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Vortex ring oscillations, the development of turbulence in vortex rings and generation of sound

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Contents

Abstract. The state of the art in describing the eigen-oscillations of a vortex ring in an ideal incompressible fluid is reviewed. To describe eigen-oscillations, the displacement field is taken as the basic dynamic variable. A vortex ring with the simplest vorticity distribution in the core and with a potential flow in the vortex ring envelope is the commonest approximation used in treating the eigen-oscillations of vortex rings of a more general form. It turns out that allowing for even a very weak degree of core smoothing causes many oscillation modes to lose their stability. It is shown that the instability effect is determined by the sign of the vibration energy. The energies of the ring eigen-oscillations are calculated and two kinds of eigen-oscillations, those with a negative energy and those with a positive energy, are identified, of which it is the former which become unstable when the core vorticity is smoothed. The multiple instabilities of vortex ring oscillations together with the details of the spatial structure of its eigen-oscillations suggest that it is the nonlinear evolution of precisely these processes which might be the origin of vortex ring turbulence. A new method for the study of unsteady processes in turbulent vortex rings, which utilizes the experimental diagnostics of the ring's sound field, is presented. The structure of the sound field strongly supports the proposed model of the turbulent vortex ring.

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1. Introduction

The vortex ring is a well known and very popular object of fluid dynamics. Investigations of vortex rings began in the last century when a vortex ring was considered as a model of the vortex theory of atoms then under development [1 – 3]. Though the quantum theory has made many of the ideas developed in that period insignificant, the vortex ring seems to remain one of the most interesting and convenient objects for investigations in hydrodynamics [4]. Really, the vortex ring is suitable for experimental research and at the same time its behaviour can be described within the limits of the main equations of continuous medium. The most important feature is that once generated, this vortex develops only under the effect of its own dynamics and is not affected by rigid boundaries. This permits the use of the vortex ring for investigations of many problems of hydroaerodynamics in a pure form. In the present work we restrict ourselves to considering a certain range of problems associated with high-frequency oscillations of this vortex. It appeared that the rigorous description of disturbance dynamics permitted the association of such problems as flow stability and mechanisms of energy exchange between the mean flow and separate modes, transition to turbulence in the vicinity of the vortex ring core and the absence of turbulence in the core, sound generation by turbulence and the contribution of vortex core eigen-oscillations to sound radiation.

A vortex ring is a vortex torus, which moves together with an ellipsoidal volume of fluid called the ring envelope. A detailed description of steady vortex rings can be found, for example, in Refs [5 – 7]. These works primarily address the question about self-translation of a vortical filament rolled up into a ring. The vortex ring description as a thin annular filament in an ideal fluid turns out to be in a certain qualitative agreement with the well known hydrodynamical phenomenon and permits the relation of the mean vortex parameters — its size and circulation — and the translation velocity. These parameters, as follows from the ideal fluid equations remain unchanged during the vortex translation, i.e. such a vortex is preserved without change in form and velocity. The limitation of such a consideration becomes evident from the fact that the real ring noticeably expands in its movement. Accounting for viscosity [8, 9] one could reduce the discrepancies between the theoretical model and the real phenomenon. At the same time in the early 1970's it became quite clear that the ideal model, even accounting for viscosity, does not take the principal features of vortex ring evolution into consideration.

Numerous experimental investigations [10 – 15] have shown that there exist two qualitatively different flow regimes — laminar and turbulent ones. The critical Reynolds number Re_0 based on the initial radius and velocity of the vortex ring is equal to about 10^3. At small Re-numbers a vortex has a characteristic and apparently seen spiral structure [16, 17]. At Re-numbers exceeding Re_0 the flow character fundamentally changes and the flow becomes turbulent. The main feature of such a flow is that its structure appears to be close to universal and is independent of the peculiarities of the vortex formation process. In this case the flow is divided into two regions: the laminar core where the vorticity is concentrated and the envelope region where the fluid particles are in chaotic motion (Fig. 1).

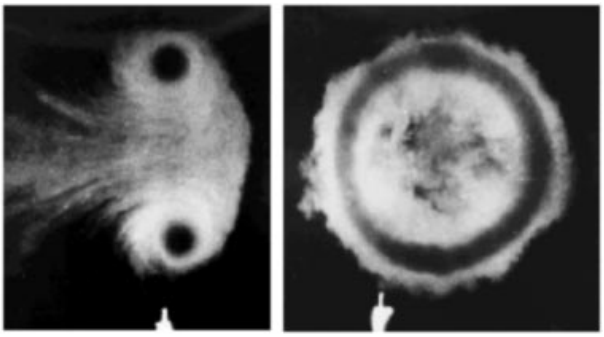

Figure 1. Photos of turbulent vortex ring (side view and front view) made with the use of a luminous plane.

The most important and interesting phenomenon is that the boundary between the turbulent and laminar regions remain sharp, despite the fact that the ring exists for a long time [18 – 20]. The turbulent flow regime peculiarities found in experiments made possible the formulation of the semi-empirical self-similar theory [11, 21, 22] describing the evolution of mean parameters (radius, velocity, vorticity etc.), on the assumption that the vorticity distribution in the core is close to uniform (solid-body rotation in the core). Measurements of the translation velocity and of the geometrical parameters of the ring confirmed the self-similar character of vortex development. However, direct reliable measurements of vorticity in the vortex core have practically not been made. We mention the works [23, 24] (the data presented there are of a preliminary character), and the new measurements based on the *PIV* technique [25]. The latter are related to not very high-velocity rings.

The limited volume of experimental data on the mean vorticity structure in the core and all the more on the unsteady processes in the vortex, on the one hand — is accompanied by the lack of theoretical investigations of the vortex ring stability problem, on the other. For a long time it has been accepted — possibly based on Kelvin's work [2] on the vortex column (Rankin's vortex) stability — that the majority of modes corresponding to vortex ring cross-section deformations are stable. In the literature the instability only of one mode (bending) has been considered [26 – 29]. The methods used in this case are based on additional simplifying assumptions (see Section 3.1) and the comparison with experiment is of a qualitative character.

The present-time view on the vortex ring turbulence structure is based on the idea of rotating flow 'elasticity', a qualitative concept of turbulence suppression in the localized vortex cores [30, 31] and an analogy between the effects of stratification and rotation. It is clear that substantial progress in understanding the turbulent flow regime requires not only qualitative discussions, but also a dynamical description of oscillating regimes and instability mechanisms which make possible the energy transfer into separate modes, as well as the development of diagnostic methods for delicate and complex processes in the vortex ring core which would be based, if possible, on non-contact procedures, since the processes are very sensitive to any interventions. Therefore we can state with assurance that the description of vortex ring turbulence is an interesting and important problem of turbulence itself and its study will require significant efforts.

The present work considers recent results relating to the description of vortex ring eigen-oscillations in an ideal incompressible fluid. These oscillations have a close analogue — Kelvin's oscillations of Rankin's cylindrical vortex. The similarity of the mean flow of a thin vortex ring with an almost circular cross-section, small in comparison with the vortex radius, and a cylindrical vortex with circular cross-section and infinite curvature radius seems to be the main cause why the vortex ring oscillations (this task is much more computationally complex) have not been considered in a complete form (see the review of this problem in Section 3.1). It appeared however that many vortex ring modes were different from the respective oscillations of the cylindrical vortex even in the leading approximation. The change in eigen-oscillation structure appeared not only an unexpected curious fact. This difference involved a number of important consequences: from multiple instabilities to peculiarities of acoustic radiation by vortices.

Since the application of perturbation method to the problem of vortex ring oscillations is not completely trivial, the most economical and convenient procedure is required. Section 2 is devoted to describing the approach developed in [32] for the description of thin vortex ring oscillations based on using the displacement field as the principal dynamical variable. The present work touches upon some important questions in describing the disturbance energy in vortical flows, based on Arnold's theorem [33 – 35]. The possibility of calculation of this non-linear quantity based only on the linear dynamics of disturbances (as a consequence of the fact

that the first variation of energy equals to zero), has great importance. The oscillation energy and the sign of this energy in particular, appear to determine the instability mechanism considered in the present work.

Section 3 considers long-wave oscillations of a thin vortex ring with a potential envelope and the simplest vorticity distribution in the core. The resemblances and differences between the oscillations of such a vortex ring and oscillations of a cylindrical vortex are analyzed.

A vortex ring with the simplest vorticity distribution in the core and a potential flow in the envelope is the basic approximation in the problem of oscillations of vortex rings of a more general type. It appears that accounting for the core smoothing (even very slight) makes many oscillations unstable. If the vorticity in the envelope is small it can be taken into account analytically and the increments of unstable harmonics can be calculated. This task is considered in Section 4. The multiple instabilities of vortex ring oscillations and the structures of unstable disturbances show that the vortex ring turbulence could be exactly a consequence of the non-linear development of these processes. From this point of view, it seems more correct to speak not of turbulence suppression in the core, but of turbulence generation in the ring envelope.

In Section 5 a new method of investigation of unsteady processes in a turbulent vortex ring is considered. It is based on experimental diagnostics and a theoretical description of its sound field. The peculiarities of the measured sound field strongly support the turbulent vortex ring model developed in the present work. Thus, the vortex ring turns out to be a very convenient model for investigation of the totality of unsteady processes which take place in three-dimensional vortex flows.

2. The displacement field as a way to describe vortex dynamics in an incompressible fluid

The velocity or vorticity field is traditionally used to describe disturbances in a vortex flow. Recently another approach has also been used in the problem of vortex disturbances, which is based on using the displacement field [36] as the principal function. This field which directly describes each vortex filament deformation, was first used for describing the evolution of vortical flow disturbances in Ref. [37]. This approach was applied to investigation of small-amplitude oscillations of a vortex ring in Ref. [38] where the bulging modes of the vortex ring were first correctly described, and in Ref. [32] where other oscillations were explored.

2.1 Displacement field evolution in arbitrary vortex flows. Displacement field as a new dynamic variable

Consider an approach to disturbance description based on the displacement field $\boldsymbol{\varepsilon}(\mathbf{r})$ as the principal function. We restrict ourselves to such disturbances at which the disturbed flow is isovortical to the reference steady flow [39, 40]. Such disturbances can be presented as a result of displacement of fluid particles with a frozen vorticity field.

2.1.1. Condition of isovorticity. Consider an arbitrary divergence -free (solenoidal) vector field $\boldsymbol{\eta}(\mathbf{r})$, $\nabla\boldsymbol{\eta} = 0$. Let this field define a set of transforms of space into itself, according to the following formula:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\eta}(\mathbf{r}) , \tag{2.1}$$

where t is some parameter. Transform (2.1) can be presented as the motion of fluid particles with a 'velocity field' $\boldsymbol{\eta}$. For such a transform each particle, which had initially the coordinate $\mathbf{r}_0$, will come to the point determined by the solution of Eqn (2.1) with the initial condition $\mathbf{r}|_{t=0} = \mathbf{r}_0$. Since the field $\boldsymbol{\eta}$ is solenoidal, such a transform preserves volume, according to Liouville's theorem. It is obvious that all the conceivable volume-preserving displacements of fluid particles are given by the solution of Eqn (2.1) at some $\boldsymbol{\eta}(\mathbf{r})$. For transforms close to identical one, t is a small parameter. In the linear approximation in t, the solution of Eqn (2.1) is given by equality $\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\eta}(\mathbf{r}_0)t + O(t^2)$. Designate the transform part, linear in parameter t, through $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\eta}(\mathbf{r}_0)t , \tag{2.2}$$

and call this value a *displacement field*.

Consider now some reference vector field $\mathbf{a}_0$, $\nabla\mathbf{a}_0 = 0$. Let this field remain 'frozen' for the fluid particle motion given by transform (2.1), i.e. the field $\mathbf{a}_0$ is transformed into the field $\mathbf{a}$ in such a way, that the intensities of any vector tubes of the field $\mathbf{a}$ remain unchanged. According to Fridman's theorem [41, 42], the necessary and sufficient condition of preserving the vector tube intensity (i.e. of the field $\mathbf{a}$ being frozen in the field $\boldsymbol{\eta}$) is the equality to zero of the following expression

$$\frac{\partial\mathbf{a}}{\partial t} + \nabla\times(\mathbf{a}\times\boldsymbol{\eta}) = 0 . \tag{2.3}$$

At small values of t we present the field $\mathbf{a}$ as follows:

$$\mathbf{a} = \mathbf{a}_0 + t\left.\frac{\partial\mathbf{a}}{\partial t}\right|_{t=0} + \frac{t^2}{2}\left.\frac{\partial^2\mathbf{a}}{\partial t^2}\right|_{t=0} + \ldots = \mathbf{a}_0 + \delta\mathbf{a} + \delta^2\mathbf{a} + \ldots$$

Using Eqn (2.3) we get

$$\left.\frac{\partial\mathbf{a}}{\partial t}\right|_{t=0} = \nabla\times(\boldsymbol{\eta}\times\mathbf{a}_0) ,$$

$$\left.\frac{\partial^2\mathbf{a}}{\partial t^2}\right|_{t=0} = \nabla\times\left(\boldsymbol{\eta}\times\frac{\partial\mathbf{a}}{\partial t}\right)\Bigg|_{t=0} = \nabla\times\big(\boldsymbol{\eta}\times\nabla\times(\boldsymbol{\eta}\times\mathbf{a}_0)\big) .$$

Then for the first and the second variations, with account for Eqn (2.2), we obtain

$$\delta\mathbf{a} = \nabla\times(\boldsymbol{\varepsilon}\times\mathbf{a}_0) , \qquad \delta^2\mathbf{a} = \frac{1}{2}\nabla\times\big(\boldsymbol{\varepsilon}\times\nabla\times(\boldsymbol{\varepsilon}\times\mathbf{a}_0)\big) . \tag{2.4}$$

These expressions will be used below for calculations of vorticity disturbances and vorticity energy.

If the field $\mathbf{a}$ means the vorticity $\boldsymbol{\Omega}$, then the condition that it is 'frozen' is called the *isovorticity condition*. It is obvious that the preservation of vector tube intensity for the vorticity field is equivalent to preserving the circulation over any fluid contour. Since transform (2.1) exhausts all the transforms of space into itself which preserve the volume, Eqns (2.4) describe all the close fields 'isovortical' to the reference field $\boldsymbol{\Omega}_0$.

2.1.2 Obtaining the main system of equations. In the linear approximation, under the condition that vortex lines are frozen into the displacement field $\boldsymbol{\varepsilon}$, the vorticity disturbance, according to Eqn (2.4), is expressed as

$$\boldsymbol{\Omega} = \nabla\times(\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0) . \tag{2.5}$$

Here and below we shall omit δ in the indication of the first field variation in the cases when we are interested only in disturbances in the linear approximation.

Since the vorticity field remains isovortical in its evolution, the disturbed state can be presented at any moment as a result of the effect of some displacement field $\boldsymbol{\varepsilon}$ (Fig. 2), i.e. the vorticity field evolution can be related to the evolution of the field $\boldsymbol{\varepsilon}$. This means that the displacement field $\boldsymbol{\varepsilon}(\mathbf{r}, t)$ can be used for a description of the disturbance evolution. It is obvious that at given $\boldsymbol{\Omega}$ the field $\boldsymbol{\varepsilon}$ is determined ambiguously. Any trivial displacement $\boldsymbol{\varepsilon}^t$ which is the solution of equation $\nabla \times (\boldsymbol{\varepsilon}^t \times \boldsymbol{\Omega}_0) = 0$ can be added to it. According to Eqn (2. 5), a trivial displacement is a fluid particle displacement which causes no vorticity field disturbance at all.

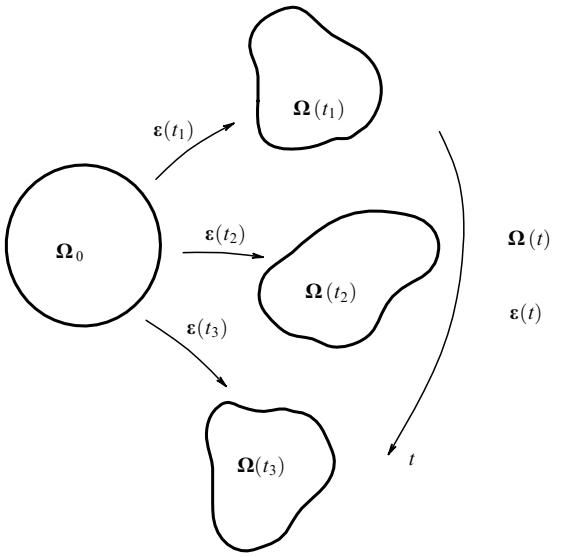

Figure 2. Evolution of the vorticity field $\boldsymbol{\Omega}(t)$ and displacement field $\boldsymbol{\varepsilon}(t)$.

We get the equation for the displacement field $\boldsymbol{\varepsilon}$ from the equations of ideal incompressible fluid dynamics. Small vorticity disturbances in an unbounded flow are described by a system of equations:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \mathbf{V}_0) + \nabla \times (\boldsymbol{\Omega}_0 \times \mathbf{v}) = 0\,, \tag{2.6}$$

$$\mathbf{v}(\mathbf{r}) = \nabla \times \frac{1}{4\pi} \int \frac{\boldsymbol{\Omega}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}\mathbf{r}'\,, \tag{2.7}$$

where $\mathbf{V}_0$, $\boldsymbol{\Omega}_0$ are the steady fields of velocity and vorticity, respectively; and $\mathbf{v}$, $\boldsymbol{\Omega}$ are the disturbances of these fields. Eqn (2.6) is the linearized Helmholtz equation and Eqn (2.7) connects the velocity and vorticity disturbances at each moment of time, according to the Biot – Savart law.

The system of equations describing the displacement field evolution can be obtained from the system of linearized Eqns (2.6) and (2.7) with the direct substitution of Eqn (2.5). Using the vector identity

$$\nabla \times \big[\mathbf{a} \times \big[\nabla \times (\mathbf{b} \times \mathbf{c})\big]\big] + \nabla \times \big[\mathbf{b} \times \big[\nabla \times (\mathbf{c} \times \mathbf{a})\big]\big] + \nabla \times \big[\mathbf{c} \times \big[\nabla \times (\mathbf{a} \times \mathbf{b})\big]\big] = 0$$

and relation $\nabla \times (\boldsymbol{\Omega}_0 \times \mathbf{V}_0) = 0$ which is the Helmholtz equation for steady flow, we get from Eqn. (26):

$$\nabla \times \left[\left(\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + \nabla \times (\boldsymbol{\varepsilon} \times \mathbf{V}_0) - \mathbf{v}\right) \times \boldsymbol{\Omega}_0\right] = 0\,.$$

This equation is equivalent to the following:

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + \nabla \times (\boldsymbol{\varepsilon} \times \mathbf{V}_0) - \mathbf{v} = \mathbf{F}\,, \tag{2.8}$$

where $\mathbf{F}(\mathbf{r}, t)$ is an arbitrary function satisfying the condition $\nabla \times (\mathbf{F} \times \boldsymbol{\Omega}_0) = 0$, and the field $\mathbf{v}$ is expressed through $\boldsymbol{\varepsilon}$ with the use of Eqns (2.5) and (2.7). It is easy to show that the arbitrary function $\mathbf{F}$ can always be compensated by adding some trivial displacement. Really, consider the field $\boldsymbol{\varepsilon}'(\mathbf{r}, t)$ which satisfies the initial zero condition and Eqn (2.8) with the same function $\mathbf{F}$. The field $\boldsymbol{\varepsilon}'$ at any following moment of time will be a trivial displacement, since it is the solution of the problem in which the steady flow has not received the initial disturbance. Therefore, the solutions of Eqn (2.8) with the same initial conditions and different right-hand sides differ from one another only by the trivial displacement. We introduce the difference $\boldsymbol{\xi} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'$ into our consideration. This value is the difference of the fluid particle position in the disturbed and undisturbed flows at the time t after the moment of disturbance, i.e. it is the Lagrangian displacement of the fluid particle [36, 43]. The right-hand side of Eqn (2.8) for this quantity is exactly equal to zero. Thus, we get the well known equation describing the evolution of Lagrangian displacement field [36]:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\boldsymbol{\xi} \times \mathbf{V}_0) - \mathbf{v} = 0\,, \tag{2.9a}$$

$$\mathbf{v} = \nabla \times \frac{1}{4\pi} \int \frac{\nabla' \times (\boldsymbol{\xi} \times \boldsymbol{\Omega}_0)}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}\mathbf{r}'\,, \tag{2.9b}$$

where ∇' is the differential operator with respect to the variable $\mathbf{r}'$.

Making use both of the vorticity field and the displacement field for describing disturbances allows the problem to be localized, i.e. the solution can only be sought for within the region occupied by the vortex. The behaviour of the field $\mathbf{v}$ in the region outside the vortex and, in particular, its decay at infinity, is automatically taken into account in the integral expression (2.9b). An important difference of Eqn (2.9a) from the Helmholtz equation (2.6), equivalent to it, is the separation of the integral term $\mathbf{v}$ in the pure form. It can easily be transformed into a differential form with using the curl operation and relation (2.5).

Equation (29a) can easily be extended to the case of vortices moving as a whole with the velocity $\mathbf{V}_\infty$ (e.g. vortex ring) or rotating as a whole with angular velocity ω (e.g. two-dimensional elliptical Kirchhoff vortex). The basic flow will be steady for such flows in the coordinate system moving and rotating together with the vortex and the disturbances will be described by the Helmholtz equation which in this coordinate system is expressed as

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \mathbf{U}_0) + \nabla \times (\boldsymbol{\Omega}_0 \times \mathbf{v}) = 0\,,$$

where $\mathbf{U}_0 = \mathbf{V}_0 - \mathbf{V}_\infty - \boldsymbol{\omega} \times \mathbf{r}$, the velocity field $\mathbf{V}_0$ decays at infinity and is connected with the reference steady vorticity

field $\mathbf{\Omega}_0$ by the Biot–Savart integral:

$$\mathbf{V}_0(\mathbf{r}) = \nabla \times \frac{1}{4\pi} \int \frac{\mathbf{\Omega}_0(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' . \tag{2.10}$$

In this case the rotation and displacement of the vortex as a whole are separated from the small displacement field ξ, and Eqn (2.9a) describing the displacement field evolution takes the form

$$\frac{\partial \xi}{\partial t} + \nabla \times (\xi \times \mathbf{U}_0) - \mathbf{v} = 0 . \tag{2.11}$$

2.2 Evolution of the displacement field in localized vortices

2.2.1 The system of governing equations. Consider a situation when the vorticity is concentrated in a finite region of space. As is known, the vector field in the bounded region M can be determined in terms of the normal component at the boundary $G(M)$, and the curl and divergence of this field over the whole region M. In the case of a nonsimply connected region for unambiguous determination of the vector field, it is also necessary to specify the circulation along a closed contour C not reduced to zero. Let us consider the whole expression $\partial\xi/\partial t + \nabla \times (\xi \times \mathbf{V}_0) - \mathbf{v}$ as the *vector field* and the region in which the vorticity is concentrated as the *region* M. Then we obtain that Eqn (2.9a) is equivalent to the system of equations

$$\frac{\partial}{\partial t} \nabla \times \xi + \nabla \times \left[\nabla \times (\xi \times \mathbf{V}_0)\right] - \nabla \times (\xi \times \mathbf{\Omega}_0) = 0, \quad \mathbf{r} \in M , \tag{2.12a}$$

$$\nabla \xi = 0 , \qquad \mathbf{r} \in M , \tag{2.12b}$$

$$\left(\frac{\partial \xi}{\partial t} + \nabla \times (\xi \times \mathbf{V}_0) - \mathbf{v}\right) \cdot \mathbf{n} = 0 , \qquad \mathbf{r} \in G(M) , \tag{2.12c}$$

$$\oint_C \left(\frac{\partial \xi}{\partial t} + \nabla \times (\xi \times \mathbf{V}_0) - \mathbf{v}\right) \mathrm{d}\mathbf{l} = 0 , \tag{2.12d}$$

where $\mathbf{n}$ is the normal to the surface $G(M)$. An important advantage of the system of Eqns (2.12) in comparison with Eqn (2.9a) is that the integral term $\mathbf{v}$ is excluded from the equation over the whole region inside the vortex and is to be calculated only at the boundary $G(M)$.

2.2.2 Transform of equation for velocity evaluation. Equation (2.12c) contains the normal velocity disturbance $v^n = (\mathbf{v} \cdot \mathbf{n})$ at the boundary $G(M)$ for a given displacement field $\boldsymbol{\varepsilon}$. Therefore the system of equations (2.12) must be supplemented with expression (2.9b) from which the velocity disturbance $\mathbf{v}$ is found. We transform this expression into a form more convenient for calculations. Making use of integration by parts, we get a chain of equalities

$$\begin{aligned} \mathbf{v} &= \nabla \times \frac{1}{4\pi} \int_M \frac{\nabla' \times \mathbf{B}}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' = \frac{1}{4\pi} \int_M \frac{\nabla' \times (\nabla' \times \mathbf{B})}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' \\ &= \frac{1}{4\pi} \int_M \frac{\nabla'(\nabla'\mathbf{B}) - \nabla'^2 \mathbf{B}}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' = \mathbf{B} + \nabla \frac{1}{4\pi} \int_M \frac{\nabla' \mathbf{B}}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' , \end{aligned} \tag{2.13}$$

where $\mathbf{B} = \xi \times \mathbf{\Omega}_0$. Hence it follows, that in the region outside the vortex the velocity disturbance $\mathbf{v}$ can be presented as the field produced by sources with intensity $Q(\mathbf{r}) = -\nabla(\xi \times \mathbf{\Omega}_0)$. Indeed, in this region $\mathbf{B} = 0$ and therefore

$$\mathbf{v} = \nabla \Phi , \qquad \Phi = -\frac{1}{4\pi} \int_M \frac{Q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, \mathrm{d}\mathbf{r}' . \tag{2.14}$$

Since the normal component of the velocity is continuous at the vortex boundary, both the external and the internal limit of expression (2.13) can be used for calculating the value v^n in Eqn (2.12c). We shall use the external limit, proceeding from Eqn (2.14).

Note also, that in calculations of the field $\mathbf{v}$ the intensity Q in Eqn (2.14) can be given in different ways. Really, the field outside the region M does not change on substituting $Q \to Q + Q'$ where $Q' = \nabla^2 G$ is an arbitrary function identically equal to zero outside the region M. Thus, the velocity disturbance outside the vortex can be found from Eqn (2.14) with an intensity of a more general kind

$$Q = -\nabla\left[(\xi \times \mathbf{\Omega}_0) + \nabla G\right] , \tag{2.15}$$

where G is an arbitrary function different from zero only in the region occupied by the vortex.

In particular, one can select the solution of the equation $\nabla^2 G = -\nabla(\xi \times \mathbf{\Omega}_0)$ for $\mathbf{r} \in M$ as the function G on condition that $G = 0$ at the boundary $\mathbf{r} \in G(M)$. Then the volume intensity Q becomes the surface one. This transform sometimes substantially facilitates calculations.

2.2.3 Some comments on the system of governing equations. We dwell on the physical meaning of the equation set (2.12), (2.14). Since the vortex boundary displacement is equal to $(\xi \cdot \mathbf{n})$ for $\mathbf{r} \in G(M)$, Eqn (2.12c) describes the disturbed vortex boundary evolution. The vortex boundary evolution is related to the disturbance shape inside the vortex through the term $(\mathbf{v} \cdot \mathbf{n})$ which is expressed as the integral (2.14) over the whole vortex volume. In its turn, the disturbance evolution inside the vortex is described by Eqn (2.12a), (2.12b). The internal disturbance dependence on the boundary shape is expressed in the fact that Eqn (2.12c) serves as a boundary condition for Eqns (2.12a), (2.12b). Thus, the equation system (2.12), (2.14) describes the disturbances of the three-dimensional vortex as a joint evolution of the vortex boundary and of disturbances inside the vortex region. Such an approach can be considered an extension of the contour dynamics method applied for describing the evolution of the boundary of a two-dimensional vortex of uniform vorticity [44, 45] to the case of three-dimensional disturbances of localized vortices with arbitrary vorticity.

2.3 Energy of disturbances in vortex flows

The disturbance energy is one of the most important flow characteristics which in many cases permits not only an understanding, but also a description of the characteristic peculiarities of the system behaviour. The energy of the disturbed flow (surely, positive) can be greater or smaller than the energy of the basic flow. Therefore it seems reasonable to speak about the positive or negative *energy of disturbances*, according to the sign of the energy difference of the disturbed and reference states. Disturbances with an energy larger than the reference state energy will be called *disturbances with positive energy*, since it is necessary to introduce energy into the system for their generation. Disturbances with an energy smaller than the reference state energy will be called *disturbances with negative energy*, since

for their production energy must be taken away. The notion of disturbances (waves) of negative energy is widely used not only in hydrodynamics [46, 47], but in the physics of plasma [48, 49], acoustics [50, 51], oceanography [52, 53] etc.

For such complex flows as a vortex ring, the search for the oscillation energy requires cumbersome calculations. In this connection, we reduce some expressions for the disturbance energy to the simplest and most convenient form. Following the well known works of Arnold [33, 34], we get a general expression for the second variation of the kinetic energy functional on the set of isovortical flows. Since, according to Arnold's theorem, the first variation is equal to zero, the second variation will present the required difference of the disturbed and steady flow energy. A detailed review of the problem and supplements can be found, for example, in Ref. [35].

For the case of flows which are steady in coordinate systems moving or rotating with constant velocities, the expression obtained requires some modification, since in this case a more complex functional presenting some generalized flow energy in moving or rotating coordinate systems turns out to be extremal [54, 43].

2.3.1 Energy of isovortical disturbances and Arnold's theorem. First consider the case when the main steady flow has a velocity field $\mathbf{V}_0$ decaying at infinity in the fixed coordinate system. The kinetic energy of an infinite flow of incompressible fluid with velocity field $\mathbf{V}$ is determined by the functional

$$T = \frac{1}{2}\int \mathbf{V}^2\,\mathrm{d}\mathbf{r}\,, \tag{2.16}$$

where the integration is performed over the whole space and the density is taken to be equal to unity. For three-dimensional flows, if the vorticity field decreases fast enough and the integral $\int(\mathbf{r}\times\mathbf{\Omega})\,\mathrm{d}\mathbf{r}$ converges, the velocity $\mathbf{V}$ decreases at infinity as r^3 and integral (2.16) does exist. This condition is known to be fulfilled for flows with localized vortices.

We present the velocity field as $\mathbf{V} = \mathbf{V}_0 + \delta\mathbf{v} + \delta^2\mathbf{v} + \ldots$, where $\delta\mathbf{v}$ and $\delta^2\mathbf{v}$ are the first and the second velocity variations, respectively. Then the disturbance energy, i.e. the energy difference between the disturbed and steady flows in the first and the second approximations, will be as follows:

$$\delta T = \int \mathbf{V}_0\,\delta\mathbf{v}\,\mathrm{d}\mathbf{r}\,, \tag{2.17}$$

$$\delta^2 T = \frac{1}{2}\int\left[(\delta\mathbf{v})^2 + 2\mathbf{V}_0\,\delta^2\mathbf{v}\right]\mathrm{d}\mathbf{r}\,. \tag{2.18}$$

We will show that the integral (2.17) becomes zero on the set of isovortical flows and calculate the second variation of energy (2.18) which exactly presents the disturbance energy.

According to Eqns (2.4), the first and second variations of the vorticity field for an arbitrary displacement field are

$$\delta\mathbf{\Omega} = \nabla\times(\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0)\,,$$

$$\delta^2\mathbf{\Omega} = \frac{1}{2}\,\nabla\times\big(\boldsymbol{\varepsilon}\times\nabla\times(\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0)\big)\,. \tag{2.19}$$

Making use of the fact that the velocity and vorticity field variations are connected with the relations $\delta\mathbf{\Omega} = \nabla\times\delta\mathbf{v}$ and $\delta^2\mathbf{\Omega} = \nabla\times\delta^2\mathbf{v}$, we get

$$\delta\mathbf{v} = (\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0) + \nabla\varphi_1\,,$$

$$\delta^2\mathbf{v} = \frac{1}{2}\big(\boldsymbol{\varepsilon}\times\nabla\times(\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0)\big) + \nabla\varphi_2\,, \tag{2.20}$$

where φ_1 and φ_2 are to be chosen in such a way that the fields $\delta\mathbf{v}$ and $\delta^2\mathbf{v}$ are divergence-free. Consider integral (2.17). With account for Eqn. (2.20), we get

$$\delta T = \int \mathbf{V}_0\,\delta\mathbf{v}\,\mathrm{d}\mathbf{r} = \int \mathbf{V}_0(\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0)\,\mathrm{d}\mathbf{r} + \int \mathbf{V}_0\nabla\varphi_1\,\mathrm{d}\mathbf{r}\,. \tag{2.21}$$

Using integration by parts and making use of the solenoidal character of the velocity field $\mathbf{V}_0$, we get that the second integral in (2.21) becomes zero, i.e.

$$\delta T = \int\big[\mathbf{V}_0(\boldsymbol{\varepsilon}\times\mathbf{\Omega}_0)\big]\,\mathrm{d}\mathbf{r} = \int\big[\boldsymbol{\varepsilon}(\mathbf{\Omega}_0\times\mathbf{V}_0)\big]\,\mathrm{d}\mathbf{r}\,. \tag{2.22}$$

In Eqn (2.22) the field $\boldsymbol{\varepsilon}$ is not arbitrary and satisfies the condition of zero divergence. This field can be presented as $\boldsymbol{\varepsilon} = \nabla\times\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is already an arbitrary vector field. After simple transforms we obtain from Eqn (2.22):

$$\delta T = \int\big[\boldsymbol{\beta}\nabla\times(\mathbf{\Omega}_0\times\mathbf{V}_0)\big]\,\mathrm{d}\mathbf{r}\,. \tag{2.23}$$

For a steady flow Helmholtz's equation is as follows

$$\nabla\times(\mathbf{\Omega}_0\times\mathbf{V}_0) = 0\,, \tag{2.24}$$

and from Eqns (2.23) and (2.24) we obtain $\delta T = 0$. Thus, we derived the well known *Arnold's variational principle*: a steady flow of ideal fluid is the point of the conditional extremum of the kinetic energy functional on the set of isovortical flows. Due to the arbitrary character of the field $\boldsymbol{\beta}$, it is obvious that the inverse statement is also valid.

We determine the second variation of the energy $\delta^2 T$. The first term in Eqn (2.18) is left unchanged, the second, with account for Eqn (2.20), is rewritten in a form similar to Eqn (2.22). The part of integral (2.18) connected with $\nabla\varphi_2$ becomes zero after integration by parts. With account for Eqn (2.19) we obtain for $\delta^2 T$,

$$\delta^2 T = \frac{1}{2}\int\big[(\delta\mathbf{v})^2 + \delta\mathbf{\Omega}(\mathbf{V}_0\times\boldsymbol{\varepsilon})\big]\,\mathrm{d}\mathbf{r}\,. \tag{2.25}$$

We emphasize that the second variation of energy is expressed through linear flow variations. Therefore it is sufficient to know the disturbance dynamics only in a linear approximation for energy calculations.

2.3.2 Disturbance energy in a coordinate system moving or rotating with constant velocity. It was assumed in deriving Eqn (2.25) that the steady velocity field $\mathbf{V}_0$ decreased at infinity. Flows which are steady in the moving coordinate system (e.g. vortex ring) obviously do not comply with Eqn (2.24), therefore Arnold's variational principle cannot be applied to them in the pure form. The same can be said about flows which are steady in the rotating coordinate system (e.g. Kirchhoff's vortex). The velocity and vorticity fields for such flows satisfy Helmholtz's equation in the following form:

$$\nabla\times(\mathbf{U}_0\times\mathbf{\Omega}_0) = 0\,, \tag{2.26}$$

where $\mathbf{U}_0 = \mathbf{V}_0 - \mathbf{V}_\infty - \boldsymbol{\omega} \times \mathbf{r}$, $\mathbf{V}_0$ is the velocity field decaying at infinity and connected with the vorticity field $\boldsymbol{\Omega}_0$ through the Biot–Savart integral (2.10); and $\mathbf{V}_\infty$ and $\boldsymbol{\omega}$ are the translation velocity and angular velocity of vortex in the fixed coordinate system, respectively.

We introduce the integrals of vortex momentum $\mathbf{p}$ and vortex angular momentum $\mathbf{M}$ [6] into the consideration in addition to the kinetic energy integral T (2.16):

$$\mathbf{p} = \frac{1}{2}\int (\mathbf{r} \times \boldsymbol{\Omega})\, \mathrm{d}\mathbf{r}\,, \tag{2.27}$$

$$\mathbf{M} = \frac{1}{3}\int \big(\mathbf{r} \times (\mathbf{r} \times \boldsymbol{\Omega})\big)\, \mathrm{d}\mathbf{r}\,. \tag{2.28}$$

According to the formulas of Hamiltonian transforms in classical mechanics, the functional

$$T' = T - \mathbf{V}_\infty \mathbf{p} - \boldsymbol{\omega}\mathbf{M} \tag{2.29}$$

presents the flow energy in the coordinate system moving with the constant velocity $\mathbf{V}_\infty$ and rotating with constant angular velocity $\boldsymbol{\omega}$.

We find the first variation of the generalized energy: $\delta T' = \delta T - \mathbf{V}_\infty \delta\mathbf{p} - \boldsymbol{\omega}\delta\mathbf{M}$. We shall use Eqn (2.22) for δT. Let us consider the variations of the momentum $\delta\mathbf{p}$ and of the angular momentum $\delta\mathbf{M}$. With account for Eqn (2.19) on the isovortical layer, we have

$$\mathbf{V}_\infty \delta\mathbf{p} = \int \big[\boldsymbol{\varepsilon}(\boldsymbol{\Omega}_0 \times \mathbf{V}_\infty)\big]\, \mathrm{d}\mathbf{r}\,, \tag{2.30}$$

$$\boldsymbol{\omega}\delta\mathbf{M} = \int \big[\boldsymbol{\varepsilon}\big(\boldsymbol{\Omega}_0 \times (\boldsymbol{\omega} \times \mathbf{r})\big)\big]\, \mathrm{d}\mathbf{r}\,. \tag{2.31}$$

Substituting Eqns (2.30) and (2.31) together with Eqn (2.22) into Eqn (2.29), we finally obtain:

$$\delta T' = \int \big[\boldsymbol{\varepsilon}(\boldsymbol{\Omega}_0 \times \mathbf{U}_0)\big]\, \mathrm{d}\mathbf{r}\,. \tag{2.32}$$

Using the same reasoning as to derive Eqn (2.23) and taking into account the Helmholtz equation (2.26) we obtain that the integral (2.32) becomes zero. Hence, the functional of the generalized energy (2.29) is extreme at flows which are steady in the coordinate system moving with constant velocity $\mathbf{V}_\infty$ and rotating with constant angular velocity $\boldsymbol{\omega}$.

For the second variation we get:

$$\delta^2 T' = \frac{1}{2}\int \big[(\delta\mathbf{v})^2 + \delta\boldsymbol{\Omega}(\mathbf{U}_0 \times \boldsymbol{\varepsilon})\big]\, \mathrm{d}\mathbf{r}\,, \tag{2.33}$$

which naturally agrees with Eqn (2.25) at $\mathbf{V}_\infty = \boldsymbol{\omega} = 0$. Making use of the equality $\int (\delta\mathbf{v})^2\, \mathrm{d}\mathbf{r} = \int (\delta\boldsymbol{\Omega} \cdot \delta\mathbf{A})\, \mathrm{d}\mathbf{r}$, where $\delta\mathbf{A}$ is the vector potential of disturbances determined by the formula $\nabla \times \delta\mathbf{A} = \delta\mathbf{v}$, we get:

$$\delta^2 T' = \frac{1}{2}\int \big[\delta\boldsymbol{\Omega}(\delta\mathbf{A} - \boldsymbol{\varepsilon} \times \mathbf{U}_0)\big]\, \mathrm{d}\mathbf{r}\,. \tag{2.34}$$

2.3.3 Expression for the disturbance energy in terms of the displacement field. Using Eqn (2.5) and the transform formula for the divergence from the vector product, we present the integrand in Eqn (2.34) in the form:

$$\delta\boldsymbol{\Omega}(\delta\mathbf{A} - \boldsymbol{\varepsilon} \times \mathbf{U}_0) = \nabla\big[(\boldsymbol{\varepsilon} \times \boldsymbol{\Omega}_0) \times (\delta\mathbf{A} - \boldsymbol{\varepsilon} \times \mathbf{U}_0)\big] + (\boldsymbol{\varepsilon} \times \boldsymbol{\Omega}_0)\big(\delta\mathbf{v} - \nabla \times (\boldsymbol{\varepsilon} \times \mathbf{U}_0)\big)\,.$$

Making use of Eqns (2.9a) or (2.10), with account for the integral of the first term being equal to zero, we get the following expression:

$$\delta^2 T = \frac{1}{2}\int \left[\boldsymbol{\Omega}_0 \left(\frac{\partial\boldsymbol{\varepsilon}}{\partial t} \times \boldsymbol{\varepsilon}\right)\right] \mathrm{d}\mathbf{r}\,. \tag{2.35}$$

Equation (2.35) is valid for arbitrary displacement fields $\boldsymbol{\varepsilon}$, including Lagrangian displacement field $\boldsymbol{\xi}$. If the frequencies and the form of vortex eigen-oscillations are found, Eqn (2.35) appears to be more convenient for calculations of the oscillation energy than Eqn (2.34).

3. Vortex ring eigen-oscillations

This section considers long-wave eigen-oscillations of a thin vortex ring with piecewise-uniform and piecewise-isochronous vorticity profiles in an ideal incompressible fluid. These vorticity distributions appear to be the simplest for obtaining analytical solutions. A complete set of three-dimensional eigen-oscillations for vortex rings with such vorticity profiles was found in Ref. [32] in the form of asymptotic expansions in terms of the small parameter μ characterizing the ring thickness. These solutions will be used in Section 4 as the starting point for exploring oscillations of vortex rings with more general profiles (smoothed vorticity profile).

3.1 Review of different approaches to the problem of vortex ring oscillations

As a consequence of the extreme complexity of the problem, all the theoretical solutions of vortex ring eigen-oscillation problem are limited to the case of a thin vortex ring ($\mu \ll 1$, μ is the ratio of the vortex ring cross-section to the ring radius). However, until recent times, even for this case only those modes were found the form of which permits simplifying the problem still more. For this purpose the disturbance axisymmetry, short-wave approximation or *a priori* assumptions relating to the oscillation shape were used. We describe the main approaches based on using such simplifications.

The simplest problem is that relating to axisymmetric oscillations. It reduces to determining the vortex boundary disturbances. In this case it is possible to study both small-amplitude [55] and the non-linear disturbances [56].

In the case of short-wave three-dimensional oscillations [28] the wave length is a supplemental small parameter, the presence of which permits solutions to be obtained, ignoring the mutual interaction of the disturbances in the vortex ring regions far away from one another.

For arbitrary three-dimensional oscillations there are two well known cases in which the problem of oscillations of a vortex ring with small parameter μ is reduced to the analysis of the limiting case $\mu = 0$. This limit can be realized in two different ways. In the first case the limit is achieved at finite ring radius and for the vortex cross-section dimension tending to zero. In this case the disturbance structure in the core is set *a priori*. In the second case the limit $\mu = 0$ is achieved at finite vortex cross-section dimension and infinite ring radius and this leads to neglecting the curvature of the vortex ring mean line.

The first approach corresponds to the method of local approximation [57] and the more exact version developed in Refs [58–60]. According to this method, the self-induced velocity in any vortex section is calculated in such a way as if the whole vortex were a filament of infinitely small thickness, excluding the vicinity of the section considered, where the

flow structure coincides with the flow structure in the fitted steady vortex ring. The complete solution shows that this method permits a correct solution to be obtained for one of the families of oscillations accompanied by bending disturbances of the vortex mean line (bending modes). However, the evolution of the mean line at each moment of time for the oscillations of other families is determined not only by this mean line form, but also by the disturbance structure inside the vortex core.

In the second approach the limiting value $\mu = 0$ is achieved by replacing a thin vortex ring with a cylindrical vortex [61, 62]. The basis for this approach is the fact that the local flow structure in a thin vortex ring is close to the flow structure in a cylindrical vortex. A more sophisticated treatment shows that the oscillations of these vortices really have a spectrum similar in structure. However, the oscillation shapes of the vortex ring, obtained in the cylindrical vortex approximation, appear incorrect. Unlike the eigen-oscillations of the cylindrical vortex, the vortex ring eigen-oscillations in the leading approximation can be not a single angular harmonic, but a sum of two angular harmonics.

To solve the problem of thin vortex ring oscillations and to avoid the difficulties indicated, the following procedure was developed in Ref. [32]. The task of finding the eigen-oscillations was divided into three simpler tasks. As the first step, a set of basic disturbances was built, and the eigen-oscillations were expanded in these disturbances. At the second stage the Biot–Savart integral was calculated for each of the basic disturbances. At the third stage the system of algebraic equations determining the eigen-oscillation frequencies and shapes was solved. Such an approach permitted simultaneously finding the disturbance structure inside the vortex, the boundary displacement form and eigen-frequencies, and evaluating the value of rejected terms in each approximation. To realize this procedure it appeared reasonable to use the displacement field $\boldsymbol{\varepsilon}$. The main elements of this procedure and the principal results are given below.

3.2 Oscillations of a cylindrical vortex

In the limiting case $\mu = 0$ ($R \to \infty$, $\rho_0 = \text{const}$, where R is the ring radius, ρ_0 is the vortex cross-section radius) the thin vortex ring with a piecewise-uniform vorticity profile transforms into the cylindrical Rankin vortex with a constant vorticity Ω_0. The dependence of the dimensionless angular velocity U_0 on the radial coordinate ρ for Rankin's vortex is presented in Fig. 3.

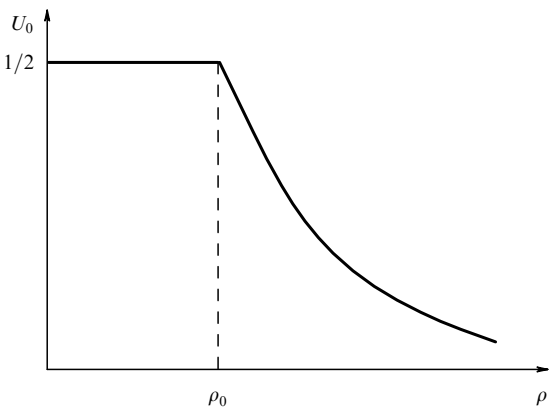

Figure 3. Angular velocity dependence on the radial coordinate for Rankin's vortex.

The solution for small oscillations of the cylindrical vortex was obtained by Kelvin [2] and is well known (e.g. Ref. [63]). We shall give here a short review of these oscillations, keeping in mind that some of their properties appear to be similar to those of vortex ring oscillations.

The eigen-oscillations of Rankin's vortex in the cylindrical coordinates ρ, φ, z with the axis $\mathbf{e}_z$ along the vortex axis are as follows:

$$V^i = V^i(\rho)\exp(\mathrm{i}m\varphi + \mathrm{i}kz - \mathrm{i}\omega t)\,, \qquad m = 0, 1, 2, \ldots \quad (3.1)$$

The possibility of seeking the solution in such a form is connected with the cylindrical vortex symmetry relative to translations along the z-coordinate and rotations around the axis $\mathbf{e}_z$. Note that for the vortex ring there is only one symmetry connected with rotations through the angle θ (the analogue to the symmetry of a cylindrical vortex relative to translations along the z-axis).

Consider only the case of $k \geqslant 0$, $m \geqslant 0$, since it can be easily extended to the case of $k < 0$, $m \leqslant -1$. We restrict ourselves to the case of long-wave oscillations, i.e. we assume that $k\rho_0 \ll 1$.

We select the dimensionless variables in which the vortex cross-section radius $\rho_0 = 1$, and the steady vorticity field $\Omega_0 = 1$ (the angular velocity in the core becomes $U_0 = 1/2$).

3.2.1 Dispersion relation. To find the cylindrical vortex eigen-oscillations, the equations for velocity disturbances in the regions inside and outside the vortex are solved separately and then their solutions are matched at the vortex boundary. The disturbances inside the vortex satisfy Helmholtz's equation (2.6) and the condition of zero divergence for the velocity field. Excluding the velocity components v^ρ and v^φ from these equations, we come to a Bessel equation for the component v^z

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\,\frac{\mathrm{d}v^z}{\mathrm{d}\rho}\right) + \left(a^2 - \frac{m^2}{\rho^2}\right)v^z = 0\,,$$

where $a = \sqrt{1-\omega'^2}\,k/\omega'$, $\omega' = \omega - m/2$. The solution of this equation finite at zero is the Bessel function $v^z = J_m(a\rho)$. Using this solution, one can easily obtain the other components of the velocity field.

The flow is potential outside the vortex. The velocity potential Φ satisfies the Laplace equation. Its solution, decreasing at infinity, takes on the form:

$$\Phi = AH_m^{(1)}(\mathrm{i}k\rho)\,,$$

where $H_m^{(1)}$ is the Hankel function of the first kind. The constant A, together with eigen-frequencies, is determined from the conditions of continuity of the normal velocity component v^ρ and pressure p at the boundary $\rho = 1$. These conditions lead to the dispersion equation

$$\frac{\omega' k J_{m+1}(a)}{\sqrt{1-\omega'^2}\,J_m(a)} + \frac{\mathrm{i}kH_{m+1}^{(1)}(\mathrm{i}k)}{H_m^{(1)}(\mathrm{i}k)} - \frac{m}{1+\omega'} = 0\,. \quad (3.2)$$

This equation is valid for an arbitrary k, but further we shall be interested only in the case of $k \ll 1$.

3.2.2 Analysis of dispersion relation for $k \ll 1$. For long-wave oscillations ($k \ll 1$) the dispersion equation (3.2) has solutions in two regions of parameter a: $a = O(1)$ and $a \ll 1$. At

$a = O(1)$ for each $m = 0, 1, 2, \ldots$ there is a family of solutions:

$$\omega_j = \frac{m}{2} + \frac{k}{a_j} + O(k^2)\,, \qquad j = \pm 1, \pm 2, \ldots \tag{3.3}$$

where a_j are the zeros of Bessel function $J_m(a)$. We shall call these eigen-oscillations Bessel modes. The eigen-frequencies ω_j have the concentration point $\omega = m/2$ and are located to the right and to the left of it. The modes with frequencies located to the right of the concentration point ($a_j > 0$) are called *advancing*, since their angular velocity is larger than the maximum angular velocity of the flow at the core boundary ($\omega_j/m > 1/2$). On the contrary, the modes whose frequencies are located to the left of the concentration point ($a_j < 0$) are called *lagging*. There is an important difference between these modes. The phase velocity of the lagging modes is smaller than the maximum angular velocity of the mean flow (see Fig. 3) and therefore it can coincide with this angular velocity on some streamline ρ_c (in the presence of variable vorticity outside the vortex core such a line will correspond to the so-called critical layer; see Section 4). These lines are in the region outside the vortex, and the closer they are to the vortex boundary, the closer the eigen-oscillation eigen-frequency is to the concentration point. On the contrary, the phase velocity of the advancing modes is larger than the angular velocity of the mean flow and such lines for them are absent.

For each $m = 1, 2, \ldots$ Eqn (3.2) has one more solution for $a \ll 1$:

$$\omega = \frac{m-1}{2} + O(k^2)\,, \qquad m \geqslant 2\,, \tag{3.4a}$$

$$\omega = -\frac{k^2}{4}\left(\ln\frac{2}{k} - C + \frac{1}{4}\right) + O(k^4 \ln k)\,, \qquad m = 1\,, \tag{3.4b}$$

where $C \approx 0.58$ is Euler's constant [64]. The respective eigen-oscillations are called *isolated modes*; the isolated mode with $m = 1$ is also called a *bending mode*. Note, that the bending mode frequency is negative. This means that this oscillation angular phase velocity is directed against the flow.

Thus, close to each frequency value $\omega = l/2$, $l = 0, 1, 2, \ldots$, there exist modes of two types: Bessel modes which are of the harmonic type $\exp(\mathrm{i}m\varphi)$ with $m = l$ and one isolated mode which is the next harmonic with $m = l + 1$ (Fig. 4a). We see that long-wave eigen-oscillations of the cylindrical vortex with a given k can be classified both according to the harmonic number m and the number l which characterizes the nearest concentration point of the eigen-frequencies $\omega = l/2$.

As we shall see, there exists only the second possibility for the vortex ring. The eigen-frequencies of the vortex ring as well as those of the cylindrical vortex are close to $\omega = l/2$ and accordingly these oscillations can be characterized by the number l. At the same time the number m of a harmonic in the vortex cross-section does not characterize the vortex ring eigen-oscillations, since these occur as a sum of harmonics $\exp(\mathrm{i}m\varphi)$ with different m, in contrast to the cylindrical vortex for which the eigen-oscillations are separate harmonics in φ.

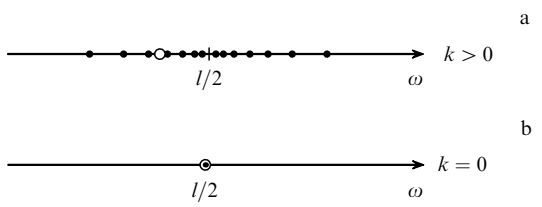

Figure 4. Eigen-frequencies of a cylindrical vortex close to the value $\omega = l/2$: $k > 0$ (a) and $k = 0$ (b); • Bessel modes ($m = l$); ∘ isolated mode ($m = l + 1$).

3.2.3 Eigen-oscillation shapes. It follows from (3.1) that the disturbed vortex boundary is of corrugated structure. This structure in each cross-section $z = z_0$ has m lobes turned around the cylinder axis through an angle depending on z, according to the expression $\exp(\mathrm{i}m\varphi + \mathrm{i}kz)$ (Fig. 5).

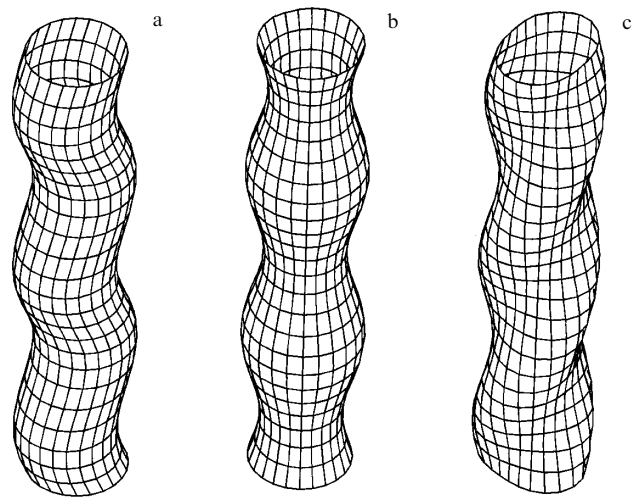

Figure 5. Disturbance shape of the cylindrical vortex boundary: bending mode, $m = 1$ (a); bulging mode, $m = 0$ (b); Bessel modes and isolated mode at $m = 2$ (c).

Bessel modes with $m = 0$ have a characteristic barrel-like shape and, according to this, are called *bulging modes* (Fig. 5b). Isolated modes with $m = 1$, as was indicated above, are called *bending modes*, as these oscillations are reduced to a periodical displacement of the vortex mean line (Fig. 5a).

Bessel and isolated modes with the same number m have an identical shape of vortex boundary disturbance (Fig. 5c). However, the disturbance structures inside the vortex are different for these oscillations. The isolated modes have a power dependency on the coordinate ρ and Bessel oscillations are of an oscillating character in the radial direction.

The difference between the Bessel and isolated modes also manifests itself through the relation between the disturbance amplitude outside the vortex and the vortex boundary displacement amplitude. For isolated oscillations these amplitudes are of the same order. In contrast, Bessel oscillations appear to be localized mostly inside the vortex. For these oscillations the disturbance amplitude outside the vortex appears to be of order $O(k)$ relative to the boundary displacement amplitude. This is connected with the fact that according to (3.1) the phase velocity of disturbances for Bessel oscillations appears to be close to the basic flow velocity at the vortex boundary, i.e. these disturbances are seen as 'adhered' to the basic flow at the vortex boundary and only slightly perturb the external flow region. This property has a number of serious consequences. In particular, with account for the medium compressibility, the vortex oscillations generate

sound radiation. These 'adhered' oscillations in this case appear to be inefficient sound sources. On the contrary, the non-'adhered' oscillations radiate sound efficiently and are of greater interest from this standpoint.

In the case of two-dimensional oscillations ($k = 0$) the isolated modes are reduced to oscillations of a plane circular vortex [5, 63] with frequency $\omega = (m-1)/2$. The frequencies of all Bessel modes appear to be $\omega = m/2$, i.e. these modes become degenerate (Fig. 4b). In this case the eigen-oscillation with frequency $\omega = m/2$ will be any disturbance of the vortex core of the form

$$\Omega^{\rho} = 0\,, \qquad \Omega^{\varphi} = 0\,,$$

$$\Omega^{z} = \left[f(\rho) - \delta(\rho-1)\int_0^1 f(\rho)\rho^{m+1}\,\mathrm{d}\rho\right]\exp(\mathrm{i}m\varphi)\,,$$

where $f(\rho)$ is an arbitrary function, and δ is Dirac's delta-function. Despite the fact that the vorticity disturbance in the core (the first term) is followed by the boundary displacement with an amplitude of order $O(1)$ (the term with the δ-function), it is easy to check that such oscillations produce no disturbances in the external region ($\rho > 1$) at all, since their phase velocity exactly equals $1/2$, i.e. these disturbances exactly follow the basic flow [65] at the vortex boundary. These modes together with the isolated mode make up a complete set of disturbances in terms of which any two-dimensional disturbance of the vortex core can be expanded.

3.3 Three-dimensional oscillations of vortex ring

We return to the consideration of vortex ring oscillations. In three-dimensional (non-axisymmetric) oscillations of the vortex ring one can separate two principal difficulties. The first is connected with selecting the simplest steady flow. Even if the vorticity is completely localized within the boundaries of the toroidal core region, the question relating to the simplest vorticity profile, from the standpoint of obtaining an analytical solution, is not trivial.

It is known that for thin vortex rings there exist an infinite number of different vorticity distributions in the core cross-section, for which the flow is steady in the coordinate system moving together with the vortex ring [66]. The simplest distribution among them seems to be the so-called uniform one for which the vorticity amplitude $\mathbf{\Omega}$ is proportional to the distance from the vortex symmetry axis ξ ($\Omega/\xi = \text{const}$). Note that a steady flow with such a vorticity distribution exists not only in the case of a thin ($\mu \ll 1$) ring, but also for rings of arbitrary thickness [67], including Hill's vortex as a limiting case [42].

The uniform vorticity distribution is characterized by the fact that the axisymmetric boundary disturbances produce no vorticity disturbances in the core. Therefore it is very convenient for studying axisymmetric oscillations (Sections 3.3.1, 3.3.2). On the other hand, for such a vorticity distribution the period of fluid particle motion on streamlines in the vortex core does not appear to be identical for different streamlines (i.e. particles do not move isochronously). The non-isochronism can be easily understood from considering the limiting case of Hill's vortex, for which the period of fluid particle movement tends to infinity on approaching the vortex boundary. This non-isochronism leads to the appearance of disturbances of the continuous spectrum arising at flow oscillations. This fact does not prevent the study of axisymmetric oscillations, for which the continuous spectrum disturbances can be described rather simply [32]. However in the case of three-dimensional (non-axisymmetric) oscillations the presence of the continuous spectrum makes the task very complex. Therefore the most suitable vorticity distribution for describing three-dimensional oscillations is one for which the periods of fluid particle movement on streamlines are identical (an isochronous vorticity distribution; see Sections 3.3.3 – 3.3.6). In the case of a cylindrical vortex the uniform vorticity distribution ($\Omega_0 = \text{const}$) is simultaneously isochronous. For a thin vortex ring these distributions coincide only in the first two approximations in μ and differ in the terms of order $O(\mu^2)$. The expressions for the isochronous vorticity distribution in the vortex ring were obtained in Ref. [68] and are presented below.

The second difficulty is connected with the fact that it is not known beforehand for three-dimensional oscillations in what form the solution ought to be sought. This is their distinction from cylindrical vortex oscillations (Section 3.2) and from two-dimensional (axisymmetric) vortex ring oscillations, for which the general form of the solution can easily be guessed. In particular, using the cylindrical vortex modes as the leading approximation appears to be unsatisfactory. Therefore a special procedure is built for the vortex ring, which is considered below.

3.3.1 Steady flow with a uniform vorticity distribution. Consider the cylindrical coordinates r, θ, z with the axis $\mathbf{e}_z$ along the ring axis and the polar coordinates ρ, φ in the core cross-section with the centre at the stagnation point (Fig. 6). These coordinates are connected by the relations $r = R - \rho\cos\varphi$ and $z = \rho\sin\varphi$, where R is the distance from the ring axis to the stagnation point. We also make use of the coordinate s connected with the angular coordinate θ by the relation $s = R\theta$.

The vorticity magnitude for the uniform distribution is proportional to the distance from the ring symmetry axis, i.e. $\mathbf{\Omega}_0 = \mathbf{e}_s\Omega_0 r/R$ ($\Omega_0 = \text{const}$). The contravariant s-component of the vorticity field is constant $\Omega_0^s = \Omega_0$. The steady velocity field $\mathbf{U}_0$ can be presented as $\mathbf{U}_0 = \mathbf{V}_0 - \mathbf{V}_\infty$, where the velocity field component $\mathbf{V}_0$ decaying at infinity is determined by the Biot – Savart law (2.10), $\mathbf{V}_\infty$ is the ring velocity in the fixed coordinate system directed along the z-axis and equal to

$$V_\infty^z = \frac{\Omega_0 a\,\mu}{4}\left(\ln\frac{8}{\mu} - \frac{1}{4}\right).$$

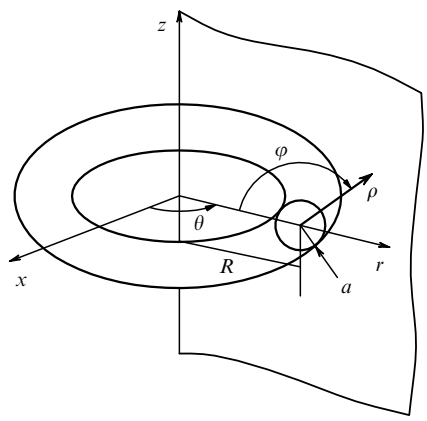

Figure 6. Coordinate system.

We exactly determine the small parameter μ which characterizes the ring thickness. This parameter is the ratio of the core cross-section dimension a to the ring radius R: $\mu = a/R$. Since the cross-section boundary is close to circular, differing only in terms of order $O(\mu^2)$, a is determined as the radius of the circle with area πa^2, exactly equal to the core cross-section area.

The contra-variant components of the steady velocity field inside the vortex ring core and the core boundary shape are as follows:

$$V_0^\rho = -\mu\Omega_0 \frac{5\rho^2}{16a}\sin\varphi + \mu^2\Omega_0\left[\left(\frac{3}{8}\ln\frac{8}{\mu} - \frac{15}{64}\right)\rho - \frac{\rho^3}{16a^2}\right] \times \sin 2\varphi + O(\mu^3)\,, \tag{3.5a}$$

$$V_0^\varphi = \frac{\Omega_0}{2} + \mu^2\Omega_0\frac{5}{16} - \mu\Omega_0\frac{7\rho}{16a}\cos\varphi + \mu^2\Omega_0\left[\left(\frac{3}{8}\ln\frac{8}{\mu} - \frac{15}{64}\right) - \frac{\rho^2}{32a^2}\right]\cos 2\varphi + O(\mu^3)\,, \tag{3.5b}$$

$$\rho = a\left[1 + \frac{5}{8}\mu\cos\varphi - \frac{25}{256}\mu^2 + \left(\frac{161}{256} - \frac{3}{8}\ln\frac{8}{\mu}\right)\mu^2\cos 2\varphi\right] + O(\mu^3)\,, \tag{3.5c}$$

where the unified expression $O(\mu^n)$ is used, for the sake of brevity, for indicating the terms of order $\mu^n \ln\mu$ and μ^n. Note, that the contra-variant velocity φ component is of angular velocity dimensionality.

Note that for describing the steady flow in the vortex ring, the polar coordinates ρ_c, φ_c with another origin located at the ring cross-section centre can be used. It is closer to the ring axis than the stagnation point by the value $\Delta\xi = (5a/8)\mu + O(\mu^3)$ [68] can be used. Equations (3.5) expressed in the coordinates ρ_c, φ_c for $a = 1$ were used in [28]. In particular, the boundary shape (3.5c) in these coordinates is simpler. It is expressed as follows:

$$\rho_c = a\left[1 + \mu^2\left(-\frac{3}{8}\ln\frac{8}{\mu} + \frac{17}{32}\right)\cos 2\varphi_c + O(\mu^3)\right].$$

For convenience of calculations in Ref. [32] the curvilinear coordinates σ, ψ are determined in the ring core cross-section, which coincide in the leading approximation with the coordinates ρ, φ respectively. The coordinates $\sigma(\rho,\varphi)$ and $\psi(\rho,\varphi)$ are selected in such a way that the relations $V_0^\sigma = 0$, $V_0^\psi = V_0^\psi(\sigma)$, $\sqrt{|g|} = \sigma$ are satisfied, where V_0^σ, V_0^ψ are the contravariant components of the velocity field, g_{ij} is the metrical tensor in the coordinate system σ, ψ, s. The lines $\sigma = \text{const}$ correspond to streamlines, since $V_0^\sigma = 0$. The condition $\sqrt{|g|} = \sigma$ is selected to ensure that the differential operators in the coordinate system σ, ψ, s have the simplest form. The specific expressions for the coordinates σ, ψ, s and the metric tensor g_{ij} with an accuracy up to μ^2 are presented in Ref. [32]. Contravariant components of the steady velocity field in these coordinates are

$$V_0^\sigma = 0\,, \qquad V_0^\psi = \frac{1}{2} - \mu^2\frac{21}{64}\sigma^2 + O(\mu^3)\,. \tag{3.6}$$

For convenience, we use dimensionless variables from here on. As the time scale we select Ω_0^{-1} and the length scale $a\left[1 + (5/16)\mu^2 + O(\mu^3)\right]$. The length scale is selected to ensure that the vortex boundary corresponds to the line $\sigma = 1$.

Calculating the rotation period of fluid particles on the streamlines, we obtain:

$$T = \oint\frac{\mathrm{d}l}{|\mathbf{V}_0|} = \frac{2\pi}{V_0^\psi(\sigma)}\,. \tag{3.7}$$

It is evident from Eqns (3.6) and (3.7), that the vortex ring with uniform vorticity appears to be non-isochronous, i.e. the rotation periods of fluid particles in the vortex core on different streamlines $\sigma = \text{const}$ turn out to be different. However this difference is of order $O(\mu^2)$, i.e. the uniform vorticity distribution for the thin vortex ring differs only slightly from the isochronous one.

3.3.2 Axisymmetric oscillations of a vortex ring with uniform vorticity. The family of axisymmetric oscillations of a vortex ring with uniform vorticity is described in Ref. [32]. Their analogue are two-dimensional oscillations of a cylindrical vortex (Section 3.2.3). The family of axisymmetric oscillations consists of isolated modes which are reduced exclusively to vortex boundary disturbances, and of continuous spectrum oscillations including vorticity disturbances in the core. The isolated mode frequency is expressed as

$$\omega = \frac{l}{2} + \mu^2\left[\frac{6l^3 + 18l^2 + 14l + 3}{16l(l+1)(l+2)} - \frac{21}{64}(l+1)\right] + O(\mu^3)\,,$$

$$l = 1, 2, \ldots$$

In contrast to the continuous spectrum of the cylindrical vortex degenerating in the two-dimensional case into the point $\omega = l/2$, the frequencies of axisymmetric oscillation belonging to the continuous spectrum of the vortex ring occupy the frequency region $\omega/l = (V_{0,\min}^\psi, V_{0,\max}^\psi)$ (Fig. 7).

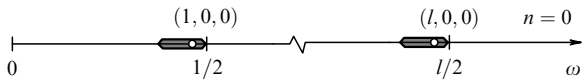

Figure 7. Mutual arrangement of the continuous spectrum and isolated mode (○) close to each value of $l/2$, $l \geqslant 1$.

The appearance of the continuous spectrum for the non-isochronous flow is connected with the fact that V_0^ψ depends on σ and the solutions of Eqn (2.12a) have a singularity at $-\omega + mV_0^\psi(\sigma) = 0$, where m is an integer.

3.3.3 Steady vortex ring with isochronous movement of fluid particles. The condition of isochronism for the vortex ring with uniform vorticity is not satisfied for the terms of order $O(\mu^2)$. In the general case the search for a vorticity distribution which satisfies the isochronism condition is a complex problem. However, if one restricts oneself to the terms of order $O(\mu^2)$, the uniform vorticity distribution can be easily modified to satisfy this condition. With accuracy up to μ^2, the steady vortex ring with isochronous flow has the vorticity and velocity fields:

$$\Omega_0^s = 1 + \mu^2\frac{21}{16}\sigma^2 + O(\mu^3)\,, \qquad V_0^\sigma = 0\,, \qquad V_0^\psi = \frac{1}{2}\,.$$

Since V_0^ψ is independent of σ, this flow, according to Eqn (3.7), is isochronous in the vortex core. The vortex ring with isochronous flow in the core represents a special choice because its ψ-component of the velocity is independent of σ

in any approximation. Hence such a vortex ring has the oscillation of the discrete spectrum only. In this aspect, it is the isochronous vortex ring, and not the uniform one, that is the simplest for studying three-dimensional oscillations. This flow was used in Ref. [32] as the principal one in investigations of three-dimensional oscillations.

3.3.4 Main equations for describing vortex ring oscillations. The vortex ring symmetry relative to the z axis permits seeking for eigen-oscillations in the following form:

$$\xi^i(\mathbf{r}, t) = \xi^i(\sigma, \psi) \exp(\mathrm{i}n\theta - \mathrm{i}\omega t) .$$

Thus, to find the vortex ring disturbance shape, it is necessary to find the vector amplitude function of two variables $\xi^i(\sigma, \psi)$. That is why the problem of vortex ring oscillations is much more complex than that of cylindrical vortex oscillations. The latter reduces to determining the function of one variable ρ.

To find the oscillations, two related tasks are to be solved: to solve the system of differential equations (2.12a), (2.12b) and to calculate the integral term in the boundary condition (2.14). In Ref. [32] a solution method was proposed for the case of long-wave oscillations ($n = O(1)$), which permits these tasks to be separated. Principally this method consists of the following steps.

The amplitude functions are expanded in terms of a set of basic disturbances

$$\boldsymbol{\xi}(\sigma, \psi) = \sum_{m=-\infty}^{\infty} C_m \boldsymbol{\xi}_{(m)}(\sigma, \psi) . \tag{3.8}$$

The basic disturbances $\boldsymbol{\xi}_{(m)}(\sigma, \psi, \omega)$ are built in such a way that each is a solution of Eqns (2.12a), (2.12b) and the whole set of basic disturbances is a complete system of functions at the boundary (in other words, the boundary values of σ-components of these fields are to be a complete system of linearly independent functions in the segment $0 \leqslant \psi \leqslant 2\pi$. Though the basic disturbances individually do not satisfy condition (2.12c) at the vortex boundary, any solution of the system of equations (2.12a), (2.12b) with an arbitrary boundary condition (including the eigen-oscillation) can be presented as expansion (3.8).

The next step is the calculation of the velocity field at the vortex boundary for each basic disturbance. For this purpose, according to the Biot – Savart law, integral (2.14) is calculated. Note, that presenting the velocity field in such a way ensures that the disturbance decreases at infinity.

Finally, after the substitution of Eqn (3.8) and expressions for the velocity field at the vortex boundary calculated at the previous step into the boundary condition (2.12c), a system of linear equations relative to the coefficients C_m of expansion (3.8) is obtained. The condition that the determinant of this system is equal to zero gives the dispersion equation on the basis of which the eigen-frequencies are found. Further, after substituting each of the eigen-frequencies in the system of algebraic equations, the coefficients C_m of eigen-oscillation expansion in terms of basic disturbances are found.

For a thin vortex ring the problem possesses the small parameter μ, which permits the solution at each of these stages to be obtained with the use of successive approximations. It is to be taken into account that the eigen-frequencies are degenerate for long-wave oscillations, i.e. in the first approximations in μ some eigen-frequencies coincide. The shape of the respective eigen-oscillations in these approximations remains indefinite and higher approximations are to be used for their determination, in which the mean flow in the vortex ring is already different from the mean flow in the cylindrical vortex. Therefore the shapes of some eigen-oscillations of the thin vortex ring and the cylindrical vortex are already different in the leading approximation despite the proximity of mean flows.

3.3.5 Dispersion equation and eigen-frequencies. Consider the cases of large ($l \geqslant 1$) and small ($l = 0$) frequencies separately. In the case of $l \geqslant 1$ the dispersion equation is

$$\left[\omega' - \mu^2\left(\frac{6l^2 + 11l + 6}{32l(l+1)(l+2)} - \frac{n^2}{4l(l+2)}\right) + O(\mu^3)\right] \frac{J_l(a_0)}{J_{l+1}(a_0)} = -\frac{(3l+2)^2(l+1)\omega'^3}{4n^3\mu} + O(\mu^2\omega', \mu^4) , \tag{3.9}$$

where $a_0 = (\mu n/\omega')\left[1 + O(\omega')\right]$, $\omega' = \omega - l/2$. The dispersion equation (3.9) is a transcendental equation, the roots of which determine the system eigen-frequencies. Since the right-hand side is small, these roots are located close to those values at which one of the multipliers in the left-hand side of the equation becomes zero.

The second multiplier in Eqn (3.9) $J_l(a_0)/J_{l+1}(a_0)$, which is a ratio of Bessel functions, becomes zero at an infinite number of points corresponding to the zeros of Bessel function J_l. These zeros correspond to an infinite family of eigen-oscillations with the frequencies

$$\omega = \frac{l}{2} + \frac{\mu n}{a_j}\left[1 + O(\mu)\right] , \tag{3.10}$$

where $J_l(a_j) = 0, j = \pm 1, \pm 2, \ldots$ The eigen-frequencies have a concentration point $l/2$ and are located on each side of it, according to the sign of a_j. The oscillations of this family are called *Bessel oscillations*. Similar to the case of cylindrical vortex, the modes with frequencies located to the right of the concentration point ($a_j > 0$) are called *advancing*. And on the contrary, the modes with frequencies located to the left of it ($a_j < 0$) are called *lagging*.

At any value of $l \geqslant 1$ there exists one more eigen-oscillation corresponding to the first multiplier becoming zero in the dispersion equation. This oscillation has the following frequency:

$$\omega = \frac{l}{2} + \mu^2\left(-\frac{n^2}{4l(l+2)} + \frac{6l^2 + 11l + 6}{32l(l+1)(l+2)}\right) + O(\mu^3) . \tag{3.11}$$

Such oscillations are called *isolated*. Depending on the relation of l and n values, the eigen-frequency can be located both to the right and to the left of the accumulation point $l/2$.

For small frequencies ($l = 0$) the dispersion equation is

$$\left[\omega^2 - \frac{\mu^4}{16} A_n B_n + O(\mu^5, \mu^3\omega)\right] \frac{J_0(a_0)}{J_1(a_0)} = O(\omega^3, \mu^6) , \tag{3.12}$$

where

$$A_n = (n^2 - 1)\ln\frac{8}{\mu} + \frac{n^2 + 5}{4} - \frac{4n^2 - 1}{2} S_n ,$$

$$B_n = n^2 \ln\frac{8}{\mu} + \frac{n^2}{4} - \frac{4n^2 - 3}{2} S_n , \qquad S_n = \sum_{k=1}^{n} \frac{1}{2k - 1} .$$

Equation (3.12) has an infinite number of roots close to the zeros of Bessel function J_0. These roots correspond to the family of eigen-oscillations with the frequencies:

$$\omega = \frac{\mu n}{a_j}\left[1 + O(\mu)\right], \tag{3.13}$$

where $J_0(a_j) = 0, j = 1, 2, \ldots$ The eigen-frequencies have the concentration point $\omega = 0$ and are located to the right of it. Bessel oscillations of this type are called *bulging modes* (similarly to the cylindrical vortex oscillations). Note, that values $a_j < 0$ give no new solutions for bulging modes, contrary to the case of $l > 0$.

At $l = 0$ there exists one more eigen-oscillation corresponding to the first multiplier in Eqn (3.12) becoming zero. This is the isolated mode (the so-called *bending mode* which was the subject in Section 3.1 and in the Introduction) with the frequency:

$$\omega = \frac{\mu^2}{4}\sqrt{A_n B_n} + O(\mu^3). \tag{3.14}$$

Note that this expression, starting with a certain number n, becomes imaginary [26]. However, the value of n, at which the instability occurs, turns out to be large and the condition of applicability of Eqn (3.14) is broken. The correct procedure for describing a short-wave instability is given in Refs [28, 63]. Note also that when the instability develops, the frequency becomes purely imaginary, i.e. such an instability is not of an oscillating character.

The spectrum of thin vortex ring oscillations is shown in Fig. 8. One can see that the structures of oscillation spectra of the thin vortex ring and cylindrical vortex appear to be similar. Really, for fixed wave number ($k = \mu n = \text{const}$) and the ring curvature tending to zero ($\mu \to 0$) expressions (3.10), (3.11) exactly coincide with (3.3), (3.4a), and expressions (3.13), (3.14) coincide with (3.3), (3.4b).

Thus, the long-wave three-dimensional eigen-oscillations of the isochronous vortex ring are characterized by three integers: the frequency number l, the azimuthal number n and the radial number j. Indeed, the vortex ring oscillations have frequencies in the vicinity of $\omega = l/2$ and have a definite number n of azimuthal harmonic. Besides, Bessel and bulging oscillations are also different in the radial number $j \neq 0$ characterizing the disturbance shape in the vortex core cross-section. The value of j for isolated oscillations is assumed to be zero.

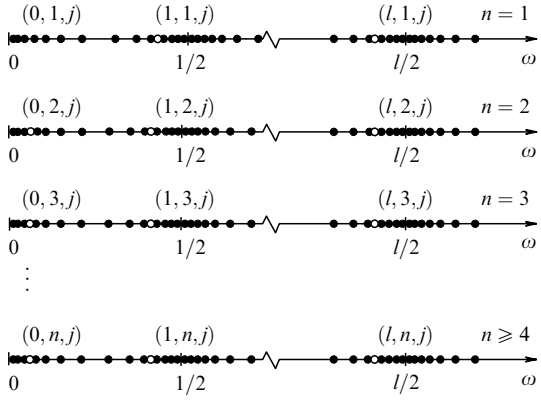

Figure 8. Vortex ring eigen-frequencies localized for each n close to values of $l/2$: Bessel (and bulging) modes (•); isolated modes (○).

3.3.6 Eigen-oscillation shapes. The oscillation shape is determined by the coefficients C_m of expansion (3.8) in terms of basic displacements $\boldsymbol{\xi}_{(m)}$ and by the form of the basic displacements themselves. The expressions for C_m and $\boldsymbol{\xi}_{(m)}$, for all the long-wave oscillations are presented in Ref. [32]. We present here only the expressions for the normal component of the displacement field at the vortex boundary $\xi^\sigma|_{\sigma=1}$, characterizing the deformation shape of the vortex boundary for each oscillation.

The vortex boundary disturbance $\xi^\sigma|_{\sigma=1}$ for Bessel oscillations for $l \geqslant 1$, $\omega'/\mu = O(1)$, is as follows (Fig. 9a):

$$\xi^\sigma\Big|_{\sigma=1} = \exp(\mathrm{i}l\psi) - \frac{(3l+2)(l+1)}{2na_j}\exp\left[\mathrm{i}(l+1)\psi\right] + O(\mu). \tag{3.15}$$

We see that the vortex ring boundary deformation in the leading approximation is a combination of two harmonics $\exp(\mathrm{i}l\psi)$ and $\exp\left[\mathrm{i}(l+1)\psi\right]$, while the Bessel oscillations of the cylindrical vortex have the shape of the lth harmonic.

Oscillations of these vortices are different still further in the region outside the vortex. Since the phase velocity of disturbances $\exp(\mathrm{i}l\psi - \mathrm{i}\omega t)$ is close to the flow mean velocity (the lth harmonic is 'adhered' to the mean flow), this harmonic produces disturbances outside the vortex less efficiently than the harmonic $l + 1$ (see Section 3.2.3). As a result, the external region for the cylindrical vortex appears weakly disturbed and for the vortex ring it is disturbed, since

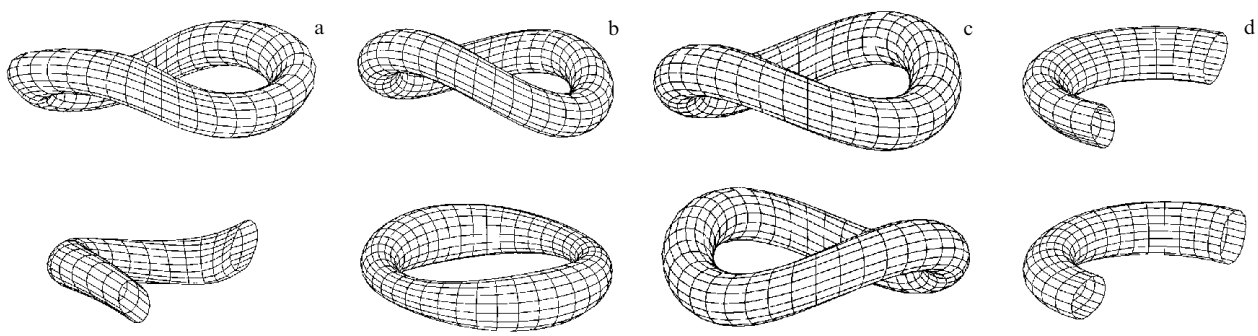

Figure 9. Disturbance shape of vortex core boundary: Bessel modes $l = 1, n = 2, j \geqslant 1$ (a); bulging modes (two phases of oscillations), $l = 0, n = 2, j \geqslant 1$ (b); bending mode $l = 0, n = 2, j = 0$ (two phases of oscillations) (c); isolated mode $l = 1, n = 1, j = 0$ (d) in comparison with an axisymmetric one.

in this region the harmonic $l+1$ dominates which is altogether absent in the cylindrical vortex case. In the polar coordinates, with ρ and φ relating to the vortex section centre, the expressions for the velocity outside the vortex for the lth Bessel oscillations of the vortex ring are

$$v^{\rho} = -\mathrm{i}\,\frac{(3l+2)(l+1)\exp\big[\mathrm{i}(l+1)\varphi\big]}{4na_j\,\rho^{l+2}}\big(1+O(\mu)\big)\,,$$

$$v^{\varphi} = -\frac{(3l+2)(l+1)\exp\big[\mathrm{i}(l+1)\varphi\big]}{4na_j\,\rho^{l+3}}\big(1+O(\mu)\big)\,. \quad (3.16)$$

Thus, even weak differences in the mean flows of the cylindrical vortex and vortex ring (curvature of vortex lines and vorticity structure in the cross-section) lead to qualitative variations in oscillation properties. These variations are connected with eigen-frequency degeneration (see remark at the end of Section 3.3.4). As for the limit $\mu \to 0$, it is necessary to take into account that for the curvature tending to zero the vortex ring oscillations must transform into cylindrical vortex oscillations on condition of constant wave length ($\mu n = \mathrm{const}$). This means that in the case of a transform into cylindrical vortex oscillations (Section 3.2) the correct limit is $\mu \to 0$, $n \to \infty$. In this case expression (3.15) really transforms into the expression for Bessel oscillations of the cylindrical vortex.

All said above on Bessel oscillations with $l \geqslant 1$ relates in full measure to bulging modes (Bessel oscillations with $l = 0$). At $\omega/\mu = O(1)$ the vortex boundary disturbance shape (Fig. 9b) is

$$\xi^{\sigma}\Big|_{\sigma=1} = \exp(\mathrm{i}0\psi) - \frac{1}{na_j}\big(\exp(\mathrm{i}\psi) - \exp(-\mathrm{i}\psi)\big) + O(\mu)\,. \quad (3.17)$$

The bulging modes with $n = 1$ present an interesting and, at first glance, unexpected version of vortex ring motion. Let us turn from the travelling waves of type $\exp(\mathrm{i}n\theta - \mathrm{i}\omega t)$ to the standing waves $\cos(n\theta)\exp(-\mathrm{i}\omega t)$. This is easy to do by taking the half-sum of two travelling waves with wave numbers n of opposite signs. In the case $n = 1$, this standing barrel-like mode has the form

$$\xi^{\sigma}\Big|_{\sigma=1} = \mathrm{Re}\left\{\left[\exp(\mathrm{i}0\psi) - \frac{1}{a_j}\big(\exp(\mathrm{i}\psi) - \exp(-\mathrm{i}\psi)\big)\right] \times \cos\theta\exp(-\mathrm{i}\omega t)\right\}$$
$$= \left(\cos\omega t - \frac{2}{a_j}\sin\psi\sin\omega t\right)\cos\theta\,. \quad (3.18)$$

At $t = 0$ the disturbance is determined by the first term in Eqn (3.18) and has the characteristic barrel-like shape (Fig. 10a) with different areas of the core cross-section at different θ, but within one fourth of the period [at $t = \pi/(2\omega)$] this disturbance is determined by the second term in Eqn (3.18) and reduces to a simple inclination of the ring plane (Fig. 10b). Really, in a linear approximation in disturbances the term with $\sin\psi$ is equivalent to the vortex cross-section drift as a whole along the z-axis. Then the multiplier $\cos\theta$ will correspond to the ring plane inclination under the angle determined by the multiplier $(2/a_j)\sin\omega t$ relative to the undisturbed position. At first glance, this seems to be impossible, since the undisturbed ring momentum is

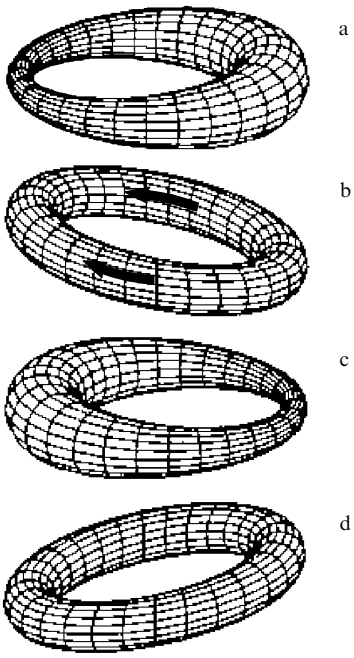

Figure 10. Bulging standing mode, $l = 0$, $n = \pm 1$, $j \geqslant 1$; the barrel-like deformation is accompanied by a bending deformation in the same approximation with a shift in phase by $\pi/2$.

directed along ring axis and without account for the internal structure of disturbances the flow momentum at such oscillations would not be preserved.

The answer to this apparent paradox is that the momentum disturbance in the second phase of oscillations is connected not only with the ring axis inclination [38], but also with the change in the internal vortex structure, i.e. the momentum disturbance has two components. Using expression (2.26), we obtain that the first component connected with the ring axis inclination is

$$\delta^2 P = \mathbf{e}_x\,\frac{2\pi^2}{\mu^2}\,\omega\sin\omega t\,,$$

where $\mathbf{e}_x$ is a unit vector in the ring plane, corresponding to the azimuthal angle $\theta = 0$. The second contribution to the momentum is associated with the fluid transport inside the vortex ring (fluid flow is indicated in Fig. 10b by arrows) which is characterized by the fluid flux $\Pi = (2\pi/a_j)\sin\theta\sin\omega t$ through the core cross-section. This secondary flow leading to a change in the core cross-section area has, evidently, a momentum directed oppositely to that associated with the ring axis inclination. Its calculations give the following result:

$$\delta^2 P = -\mathbf{e}_x\,\frac{2\pi^2}{\mu^2}\,\omega\sin\omega t\,,$$

i.e. the contributions of two components are equal in value and opposite in sign. Thus, the contributions of two disturbance components to the momentum (ring plane inclination and flow in the ring core along its mean line) exactly compensate each other. This example is of great importance, since it demonstrates in the leading approximation the presence of two harmonics simultaneously in vortex

ring eigen-oscillations and permits an independent verification of the correctness of calculations for this specific case.

For isolated oscillations with $l \geqslant 1$ the vortex boundary disturbance (Fig. 9c, d) is:

$$\xi^{\sigma}\Big|_{\sigma=1} = \exp\big[\mathrm{i}(l+1)\psi\big] - \frac{\mu}{4}\exp(\mathrm{i}l\psi) - \frac{(2l+3)\mu}{4(l+1)}\exp\big[\mathrm{i}(l+2)\psi\big] + O(\mu^2)\,, \qquad l \geqslant 1\,, \quad (3.19a)$$

$$\xi^{\sigma}\Big|_{\sigma=1} = \cos\psi - \mathrm{i}\left(\frac{B_n}{A_n}\right)^{1/2}\sin\psi + O(\mu)\,, \qquad l = 0\,. \quad (3.19b)$$

In contrast to Bessel (bulging) oscillations, the isolated (bending) modes of the vortex ring in the leading approximation coincide with the isolated modes of the cylindrical vortex not only in frequency, but also in shape.

A comparison between Bessel and isolated modes with neighbouring numbers l and identical n gives one more example of the idea that the vortex ring oscillation dynamics is determined not only by the global shape of the ring mean line deformation, but also by the disturbance structure inside the core. Thus, for example, the Bessel mode with $l = 1, n = 2$ and the bending mode with $l = 0$, $n = 2$ in successive phases are given in Fig. 11. It is evident that the bending deformation of the ring mean line for these oscillations coincides. However, in the Bessel mode this bend is additionally followed by an elliptical deformation of the core cross-section. As a result, these modes, which are alike in respect to the mean line deformation, have frequencies different by several orders in μ.

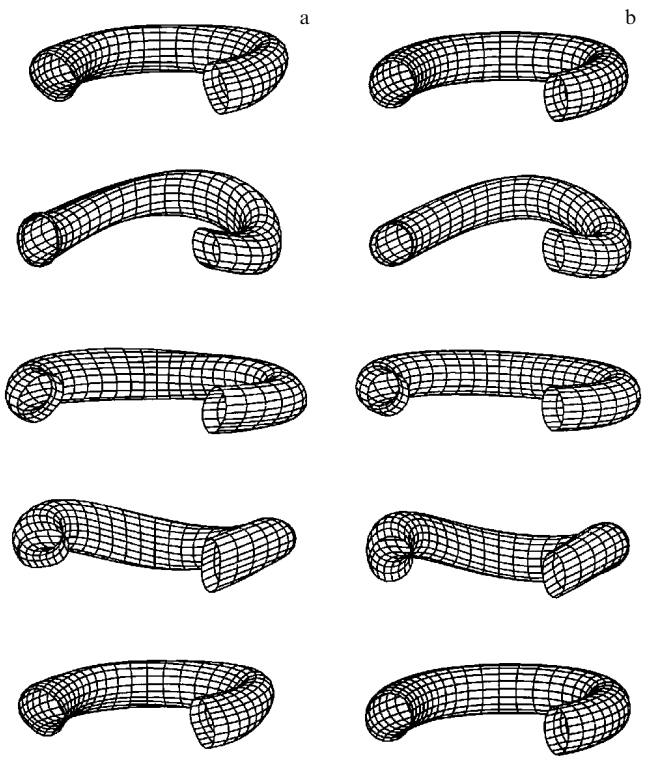

Figure 11. Four phases (one period) of eigen-oscillations for fast Bessel modes $(1, 2, j)$ (a) and a slow bending mode $(0, 2, 0)$ (b). For the Bessel mode the bend is followed by a rotation of the elliptical core deformation. The same mean line bend corresponds to eigen-oscillations, the frequencies of which are different by several orders.

3.4 Disturbance energy

We find the disturbance energy for the vortex ring oscillations described above. To calculate the energy we shall use expression (2.35). The vortex ring eigen-oscillations will be presented in the form $\boldsymbol{\varepsilon} = (\mathbf{a} + \mathrm{i}\mathbf{b})\exp(\mathrm{i}n\theta - \mathrm{i}\omega t)$ where $\mathbf{a}$ and $\mathbf{b}$ are real-valued vectors with components depending on the coordinates σ, ψ. Integration over θ leads to the following expression

$$E = \frac{\pi\omega}{\mu}\int_0^{2\pi}\int_0^1 \sigma^2(a^{\psi}b^{\sigma} - a^{\sigma}b^{\psi})\,\mathrm{d}\sigma\,\mathrm{d}\psi\,. \qquad (3.20)$$

After integrating over ψ, only contributions from the products of identical azimuthal harmonics remain.

For Bessel oscillations the main contribution to the energy is made by the lth ψ-harmonic. Calculating integral (3.20) for the case $l \geqslant 1$, we get:

$$E = \frac{\pi^2 l}{2\mu^2 n}\,a_j + O(\mu^{-1})\,. \qquad (3.21)$$

The energy sign depends on the sign of a_j. For advancing Bessel modes $(a_j > 0)$ the energy is positive and for lagging ones $(a_j < 0)$ it is negative.

For bulging modes $(l = 0)$ the energy is positive:

$$E = \frac{\pi^2}{\mu} + O(1)\,. \qquad (3.22)$$

For isolated oscillations (including an axisymmetric mode) the main contribution is made by the ψ-harmonic $(l + 1)$. Calculating integral (3.20) for the case $l \geqslant 1$ we find that the energy is negative and is

$$E = -\frac{\pi^2}{2\mu}\frac{l}{l+1} + O(1)\,. \qquad (3.23)$$

And finally for bending modes the energy is positive and is determined by the expression:

$$E = \frac{\pi^2\mu}{4}\,B_n + O(\mu^2)\,. \qquad (3.24)$$

Thus, the following situation persists for a vortex ring. All the oscillations with frequencies close to zero (bulging and bending modes $l = 0$) have positive energy. All the isolated oscillations with $l \geqslant 1$ have negative energy. The Bessel modes can have either positive or negative energy according to whether the mode is advancing or lagging. Note, that the isolated mode with negative energy can be among the Bessel modes with positive energy as well as among those with negative energy, depending on the relation between the numbers n and l in formula (3.11).

Expressions (3.21)–(3.24) are obtained for travelling azimuthal waves of the type $\exp(-\mathrm{i}\omega t + \mathrm{i}n\theta)$. For $\exp(-\mathrm{i}\omega t)\cos(n\theta)$ type standing waves the integration over θ in Eqn (3.20) gives half the value. This means that the standing wave energy appears to be equal to half the running wave energy.

Similar calculations for the cylindrical vortex show that the energy of the respective oscillations per unit length in the longitudinal direction is determined by expressions (3.21)–(3.24) divided by the ring length $2\pi/\mu$.

As it will be shown in the further analysis, the sign of oscillation energy determines the eigen-oscillation stability or

the instability when vorticity profiles of a more general type are considered.

4. Instability of a vortex ring with a smoothed vorticity profile and transition to turbulence

Vortex ring oscillations were examined in Section 3 for the cases when the vorticity was concentrated in a thin toroidal core and this core was surrounded by a potential flow. It is of interest to understand the effect of adding a weak vorticity to the flow around the vortex core (in the ring envelope) on the oscillation properties. Such smoothing of the vorticity profile also corresponds to the physics of the process. Really, as the vortex ring moves, the vorticity must penetrate from the vortex core into the surrounding flow as a result of viscous diffusion.

To clarify the mechanism of the vorticity effect on the oscillation properties, first consider a simple oscillator — a circular cylinder able to make its own elastic oscillations — in a two-dimensional circulating flow (potential or vortical). This task may have both an exact solution (Sections 4.1.2, 4.1.3) and an approximate one (Section 4.1.4) based on considering the energy balance in the system.

It appears that the appearance of monotonically decreasing vorticity in the circular flow streamlining the oscillating cylinder boundary could lead to instability. This instability mechanism can easily be understood in the case of weak vorticity when the task can be solved by perturbation methods. It turns out that instability of this type occurs each time when two conditions are simultaneously fulfilled: first, the oscillator oscillations are of positive energy if the flow around the cylinder is potential; second, these oscillations are accompanied by the appearance of a critical layer, when the monotonically decreasing vorticity is present in the flow.

Vortex flows in which the role of the oscillator is played by the oscillating boundary of the vortex core can also possess a similar instability. It appears that the simplest vortex for which such an instability can be realized is a vortex ring (Section 4.2). In this case the stability loss in the vortex ring occurs simultaneously for a set of modes having positive energy. In Section 4.2.2 it is shown that the critical layers appear to fill up the whole region from the core region to the ring envelope boundary. At the same time the critical layers are absent inside the vortex core. In the critical layers the amplitudes of fluid particle displacements achieve large magnitudes and this leads to intensive fluid mixing in the envelope region. It is shown that simultaneously with that process vorticity field intensification also occurs in the vicinity of each critical layer. This result agrees qualitatively with the experimental data on the turbulent vortex ring structure [18, 19] establishing a sharp boundary between the laminar vortex core and the turbulent ring envelope. In Section 4.3 the possible effect of non-linearity and viscosity on the processes investigated is examined.

4.1 Effect of monotonically decreasing vorticity on oscillation properties. Instability of oscillating cylinder in circulating flow of ideal fluid

First of all we consider a simple problem where the characteristic peculiarities of instability are displayed — a rigid cylinder in a circulating flow. Cylinder oscillations are supported not by the vorticity dynamics, but directly by an elastic spring. It is assumed that the circulating flow has a monotonically decreasing vorticity profile. It appears, that such a simple system can become unstable. The problem can be solved without any approximations and the increment can be obtained directly from the exact dispersion relation [69]. At the same time, if the vorticity is small, the solution can be obtained from considering the energy balance in the system. Such an approach permits not only finding the correct expression for the increment, but also understanding the physical mechanism of the instability. The instability is based on the possibility of energy transfer from disturbances in the critical layer to the cylinder oscillations. The total energy in this case does not change, since a simultaneous increase of disturbances with energy of different signs occurs. This mechanism of instability is discussed in detail in Section 4.1.4.

4.1.1. Dispersion relationship. Cylinder oscillations in a circulating flow are a special case of arbitrary body motion in a fluid, which has been studied in detail starting with the works of Kelvin and Tait [5, 70, 71]. At the same time, as is noted in Ref. [5], this problem becomes extremely complex for an arbitrary vortical flow and, generally speaking, it permits a general solution only in the case of a uniformly rotating fluid. From the results obtained in this direction, we note works [72, 73] relating to sphere motion stability in a non-uniform flow of ideal fluid and work [74] relating to stability of co-axial cylinders with a uniformly rotating fluid between them. Note also a new approach to the problem of stability of a 'body+fluid' system, developed in Ref. [75] and based on extending the Arnold theorems to the case of arbitrary rigid body motion in a vortical flow. Such an approach permits formulating general criteria of flow stability under rather wide assumptions, but it gives no criteria of the system instability and this is characteristic of all the works in this direction.

Consider a simple oscillator in a two-dimensional flow with circulation. The system consists of a circular cylinder with unit radius and mass M, which can be displaced in the plane with Cartesian coordinates x, y (Fig. 12). The returning force is characterized by the rigidity of the spring χ. The cylinder is in a flow of an incompressible ideal fluid with unit density. The mean flow in the cylindrical coordinate system has angular velocity $U_0(\rho)$ and vorticity $\Omega_0(\rho)$, related by the relationship $\Omega_0(\rho) = 2U_0 + \rho U_0'$. It is assumed that $U_0(\rho)$ and $\Omega_0(\rho)$ are the monotonically decreasing functions. The problem is solved in the linear approximation in disturbance amplitudes.

As is known [76], a plane circular flow with a monotonically decreasing vorticity profile near a fixed cylinder is stable relative to two-dimensional disturbances. The oscilla-

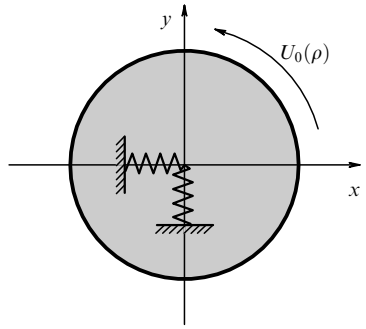

Figure 12. Oscillator (circular cylinder on springs) in a circulating flow.

tor is evidently also stable in the absence of a circulating flow. At the same time, as will be shown below, the joint oscillations of this system over a wide range of parameters appear to be unstable.

To calculate the disturbances produced by the oscillating cylinder in the flow, we use the equations for the displacement field $\boldsymbol{\xi}$ (2.9a). Applying the curl operation to Eqn (2.9a) and using the condition that the displacement field is divergence-free, for $\exp(-\mathrm{i}\omega t + \mathrm{i}m\varphi)$ type disturbances we get the equation for the ρ-component of the field $\boldsymbol{\xi}$ subject to the conditions of impermeability at the boundary of the oscillating cylinder and of disturbance decay at infinity:

$$\frac{\mathrm{d}^2\xi^\rho}{\mathrm{d}\rho^2} + \left(\frac{3}{\rho} + \frac{2mU_0'}{mU_0 - \omega}\right)\frac{\mathrm{d}\xi^\rho}{\mathrm{d}\rho} + \frac{1-m^2}{\rho^2}\,\xi^\rho = 0\,,$$

$$\xi^\rho\Big|_{\rho=1} = \xi_0\,, \qquad \xi^\rho\Big|_{\rho=\infty} = 0\,. \tag{4.1}$$

This equation is an analogue of the well-known Rayleigh equation for flows with circular streamlines . It is usually written for the stream function A [76] as follows:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}A}{\mathrm{d}\rho} - \left(\frac{m^2}{\rho^2} + \frac{m\Omega_0'}{\rho(mU_0 - \omega)}\right)A = 0\,. \tag{4.2}$$

Equations (4.1) and (4.2) have a singularity in the critical layer $\rho = \rho_\mathrm{c}$ determined from the relationship $U_0(\rho_\mathrm{c}) = \omega/m$. These equations are equivalent with account for the relationship $v^\rho = \mathrm{i}mA/\rho = (-\mathrm{i}\omega + \mathrm{i}mU_0)\xi^\rho$ which follows from the stream function definition and Eqn (2.9a).

The advantage of Eqn (4.1) for describing the disturbance is displayed for the harmonics $m = \pm 1$. In this case the latter term on the left-hand side of this equation becomes zero and this permits writing at once the general solution:

$$\xi^\rho(\rho) = C_1 + C_2 I(\rho)\,, \tag{4.3}$$

$$I(\rho) = \int_\rho^\infty \frac{\mathrm{d}\rho}{\rho^3(mU_0 - \omega)^2}\,.$$

The possibility of obtaining the general solution in the case $m = \pm 1$ is shown in Ref. [76]. The first term in Eqn (4.3) corresponds to the constant vector field $\boldsymbol{\xi}$ with the contravariant components $\xi^\rho(\rho,\varphi) = C_1\exp(\pm\mathrm{i}\varphi)$, $\xi^\varphi(\rho,\varphi) = (\pm\mathrm{i}/\rho)C_1\exp(\pm\mathrm{i}\varphi)$ and is a simple displacement of the flow as a whole. The presence of this elementary disturbance among the solutions permits obtaining the general solution in an analytical form for a flow with an arbitrary velocity profile $U_0(\rho)$.

Further we restrict ourselves to the case $m = 1$. This case for $\omega > 0$ corresponds to anticlockwise cylinder rotation (see Fig. 12). Constants C_1 and C_2 in the solution (4.3) can be found from the boundary condition (4.1). As a result, we find that the disturbances generated by the oscillating cylinder in the flow are

$$\xi^i(\rho,\varphi,t) = \xi^i(\rho)\exp(-\mathrm{i}\omega t + \mathrm{i}\varphi)\,,$$

$$\xi^\rho(\rho) = \frac{I(\rho,\omega)}{I_0(\omega)}\,\xi_0\,, \tag{4.4}$$

where $I_0(\omega)$ is the value of function $I(\rho,\omega)$ when $\rho = 1$. The component ξ^φ can be found from the condition of incompressibility of the field $\boldsymbol{\xi}$.

To obtain the equation describing the system oscillations, we find the pressure disturbance p_b at the oscillating cylinder boundary. Using the Euler equations, we express the pressure disturbance through the displacement field:

$$p = (\omega^2 - U_0^2)\,\rho\xi^\rho + (U_0 - \omega)^2\rho^2\,\frac{\mathrm{d}\xi^\rho}{\mathrm{d}\rho}\,. \tag{4.5}$$

The pressure at any point is equal to $P_0 + p$, where P_0 is the pressure in the steady flow, and p is the pressure disturbance (4.5). Since we are interested in the pressure not at a fixed space point but on the moving cylinder surface, we get for this value in the linear approximation $p_\mathrm{b} = \xi_0\,\mathrm{d}P_0/\mathrm{d}\rho + p$, where $\mathrm{d}P_0/\mathrm{d}\rho$ and p are taken at the undisturbed cylinder boundary $\rho = 1$. Using the exact equation $\mathrm{d}P_0/\mathrm{d}\rho = \rho U_0^2$ and Eqn (4.5) we obtain

$$p_\mathrm{b} = \left[\omega^2 - \frac{1}{I_0(\omega)}\right]\xi_0\,. \tag{4.6}$$

The force acting on the cylinder is composed of the elastic force of the spring and the pressure force which is found by integration of Eqn (4.6) over the cylinder boundary. The balance of forces acting on the cylinder is expressed by $(-M\omega^2 + k)\xi_0 = -\pi p_\mathrm{b}$. Hence we obtain the dispersion relationship

$$D(\omega) = \omega_0^2 - \omega^2 + \gamma\left(\omega^2 - \frac{1}{I_0(\omega)}\right) = 0\,, \tag{4.7}$$

$$I_0(\omega) = \int_1^\infty \frac{\mathrm{d}\rho}{\rho^3\left[U_0(\rho) - \omega\right]^2}\,, \tag{4.8}$$

where $\gamma = \pi/M$ is the ratio of the mass of fluid displaced by the cylinder to the cylinder mass, $\omega_0^2 = \chi/M$. The integrand in Eqn (4.8) has a singularity at the point $\rho = \rho_\mathrm{c}$, where $U_0(\rho_\mathrm{c}) = \omega$.

In the specific case of potential flow the angular velocity is $U_0(\rho) = U_M/\rho^2$ and the integral of Eqn(4.8) can easily be calculated:

$$I_0 = \left[2\omega(\omega - U_M)\right]^{-1}\,. \tag{4.9}$$

Then we get from dispersion equation (4.7):

$$\omega_{1,2} = \frac{\gamma U_M}{1+\gamma} \pm \sqrt{\left(\frac{\gamma U_M}{1+\gamma}\right)^2 + \frac{\omega_0^2}{1+\gamma}}\,. \tag{4.10}$$

Thus, in the potential flow case the oscillating system possesses two real eigen-frequencies $\omega_{1,2}$ for any parameters of the system. If the fluid is at rest, we get from Eqn (4.10) $\omega_{1,2} = \pm\omega_0/\sqrt{1+\gamma}$. This result corresponds to the oscillations with account for the apparent additional fluid mass. Finally, the case $\gamma = 0$ corresponds to a weightless fluid which evidently does not affect the oscillation frequency equal to ω_0.

4.1.2 Exact solution for a specific case of vorticity of type $\boldsymbol{\Omega_0 = \Omega_M/\rho}$. The angular velocity field for this flow is $U_0 = \Gamma/\rho^2 + \Omega_M/\rho$, where the first term corresponds to a potential flow component and the second corresponds to a vorticity contribution. In the case considered integral (4.8) is calculated precisely and is equal to:

$$I_0 = \frac{2\Gamma + \Omega_M}{4Q^2(\omega - \Gamma - \Omega_M)} + \frac{\Omega_M}{8Q^3}\ln\frac{P-Q}{P+Q}\,, \tag{4.11}$$

where $P = \omega - \Omega_M/2$, $Q = \sqrt{\Gamma\omega + \Omega_M^2/4}$. The logarithmic function in the complex plane ω is made single-valued by introducing a cut $0 < \omega < U_M$, where $U_M = \Gamma + \Omega_M$. The requirement that $I_0(\omega)$ for $\omega > U_M$ takes on a positive real value selects the logarithmic function branch. Substituting Eqn (4.11) into Eqn (4.7), we get the transcendental equation determining the system eigen-frequencies. We shall not carry out a complete investigation of Eqn (4.11) and restrict ourselves to investigating the effect of the circulating flow on two eigen-frequencies of the oscillator, in a function of the parameters γ and ω_0.

At $\gamma = 0$ (a very heavy cylinder) the fluid has no influence on the oscillations and the system has two eigen-oscillations with frequencies $\omega_{1,2} = \pm\omega_0$. As γ increases (the cylinder mass decreases) the frequency ω_2 remains real and the frequency ω_1 behaves differently depending on the relationship between the values ω_0 and U_M.

If $\omega_0 < U_M$ then for $0 < \gamma < \gamma_0$, where $\gamma_0 = 1 - \omega_0^2/U_M^2$, this eigen-frequency is split into two complex-conjugated frequencies (Fig. 13), one of which corresponds to unstable oscillations and the other to decaying ones; at $\gamma = \gamma_0$ these frequencies become real and merge into one eigen-frequency $\omega_1 = U_M$; at $\gamma > \gamma_0$ this frequency is real and increases from U_M to ω_∞, where ω_∞ is the solution of the equation $\omega_\infty^2 I(\omega_\infty) = 1$.

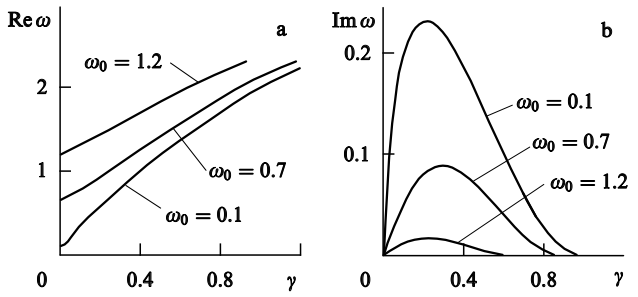

Figure 13. (a) Real part, (b) imaginary part of eigen-frequency for the unstable mode as a function of the parameter γ. The imaginary part of the decaying mode is symmetric relative to the γ-axis.

If $\omega_0 > U_M$, the eigen-frequency ω_1 as well as ω_2, remains real for all the values of γ, i.e. the oscillations considered remain stable for any cylinder mass.

Thus, for $\gamma < 1$ there exists a range of frequencies ω_0, for which instability occurs. When γ approaches unity, this range becomes narrower and for $\gamma \geqslant 1$ this instability is absent.

4.1.3 Case of weak vorticity (dispersion equation solution). We assume further that the circulating flow is weakly vortical ($\Omega_M \ll U_M$). In this case general expressions for integral (4.8) and an increment value can be obtained. With the use of the relationships $\Omega_0 = 2U_0 + \rho U_0'$, $\Omega_0' = (U_0'\rho^3)'/\rho^2$ and of integration by parts, Eqn (4.8) can be transformed:

$$I_0 = \frac{1}{2\omega(\omega - U_M)\big(1 - \Omega_M/(2U_M)\big)} - \int_0^{U_M} \frac{f(z)}{z - \omega}\,\mathrm{d}z\,, \quad (4.12)$$

where

$$f(z) = \frac{U_0\Omega_0'}{\omega U_0'^3\rho^4}\,, \qquad z = U_0(\rho)\,,$$

and the prime means derivative with respect to ρ. Since the vorticity and its derivative are assumed to be small, one can neglect the quantity $\Omega_M/(2U_M)$ in the first term and seemingly the whole second term. However, rejecting the second term, one must be careful, since the Cauchy type integral can take not only real values, but also complex ones. At the same time, the presence of even a small imaginary part in the dispersion relationship corresponds to instability, i.e. to a qualitative change in the flow dynamics.

Consider the integral term in Eqn (4.12). As is known, the Cauchy type integral has a discontinuity along the integration contour (here the integration contour is a segment of the real axis from 0 to U_M). Following Ref. [77], we separate the singularity under the integral and make an integration:

$$\int_0^{U_M} \frac{f(z)\,\mathrm{d}z}{z - \omega} = \int_0^{U_M} \frac{f(\omega)}{z - \omega}\,\mathrm{d}z + \int_0^{U_M} \frac{f(z) - f(\omega)}{z - \omega}\,\mathrm{d}z$$

$$= f(\omega)\ln\frac{\omega - U_M}{\omega} + \int_0^{U_M} \frac{f(z) - f(\omega)}{z - \omega}\,\mathrm{d}z\,.$$

In the latter equation the integral determines a regular function, real and finite for all the real values of ω, and the first term determines a multiple-valued function, the regular branch of which can be selected with the use of an analytical continuation from the real $\omega > U_M$, at which the integral $\int_0^{U_M} \mathrm{d}z/(\omega - z)$ is not singular and has a real positive value. On the cut from 0 to U_M only the imaginary part of the integral is discontinuous. Since at the analytical continuation from large ω values to the value on the cut, the point U_M is passed from above, the imaginary part of the logarithm gets the increment $+\mathrm{i}\pi$, and the imaginary part of the integral is equal to $+\mathrm{i}\pi f(\omega)$ on the upper edge of the cut. It is equal to $-\mathrm{i}\pi f(\omega)$ on the lower edge, since the point U_M is passed from below.

Now use the condition of smallness of vorticity and its derivative. Keeping only leading terms in small vorticity in the real and imaginary parts of I_0, we get for real ω

$$I_0 = \frac{1}{2\omega(\omega - U_M)} \pm \mathrm{i}\pi f(\omega) + O\left(\frac{\Omega_M}{U_M}\right),$$

$$f(\omega) = \left.\frac{\Omega_0'}{\rho^4(U_0')^3}\right|_{\rho=\rho_c(\omega)}, \quad (4.13)$$

where the critical layer coordinate ρ_c is found from the equation $U_0(\rho_c) = \omega$ and the signs $\pm$ correspond to the upper and lower edges of the cut $0 < \omega < U_M$. The appearance of the purely imaginary part in Eqn (4.13) and, respectively, in the dispersion relation shows that the oscillation frequency in the interval from 0 to U_M cannot remain real, in contrast to the case of potential flow. Note also, that Eqn (4.11) in the case of small vorticity naturally transfers into the general expression (4.13).

Since the imaginary part of I_0 is small, the imaginary part of the frequency will be also small, i.e. the eigen-frequency can be presented as $\omega = \omega_R + \mathrm{i}\delta$, $\delta/\omega_R \ll 1$. Restricting ourselves to the principal terms we get, as a result, the system of equations for finding the real ω_R and the imaginary δ parts of the eigen-frequency

$$(\gamma + 1)\omega_R^2 - 2\omega_R\gamma U_M - \omega_0^2 = 0\,, \quad (4.14a)$$

$$(\gamma + 1)\left(\omega_R - \frac{\gamma U_M}{\gamma + 1}\right)\delta = \pm 2\pi\gamma f(\omega_R)\omega_R^2(\omega_R - U_M)^2\,,$$

$$0 < \omega_R < U_M\,. \quad (4.14b)$$

If $\omega_R \leqslant 0$ or $\omega_R \geqslant U_M$ then $\delta = 0$. In Eqn (4.14b) for growing oscillations, the plus sign is to be taken and for decaying oscillations the minus sign is to be taken.

Equation (4.14a) coincides with the equation for the potential flow case and leads to two roots of type (4.10). One of these roots is positive, the other is negative. The negative root is of no interest since no imaginary part in I_0 appears and such an oscillation is always stable. Accounting for a small vorticity in the next approximation can only slightly displace this frequency along the real axis. We consider further only the root with a positive real part:

$$\omega_R = \frac{\gamma U_M}{1+\gamma} + \sqrt{\left(\frac{\gamma U_M}{1+\gamma}\right)^2 + \frac{\omega_0^2}{1+\gamma}}\,. \tag{4.15}$$

If the system parameters are such that $0 < \omega_R < U_M$, the frequency ω is complex and for calculating its imaginary part, we use (4.14b). In this case the plus sign corresponding to the upper edge of the cut is taken for increasing oscillations ($\delta > 0$) and the minus sign — for attenuating oscillations ($\delta < 0$). The solutions exist if these signs correspond to the sign of δ determined from Eqn (4.14b). Since in the case considered $\omega_R - \gamma U_M/(\gamma+1) > 0$ and $f(\omega_R) > 0$, the sign of δ obtained from (4.14b) agrees with the assumption on the sign of the right-hand side in this equation. As a result, we get

$$\delta = \pm\frac{2\pi\gamma f(\omega_R)\omega_R^2(\omega_R - U_M)^2}{\sqrt{\omega_0^2(1+\gamma) + \gamma^2 U_M}}\,. \tag{4.16}$$

Thus, the eigen-frequency ω_1 is split into a pair of complex-conjugated frequencies, one of which corresponds to an unstable oscillation and the other to an attenuating one (Fig. 14).

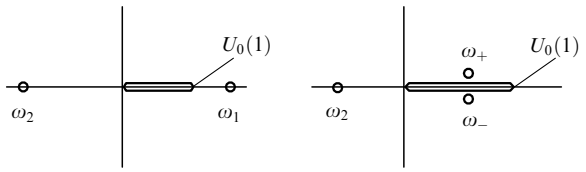

Figure 14. Split-up of eigen-frequency ω_1 at the cut.

Note, that if the signs of the left-hand and right-hand sides in Eqn (4.16) were different (e.g., in the cases $\gamma < 0$ or $\Omega_0'(\rho_c) > 0$), this would mean that there are no continuous solutions of Eqn (4.1) decreasing at infinity. Such a situation holds for Kelvin vortex oscillations in a weakly vortical flow. The decreasing vorticity in the mean flow outside the vortex core leads to displacement of the eigen-frequencies from the upper (causal) sheet of ω-plane under the cut to the non-physical sheet [78].

Neglecting the weak vorticity in the expression for the mean flow, we get from Eqn (4.16):

$$\delta = -\frac{\gamma\pi\rho_0\Omega_0'(\rho_0)(\omega_R - U_M)^2}{4U_M\big[(1+\gamma)\omega_R - \gamma U_M\big]}\,, \tag{4.17}$$

where $\rho_0 = \mathrm{Re}\,(\rho_c)$, the value of ρ_0 in the leading approximation is determined by the relationship $\omega_R = U_M/\rho_0^2$.

Note that the instability described cannot be realized for flows with $\gamma > 1$ (light cylinder). Really, the condition $\omega_R < U_M$ is to be fulfilled for the appearance of instability. This condition can easily be rewritten using Eqn (4.15) as $\omega_0^2/U_M < 1 - \gamma$, which is impossible for $\gamma \geqslant 1$, since U_M is a positive value. On the contrary, for $\gamma < 1$ (heavy cylinder) there always exists a range of oscillator parameters, for which the flow will be unstable. This result agrees with the numerical analysis of the above case of a vorticity of arbitrary value and completely corresponds to the stability criterion for a light cylinder obtained in Ref. [75].

4.1.4 Case of weak vorticity (the energy approach). Let us consider the effect of the oscillator stability loss in a circulating flow from the standpoint of energy balance in the system. In the case of a weakly vortical flow ($\Omega_0 \ll U_0$) the law of energy conservation permits us to find the imaginary increments to eigen-frequencies in a rather simple way. Furthermore, the energy approach is equally applicable to both oscillations with $m = 1$ and with arbitrary m, when a solution of Eqn (4.3) cannot be built.

Consider the disturbances of type $\exp(-\mathrm{i}\omega t + \mathrm{i}m\varphi)$ with an arbitrary harmonic number m. In the case of weak vorticity it is more suitable to use not the displacement field $\boldsymbol{\xi}$ for describing disturbances with arbitrary m, as was done above, but the stream function A, since in Eqn (4.2) the small vorticity effect is localized in the last term which is small almost everywhere except in the vicinity of the singularity. Making use of the perturbation methods, one can show that the solution satisfying the boundary condition on the cylinder surface and the condition at infinity, is [79]

$$A = A_0(\rho)\big(1 + \alpha g(\rho)\big)\,,$$
$$g(\rho) = (\rho - \rho_c)\ln(\rho - \rho_c) + g_1(\rho)\,, \qquad \alpha = \frac{\Omega_0'(\rho_c)}{\rho_c U_0'(\rho_c)} \ll 1\,, \tag{4.18}$$

where $A_0(\rho) = (U_M - \omega/m)\xi_0\,\rho^{-m}$ is the solution of the problem of cylinder oscillations with an amplitude ξ_0 in a potential flow; $g_1(\rho)$ is the function continuously differentiated on the real axis and limited as $\rho \to \infty$; the regular singularity $\rho = \rho_c$ is determined by the condition $mU_0(\rho_c) = \omega$.

Equation (4.18) shows that the solution itself remains continuous in the vicinity of the singularity point. Only the stream function derivative appears to be broken. This discontinuity is connected with the presence of a logarithmic term in $A(\rho)$ and determines the jump of the imaginary part of the velocity φ-component at the critical layer. In a potential flow this discontinuity is absent but for a vortical flow, as will be shown below, it leads to flow energy extraction from the vicinity of the critical layer and, as result, the system loses stability. We find the jump of the value $\mathrm{Im}\,v^\varphi$.

For a weakly vortical flow ($\alpha \ll 1$) the eigen-oscillation frequencies are slightly different from the eigen-frequencies in the potential flow ω_u i.e. $\omega = \omega_u + O(\alpha)$ [for oscillations with $m = 1$ the frequencies ω_u are determined by the relation (4.10)]. For the complex frequency ω the singularity point ρ_c is also complex and is located close to the real axis at the point:

$$\rho_c = \rho_0 + \mathrm{i}\,\frac{\delta}{mU_0'(\rho_0)} + O(\alpha^2)\,, \tag{4.19}$$

where $\delta = \mathrm{Im}\,\omega = O(\alpha)$, $\rho_0 = \mathrm{Re}\,\rho_c$, the streamline $\rho = \rho_0$ corresponds to the critical layer in the flow. Consider the

vicinity of the critical layer $\rho_0 - \Delta < \rho < \rho_0 + \Delta$, where $\alpha \ll \Delta \ll 1$. Since the singularity point ρ_c shifts into the complex plane, as determined by Eqn (4.19), and its imaginary part is much less than Δ, the function $\ln(\rho - \rho_c)$ in this interval gets an addition $\pm i\pi$ where the sign depends on the singularity point position either in the upper half-plane or in the lower one. Since $U_0' < 0$ the singularity is in the lower half plane for growing oscillations and in the upper half plane for decaying ones. According to this, the minus sign is selected for $\delta > 0$, and the plus sign is selected for $\delta < 0$. Using Eqn (4.18) and the equations connecting the velocity field with the stream function, we get:

$$\operatorname{Im} v^{\varphi}\Big|_{\rho=\rho_0+\Delta} - \operatorname{Im} v^{\varphi}\Big|_{\rho=\rho_0-\Delta} = \pm\frac{\pi\alpha}{m}\left|v_0^{\rho}(\rho_0)\right| + O(\alpha\Delta)\,, \tag{4.20}$$

where $v_0^{\rho} = imA_0/\rho$ is the velocity ρ-component calculated for the potential flow case and the plus and minus signs correspond to the cases $\delta > 0$ and $\delta < 0$.

Consider now the energy balance between the small region $|\rho - \rho_0| < \Delta$ and the rest of the flow. The energy flux through an arbitrary line $\rho = \text{const}$ is quadratic with respect to disturbance amplitude and is equal to [80]:

$$J = \int_0^{2\pi} \operatorname{Re}\left(p + \rho^2 U_0 v^{\varphi}\right) \operatorname{Re}\left(v^{\rho}\right)\rho\,\mathrm{d}\varphi = \frac{\pi\rho^3}{m}\operatorname{Re}\left(\omega v^{\rho^*} v^{\varphi}\right).$$

In this expression the external normal direction is selected to be positive. The energy flux from the vicinity of the critical layer is determined by the difference in J-values at the boundary of this region, i.e. $\Delta J = J|_{\rho=\rho_0+\Delta} - J|_{\rho=\rho_0-\Delta}$. For eigen-oscillations in the potential flow $\Delta J = 0$, since the system oscillation frequency ω_u and the velocity field component v_u^{φ} are real and the component v_0^{ρ} is purely imaginary. In the vortical flow case the energy flux ΔJ is distinct from zero. In the leading approximation it is determined by the jump of the quantity $\operatorname{Im} v^{\varphi}$. Using Eqn (4.20), we get:

$$\Delta J = \pm\frac{\pi^2\alpha\omega_u\rho_0^3}{m^2}\left|v_0^{\rho}(\rho_0)\right|^2 + O(\alpha^2)\,. \tag{4.21}$$

Thus, the energy flux ΔJ for a weakly vortical flow can be expressed through the ρ-component of the velocity v_0^{ρ} found from the solution of a simpler problem with $\alpha = 0$, and the vorticity effect on the system oscillations is determined only by the parameter α.

Examine now the vicinity of the critical layer as a separate subsystem which is the source or the sink of energy depending on the sign of the energy flux ΔJ. Then the remaining flow region is the other subsystem with varying energy E. It is evident that the values ΔJ and E are related by the energy balance equation $\Delta J = \mathrm{d}E/\mathrm{d}t$.

Outside the vicinity of the critical layer the shape of disturbances and their energy in the leading approximation are found from the solution with $\alpha = 0$. Since the potential disturbance energy near the oscillating cylinder is determined only by oscillation amplitudes, a slow variation of the disturbance energy E in a weakly vortical flow can occur only at the expense of amplitude variations $\varepsilon(t) = \varepsilon(0)\exp(\delta t)$. The energy E is quadratically dependent on the oscillation amplitudes, therefore the energy balance equation leads to the following expression for an oscillation increment (decrement)

$$\delta = \frac{\Delta J}{2E}\,. \tag{4.22}$$

Thus, considering the system energy balance one can find the shift δ of eigen-oscillations into the complex plane without solving the complete problem of disturbances in a vortical flow. To this end, it is sufficient to solve the problem of disturbances in the potential circulating flow with the help of which the energy flux ΔJ and energy E are determined.

Note that in contrast to the external region where the disturbance energy slightly changes on conversion from a potential flow to weakly vortical one, a strong disturbance energy change takes place in the internal region (in the vicinity of the critical layer). Really, for the potential flow the disturbance energy E' in the small vicinity of the line $\rho = \rho_c$ is proportional to the area of the region, i.e. $E' \ll E$. In the weakly vortical flow case the total energy of unstable oscillations must be equal to zero, since otherwise in the case of growing oscillation amplitude, the energy would not be preserved; in this case $E + E' = 0$, i.e. $E' \approx -E$.

Equation (4.22) is the consequence of the law of energy conservation and therefore is of universal character. The advantages of the approach described are especially important in the case of flows with a complex structure (e.g. a vortex ring). Really, for calculating the energy flux ΔJ in Eqn (4.22) it is sufficient to know the velocity disturbance in an approximation of the potential external streamline of the oscillator and to use Eqn (4.21), since the energy flux is determined only by the local flow structure in the vicinity of the critical layer. It is evidently of no importance in this case what is the cause of the oscillations, whether it is cylinder elasticity or vortex ring oscillations. In turn, the disturbance energy E depends on the oscillator type (cylinder on a spring, vortex oscillations etc.) and needs to be calculated in each special case; however, it is also sufficient to calculate this energy only in an approximation of the potential streamline.

Note that the sign of ΔJ is determined in Eqn (4.21) according to the sign of δ. In turn, the sign of δ, according to Eqn (4.22), is determined by the relation between ΔJ and E. This circumstance can lead to two essentially different situations [compare with the remark after Eqn (4.16)].

If $E > 0$, Eqns (4.21) and (4.22) have two solutions: with $\delta > 0$, $\Delta J > 0$ and with $\delta < 0$, $\Delta J < 0$. This means that in this case the appearance of the critical layer leads to splitting the eigen-frequency into a pair of complex-conjugated frequencies and the system becomes unstable. This instability mechanism is similar to the instability mechanism of Miles for wind waves on water [47, 81], when the wave on the heavy fluid surface (with positive energy) interacts with the critical layer in the non-uniform wind flow. If $E < 0$, Eqn (4.22) will be incompatible with (4.21) for the energy flux ΔJ either for positive δ, or for negative δ.

Making use of the results of Ref. [82], we find that for eigen-oscillations with $m = 1$ (the cylinder centre rotating about the equilibrium position in the potential flow) the energy E is the following:

$$E = \frac{M}{2}\left[(\gamma+1)\omega_{1,2}^2 + \omega_0^2\right]\xi_0^2\,,$$

where $\omega_{1,2}$ are the eigen-frequencies determined from Eqn (4.10). Since E is positive, the system can become unstable in the frequency region $0 < \omega < U_M$. The oscillation with ω_1 will be unstable, since the critical layer exists for this oscillation.

From Eqns (4.21) and (4.22), in the leading approximation we obtain

$$\delta = -\frac{\pi\gamma\omega_1(\omega_1 - U_M)^2\rho_0\Omega_0'(\rho_0)}{2U_M\left[(\gamma+1)\omega_1^2 + \omega_0^2\right]} . \tag{4.23}$$

With account for Eqn (4.15), this formula completely coincides with Eqn (4.17). Thus, a consideration of the energy balance in the system permits us not only to understand the instability mechanism, but also to obtain a precise expression for the increments.

4.2 Instability of Bessel and bulging oscillations of vortex ring

The instability of an elastic cylinder in a circulating flow with decreasing vorticity described above is realized at the expense of the fact that the vicinity of the critical layer appears to be an energy source for growing disturbances. As we see, this instability mechanism is of a rather general character. It is realized each time the oscillating system possesses positive energy and its oscillations occur in the presence of a critical layer. The cause of disturbances in the flow seems to be of no importance in this case, whether it is a solid cylinder or some other oscillator. The appearance of the critical layer is connected only with the relationship between the phase velocity of oscillations and the angular velocity of the mean flow.

4.2.1 Stability of Rankin's vortex. At first glance, an instability of the type examined could be realized for Rankin's cylindrical vortex considered in Section 3.2. The vortex core boundary in this flow would play the role of an oscillator in a circulating flow. The addition of weak vorticity field to the flow around the core could cause the appearance of the critical layer and real frequency shift into the complex plane. However a more careful analysis shows that the instability mechanism under consideration is not realized for any cylindrical vortex oscillations. Really, for isolated oscillations with $l \geqslant 1$, even if the critical layer appears, the energy of these oscillations is negative (see Section 3.4). The bending mode has positive energy, but its angular phase velocity is against the flow and therefore the critical layer does not appear [see (3.4b)]. It is evident that the critical layer also does not appear for bulging oscillations (angular phase velocity ω/m of these oscillations is equal to infinity) having positive energy. Bessel oscillations with $l \geqslant 1$ can be of two types: advancing and lagging with an angular phase velocity, respectively, larger or less than the flow velocity at the vortex boundary [see remark after (3.3)]. Only the advancing modes possess positive energy, but no critical layer appears for them, since their phase velocity is larger than the angular velocity in the whole flow. The lagging Bessel oscillations, on the contrary, have critical layers, but their energy is negative. Thus, Rankin's vortex with a smoothed vorticity profile has no instability described above and this could be expected, since it is known that a cylindrical vortex with a monotonically decreasing vorticity profile is stable [76, 83].

4.2.2 Instability of a vortex ring. The situation is fundamentally different for a vortex ring and this is connected with the difference in the oscillation shape of cylindrical vortex and the vortex ring. Both conditions for instability appearance described above are fulfilled for a whole family of vortex ring oscillations.

As was indicated above [see the remark after formula (3.15)], for Bessel oscillations of a vortex ring with frequency number l (including bulging modes with $l = 0$) the core boundary deformation in the leading approximation is a sum of the lth and the $(l+1)$ harmonics and the velocity disturbance outside the core is $(l+1)$ harmonic. This distinguishes Bessel oscillations of a vortex ring from similar oscillations of a cylindrical vortex which include only the l harmonic. The presence of the $(l+1)$ harmonic leads to the appearance of the critical layer for all Bessel oscillations of the vortex ring and not only for the lagging ones as is characteristic of the cylindrical vortex case.

Thus, for advancing Bessel oscillations of a vortex ring (including bulging modes) both conditions of instability appearance are fulfilled: the oscillations have a positive energy and are accompanied by the appearance of the critical layer. This means that if some monotonically decreasing vorticity is added to the flow around the vortex core, these oscillations lose stability.

A complete consideration of the spectral problem for a vortex ring with an arbitrary vorticity profile is a very complex task. However, if the vorticity in the region outside the core is small ($\alpha \ll 1$), an increment of instability can be found on the basis of considering the energy balance in the system, i.e. using the method used above for an oscillating cylinder in a circulating flow.

We note some peculiarities of the appearance of the critical layers for the vortex ring, in comparison with the cylindrical geometry flow. First, the streamlines in the vortex ring case are of circular shape only in the leading approximation in terms of the parameter $\mu\rho$. Therefore the condition of resonance interaction between the mean flow and unsteady disturbances is, generally speaking, not the condition of coincidence of the flow velocity with the disturbance phase velocity, but is the condition of coincidence of the oscillation period with the period of fluid particle rotation. Second, the vortex ring oscillations are a sum of different harmonics $\exp(im\psi)$ and each of them has its own rotation period $T_m = 2\pi m/\omega$. Thus, a multitude of critical layers corresponding to different harmonics appear for each eigen-oscillation. However, these peculiarities manifest themselves not in the leading approximation, but in higher approximations in term of μ. The dynamics of each oscillation in the leading approximation are determined only by one critical layer, the shape of which in this approximation is circular.

Thus, for calculating the instability increment it is sufficient:

(a) to have a solution for the case of potential flow outside the vortex core and on the basis of this solution to find the oscillation energy E;

(b) to find the energy flux ΔJ from the vicinity of the critical layer.

Solutions of problem (a) for vortex ring oscillations are given in Section 3 and the energies E are obtained in Section 3.5. To calculate the energy flux ΔJ in problem (b), we can use the general expression (4.21) obtained for cylinder oscillations. For this purpose, it is necessary only to find those streamlines on which the critical layers corresponding to different modes are located.

The velocity disturbance outside the core for Bessel modes, according to Eqn (3.16), is of $(l+1)$ harmonic kind. Taking into account that such disturbances possess a phase angular velocity $U_p = \omega/(l+1)$, and the angular velocity of

steady flow outside the core is

$$U_0 = \frac{1}{2\rho^2}\left[1 + O(\rho\mu)\right],$$

we obtain that the critical layers for Bessel (and bulging) modes are on the streamlines $\rho = \rho_0\left[1 + O(\mu)\right]$ where

$$\rho_0 = \sqrt{\frac{l+1}{2\omega}}, \qquad \omega = \frac{l}{2} + \frac{\mu n}{a_j} + O(\mu^2), \qquad l = 0, 1, 2, \ldots \tag{4.24}$$

It is easy to see that all the critical layers are located outside the vortex ring core and as the l-number increases they are concentrated close to the boundary. We calculate the instability increment values. To this end, we use Eqn (4.22). Equation (4.21) gives an energy flux from the vicinity of the critical layer per unit length in the coordinate s directed along the ring mean line. Multiplying this expression by the ring length $2\pi/\mu$ and using Eqn (3.16) for the disturbed velocity, we find the total energy flux:

$$\Delta J = \frac{\pi^3(3l+2)^2(l+1)^2}{16n^2a_j^2\mu\rho_0^{2l+1}}\left|\Omega_0'(\rho_0)\right|. \tag{4.25}$$

The bulging and advancing Bessel mode energy determined from Eqns (2.3), (2.4) can be presented by the unified expression:

$$E = \frac{\pi^2 a_j\,\omega}{\mu^2 n}\left[1 + O(\mu)\right]. \tag{4.26}$$

Substituting Eqns (4.25) and (4.26) into Eqn (4.22), we get the instability increment:

$$\delta = \frac{(3l+2)^2(l+1)^2\pi\mu}{2^5 n a_j^3\rho_0^{2l+1}\omega}\left|\Omega_0'(\rho_0)\right|. \tag{4.27}$$

The increment value (4.27) appears to be proportional to the derivative of vorticity in the critical layer and depends on the oscillation parameters l, n, j.

It follows from Eqn (4.24) that the critical layers of Bessel modes with $l \geqslant 1$ are located on the lines $\rho = \sqrt{(l+1)/l} + O(\mu)$ and fill the region $1 < \rho < \sqrt{2}$ adjacent to the vortex core. The higher the frequency number l, the closer the corresponding critical layer is to the core boundary. For Bessel modes with identical l and different radial numbers j the critical layers cover a region of width $\Delta\rho = O(\mu)$, according to the frequency spread of these modes.

The bulging modes have a frequency lower than that of Bessel modes with $l \geqslant 1$. Correspondingly, the critical layers of bulging modes fill a remote region $\rho = O(\mu^{-1/2})$. It follows from Eqn (4.24) that these critical layers are on the lines

$$\rho = \sqrt{\frac{a_j}{2n\mu}}\left[1 + O(\mu)\right],$$

i.e. the farther they are from the vortex core, the greater the radial number j is. For frequencies approaching zero, the critical layers approach the boundary of the ring's envelope. Thus, the critical layers fill the whole of the envelope from its borders to those of the core.

We evaluate now the characteristic values of disturbances near the critical layer. Differentiating (4.18), we obtain that the velocity field disturbances are evaluated with the function $A_0(\rho)$ and its derivative in the whole flow, i.e. with disturbance value in the potential flow. The displacement field

$$\varepsilon^\rho = -\frac{\mathrm{i}}{mV_0 - \omega}\,v^\rho,$$
$$\varepsilon^\varphi = -\frac{\mathrm{i}}{mV_0 - \omega}\,v^\varphi - \frac{V_0'}{(mV_0 - \omega)^2}\,v^\rho \tag{4.28}$$

near the critical layer is determined by velocity disturbances in the potential flow and by the oscillation increment $\delta = \operatorname{Im}\omega$. Near the critical layer the ρ-component of the displacement field reaches the value $O(\varepsilon/\mu\alpha)$ and the φ-component reaches the value $O(\varepsilon/\mu^2\alpha^2)$, where ε is the disturbance amplitude, and μ and α are small parameters, the product of which determines the increment value. This evaluation shows that the amplitude of fluid particle displacements at small amplitudes of the core oscillations can be large. The large amplitude of displacements near the critical layer leads to intensive mixing of fluid particles (see Section 4.2.3). We note that this behaviour of fluid particles is connected only with resonant properties of the flow with circular streamlines and with coincidence of the fluid particle rotation period with the oscillation period (i.e. it is also possible for potential flows, if the oscillator is forced by an external force). The weak vorticity present in the flow around the core only slightly affects velocity disturbances entering the formula for Lagrangian particle displacement (4.28) but it is taken into account indirectly through the increment value which enters the denominator of this expression.

In contrast to the velocity and displacement fields, differentiation of the potential part A_0 of the stream function (4.18) gives zero. The appearance of the vorticity disturbance is connected only with the second term in Eqn (4.18) which has a logarithmic singularity near the critical layer. On differentiation of this term, one can easily evaluate the vorticity disturbance which is of order $\varepsilon\alpha$ over the whole flow, except the critical layer, where the vorticity disturbance is intensified and is of order ε/α.

Hence the instability described above is accompanied near the critical layer by two processes: the first process is connected with intense mixing associated with large Lagrangian displacements of fluid particles and the second is connected with the vorticity field intensification.

4.3 Non-linear stage of vortex ring instability and transition to turbulence

Unstable oscillations of the vortex ring were considered above at the linear stage of their development. This type of oscillations is characterized by large fluid particle displacements in the critical layer. Therefore for increasing amplitude the non-linear effects begin to manifest themselves first of all in this region. The non-linear interaction between oscillations and the mean flow leads to decreasing the energy flux from the vicinity of the critical layer with increasing oscillation amplitude and when this amplitude reaches some limiting value, this energy flux becomes zero.

Indeed, in the vicinity of the critical layer, with account for non-linear effects, a region of finite size is formed, where the process of intensive mixing of fluid particles takes place and this leads to smoothing the mean vorticity profile. A similar situation appears in the phase space of Hamiltonian systems in the case of non-linear resonance. The intensive mixing region for such systems coincides with the *resonant layer* (see,

for example, Ref. [84]). When there are several resonant layers in the system, they can overlap, forming a whole region of stochastic motion, within which intensive transport of fluid particles along and across the main system of streamlines occurs.

Since the critical layers (4.24) are located very close to each other, it is reasonable to assume that a multitude of oscillations with different values of l, n, j for a vortex ring at the non-linear stage achieves the limiting amplitudes, forming a system of overlapping resonant layers, which are concentrated near the core boundary and fill up the whole ring envelope region. Thus, the chaotization of fluid particle motion in the vortex ring is very similar to the Lagrangian chaos which is observed for oscillating regimes of Hamiltonian systems [85, 86], but with the difference that for the vortex ring the Lagrangian chaos is accompanied by real intensification of the vorticity field disturbances. At the same time, according to Eqn (4.24), the resonant layers do not exist inside the core, i.e. the core remains laminar. This result qualitatively agrees with the above experimental data on the turbulent vortex ring structure which show that a sharp boundary between the turbulent ring envelope and the laminar core (Fig. 15) is preserved in most of the vortex ring trajectory. It would be more correct, from this point of view, to speak not of turbulence suppression in the vortex core [19], but of turbulence generation outside the core due to the formation of a large number of critical layers and intensificationof the vorticity disturbances in them. Note that the intense non-linear mixing in the resonant layer, by virtue of (4.27), can be caused by very small vortex ring core deformations. This means that the linear theory of oscillations of a vortex ring with a potential envelope is applicable to a vortex ring with a weakly vortical envelope even for intense non-linear processes in the vicinity of the critical layer.

4.3.1 The role of viscosity. Unstable oscillations of vortex ring were considered above in the approximation of ideal fluid. We evaluate the range of Reynolds numbers for which such a consideration is valid. Small disturbances in the viscous fluid are described by the Orr – Sommerfeld equation:

$$-\mathrm{i}\nu\nabla^4 A - (mU_0 - \omega)\nabla^2 A + \frac{m\Omega_0'}{\rho} A = 0\,, \qquad (4.29)$$

where

$$\nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{m^2}{\rho^2}\,.$$

The viscosity can be neglected when the first term in this equation is small in comparison with other terms over the whole flow region. In this case the Orr – Sommerfeld equation (4.29) is reduced to the Rayleigh equation (4.2), i.e. the disturbances can be described within the limits of an ideal fluid. In making the evaluations one must take into account that the viscous term in Eqn (4.29) contains senior derivatives with respect to ρ. This means that this term has the largest value in the region, where the disturbance gradients achieve the maximum values. Such a region is, as we have seen, the vicinity of the critical layer. Exactly here the viscosity has the greatest influence on the oscillation properties. This influence manifests itself in changing the energy flux ΔJ, in comparison with an inviscid solution. This, in turn, will lead to the decrease of the increment δ and finally disappearance of instability.

The solution of the Rayleigh equation (4.2) considered above will be valid for a viscous fluid on condition that, if it is substituted in the Orr – Sommerfeld equation (4.28), the viscous term appears small. The solution (4.22) near the critical layer $\rho = \rho_c$ takes on the form (4.18):

$$A \sim (\rho - \rho_c)\ln(\rho - \rho_c)\,.$$

Substituting this solution into Eqn (4.29), we obtain that the first term of this equation (the viscous term) is of order $O\big(\nu/(\rho - \rho_c)^3\big)$ and the second term is of order $O\big(U_0'(\rho_c)\big)$. The eigen-frequency for unstable oscillations is complex and the point ρ_c is also complex with the imaginary part $\mathrm{Im}\,(\rho_c) \sim \delta/U$. Therefore for real ρ in the region, the closest to the singularity point, the first term of Eqn (4.29) is of order $O(\nu U_0'^3/\delta^3)$ and the second term is of order $O(U_0')$. Thus, the viscous term in the Orr – Sommerfeld equation can be neglected provided that

$$\frac{\nu U_0'^2}{\delta^3} \leqslant 1\,.$$

Note that this condition can also be obtained from the condition that the characteristic time of viscous spreading of disturbances in the vicinity of the critical layer is large in comparison with the characteristic time of instability $1/\delta$ [88].

For a vortex ring with small vorticity outside the core (Section 4.2) the increment is determined by Eqn (4.27) and has the magnitude $\delta = O(\alpha\mu U_0)$. If the vortex ring has no distinct core and the vorticity profile is smoothly decreasing, the parameter α characterising the vorticity gradient value in

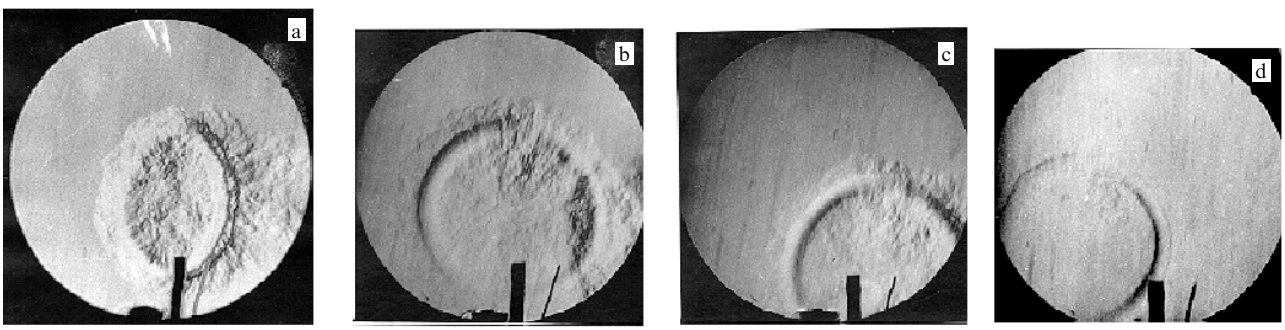

Figure 15. Vortex ring at different distances from the exit plane of the vortex generator nozzle (25, 35, 40 and 50 calibres, respectively). Visualization is performed using helium additions (according to materials of Ref. [87]). As the ring moves away from the nozzle exit plane the turbulence coloured by helium is shed into the wake revealing the laminar core.

the critical layer will no longer be small. Though for a vortex ring with $\alpha = O(1)$ no solution has been obtained, one can expect that in this case instability will also arise. This is indicated, in particular, by the solution for an oscillating cylinder in a circulating flow with a noticeable vorticity (Section 4.1.2). Thus, assuming that for the vortex ring with $\alpha = O(1)$ the instability mechanism exists and extending the above evaluation to this case, we get $\delta = O(\mu U_0)$. Note that the increment in this case also remains small and this is connected with the high energy of Bessel oscillations (as follows from the results of Section 3.5, at equal amplitudes the energy of Bessel oscillation appears $O(\mu^{-1})$ times greater than the energy in oscillations of other types). Evaluating the Reynolds number from the vortex ring radius and its translation velocity and using the above evaluations, we get that the viscous effects can be neglected at Reynolds numbers $\mathrm{Re} \geqslant \mathrm{Re}_0$, where

$$\mathrm{Re}_0 = O(\mu^{-3})\,. \tag{4.30}$$

Thus, there exists a threshold value of the Reynolds number Re_0, starting with which the mechanism of instability and chaotization of fluid particle motion in the ring envelope can manifest itself. For Reynolds numbers $\mathrm{Re} < \mathrm{Re}_0$ the viscous effects will be dominant and instability will be absent. The estimate (4.30) agrees well with the value $\mathrm{Re}_0 = 10^3$ known from the experiments related to generation of laminar and turbulent vortex rings, where rings with the characteristic value $\mu \sim 0.1$ [15] were investigated.

5. A turbulent vortex ring as a sound source. The possibility of non-contact diagnostics of unsteady processes in vortices

The photos of turbulent vortex ring presented in the previous section show that the turbulence could really be generated according to the scenario proposed, which assumes that the chaotic behaviour of fluid particles in the region of the vortex ring envelope occurs preserving the laminar flow in the core. However in addition to visualization there exists another non-contact method for diagnostics of unsteady process in vortices — their sound radiation. It is of interest to examine how the above ideas agree with the theory of sound radiation by a vortex ring and with the experimental works in this direction. Since the agreement is rather good, it seems reasonable to present a short review of these results.

5.1 Theory of sound radiation by a vortex ring

As is known, the unsteady motion of vortices in a compressible medium is accompanied by sound radiation of quadrupole character [89–91]. If the characteristic Mach number is small and the vorticity is localized in a region with a characteristic size substantially less than the sound wave length, the sound field can be expressed through the unsteady velocity field calculated in the approximation of incompressible fluid. To calculate the sound field generated by a vortex, it appeared possible to connect the sound field only with that part of the incompressible flow in which the vorticity is different from zero [92, 93]. The most suitable expression for the sound field which linearly connects the sound field with the unsteady field of vorticity was obtained in Refs [94, 95].

The theory describing the acoustic radiation of unsteady vortices is based on the fact that at small Mach numbers M there exist two spatial scales in the problem: the dimension l of the region where the vorticity is different from zero, and the sound wavelength λ. Really, at the characteristic velocity in the vortex ring u the characteristic frequency is of order u/l and the sound wavelength is $\lambda = c_0/\omega \cong c_0 l/u = l/M$ (where c_0 is the sound velocity). It follows from this evaluation that $\lambda \gg l$ for $M \ll 1$. The spatial scales l and λ determine two regions: an internal one, where the flow is determined by vorticity dynamics, and an external one, where the acoustic disturbances with wave structure are formed. The radiation solution can be obtained with the use of matching of the far asymptotics of the incompressible solution in the internal region and the near asymptotics of the wave field in the external region [96–99]. In the case of low-frequency volume oscillations or solid body oscillations, when the principal terms in the multipole expansion of the source are a monopole or a dipole, such a solution can be easily built (see Ref. [80] Ch. 8). However, in the case of acoustic radiation of unsteady vortices the principal term of the expansion is a quadrupole and a series of peculiarities appear which make the procedure of matching much more complex [96, 100].

The final expression for the sound field is

$$p = \rho_0 c_0^2 \frac{x_i x_j}{x^3} \frac{\partial^2}{\partial t^2} C_{ij}\left(t - \frac{x}{c_0}\right), \tag{5.1}$$

where p is the sound pressure, ρ_0 is the medium density,

$$C_{ij} = \frac{1}{12\pi} \frac{\mathrm{d}}{\mathrm{d}t} \int [\mathbf{\Omega} \times \mathbf{y}]_i \, y_j \, \mathrm{d}^3\mathbf{y}$$

is the quadrupole moment written in Möhring's form [94], $\mathbf{\Omega}(t)$ is the unsteady field of vorticity found in the approximation of incompressible fluid, and $\mathbf{x}$ is the radius-vector of observation point. Thus, to calculate the sound field in a weakly compressible fluid it is enough to find the dynamics of an unsteady vortex flow in an incompressible fluid and to substitute the expression obtained for $\mathbf{\Omega}(t)$ in Eqn (5.1).

On the basis of this theory the sound radiation produced by vortex ring oscillations was found in Ref. [32]. These oscillations are examined in detail in Section 3. We give here a short description of these results. The oscillations which radiate sound most efficiently can easily be distinguished. First of all, compare slow oscillations, the frequencies of which are close to zero [bending modes with $\omega = O(\mu^2 \ln \mu)$ and bulging modes with $\omega = O(\mu)$] and the fast oscillations with frequencies $\omega = O(1)$ which are close to semi-integer values of $l/2$ (see Fig. 8). Since the frequency ω enters expression (5.1) for the sound field in the third power, the sound radiation efficiency of bulging and bending oscillations is several orders lower than the efficiency of fast oscillations and they can be excluded from our consideration.

The fast oscillations, in their turn, can be divided into different types, depending on the efficiency of their acoustic radiation. The quadrupole moment is evidently different from zero only for those oscillations which have azimuthal numbers $n = 0, 1, 2$. The radiation of all the oscillations with $n \geqslant 3$ contributes in higher orders and is inefficient when $M \ll 1$. Hence, of all the fast oscillations only three types remain: axisymmetric modes and modes which look like the first and the second azimuthal harmonics. With the use of direct calculation one can show that the most efficient oscillations among all those are the oscillations with frequency number $l = 1$. The point is that the greater l, the higher the harmonics which determine the oscillation form

in the vortex cross-section. In its turn, the higher the harmonic number in the core cross-section, the larger its multipole type in the leading approximation and the smaller the contribution made by this harmonic to the quadrupole moment $C_{ij}(t)$.

Thus, the most efficient sound-radiating modes are those with $n = 0, 1, 2$ and frequency number $l = 1$. These are a set of Bessel modes of two types ($n = 1$ and $n = 2$), isolated modes of two types ($n = 1$ and $n = 2$) and axisymmetric modes ($n = 0$) (Fig. 16). All these modes have close frequencies and fill up the interval $\Delta\omega/\omega = O(\mu)$. Hence, if all the oscillations are excited in the vortex ring, its sound field must have a narrow-band spectrum with characteristic dimensionless frequency $\omega = 1/2$ corresponding to $l = 1$. In this case the peak width is determined by an interval of frequency distribution of radiating modes of the vortex core.

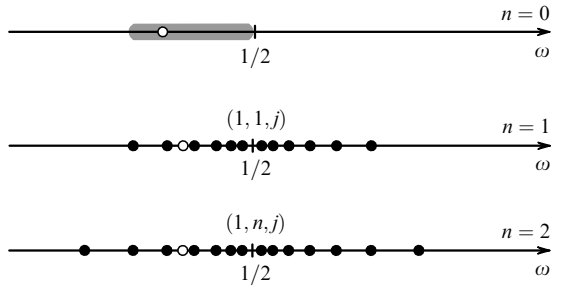

Figure 16. Spectrum of radiating modes.

It is easy to see that the above results for sound radiation can be extended to a vortex ring with a smoothed vorticity profile. Indeed, in Section 4.3 it is shown that the effect of weak vorticity surrounding the core on the oscillation form is localized in the vicinity of the critical layer. As for the rest of the flow, the addition of weak vorticity to the mean flow only slightly affects the disturbances generated by the core boundary deformation. This means that a small change of the vorticity profile slightly affects the far asymptotics of the incompressible flow oscillations and, respectively, the results of asymptotic matching with the sound field in a weakly compressible fluid. Therefore the acoustic radiation of a vortex ring with a small vorticity outside the core ($\alpha \ll 1$) and of a vortex ring with a potential flow outside the core ($\alpha = 0$) will be close in the case of equal amplitudes of core deformation. Thus, the theory of sound radiation by a vortex ring developed in Ref. [32] and described above will also be valid in the case of a weak monotonically decreasing vorticity outside the core, despite the fact that in this case the flow outside the core will be turbulent.

Though a weak vorticity outside the core has no effect on the characteristics of sound radiation from the vortex core oscillations, the presence of this vorticity is very important, from the standpoint of sound radiation, since it produces conditions for sound-generating oscillation excitation. The instability examined in Section 3 transforms a vortex ring into a real oscillator, the multiple oscillations of which are supported at the expense of energy transfer from the mean flow into unsteady fluctuations of different scales. From the sound generation standpoint, the multipole structure of oscillations appears to be a serious filter cutting off almost all the oscillations and preserving only a small fraction of the efficiently radiating modes.

5.2 Possible mechanisms of sound radiation by a turbulent vortex ring

As has already been noted, the vortex rings with high Reynolds numbers obtained experimentally appear to be turbulent and the vortex motion is accompanied by a turbulent wake. If the vortex ring turbulence is organized according to the above model, the turbulent fluctuations in the ring envelope are passively connected with the vortex core oscillations, and the sound radiation is determined only by eigen-oscillations of the laminar core, according to the theory described above. Such a radiation must be of narrow-band character with a peak frequency determined by the vortex core parameters.

If the model developed above is not realized and the ring envelope turbulence is determined by its own dynamics, one can expect that, according to the Lighthill theory [90, 91], the turbulence will be followed by broad-band sound radiation which is produced by disturbances of different scales. In this case the radiation described above which is connected with the core oscillations, will be either comparable with the turbulence sound radiation or will be substantially less than that broad-band component and the above picture of radiation will 'sink' in the vortex envelope noise.

Finally, one more sound source can appear which is connected with the fact that the vortex ring motion is followed by an intensive wake. This wake radiation could be of narrow-band character similar to aeolian tones arising on vortex separation in flows past obstacles (cylinders, spheres etc. [101]).

Thus the most probable mechanisms of radiation in real vortex rings could be associated with the following processes:

— a vortex train in the ring wake similar to von Karman's vortex train (Fig. 17a);

— small-scale turbulent fluctuations in the vortex envelope (Fig.17b);

— eigen-oscillations of the vortex core (Fig. 17c) excited at the moment of ring formation or developing due to instability.

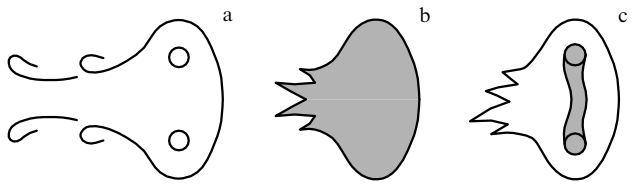

Figure 17. Possible mechanisms of sound radiation: by unsteady vorticity in the wake (a); by turbulent fluctuations in the 'envelope' (b); by vortex core modes (c).

The answer to the question of which of these three scenarios is realized in reality may be obtained experimentally.

5.3 Experimental investigation of vortex ring noise and comparison between the theory and experiments

Experiments recently carried out in the anechoic chamber (Fig. 18a) of Central Institute of Aerohydrodynamics have shown [102, 103] that acoustic radiation of a vortex ring is concentrated in a fairly narrow frequency band with the maximum close to a frequency depending on the mean parameters of vortex ring size and circulation (Fig. 18b). This fact means that the scenario of Fig. 17b is not realized,

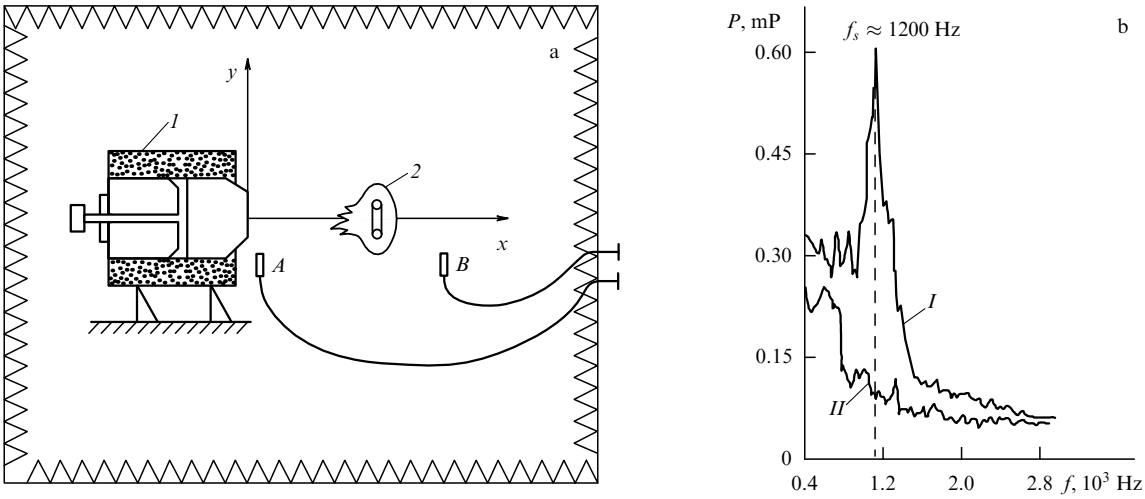

Figure 18. (a) Scheme of experiment in an anechoic chamber: (*1*) vortex generator submerged in a container with sand, (*2*) vortex ring, *A* — trigger, *B* — measuring transducer; (b) averaged spectrum of sound pressure: of vortex ring (*I*), background noise (*II*). Delay time from the moment of ring launch $\tau = 220$ ms. Initial diameter of vortex rings is 4 cm.

since small-scale turbulence radiation must have a broad-band spectrum.

We compare the characteristic frequency peak in the radiation spectrum obtained experimentally at $f_0 = 1200$ Hz with the theoretical value $\omega = \Omega_0/2$. To this end we express this frequency through the quantities measured in the experiment — the translation velocity of the vortex

$$V = \frac{\mu\Omega_0 a}{4}\left(\ln\frac{8}{\mu} - \frac{1}{4}\right),$$

the ring radius R and the vortex core radius a. Hence we have:

$$\omega = \frac{\Omega_0}{2} = \frac{2V}{R\mu^2\left(\ln(8/\mu) - 1/4\right)}.$$

Substituting the measurement results $V = 8$ m s^{-1}, $\mu = 0.12$, $R = 0.035$ m into this formula, we find that the frequency $f_0 = (\omega/2\pi)$ predicted theoretically satisfactorily corresponds to the measured value. This not only supports the scenario of Fig. 17c, but also excludes the scenario of Fig. 17a since the characteristic frequency of vortices shed off an obstacle with dimensions equal to the vortex ring dimensions, $\omega \approx 0.1V/R$ [101], is several orders lower than the measured one. The scenario of Fig. 17c is also confirmed by the peak width in the spectrum. The sound-radiating modes are to fill up the frequency interval:

$$\frac{\Delta\omega}{\omega} = \left(-\frac{4\mu}{a_1}, \frac{4\mu}{a_1}\right).$$

This value corresponds to $\Delta\omega = 300$ Hz in dimensional units and this also agrees with the experimental data.

Thus, the acoustic experiment data show that despite the fact that throughout the ring envelope, turbulent motion of fluid particles with large amplitudes occurs over a wide frequency range, the sound field is determined by small oscillations of the vortex core which cause not only the vortex ring turbulence, but sound radiation as well.

6. Conclusions

Let us sum up certain results now. The work considers oscillations of vortex rings with profiles of mean vorticity close to uniform (isochronous). It is shown that in the case of slight smoothing of the mean vorticity profile the majority of vortex ring eigen-oscillations become unstable. The energetics of the process of stability loss connected with the appearance of a number of critical layers in the ellipsoidal region surrounding the core is examined in detail. Such a mechanism becomes possible owing to the complex form of each oscillation, permitting energy exchange between the oscillations and the mean flow, which is impossible in topologically more simple vortex structures (for example in Rankin's cylindrical vortex). The vortex ring seems to be the simplest vortex in an unbounded fluid, in which such an instability can be realized.

The problem of aerodynamic sound generation by oscillations in a weakly compressible fluid is considered in short. It is shown that the appearance of the critical layer slightly affects the sound field, the amplitude of which turns out to be connected mainly with the amplitude of the vortex core boundary oscillations. In this case the instability considered leads to generation of oscillations of different space and time scales and the acoustic radiation separates a narrow range of oscillations capable of efficient sound generation. As a result, the vortex ring noise is displayed as a rather narrow peak in the spectrum, the frequency of which agrees well with the experimental one.

The presence of multiple instabilities leading to vorticity generation in the critical layers and the intensification of Lagrangian displacement of fluid particles in the vortex ring envelope permitted a hypothesis that the mechanism of turbulence generation in vortex rings at high Re-numbers (and simultaneous preservation of laminar motion in the core whose fluctuations, according to Section 4.2.2, are determined only by small oscillations of its boundary) can be associated with the processes examined above. For a more reliable answer to the questions set in the work a complete study of the non-linear problem is certainly necessary. This would result in a description of flow regimes close to self-

oscillatory. Such a consideration would allow, on the one hand, a prediction of the limiting amplitude of unsteady disturbances in the ring; on the other hand, such predictions could be compared with acoustic experiments, after the analysis of the mean relation of the zeroth, first and second azimuthal modes in sound-generating vortex oscillations. Achieving quantitative agreement in such a problem could be the determining argument in support of one or other scenario of turbulent fluctuation development in a vortex ring.

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