} - r^{2} d\varphi^{2} - dz^{2},$$

and as $r \to r_{\text{max}} = (\pi/2) \exp x^*$ it becomes of a Kasner type [24]:

$$ds^{2} = (r_{\max} - r)^{4/3} \exp\left(-\frac{4}{3}x^{*}\right)(dt^{2} - dz^{2}) - dr^{2} - (r_{\max} - r)^{-2/3} \exp\left(\frac{8}{3}x^{*}\right)d\varphi^{2}.$$
 (149)

5. Concluding remarks

At the critical point (143), the order parameter f of the global string vanishes. It becomes evident that in the Abelian Higgs model with potential (3) not only the Kasner singularity at the bound $r = r_{\text{max}}$ [38] but also the curvature of the entire metric (148) is generated by the vacuum which is 'spoiled' by the fact that the potential $V(\phi)$ stimulating spontaneous breaking of symmetry does not vanish outside the string. The deviation of the limiting value of the order parameter from unity may have major cosmological consequences because it is equivalent to the appearance of a nonzero cosmological constant in the Einstein equations.

The hypothesis of cosmic 'censorship' [39] forbids the development of bare singularities from regular original states. The formation of global strings and supermassive gauge strings in the early Universe can be reconciled in a natural way with the cosmic censorship hypothesis by assuming that the original state of nonbroken symmetry was not a regular state.

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References

- 1. Vilenkin A, Shellard E P S Cosmic Strings and Other Topological Defects (Cambridge: Cambridge Univ. Press, 1994)
- Zel'dovich Ya B, Novikov I D *Relyativistskaya Astrofizika* (Relativistic Astrophysics) (Moscow: Nauka, 1967) [Translated into English: *Stars and Relativity* (Chicago: Chicago Univ. Press, 1971)]
- 3. Kibble T W B J. Phys. A 9 1387 (1976)
- Landau L D, Lifshitz E M Phys. Zs. Sowjet. 8 153 (1935); also in Landau L D Sobranie Trudov Vol. 1 (Collected Works) (Moscow: Nauka, 1969) p. 128
- Landau L D Zh. Eksp. Teor. Fiz. 7 19 (1937) [Phys. Zs. Sowjet. 11 26 (1937)]; also in Landau L D Sobranie Trudov Vol. 1 (Collected Works) (Moscow: Nauka, 1969) p. 234
- Abrikosov A A Zh. Eksp. Teor. Fiz. 32 1442 (1957) [Sov. Phys. JETP 5 1174 (1957)]
- 7. Khalatnikov I M Zh. Eksp. Teor. Fiz. 23 253 (1952)
- Kirzhnits D A Pis'ma Zh. Eksp. Teor. Fiz. 15 745 (1972) [JETP Lett. 15 529 (1972)]
- 9. Weinberg S Phys. Rev. D 9 3357 (1974)
- Zel'dovich Ya B, Kobzarev I Yu, Okun' L B Zh. Eksp. Teor. Fiz. 67 3 (1974) [Sov. Phys. JETP 40 1 (1975)]
- 11. Zel'dovich Ia B Month. Not. R. Astron. Soc. 192 663 (1980)

- Vilenkin A Phys. Rev. Lett. 46 1169 (1981); Erratum: Phys. Rev. Lett. 46 1496 (1981)
- 13. Linde A Phys. Lett. B 327 208 (1994)
- 14. Vilenkin A Phys. Rev. Lett. 72 3137 (1994)
- 15. De Laix A, Trodden M, Vachaspati T Phys. Rev. D 57 7186 (1998)
- 16. Ginzburg V L, Landau L D Zh. Eksp. Teor. Fiz. 20 1064 (1950)
- 17. Cooper L N Phys. Rev. 104 1189 (1956)
- 18. Bronnikov K A J. Phys. A 12 201 (1979)
- Meĭerovich B É Zh. Eksp. Teor. Fiz. 112 385 (1997) [JETP 85 209 (1997)]; Meierovich B E Grav. Cosmology 3 29 (1997); Phys. Rev. D 61 024004 (2000)
- Landau L D, Lifshitz E M *Teoriya Polya* (The Classical Theory of Fields) (Moscow: Nauka, 1973) [Translated into English (Oxford: Pergamon Press, 1975)]
- 21. Meierovich B E, Podolyak E R Phys. Rev. D 61 125007 (2000)
- Bach R, Weyl H Math. Z. 13 134 (1922); Marder L Proc. R. Soc. London Ser. A 252 45 (1959); Israel W Phys. Rev. D 15 935 (1977); Sokolov D D, Starobinskii A A Dokl. Akad. Nauk SSSR 234 1043 (1977) [Sov. Phys. Dokl. 22 312 (1977)]
- 23. Andreev A F Zh. Eksp. Teor. Fiz. 65 1045 (1974) [Sov. Phys. JETP 38 517 (1974)]
- 24. Kasner E Am. J. Math. 43 217 (1921)
- 25. Vilenkin A Phys. Rev. D 23 852 (1981)
- 26. Laguna P, Garfinkle D Phys. Rev. D 40 1011 (1989)
- Landau L D, Lifshitz E M Kvantovaya Mekhanika (Quantum Mechanics: Non-Relativistic Theory) (Moscow: Nauka, 1974) [Translated into English (Oxford: Pergamon Press, 1977)]
- 28. Christensen M, Larsen A L, Verbin Y Phys. Rev. D 60 125012 (1999)
- Nielsen N B, Olssen P *Nucl. Phys. B* 61 45 (1973)
 Bogomol'nyĭ E B *Yad. Fiz.* 24 861 (1976) [*Sov. J. Nucl. Phys.* 24 449
- (1976)]
- 31. Linet B Class. Quant. Grav. 7 L75 (1990)
- 32. Harari D, Sikivie P Phys. Rev. D 37 3488 (1988)
- 33. Gregory R Phys. Lett. B 215 663 (1988)
- 34. Gibbons G W, Ortiz M E, Ruiz F R Phys. Rev. D 39 1546 (1989)
- 35. Cohen A G, Kaplan D B Phys. Lett. B 215 67 (1988)
- 36. Meierovich B E Gen. Rel. Grav. **33** 405 (2001)
- 37. Pitaevskiĭ L P Zh. Eksp. Teor. Fiz. **40** 646 (1961) [Sov. Phys. JETP **13** 451 (1961)]
- 38. Harari D, Polychronakos A P Phys. Lett. B 240 55 (1990)
- 39. Penrose R Riv. Nuovo Cimento 1 252 (1969)
- 40. Meierovich B E, Podolyak E R Grav. Cosmology 7 117 (2001)
- 41. Zel'dovich Ya B Pis'ma Zh. Eksp. Teor. Fiz. 6 883 (1967) [JETP Lett. 6 316 (1967)]