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Simple example of the development of cluster structure of a passive tracer field in random flows

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Contents

Abstract. As a follow-up to Ref. [1], the formation of particle clusters (Lagrangian description) and cluster structure of the field of a passive tracer (Eulerian description) in a random velocity field is analyzed for a simple problem amenable to an analytical solution.

1. Introduction

At the early stages of development, the diffusion of density of a passive conservative tracer $\rho(\mathbf{r},t)$, moving in a random velocity field $\mathbf{u}(\mathbf{r},t)$, is described by the continuity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r},t)\right)\rho(\mathbf{r},t)=0\,,$$

which can be rewritten in the form

$$\left(\frac{\partial}{\partial t}+\mathbf{u}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\rho(\mathbf{r},t)+\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}\,\rho(\mathbf{r},t)=0\,. \qquad (1)$$

In the general case the random field $\mathbf{u}(\mathbf{r},t)$ may contain both a solenoidal (divergence-free) component, for which $\operatorname{div}\mathbf{u}(\mathbf{r},t)=0$, and a divergent component with $\operatorname{div}\mathbf{u}(\mathbf{r},t)\neq 0$. The total mass of the tracer is conserved in the course of evolution — that is,

$$M=\int \rho(\mathbf{r},t)\,\mathrm{d}\mathbf{r}=\int \rho_0(\mathbf{r})\,\mathrm{d}\mathbf{r}=\text{const}\,.$$

This is the Eulerian description of the evolution of the density field. Equation (1) is an equation in partial derivatives of the first order, and can be solved using the method of characteristics. Defining the characteristic curves $\mathbf{r}(t)$ — the paths of particles described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t)=\mathbf{u}(\mathbf{r},t)\,,\qquad \mathbf{r}(0)=\mathbf{r}_0\,, \qquad (2)$$

we go over from Eqn (1) to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t)=-\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}\,\rho(t)\,,\qquad \rho(0)=\rho_0(\mathbf{r}_0)\,. \qquad (3)$$

The solutions of Eqns (2), (3) admit an obvious geometrical interpretation. They describe the evolution of density in the neighborhood of a fixed tracer particle, whose path is described by the equation $\mathbf{r}=\mathbf{r}(t)$. As follows from Eqn (3), the density in divergent flows varies — it is greater in the regions of compression and smaller in the regions of rarefaction of the medium. The divergence (the elementary volume of particle) $j(t)=\operatorname{Det}\|\,\partial r_i(t)/\partial r_{0j}\,\|$ is linked to the density of the particle by the equation

$$j(t)=\frac{\rho_0(\mathbf{r}_0)}{\rho(t)}\,.$$

Solutions of the set (2), (3) depend on the characteristic parameter — the initial coordinate of particle $\mathbf{r}_0$:

$$\mathbf{r}(t)=\mathbf{r}(t|\mathbf{r}_0)\,,\qquad \rho(t)=\rho(t|\mathbf{r}_0)\,, \qquad (4)$$

which will be denoted by a vertical bar. The components of vector $\mathbf{r}_0$, which uniquely determines the position of an

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arbitrary particle, are known as its Lagrangian coordinates. Then Eqns (2), (3) correspond to the Lagrangian description of the density field. The linkage between Eulerian and Lagrangian descriptions is given by the first expression in Eqn (4). Resolving it with respect to $\mathbf{r}_0$, we get a relation that expresses Lagrangian coordinates in terms of Eulerian coordinates:

$$\mathbf{r}_0 = \mathbf{r}_0(t, \mathbf{r}) \,. \tag{5}$$

Now we use Eqn (5) to eliminate the dependence on $\mathbf{r}_0$ in the last expression in Eqn (4), and return to the Eulerian description of density:

$$\rho(\mathbf{r}, t) = \rho\big(t|\mathbf{r}_0(t, \mathbf{r})\big) \,. \tag{6}$$

For the stationary velocity field $\mathbf{u}(\mathbf{r}, t) \equiv \mathbf{u}(\mathbf{r})$ Eqn (2) is simplified:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t) = \mathbf{u}(\mathbf{r}) \,, \qquad \mathbf{r}(0) = \mathbf{r}_0 \,. \tag{7}$$

Hence it follows that the stationary points $\tilde{\mathbf{r}}$, at which $\mathbf{u}(\tilde{\mathbf{r}}) = 0$, remain fixed. Depending on whether such points are stable or unstable, they will either attract or repel particles in their neighborhood. Owing to the stochastic nature of the function $\mathbf{u}(\mathbf{r})$, the positions of points $\tilde{\mathbf{r}}$ are also random.

A similar situation ought to persist in the general case of a random space-time velocity field $\mathbf{u}(\mathbf{r}, t)$.

If some points $\tilde{\mathbf{r}}$ remain stable for a sufficiently long time, then in certain realizations of the random field $\mathbf{u}(\mathbf{r}, t)$ clusters of particles ought to form in their neighborhood (that is, compact regions of elevated particle density, which are located in less dense areas). If, however, the stable points become unstable soon enough, and the particles do not have the time to rearrange, then cluster regions do not form.

Numerical simulation [2–4] indicates that the dynamic behavior of the system of particles can be considerably different depending on whether the random velocity field $\mathbf{u}(\mathbf{r})$ is divergent or divergence-free. For example, in Fig. 1, reproduced from Ref. [1], for a particular realization of the divergence-free velocity field $\mathbf{u}(\mathbf{r})$ (two-dimensional), the evolution of a system of particles homogeneously distributed in a circle is shown schematically, with the dimensionless time related to the statistical parameters of the field $\mathbf{u}(\mathbf{r})$. In this case the area enclosed by the contour is conserved, and the particles more or less uniformly fill up the distorted shape that had initially been the circle, the boundary now cusped in fractal-like fashion. When, however, the velocity field $\mathbf{u}(\mathbf{r})$ is divergent, the particles that were initially distributed uniformly within the square, eventually huddle in clusters. The results of numerical simulation for this case are shown in Fig. 1b. Observe that this is a purely kinematic phenomenon. This feature of dynamic behavior of particles disappears upon averaging over the ensemble of random velocity field realizations.

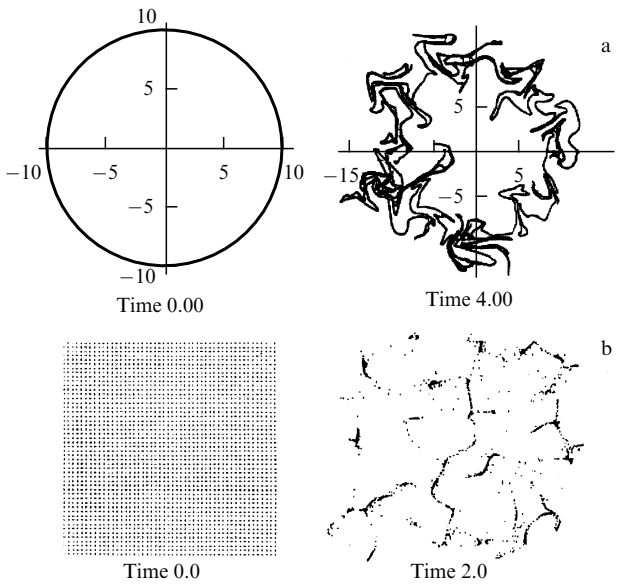

Figure 1. Results of numerical simulation of diffusion of a system of particles in solenoidal (a) and divergent (b) random velocity fields $\mathbf{u}(\mathbf{r})$.

The purpose of this paper is to demonstrate the effect of clustering of particles and passive tracer field in a random velocity field using a simple example of velocity field that makes possible analytical solution of the problem.

First of all we recall the basic ideas of statistical topography, which allow us to define and describe the process of clustering of the Eulerian density field in the Gaussian delta-correlated velocity field $\mathbf{u}(\mathbf{r}, t)$. As will be demonstrated thereafter, this general case is statistically equivalent to the problem of diffusion of tracers in a random velocity field of the form $\mathbf{u}(\mathbf{r}, t) = \mathbf{v}(t) f(\mathbf{r})$, where $f(\mathbf{r})$ is a deterministic 'quasi-periodical' function, and $\mathbf{v}(t)$ is a vector Gaussian process delta-correlated in time. Observe that the requirement of delta-correlation can be dropped in the case of a one-dimensional velocity field. Finally, we shall give an example that admits analytical solution for every realization of the random velocity field. This example illustrates the predictions of the general theory.

So, to describe the density field in the Eulerian representation, we define the indicator function

$$\Phi(t, \mathbf{r}; \rho) = \delta\big(\rho(\mathbf{r}, t) - \rho\big) \,, \tag{8}$$

localized on the surface $\rho(\mathbf{r}, t) = \rho = \text{const}$ in the three-dimensional case, or on a contour in the case of two-dimensions. The evolution of this function is described by the Liouville equation [1, 5]

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}}\right)\Phi(t, \mathbf{r}; \rho) = \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial \mathbf{r}}\,\frac{\partial}{\partial \rho}\,\rho\Phi(t, \mathbf{r}; \rho) \,,$$

$$\Phi(0, \mathbf{r}; \rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big) \,. \tag{9}$$

The one-point probability density for a solution of the dynamic equation (1) in this case coincides with the indicator function averaged over the ensemble of realizations:

$$P(t, \mathbf{r}; \rho) = \big\langle \Phi(t; \mathbf{r}; \rho) \big\rangle \,.$$

The indicator function also characterizes the geometrical structure of the density field [1, 5]. In terms of function (8) in the two-dimensional case, for example, it is possible to express such quantities as the total area of regions delimited by the level lines, where $\rho(\mathbf{r}, t) > \rho$:

$$S(t, \rho) = \int_{\rho}^{\infty} \mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\, \Phi(t, \mathbf{r}; \tilde{\rho}) \,, \tag{10}$$

and the total 'mass' of the field confined within these regions:

$$M(t,\rho) = \int_{\rho}^{\infty} \tilde{\rho}\, \mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\, \Phi(t,\mathbf{r};\tilde{\rho})\,. \tag{11}$$

The random component of the velocity field in the general case is assumed to have a nonzero divergence ($\operatorname{div}\mathbf{u}(\mathbf{r},t) \neq 0$), and is represented by a statistically homogeneous, space-isotropic and time-stationary random Gaussian field with the correlation and spectral tensors $\big(\langle \mathbf{u}(\mathbf{r},t)\rangle \equiv 0\big)$ given by

$$\big\langle u_i(\mathbf{r},t) u_j(\mathbf{r}',t')\big\rangle = B_{ij}(\mathbf{r}-\mathbf{r}', t-t')\,,$$

$$E_{ij}(\mathbf{k},t) = \frac{1}{(2\pi)^N} \int \mathrm{d}\mathbf{r}\, B_{ij}(\mathbf{r},t) \exp(-\mathrm{i}\mathbf{k}\mathbf{r})\,, \tag{12}$$

where N is the dimensionality of space, and the structure of spectral tensor of velocity field is

$$E_{ij}(\mathbf{k},t) = E^{\mathrm{s}}(k,t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) + E^{\mathrm{p}}(k,t)\,\frac{k_i k_j}{k^2}\,. \tag{13}$$

Here we denote respectively the solenoidal and the potential components of the spectral density of the velocity field by $E^{\mathrm{s}}(k,t)$ and $E^{\mathrm{p}}(k,t)$.

Usually the calculation of the statistical properties of the field is based on the assumption that the velocity field $\mathbf{u}(\mathbf{r},t)$ is delta-correlated in time (see, for example, Ref. [6]). In this case the correlation tensor (12) is approximated by

$$B_{ij}(\mathbf{r},t) = 2B_{ij}^{\mathrm{eff}}(\mathbf{r})\delta(t)\,,$$

where

$$B_{ij}^{\mathrm{eff}}(\mathbf{r}) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\, B_{ij}(\mathbf{r},t) = \int_{0}^{\infty} \mathrm{d}t\, B_{ij}(\mathbf{r},t)\,.$$

When Eqn (9) is averaged with respect to realizations of the field $\mathbf{u}(\mathbf{r},t)$, the field $\mathbf{u}(\mathbf{r},t)$ becomes correlated with the function $\Phi(t,\mathbf{r};\rho)$, which is a functional of the field $\mathbf{u}(\mathbf{r},t)$. For the Gaussian velocity field $\mathbf{u}(\mathbf{r},t)$, the split-up of correlation is based on the Furutsu–Novikov formula (see, for example, Ref. [6]):

$$\big\langle u_\alpha(\mathbf{r},t)\Phi\big[\mathbf{u}(\tilde{\mathbf{r}},\tilde{t})\big]\big\rangle = \int \mathrm{d}\mathbf{r}' \int \mathrm{d}t'\, B_{\alpha\beta}(\mathbf{r}-\mathbf{r}', t-t') \times \left\langle \frac{\delta}{\delta u_\beta(\mathbf{r}',t')}\big[\Phi\big[\mathbf{u}(\tilde{\mathbf{r}},\tilde{t})\big]\right\rangle, \tag{14}$$

which holds for arbitrary functional $\Phi\big[\mathbf{u}(\tilde{\mathbf{r}},\tilde{t})\big]$ of the Gaussian field $\mathbf{u}(\mathbf{r},t)$, and is basically a formula of integration by parts in functional space.

Averaging Eqn (9) over the ensemble of realizations of random field $\mathbf{u}(\mathbf{r},t)$ in the approximation of time delta-correlated field $\mathbf{u}(\mathbf{r},t)$, we get the equation for the probability density of the density field in the form [1]

$$\left(\frac{\partial}{\partial t} - D_0\,\frac{\partial^2}{\partial \mathbf{r}^2}\right) P(t,\mathbf{r};\rho) = D^{\mathrm{p}}\,\frac{\partial^2}{\partial \rho^2}\,\rho^2 P(t,\mathbf{r};\rho)\,,$$

$$P(0,\mathbf{r};\rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,, \tag{15}$$

where

$$D_0 = \int_0^{\infty} \mathrm{d}t \int \mathrm{d}\mathbf{k}\,\big[(N-1)E^{\mathrm{s}}(k,t) + E^{\mathrm{p}}(k,t)\big]\,,$$

$$D^{\mathrm{p}} = \int_0^{\infty} \mathrm{d}t \int \mathrm{d}\mathbf{k}\, k^2 E^{\mathrm{p}}(k,t)\,.$$

Knowing the solution of Eqn (15), we can calculate the time evolution of such functionals of the density field as — in the two-dimensional case, for example — the mean total area $\langle S(t,\rho)\rangle$ where $\rho(\mathbf{r},t) > \rho$, and the mean total mass of tracer contained within such areas $\langle M(t,\rho)\rangle$:

$$\big\langle S(t,\rho)\big\rangle = \int_{\rho}^{\infty} \mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\, P(t,\mathbf{r};\tilde{\rho})\,,$$

$$\big\langle M(t,\rho)\big\rangle = \int_{\rho}^{\infty} \tilde{\rho}\, \mathrm{d}\tilde{\rho} \int \mathrm{d}\mathbf{r}\, P(t,\mathbf{r};\tilde{\rho})\,,$$

or ($\tau = D^{\mathrm{p}} t$),

$$\big\langle S(t,\rho)\big\rangle = \int \mathrm{d}\mathbf{r}\, \Phi\left(\frac{\ln\big(\rho_0(\mathbf{r})\exp(-\tau)/\rho\big)}{2\sqrt{\tau}}\right),$$

$$\big\langle M(t,\rho)\big\rangle = \int \rho_0(\mathbf{r})\, \mathrm{d}\mathbf{r}\, \Phi\left(\frac{\ln\big(\rho_0(\mathbf{r})\exp(\tau)/\rho\big)}{2\sqrt{\tau}}\right), \tag{16}$$

where $\Phi(z)$ is the error function

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{d}y\, \exp\left(-\frac{y^2}{2}\right).$$

Hence it follows that for $\tau \gg 1$, the mean area of regions where the density is above the preset level ρ, decreases with time as

$$\big\langle S(t,\rho)\big\rangle = \frac{1}{\sqrt{\pi\tau\rho}}\exp\left(-\frac{\tau}{4}\right)\int \sqrt{\rho_0(\mathbf{r})}\,\mathrm{d}\mathbf{r}\,,$$

while the mean mass of tracer contained within such regions

$$\big\langle M(t,\rho)\big\rangle = M - \sqrt{\frac{\rho}{\pi\tau}}\exp\left(-\frac{\tau}{4}\right)\int \sqrt{\rho_0(\mathbf{r})}\,\mathrm{d}\mathbf{r}$$

tends steadily to the total mass of tracer $M = \int \rho_0(\mathbf{r})\,\mathrm{d}\mathbf{r}$. This points to the formation of clusters of tracer field in the random field of velocities — compact regions of elevated density surrounded by areas of smaller density.

If the initial density of tracer is the same throughout, $\rho_0(\mathbf{r}) = \rho_0 = \mathrm{const}$, then the probabilistic density distribution does not depend on the coordinate $\mathbf{r}$, and is log-normal with the following probability density and probability distribution function:

$$P(t;\rho) = \frac{1}{2\rho\sqrt{\pi\tau}}\exp\left\{-\frac{\ln^2(\rho\exp\tau/\rho_0)}{4\tau}\right\},$$

$$F(t;\rho) = \Phi\left(\frac{\ln(\rho\exp\tau/\rho_0)}{2\sqrt{\tau}}\right). \tag{17}$$

2. Model of velocity field

Consider now a model of the random velocity field of the form

$$\mathbf{u}(\mathbf{r},t) = \mathbf{v}(t) f(\mathbf{r})\,, \tag{18}$$

where $\mathbf{v}(t)$ is a random Gaussian stationary vector process with the parameters

$$\langle \mathbf{v}(t) \rangle = 0\,, \quad B_{ij}(t-t') = \langle v_i(t) v_j(t') \rangle \quad \left(B_v(0) = \langle \mathbf{v}^2(t) \rangle\right),$$

and $f(\mathbf{r})$ a deterministic function.

In the approximation of the delta-correlated process $\mathbf{v}(t)$, we assume that

$$B_{ij}(t-t') = 2\sigma^2 \delta_{ij} \tau_0 \delta(t-t') \qquad \left(\sigma^2 \delta_{ij} \tau_0 = \int_0^\infty \mathrm{d}\tau B_{ij}(\tau)\right), \tag{19}$$

where σ^2 is the variance of fluctuations of the velocity field, and τ_0 is the time-domain correlation radius. Then the Furutsu – Novikov formula is simplified:

$$\langle v_\alpha(t) \Phi[\mathbf{v}(\tilde{t})] \rangle = \sigma^2 \tau_0 \left\langle \frac{\delta}{\delta v_\alpha(t)} \left[\Phi[\mathbf{v}(\tilde{t})]\right] \right\rangle,$$

and the Liouville equation (9) becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(t) f(\mathbf{r}) \frac{\partial}{\partial \mathbf{r}}\right) \Phi(t, \mathbf{r}; \rho) = \mathbf{v}(t) \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \frac{\partial}{\partial \rho} \rho \Phi(t, \mathbf{r}; \rho)\,,$$

$$\Phi(0, \mathbf{r}; \rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,,$$

which we rewrite in the form

$$\frac{\partial}{\partial t} \Phi(t, \mathbf{r}; \rho) = -\mathbf{v}(t) \left\{ \frac{\partial}{\partial \mathbf{r}} f(\mathbf{r}) - \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} \Phi(t, \mathbf{r}; \rho)\,,$$

$$\Phi(0, \mathbf{r}; \rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,. \tag{20}$$

Averaging Eqn (20) over the ensemble of random processes $\mathbf{v}(t)$, we get

$$\frac{\partial}{\partial t} P(t, \mathbf{r}; \rho) = -\int_0^t \mathrm{d}t'\, B_{ij}(t-t') \times \left\{ \frac{\partial}{\partial r_i} f(\mathbf{r}) - \frac{\partial f(\mathbf{r})}{\partial r_i} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} \left\langle \frac{\delta}{\delta v_j(t')} \Phi(t, \mathbf{r}; \rho) \right\rangle,$$

$$P(0, \mathbf{r}; \rho) = \delta\big(\rho_0(\mathbf{r}) - \rho\big)\,. \tag{21}$$

Now using the equality

$$\frac{\delta}{\delta v_j(t)} \Phi(t, \mathbf{r}; \rho) = -\left(f(\mathbf{r}) \frac{\partial}{\partial r_j} - \frac{\partial f(\mathbf{r})}{\partial r_j} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right) \Phi(t, \mathbf{r}; \rho)\,,$$

which follows from Eqn (20), in the approximation of the delta-correlated process $\mathbf{v}(t)$, we get an equation in probability density of the form

$$\begin{aligned} \frac{\partial}{\partial t} P(t, \mathbf{r}; \rho) = \sigma^2 \tau_0 \Bigg\{ & \frac{\partial^2}{\partial \mathbf{r}^2} f^2(\mathbf{r}) - \left(3 + 2 \frac{\partial}{\partial \rho} \rho\right) \frac{\partial}{\partial \mathbf{r}} f(\mathbf{r}) \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \\ & + f(\mathbf{r}) \frac{\partial^2 f(\mathbf{r})}{\partial \mathbf{r}^2} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \\ & + \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \left(1 + \frac{\partial}{\partial \rho} \rho\right)^2 \Bigg\} P(t, \mathbf{r}, \rho)\,. \end{aligned} \tag{22}$$

If now the function $f(\mathbf{r})$ has the characteristic scale of variation $\tilde{k}^{-1}$ with respect to $\mathbf{r}$, and is a quasi-periodical ('fast') function, then we can also average Eqn (22) with respect to these scales and find the equation for 'slow' spatial variations

$$\frac{\partial}{\partial t} P(t, \mathbf{r}; \rho) = \sigma^2 \tau_0 \left\{ \overline{f^2(\mathbf{r})} \frac{\partial^2}{\partial \mathbf{r}^2} + \overline{\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}}} \frac{\partial^2}{\partial \rho^2} \rho^2 \right\} P(t, \mathbf{r}, \rho)\,. \tag{23}$$

Equation (23) coincides with Eqn (15), and therefore the model of the velocity field (18) for one-point statistical characteristics of the density field is statistically equivalent to the model of the Gaussian delta-correlated field $\mathbf{u}(\mathbf{r}, t)$. Therefore, in the model of the fluctuations of the velocity field (18) the tracers should also exhibit clustering if

$$\mathbf{v}(t) \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \neq 0\,.$$

For two-point statistical characteristics these models will obviously be statistically equivalent too.

Observe that for the one-dimensional case the random field of velocities $u(x, t) = v(t) f(x)$ is always potential, and the assumption that the process $v(t)$ is delta-correlated is not necessary. Indeed, the structure of solution of Eqn (20) in this case is

$$\Phi(t, x, \rho) = \Phi\big(T(t), x, \rho\big)\,,$$

where the new 'random' time is $T(t) = \int_0^t \mathrm{d}\tau\, v(\tau)$, and the function $\Phi(T, x, \rho)$ as a function of its variables is defined by the deterministic equation

$$\frac{\partial}{\partial T} \Phi(T, x; \rho) = -\left\{ \frac{\partial}{\partial x} f(x) - \frac{\mathrm{d}f(x)}{\mathrm{d}x} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} \Phi(T, x; \rho)\,,$$

$$\Phi(0, x; \rho) = \delta\big(\rho_0(x) - \rho\big)\,. \tag{24}$$

Hence

$$\begin{aligned} \frac{\delta}{\delta v(t')} \Phi(t, x; \rho) &= \frac{\partial \Phi(T, x; \rho)}{\partial T} \frac{\delta}{\delta v(t')} T(t) \\ &= \frac{\partial \Phi(T, x; \rho)}{\partial T} \theta(t - t') \\ &= -\theta(t - t') \left\{ \frac{\partial}{\partial x} f(x) - \frac{\mathrm{d}f(x)}{\mathrm{d}x} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} \Phi(t, x; \rho)\,, \end{aligned}$$

where the Heaviside function $\theta(t)$ is 1 for $t > 0$, and 0 for $t < 0$; then Eqn (21) takes on the form of a closed equation

$$\begin{aligned} \frac{\partial}{\partial t} P(t, x; \rho) = & \int_0^t \mathrm{d}t'\, B(t - t') \left\{ \frac{\partial}{\partial x} f(x) - \frac{\mathrm{d}f(x)}{\mathrm{d}x} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} \\ & \times \left\{ \frac{\partial}{\partial x} f(x) - \frac{\mathrm{d}f(x)}{\mathrm{d}x} \left(1 + \frac{\partial}{\partial \rho} \rho\right) \right\} P(t, x; \rho)\,, \end{aligned}$$

$$P(0, x; \rho) = \delta\big(\rho_0(x) - \rho\big)\,,$$

which, after averaging with respect to the fast space variable, becomes

$$\begin{aligned} & \frac{\partial}{\partial t} P(t, x; \rho) \\ & \quad = \int_0^t \mathrm{d}\tau\, B(\tau) \left\{ \overline{f^2(x)} \frac{\partial^2}{\partial x^2} + \overline{\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2} \frac{\partial^2}{\partial \rho^2} \rho^2 \right\} P(t, x; \rho)\,. \end{aligned}$$

If the initial density distribution does not depend on x — that is, $\rho_0(x) = \rho_0$ — then $P(t, x; \rho) = P(t; \rho)$, and this

function is defined by the equation

$$\frac{\partial}{\partial t}\,P(t;\rho)=D(t)\,\frac{\partial^2}{\partial\rho^2}\,\rho^2 P(t,x;\rho)\,,$$

$$D(t)=\overline{\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2}\int_0^t \mathrm{d}\tau\,B(\tau)\,.$$

3. A simple example

In the simplest case, the function $f(\mathbf{r})$ is a function of one variable, and we select it in the form

$$f(\mathbf{r})=\sin 2(\mathbf{kr})\,. \tag{25}$$

Note that such a function $f(\mathbf{r})$ corresponds to the first term in the Fourier expansion of arbitrary $f(\mathbf{r})$, used in numerical simulations of the problem [3, 4].

3.1 Lagrangian description of the model

In this case the Lagrangian equations (2), (3) are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}(t)=\mathbf{v}(t)\sin 2(\mathbf{kr})\,,\qquad \mathbf{r}(0)=\mathbf{r}_0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t)=-2\big(\mathbf{kv}(t)\big)\cos 2(\mathbf{kr})\rho(t)\,,\qquad \rho(0)=\rho_0(\mathbf{r}_0)\,.$$

For such a model, the motion of a particle in the direction of vector $\mathbf{k}$ and in the perpendicular direction can be separated. So, if we align the x-axis with the direction of vector $\mathbf{k}$, then the equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x(t)=v_x(t)\sin(2kx)\,,\qquad x(0)=x_0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{R}(t)=\mathbf{v}_{\mathbf{R}}(t)\sin(2kx)\,,\qquad \mathbf{R}(0)=\mathbf{R}_0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t)=-2kv_x(t)\cos(2kx)\rho(t)\,,\qquad \rho(0)=\rho_0(\mathbf{r}_0)\,. \tag{26}$$

The solution of the first equation in Eqns (26) is

$$x(t)=\frac{1}{k}\arctan\big[\exp\big(T(t)\big)\tan(kx_0)\big]\,, \tag{27}$$

where

$$T(t)=2k\int_0^t \mathrm{d}\tau\,v_x(\tau)\,.$$

Using the equalities that follow from Eqn (27)

$$\sin(2kx)=\sin(2kx_0)\,\frac{1}{\exp\big(-T(t)\big)\cos^2(kx_0)+\exp\big(T(t)\big)\sin^2(kx_0)}\,,$$

$$\cos(2kx)=\frac{1-\exp\big(2T(t)\big)\tan^2(kx_0)}{1+\exp\big(2T(t)\big)\tan^2(kx_0)}\,,$$

we rewrite the last two equations in Eqns (26) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{R}(t|\mathbf{r}_0)=\sin(2kx_0)\,\frac{\mathbf{v}_{\mathbf{R}}(t)}{\exp\big(-T(t)\big)\cos^2(kx_0)+\exp\big(T(t)\big)\sin^2(kx_0)}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t|\mathbf{r}_0)=-2kv_x(t)\,\frac{1-\exp\big(2T(t)\big)\tan^2(kx_0)}{1+\exp\big(2T(t)\big)\tan^2(kx_0)}\,\rho(t|\mathbf{r}_0)\,.$$

Hence

$$\mathbf{R}(t|\mathbf{r}_0)=\mathbf{R}_0+\sin(2kx_0)\times\int_0^t \mathrm{d}\tau\,\frac{\mathbf{v}_{\mathbf{R}}(\tau)}{\exp\big(-T(\tau)\big)\cos^2(kx_0)+\exp\big(T(\tau)\big)\sin^2(kx_0)}\,,$$

$$\rho(t|\mathbf{r}_0)=\rho_0(\mathbf{r}_0)\big[\exp\big(-T(t)\big)\cos^2(kx_0)+\exp\big(T(t)\big)\sin^2(kx_0)\big]\,. \tag{28}$$

Then the divergence (the elementary volume of a particle) is

$$j(t|\mathbf{r}_0)=\frac{1}{\exp\big(-T(t)\big)\cos^2(kx_0)+\exp\big(T(t)\big)\sin^2(kx_0)}\,.$$

Thus, if the initial parameter x_0 is such that

$$kx_0=n\,\frac{\pi}{2}\,, \tag{29}$$

where $n=0,\pm1,\ldots$, then the particles are fixed and $\mathbf{r}(t)\equiv\mathbf{r}_0$.

Equations (29) in the general case define planes, or points in the case of one dimension. They correspond to the zeros of the velocity field. The stability of these points, however, depends on the sign of the function $\mathbf{v}(t)$, which may change in the course of evolution. One may expect that in the course of evolution the 'particles' — the Lagrangian paths — will condense in the neighborhood of these points if $v_x(t)\neq 0$, which is what is meant by the clustering of particles. In the neighborhoods of these points we have

$$\rho(t|\mathbf{r}_0)\sim\rho_0(\mathbf{r}_0)\exp\big(\pm T(t)\big)\,,\qquad j(t|\mathbf{r}_0)\sim\exp\big(\mp T(t)\big)\,,$$

and all the statistical moment functions of these quantities increase with time.

For a divergence-free velocity field, when $v_x(t)=0$, and therefore $T(t)\equiv 0$, we have

$$x(t|x_0)\equiv x_0\,,\qquad \mathbf{R}(t|\mathbf{r}_0)=\mathbf{R}_0+\sin 2(kx_0)\int_0^t \mathrm{d}\tau\,\mathbf{v}_{\mathbf{R}}(\tau)\,,$$

$$\rho(t|\mathbf{r}_0)=\rho_0(\mathbf{r}_0)\,,\qquad j(t|\mathbf{r}_0)=1\,.$$

3.2 Eulerian description of the model

To go over to the Eulerian description, we use Eqn (27) to resolve Eqn (28) with respect to the characteristic parameter $\mathbf{r}_0$:

$$x_0=\frac{1}{k}\arctan\big[\exp\big(-T(t)\big)\tan(kx)\big]\,,$$

$$\mathbf{R}_0=\mathbf{R}-\sin(2kx)\int_0^t \mathrm{d}\tau\times\frac{\mathbf{v}_{\mathbf{R}}(\tau)}{\exp\big[T(t)-T(\tau)\big]\cos^2(kx)+\exp\big[-T(t)+T(\tau)\big]\sin^2(kx)}\,.$$

Now we can rewrite the density field as

$$\rho(\mathbf{r},t)=\rho_0(\mathbf{r}_0)\,\frac{1}{\exp\big(T(t)\big)\cos^2(kx)+\exp\big(-T(t)\big)\sin^2(kx)}\,. \tag{30}$$

For a divergence-free velocity field, when $v_x(t) = 0$, $T(t) \equiv 0$, we have

$$\mathbf{r}_0 = \mathbf{r} - \sin(2kx) \int_0^t \mathrm{d}\tau\, \mathbf{v}(\tau)\,,$$

$$\rho(\mathbf{r}, t) = \rho_0 \left(\mathbf{r} - \sin(2kx) \int_0^t \mathrm{d}\tau\, \mathbf{v}(\tau) \right).$$

In the particular case when the initial density distribution does not depend on x — that is, $\rho_0(\mathbf{r}_0) = \rho_0$ — Eqn (30) is simplified:

$$\frac{\rho(\mathbf{r}, t)}{\rho_0} = \frac{1}{\exp\big(T(t)\big)\cos^2(kx) + \exp\big(-T(t)\big)\sin^2(kx)}\,. \quad (31)$$

From Eqn (31) it follows that after averaging with respect to the fast spatial variables we have

$$\overline{\frac{\rho(\mathbf{r}, t)}{\rho_0}} = 1$$

and this quantity does not depend on the random time $T(t)$.

In a similar way we find that

$$\overline{\left(\frac{\rho(\mathbf{r}, t)}{\rho_0}\right)^2} = \frac{1}{2}\Big(\exp\big(T(t)\big) + \exp\big(-T(t)\big)\Big)\,,$$

and therefore for a Gaussian random process $v_x(t)$ we have

$$\left\langle \overline{\left(\frac{\rho(\mathbf{r}, t)}{\rho_0}\right)^2} \right\rangle = \big\langle \exp\big(T(t)\big) \big\rangle = \exp\left\{ \frac{1}{2} \big\langle T^2(t) \big\rangle \right\},$$

in accordance with the log-normal distribution of probabilities (17).

As far as the structure of the density field (31) itself is concerned, it is obvious that the density field is minor everywhere except in the neighborhood of the points $kx = n\pi/2$, where $\rho(\mathbf{r}, t)/\rho_0 = \exp\big(\pm T(t)\big)$ and hence the field is sufficiently large with the right sign of the random time $T(t)$.

So we see that for the problem under consideration the cluster structure of the density field in the Eulerian description is formed in the neighborhood of the points

$$kx = n\,\frac{\pi}{2} \qquad (n = 0, \pm 1, \pm 2, \ldots)\,.$$

4. Numerical simulation

For the purposes of numerical simulation of the two-dimensional problem we assume that the vector process $\mathbf{v}(t)$ is a Gaussian delta-correlated process with parameters (19). In dimensionless coordinates and time

$$\mathbf{r} \to k\mathbf{r}\,, \qquad t \to k^2\sigma^2\tau_0 t\,, \quad (32)$$

the Lagrangian equations for the coordinates of particle (26) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x(t) = v_x(t)\sin(2x)\,, \qquad x(0) = x_0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y(t) = v_y(t)\sin(2x)\,, \qquad y(0) = y_0\,, \quad (33)$$

where the correlation functions of the velocity field are given by

$$\big\langle v_x(t) v_x(t') \big\rangle = \big\langle v_y(t) v_y(t') \big\rangle = 2\delta(t - t')\,.$$

Then the x-coordinate of the Lagrangian particle (27) and the Eulerian density field (31) are

$$x(t) = \arctan\Big[\exp\big(T(t)\big)\tan x_0\Big]\,,$$

$$\frac{\rho(\mathbf{r}, t)}{\rho_0} = \frac{1}{\exp\big(T(t)\big)\cos^2 x + \exp\big(-T(t)\big)\sin^2 x}\,, \quad (34)$$

where

$$T(t) = 2\int_0^t \mathrm{d}\tau\, v_x(\tau)\,. \quad (35)$$

Observe that the initial stochastic equations (33) and expressions (34) and (35) do not formally involve the parameters σ^2 and t_0. These parameters only enter the definition of dimensionless time (32). This is a consequence of the earlier discussed independence of the diffusion of particles from the model of a random velocity field in the one-dimensional problem.

Figure 2a shows a fragment of the realization of the random process $T(t)$, obtained by numerical integration of Eqn (35) for one particular realization of the random process $v_x(t)$, used for numerical simulation of time evolution of the coordinates of four particles $x(t)$ $(x \in (0, \pi/2))$ with the initial coordinates $x_0(i) = (\pi/2)(i/5)$ $(i = 1, 2, 3, 4)$ (Fig. 2b), and an Eulerian density field (Fig. 3).

From Fig. 2b we see that the particles at the time $t \approx 4$ form a cluster in the neighborhood of the point $x = 0$. At the time $t \approx 16$ this cluster falls apart, and a new cluster is formed in the neighborhood of the point $x = \pi/2$. At $t \approx 40$ a cluster is formed again in the neighborhood of the point $x = 0$, and so on. The particles in clusters remember their history, and in the interim move apart to considerable distances (Fig. 2c).

We see that in this example the cluster as an entity does not move from one point in space to another — instead, one cluster breaks up and a new one is formed. The lifetime of the cluster is much greater than the transition time. This is apparently a specific property of the employed model of the velocity field, related to the stationary positions of points (29).

The diffusion of particles with respect to y-axis is not associated with formation of clusters.

Figure 3a – d shows the space-time evolution of an Eulerian density field $1 + \rho(\mathbf{r}, t)/\rho_0$, calculated according to Eqn (34) (one is added to avoid problems with logarithms when the density is close to zero). In these diagrams we clearly see the gradual concentration of the field in the close neighborhood of the points $x \approx 0$ and $x \approx \pi/2$, which implies the formation of clusters. For example, Fig. 3a, b shows the time sequence $(t = 1 - 10)$ of cluster formation in the neighborhood of the point $x \approx 0$. Figure 3c, d shows the time sequence $(t = 16 - 25)$ of the drift of the density field from the neighborhood of the point $x \approx 0$ to the neighborhood of the point $x \approx \pi/2$ — that is, disintegration of the cluster at $x \approx 0$ and formation of a new cluster at $x \approx \pi/2$. Then this process is repeated in time. As seen from the diagrams, the lifetime of clusters in this model is of the same order as the time of their formation.

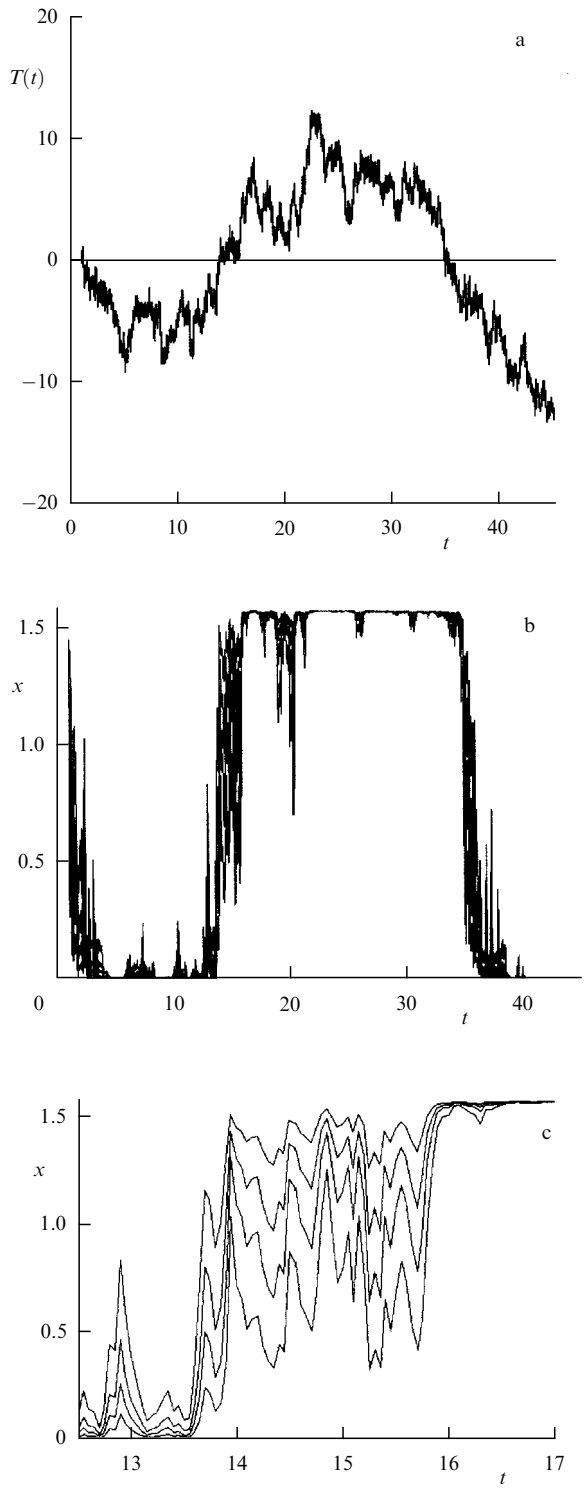

Figure 2. Fragment of realization of random process $T(t)$ (a), obtained by numerical integration of Eqn (35) for one realization of the random process $v_x(t)$, and used for the calculation of the time evolution of the x-coordinates of four particles (b, c).

5. Conclusion

We have discussed a simple model of the diffusion of tracers (particles and Eulerian density field) in a random velocity field, which clearly illustrates the process of cluster formation. Of course, the value of this model is somewhat compromised by the fact that the points where the clusters form are fixed in space.

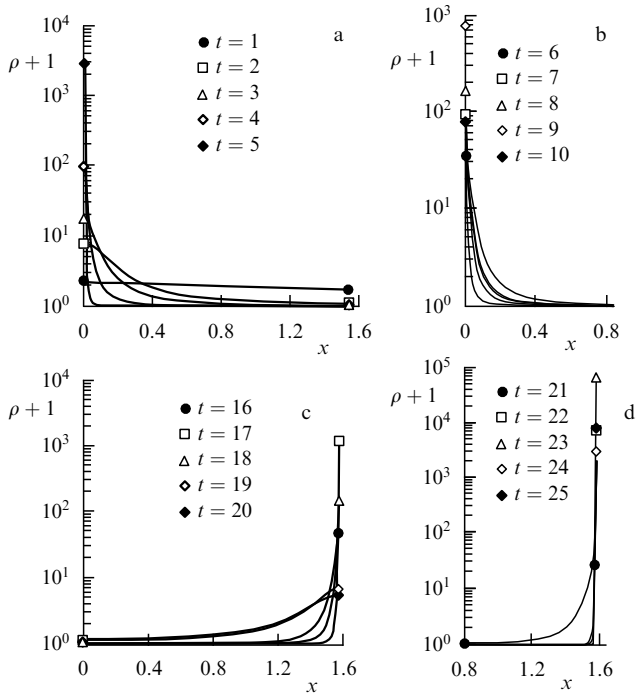

Figure 3. Space-time evolution of Eulerian density field.

However, this model helps to understand the basic distinction between diffusion in divergent and divergence-free velocity fields. In the divergence-free (incompressible) velocity fields, the particles (and hence the density field) do not have the time to drift towards the attraction sites while the latter still exist, and only slightly fluctuate about their initial positions. In the divergent (compressible) velocity field, the particles are able to move towards the attraction sites over the lifetime of the latter (which is the same as before), because the process of attraction is exponentially accelerated, which is obvious from expressions (34).

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References

1. Klyatskin V I, Gurarie D *Usp. Fiz. Nauk* **169** 171 (1999) [*Phys. Usp.* **42** 165 (1999)]
2. Saichev A I, Woyczynski W A, in *Stochastic Models in Geosystems* (The IMA Volumes in Mathematics and Its Applications, Vol. 85, Eds S A Molchanov, W A Woyczynski) (New York: Springer, 1997) p. 359
3. Zirbel C L, Cinlar E, in *Stochastic Models in Geosystems* (The IMA Volumes in Mathematics and Its Applications, Vol. 85, Eds S A Molchanov, W A Woyczynski) (New York: Springer, 1997) p. 459
4. Koshel' K V, Aleksandrova O V *Izv. Ross. Akad. Nauk Fiz. Atm. Okeana* **35** 658 (1999) [*Izv., Atm. Oceanic Phys.* **35** 578 (1999)]
5. Klyatskin V I, Saichev A M *Zh. Eksp. Teor. Fiz.* **111** 1207 (1997) [*JETP* **84** 716 (1997)]
6. Klyatskin V I *Stokhasticheskie Uravneniya i Volny v Sluchaĭno-neodnorodnykh Sredakh* (Stochastic Equations and Waves in Randomly Inhomogeneous Media) (Moscow: Nauka, 1980) [*Ondes et équations stochastiques dans les milieux aléatoirement non homogénes* (Besancon Cedex, France: Les éditions de physique, 1985)]