

LETTERS TO THE EDITORS

PACS numbers: **01.40.–d, 03.30.+p**

What is mass?

R I Khrapko

DOI: 10.1070/PU2000v043n12ABEH000778

Editor's note. The paper by L B Okun' published in *Usp. Fiz. Nauk* **158** 512 (1989) [*Sov. Phys. Usp.* **32** 629 (1989)] discusses in considerable detail the concept of a mass in non-relativistic and relativistic physics. The author argues that only the notion of mass m entering the famous relation $mc^2 = \sqrt{E^2 - c^2p^2}$ should be used. This mass is sometimes referred to as the rest mass and denoted by the symbol m_0. In 1989, I was not Editor-in-Chief of *Uspekhi Fizicheskikh Nauk* journal and had no case to give due attention to the paper by L B Okun' both for lack of time and in the conviction of being fairly well acquainted with the fundamentals of relativity theory. Now that I have read the paper for reasons apparent from the forthcoming material I can highly appreciate its methodological, pedagogical, and historical value. Above all, I agree with L B Okun' that in the teaching and application of the theory of relativity one should introduce only the mass m and avoid the notion of any relativistic mass. But I do not think that the introduction and use of a relativistic mass (e.g. mass $m_0/\sqrt{1 - v^2/c^2}$) can do any harm and necessarily suggest a failure to understand the theory of relativity. All this may seem a matter of taste for those knowing the crux of the problem; then, there is no point at issue to settle. However, a letter to the editor from R I Khrapko, a lecturer in a Moscow institution of higher learning, indicates that there is still no unanimity on the question of mass. A solidarity of opinion concerning such issues as that is hardly possible at all, and it is difficult to say in advance when and where the debate once initiated will be resolved. Certain members of the Editorial Board spoke to the effect that it is high time to stop and objected to the publication of the letter by R I Khrapko. In my opinion, the publication of this letter in *Uspekhi Fizicheskikh Nauk* together with the answer from L B Okun' is justified by the importance of the problem and the long history of its discussion. It will be of benefit to everybody, especially teachers, and promote a deeper understanding of the matter of dispute. Besides, there has been no critical note on the paper by L B Okun' published in this journal till now; thus, there has been no discussion. It is my belief that we should publish letters from our readers, including arguable ones, with the Editorial Board bearing only partial responsibility for the opinions stated by their authors.

21 June 2000 *V L Ginzburg*

R I Khrapko Moscow Aviation Institute,
Volokolamskoye shosse 4, 125871 Moscow, Russian Federation
Tel. (7-095) 158-42 71
E-mail: tahir@k804.mainet.msk.su, for Khrapko

Received 18 April 2000, revised 14 June 2000
Uspekhi Fizicheskikh Nauk **170** (12) 1363 – 1366 (2000)
Translated by Yu V Morozov, edited by M S Aksent'eva

Does the mass of bodies depend on their velocity? Is the mass additive if separate bodies are joined together to form a composite system? Is the mass of an isolated system conserved? Different teachers of physics and specialists give different answers to these questions because there is no general agreement on the definition of mass. We shall show that the notion of the velocity-dependent relativistic mass should be given preference over that of the rest mass.

One of the achievements of the special theory of relativity is the statement about the equivalence of mass and energy in a sense that the mass of a body increases with its energy including kinetic energy; therefore, the mass depends on the velocity of the body. This relationship is unambiguously interpreted in the works of renowned physicists.

Max Born (1962): "The mass of one and the same body is a relative quantity. It is to have different values according to the system of reference from which it is measured, or, if measured from a definite system of reference, according to the velocity of the moving body. It is impossible that mass is a constant quantity peculiar to each body" [1].

Richard Feynman (1965): "Because of the relation of mass and energy the energy associated with the motion appears as an extra mass, so things get heavier when they move. Newton believed that this was not the case, and that the masses stayed constant" [2].

Statements to the same effect can be also found in textbooks.

S P Strelkov (1975): "The dependence of mass on energy is a principal proposition of Einstein's mechanics" [3].

However, recently there had been a return to the Newton's belief. According to this belief the mass the mass of a body does not change with increasing velocity and remains equal to the rest mass. L B Okun' is a dedicated mouthpiece of this tendency [4, 5]. Earlier, a similar viewpoint was advocated in the book [6].

L B Okun' (1989): "The mass that increases with speed — that was truly incomprehensible. The mass of a body m does not change when it is in motion and, apart from the factor c, is equal to the energy contained in the body at rest. The mass m does not depend on the reference frame. At the end of the twentieth century one should bid farewell to the concept of mass dependent on velocity. This is an absolutely simple matter!" [4].

J Wheeler at al. (1966): "The concept of relativistic mass is subject to misunderstanding ..." ([6], p. 137).

This opinion is shared by the authors of certain textbooks for university students published abroad.

R Resnick et al. (1992): "The Concept of Mass" by Lev B Okun (see Ref. [5] of this letter) summarizes the views held by many physicists and adopted for use in this book." But' '...there is not universal agreement on the interpretation of Eq. 35." (Formula (35) is $E_0 = mc^2$ in Ref. [7]).

"This equation tells us that ... a particle of mass m has associated with it a rest energy E_0...." Nevertheless "Eq. 35 asserts that energy has mass " [7].

A serious confusion that arose from the reversion to the Newtonian concept of mass is reflected in the following dialogue:

"Schoolboy: "Does mass really depend on velocity, dad?"

Father physicist: "No! Well, yes... Actually, no, but don't tell your teacher." The next day the son dropped physics" [8].

We hope that we shall succeed in this letter to formulate a rational approach to the definition of mass.

There are two different definitions of the inertial mass, coincident in the non-relativistic context.

Definition 1. "In ordinary language the word *mass* denotes something like amount of substance. ... The concept of substance is considered self-evident." (See [1] p. 33.) More precisely: mass is defined "... as a number attached to each particle or body obtained by comparison with a standard body whose mass is define as unity" [9].

Definition 2. Mass is a measure of the inertia of a body, i.e. the coefficient of proportionality in the formula

$$\mathbf{F} = m\mathbf{a} \tag{1}$$

or in the formula

$$\mathbf{p} = m\mathbf{v}. \tag{2}$$

Because $\mathbf{F}$, $\mathbf{a}$, $\mathbf{p}$ and $\mathbf{v}$ have indisputable operational definitions [1], formulas (1) and (2) give the operational definition of mass. These formulas will be used to make the aforementioned comparison [see Def. (1)] in order to obtain the number m attached to a body.

However, the attached number determined by formulas (1) and (2) using the operational definitions of $\mathbf{F}$, $\mathbf{a}$, $\mathbf{p}$, $\mathbf{v}$ for one and the same body, i.e. for the same 'amount of substance', turns out to be dependent on the speed of the body; when the body has a speed, it also depends on the choice of the formula, (1) or (2). Therefore, the definition of mass for a body in motion splits in three. 'The amount of substance' specified by the attached number from Def. (1) is no longer a measure of a inertia of the moving body.

(a) In order to determine the 'amount of substance', i.e. the attached number from Def. (1), the body must be stopped and formula (1) or (2) used for a low speed. The number received by this method is called the rest mass. By definition, this mass does not change when the body undergoes acceleration.

(b) If the body is not stopped to measure its mass, formula (1) is known to give no unambiguous result. Because the force and acceleration are not properties of the body, the coefficient in formula (1) depends on the direction of the force relative to the body's velocity. As a matter of fact, this coefficient becomes a tensor. Therefore, the definition of the mass by formula (1) is completely inadequate. It is even not worth considering if the body's speed is not sufficiently low.

[1] For the operational definition of momentum, see [10]. Here is an extract from this work: "The meaning of the operational definition consists in the identification of two English equivalents of the Russian term 'opredelenie': 'definition' and 'determination'. The operation used to define a momentum is essentially as follows. When a certain obstacle causes a moving particle to stop, a force $\mathbf{F}(t)$ is measured with which the particle acts on the obstacle during retardation. The particle's initial momentum equals the integral $\mathbf{p} = \int \mathbf{F}(t)\,\mathrm{d}t$, by definition. It is postulated that this integral is independent of retardation characteristics, i.e. the form of the function $\mathbf{F}(t)$."

(c) In contrast, formula (2) is valid at any speed including that of light. For this reason, it and only it gives the operational definition of the mass of a moving body. Such a mass is a measure of the inertia of a moving body [2]. It is called the *relativistic mass*.

At this point, a problem arises. Which of the two masses, the rest mass of a) or the relativistic mass of c), is to be called simply *mass* and denoted by the letter m without a subscript and thus regarded as the 'chief' mass. This is not a matter of terminology. The problem has serious psychological and methodological implications.

It can be resolved through the comparison of the properties of different masses. The rest mass will be denoted by the symbol m_0 and the relativistic mass by the symbol m (otherwise, the latter will have no simple designation at all).

If two particles having momenta $\mathbf{p}_1 = m_1\mathbf{v}_1$ and $\mathbf{p}_2 = m_2\mathbf{v}_2$ join together into a single whole system, the momenta are known to add up so that $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$. Moreover, the four-dimensional momenta are also summed giving $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$. The 4-momentum $\mathbb{P}$ is by definition tangential to the world line of a particle in Minkowski space and its spatial component equals an ordinary momentum $\mathbf{p}$. Hence, the time component is equal to the relativistic mass m:

$$\mathbb{P} = \{m, \mathbf{p}\}\,.$$

This assertion is illustrated by a two-dimensional plot (Fig.1), which shows the world line (left) and 4-momentum tangential to it (right).

This immediately leads to the conclusion that the relativistic masses are simply summed up: $m = m_1 + m_2$, when particles join together into a system.

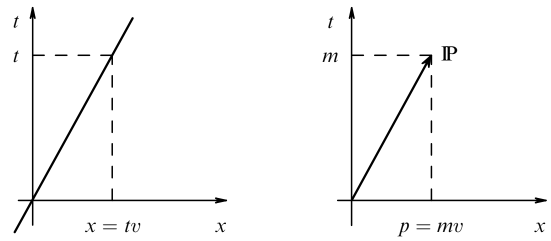

Figure 1.

Things differ when rest masses come into question. In the 4-dimensional sense, the rest mass of a particle is the modulus of its 4-momentum (to an accuracy of c):

$$m_0 = \sqrt{m^2 - \frac{p^2}{c^2}}\,.$$

Therefore, the rest mass of a pair of bodies with rest masses m_{01}, m_{02} is not equal to the sum $m_{01} + m_{02}$ but is determined by a complicated expression dependent on momenta $\mathbf{p_1}$, $\mathbf{p_2}$ [4]:

$$m_0 = \left[\left(\sqrt{m_{01}^2 + \frac{p_1^2}{c^2}} + \sqrt{m_{02}^2 + \frac{p_2^2}{c^2}}\right)^2 - \frac{(\mathbf{p}_1 + \mathbf{p}_2)^2}{c^2}\right]^{1/2}. \tag{3}$$

[2] It appears appropriate to cite M Born once again: "In physics, however, as we must very strongly emphasize, the word *mass* has no meaning other than that given by formula (6)" [Formula (2) of this letter]. (See [1] p. 33.)

A similar formula for the rest mass is presented in [6] ($c = 1$):

$$M^2 = (E_{\rm system})^2 - (p^x_{\rm system})^2 - (p^y_{\rm system})^2 - (p^z_{\rm system})^2 . \quad (4)$$

It follows from formulas (3) and (4) that the rest mass is lacking the property of additivity. We think that physicists do not mean the rest mass when they speak about beauty as a criterion for truth.

The thing is that both the relativistic mass (a time component of 4-momentum) and the rest mass (its modulus) obey the conservation law. This is ascertained in [4]. However, it is not so simple to accept that a non-additive quantity is conserved. Indeed, according to (3) and (4), the rest mass of a system does not change as a result of particle collisions or nuclear reactions. However, as soon as a system of two moving bodies is mentally divided into two separate bodies, the rest mass will change because the rest mass of the pair is not equal to the total mass of the bodies that make up the system. In our opinion, the use of non-additive notions entails a serious intellectual burden: a pair of photons, each having no rest mass, does have a rest mass.

Another very difficult question is: ''Does energy have a rest mass?'' The correct answer may be as follows: the energy of two photons will have a rest mass when they move in opposite directions. A system of two photons will have zero rest mass if they move in the same direction [4], p. 632 [3]. Thus, it appears that even the authors of the textbook [7] failed to solve the problem.

Furthermore, photons moving in the same direction have no rest mass while the rest mass of the body which emitted them decreases. Therefore, it may be suggested that some of the body's rest mass has been converted into the massless energy of photons. However, according to (3), (4) the rest mass of the system constituted by the body and the photons has been conserved during radiation!

Unable to bear such an intellectual burden, the advocates of the rest mass concept refuse to adopt the law of conservation of the rest mass of a system, in defiance of the formulas (3), (4). Now, they state that ''rest mass of final system increases in an inelastic encounter'' ([6], p. 121). In contrast, nuclear reactions lead to 'the mass defect'. For example, in the synthesis of deuteron, $\mathrm{p} + \mathrm{n} = \mathrm{D} + 0.2$ MeV, its rest mass is less than that of the neutron and proton.

At the same time, it follows from formulas (3), (4) that there must be no rest mass 'defect' during nuclear reactions. In our example, the allegedly lacking rest mass of the system at stage D + 0.2 MeV is actually provided by a massless γ-quantum with the energy of 0.2 MeV. This disturbs the additivity of the system's rest mass.

It is easy to understand why the schoolboy dropped physics in the face of such a confusion concerning the rest mass.

For all that, many physicists consider the rest mass to be the 'chief' one and denote it by the symbol m instead of m_0. Simultaneously, they discriminate against the relativistic mass and leave it without notation. This causes an additional confusion making it sometimes difficult to understand which mass is really meant. This situation is exemplified by the statement from [7] cited above.

These physicists agree that the mass of a gas in a state of rest increases upon heating because the energy contained in it grows. However, there seems to exist a psychological barrier which prevents relating this rise to a larger mass of individual molecules due to their high thermal velocity.

The said physicists sacrifice the concept of a mass as a measure of inertia to a label attached to each particle and bearing information about a constant 'amount of substance', just because such a label is in line with the deeply ingrained Newtonian concept of mass. For them, radiation that ''transmits inertia'' (according to A Einstein [11]) has no mass.

The main psychological problem is how to establish the identity between mass and energy (which varies) and regard these two entities as one. It is easy to accept that $E_0 = m_0 c^2$ for a body at rest. The authors of Ref. [6] entitled Chapter 13 as ''The equivalence of energy and rest mass'' [4]. It is more difficult to admit that the formula $E = mc^2$ is valid for any speed. The exquisite formula $E = mc^2$ is described by L B Okun' as 'ugly' [4].

Thus, the relativistic mass has a natural operational definition based on the formula $\mathbf{p} = m\mathbf{v}$. It is additive and obeys the law of conservation. Also, it is equivalent to both energy and gravitational mass. It should be referred to as mass and denoted by the letter m.

The rest mass is not conserved or lacks the property of additivity [5]. It is not equivalent to energy. It should be denoted as m_0 and used with caution especially if the notion is applied to a system of bodies.

The relativistic mass together with momentum are transformed as coordinates of an event during transition to a new inertial laboratory:

$$m = \frac{m' + p'v/c^2}{\sqrt{1 - v^2/c^2}} , \quad p = \frac{p' + m'v}{\sqrt{1 - v^2/c^2}} .$$

Specifically, if $P' = 0$ then $m' = m_0$, and

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} , \quad P = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} .$$

Transition from the rest mass to the relativistic one in the relativistic theory appears to encounter the same psychological problems as transition from proper to relative time.

It is worthwhile to note in conclusion that if instead of the coordinates t, x, ... we use the coordinates t', x', ... the relativistic mass m and the rest mass m_0, which are both scalars, will be expressed by the formulas

$$mc = u^{i'} P^{j'} g_{i'j'} , \quad m_0 c = \sqrt{P^{i'} P^{j'} g_{i'j'}} ,$$

which are valid for the curved space of GTR. Here, $u^{i'}$, $P^{j'}$ and $g_{i'j'}$ are the unit vector of the experimentalist, 4-momentum of the body, and metric tensor of the new coordinates respectively. It is assumed that for the initial coordinates $t, x, \ldots, u^i = \delta^i_0, g_{00} = 1, g_{11} = -1, \ldots$

A photon has no rest mass-energy, hence no proper frequency. But its mass-energy and frequency can be measured in experiment as $E = h\nu = cu^i P^j g_{ij}$ and prove to be of any value depending on the experimenter's speed.

[3] Pages in the earlier paper of L B Okun' (referred to as [4] in R I Khrapko's letter and as [7] in the answer of L B Okun') are given according to its English version [see *Sov. Phys. Usp.* **32** (7) 1989]. (*Translator's note.*)

[4] The title is characteristically ambiguous implying the equivalence between the rest energy and the rest mass.

[5] Here, the advocates of the rest mass concept contradict themselves; at first, they justly maintain that the rest mass is conserved but not additive, then they say that it is additive but not conserved.

I thank G S Lapidus whose comments helped to improve the text of this paper. This topic is elaborated in *physics/0103008*.

References

1. Born M *Einstein's Theory of Relativity* (New York: Dover Publ., 1962) p. 269 [Translated into Russian (Moscow: Mir, 1964)]
2. Feynman R P *The Character of Physical Law* (London: Cox and Wyman, 1965) p. 76 [Translated into Russian (Moscow: Mir, 1968)]
3. Strelkov S P *Mekhanika* 3d ed. (Mechanics) (Moscow: Nauka, 1975) p. 533
4. Okun' L B *Usp. Fiz. Nauk* **158** 511 (1989) [*Sov. Phys. Usp.* **32** 629 (1989)]
5. Okun L B "The concept of mass" *Phys. Today* **42** (6) 31 (1989)
6. Taylor E F, Wheeler J A *Spacetime Physics* (San Francisco: W.H. Freeman, 1966)
7. Resnick R, Halliday D, Krane K S *Physics* Vol. 1 (New York: Wiley, 1992) p. 166, 167
8. Adler C G "Does mass really depend on velocity, dad?" *Am. J. Phys.* **55** 739 (1987)
9. Alonso M, Finn E J *Physics* (Wokingham, England: Addison-Wesley, 1992) p. 96
10. Khrapko R I, Spirin G G, Razorenov V M *Mekhanika* (Mechanics) (Moscow: Izd. MAI, 1993)
11. Einstein A "Ist die Trägheit eines Körpers von seinem Energiegehalt abhängig?" *Ann. Phys.* (Leipzig) **18** 639 (1905) [Translated into Russian: in *Printsip Otnositel'nosti* (Principles of Relativity) (Eds V K Frederiks, D D Ivanenko) (Moscow: ONTI, 1935) p. 175]

Reply to the letter "What is mass?" by R I Khrapko

L B Okun'

In my opinion, there are a few false statements in the letter of R I Khrapko. I shall consider them in my answer organized as an alternation of R I Khrapko's assertions (Kh) and my comments (O).

Let us begin from the very first paragraph.

Kh: "Does the mass of bodies depend on their velocity? Is the mass additive if separate bodies are joined together to form a composite system? Is the mass of an isolated system conserved? Different teachers of physics and specialists give different answers to these questions because there is no general agreement on the definition of mass."

O: The author is right that different teachers give different answers to these questions. As regards active specialists they answer in perfect unison insofar as their scientific work is concerned: the mass is independent of velocity, it is not additive, the mass of an isolated system is conserved. In fact, there is no disagreement among researchers on the definition of mass.

However, the specialists are not equally consistent when they come to use contemporary scientific terminology in their papers and books intended to reach a broad audience. Not infrequently, they prefer archaic terms which were current at the beginning of the 20th century when the theory of relativity was being constructed. At that time, the language of relativistic theory was not yet completely formulated, and its creators did not hesitate to use non-relativistic expressions for physical quantities in their works.

Kh: "We shall show that the notion of the velocity-dependent relativistic mass should be given preference over that of the rest mass."

O: According to modern terminology, both terms, 'relativistic mass' and 'rest mass', are obsolete. They should not be used at all, and 'preference should be given' simply to mass m avoiding any attributes or other additional words in its notation. Such a mass is defined by the relation

$$m^2 = \frac{E^2}{c^4} - \frac{\mathbf{p}^2}{c^2}\,, \tag{1}$$

where E is the total energy of a free body, $\mathbf{p}$ is its momentum, and c is the velocity of light. This mass does not change upon the transition from one inertial system to another. This is easy to see using the Lorentz transformations for E and $\mathbf{p}$:

$$E \to (E' + \mathbf{v}\mathbf{p}')\gamma\,, \tag{2}$$

$$p_x \to \left(p'_x + \frac{vE'}{c^2}\right)\gamma\,, \tag{3}$$

$$p_y \to p'_y\,, \tag{4}$$

$$p_z \to p'_z\,, \tag{5}$$

where $\mathbf{v}$ is the velocity of one reference frame relative to another, $v = |\mathbf{v}|$, and $\gamma = 1/\sqrt{1 - v^2/c^2}$; as usual, we assume that vector $\mathbf{v}$ is directed along the x axis. Thus, the mass m is a Lorentz invariant, unlike E and $\mathbf{p}$ which are components of a 4-dimensional vector.

The physical meaning of the mass was discovered by Einstein in 1905 when he introduced the notion of rest energy into physics. Indeed, relation (1) for a body at rest ($\mathbf{p} = 0$) gives

$$m = \frac{E_0}{c^2}\,. \tag{6}$$

Thus, the mass is proportional to the rest energy. If the speed of light c is taken to be the unit speed, i.e. $c = 1$, the mass of a body is equal to its rest energy. It is the rest energy, 'dormant' in massive bodies, that is released in part during chemical and especially nuclear reactions.

The relativity principle was first formulated by Galileo who illustrated it by the fact that for a person shut in the cabin of a ship it is impossible to tell from any physical experiment whether the ship is standing still or moving uniformly and rectilinearly relative to the shore. Einstein's relativistic theory added optical and electrodynamic experiments to the experiments of Galileo. The quintessence of these experiments was the assertion that there exists in nature a limiting maximum speed c equalling the velocity of light.

By applying the Lorentz transformations (2)–(5) to a body at rest, one immediately arrives at the formulas that connect the energy and momentum of a body to its velocity:

$$E = mc^2\gamma\,, \tag{7}$$

$$\mathbf{p} = m\mathbf{v}\gamma = \frac{E}{c^2}\,\mathbf{v} \tag{8}$$

L B Okun' State Scientific Center of Russian Federation
'Institute of Theoretical and Experimental Physics',
ul. B Cheremushkinskaya 25, 117259 Moscow, Russian Federation
E-mail: okun@heron.itep.ru

Received 15 November 2000
Uspekhi Fizicheskikh Nauk **170** (12) 1366–1371 (2000)
Translated by Yu V Morozov; edited by M S Aksent'eva