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Golden section in the Carnot cycle

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Abstract. Some aspects of classical thermodynamics are analyzed for the presence of duality and of the golden section.

The golden section presents one of the most dazzling manifestations of the harmony in Nature. It arises as the result of solving the problem of division of a whole into two unequal parts, so that the lesser (a) of the two relates to the greater (b) as the greater is to the sum of both [1, 2]:

$$\frac{a}{b} = \frac{b}{a+b} \,. \tag{1}$$

Requirement (1) can be satisfied on fulfilling the condition

$$\frac{b}{a+b} = \varphi \,, \tag{2}$$

where

$$\varphi = \frac{\sqrt{5}-1}{2} \equiv 0.6180\ldots \tag{3}$$

Here, the quantity φ is the unique positive root of the equation

$$\varphi^2 + \varphi = 1 \tag{4}$$

and is known as the golden section.

The interest in the proportional division of a whole into two parts in compliance with relation (1) first arose in antiquity (Pythagoras, Plato, Euclid) [3, 4]. In the Middle Ages, this problem was studied by Italian mathematician Fibonacci [5]. The Renaissance promoted the proportion (1)–(3) to the rank of the supreme aesthetic principle: Johann Kepler called it the 'invaluable treasure', and Leonardo gave it the name of the golden section (*sectio aurea*), which survives to this day [6]. In the 19th century the German scientist Zeising 'rediscovered' the golden section in an attempt to formulate the universal law of proportion: "So that the whole divided into two unequal parts appeared beautiful in shape, the smaller part must be to the bigger part as the bigger part is to the whole" [7]. Zeising and his successors showed that the proportions of the golden section (or those closely related to them) are found in human body, in antique temples and statues, in the morphology of plants and minerals, and in musical chords [1, 7]. In the 20th century, the interest in the golden section was revived again. It was discovered in (or applied to) astronomy [6, 8], biology [9], psychology [10], computer science [11], music [2, 12], architecture [2, 6, 13], and other liberal arts and exact sciences. In this way, the golden section occupied the minds and hearts of many prominent scholars of the past, and continues to excite our contemporaries.

The international Fibonacci Association holds annual conferences devoted to the study of Fibonacci numbers and their applications. Of special interest is the search for the golden section in those fields of scientific knowledge where it has not yet been discovered — for example, in theoretical physics. The most promising in this respect is apparently that branch of theoretical physics which (like the golden section itself) is especially fundamental and harmonious. What we mean is of course thermodynamics — the science of thermal motion of matter [14, 15], whose special role in the scientific study of the Universe was emphasized by Albert Einstein: "Thermodynamics made a deep impression upon me. This is the only general theory in physics of which I am certain that it will never be refuted within the limits of applicability of its basic concepts" [16].

An outstanding role in the formulation and development of thermodynamics belongs to the Carnot cycle. This cycle was proposed in the early theoretical study of heat engines aimed at increasing their efficiency [17], and is the cornerstone in the foundation of classical thermodynamics. Its author — French physicist and military engineer Nicolas Leonard Sadi Carnot (1796 – 1832) — was the founder and the pioneer of the science of thermodynamics. He and his successors used the Carnot cycle for establishing the universal thermodynamic laws and for deriving many concrete results [14, 18].

We know that a heat engine is a machine for cyclic conversion of heat into work. The quality of a heat engine is measured by its efficiency η, which is the ratio of work A done by the engine in one cycle to the heat Q_1 absorbed by the engine during that cycle [19]:

$$\eta = \frac{A}{Q_1} \,. \tag{5}$$

To ensure the periodical operation of the heat engine, its working medium (gas) at the end of each cycle must be brought back to the initial thermodynamic state. According to the second law of thermodynamics [14], this requires that some absorbed heat (denoted Q_2) should be given away to surrounding bodies, so that the work done by the heat engine is given as [19]

$$A = Q_1 - Q_2 \,. \tag{6}$$

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This implies that a heat engine must have not only a heater (hot body) but also a cooler (cold body). The basic idea of Sadi Carnot is that the heat engine delivers work not by absorbing heat, but rather because of heat transfer from hot body to cold body [17]. In other words, the conversion of heat to work requires a two-pole system, 'heater – cooler'. The temperature of the heater we shall denote by T_1, and the temperature of the cooler by T_2 ($T_2 < T_1$).

Another fundamental idea of Sadi Carnot is the possible reversal of the thermal cycle, so that the heat engine becomes a refrigerator which (unlike the heat engine) consumes (rather than supplies) work but in doing so it transfers heat from the cooler to a heater, thus creating a temperature difference [15, 17]. As the purpose of the refrigerating machine consists in cooling, it would be natural to define its efficiency λ as the ratio of the heat Q_2 taken out of the refrigerator to the work A expended on doing this [19]:

$$\lambda = \frac{Q_2}{A} . \tag{7}$$

Definitions (5) and (7) are based on the same principle — the efficiency is the ratio of the gain desired to the expense required.

So, one and the same machine can work as both a heat engine and a refrigerator. Accordingly, a heat engine is a dual object in the sense that potentially it always contains a refrigerating machine as well (in other words, we are obviously dealing with two structures in one body).

Now let us express explicitly the efficiency η of the ideal heat engine (operating on the Carnot cycle), and the efficiency λ of the reverse refrigerating machine (operating according to the inverse Carnot cycle). Integration of the mathematical formula expressing the second law of thermodynamics:

$$\delta Q = T \mathrm{d}S \tag{8}$$

as applied to the Carnot cycle gives [14]

$$Q_1 = (S_2 - S_1)T_1 , \qquad Q_2 = (S_2 - S_1)T_2 , \tag{9}$$

where S_1 and S_2 are respectively the minimum and maximum values of entropy attained with the Carnot cycle. Substituting Eqns (6) and (9) into (5) and (7) we get

$$\eta = 1 - \tau , \tag{10}$$

$$\lambda = \frac{\tau}{1 - \tau} , \tag{11}$$

where τ is the ratio of temperatures of the cooler and heater for the ideal heat engine:

$$\tau = \frac{T_2}{T_1} . \tag{12}$$

Eliminating τ from expressions (10) and (11), we find

$$(1 + \lambda)\eta = 1 , \tag{13}$$

which links together the two branches (heating and cooling machines) of the dual object under consideration (an ideal heat engine).

If none of the two branches is predominant (that is, when the ideal heat engine is equally efficient on operating both ways), then the efficiencies of direct and inverse Carnot cycles are equal, viz.

$$\lambda = \eta , \tag{14}$$

and then equation (13) becomes

$$\eta^2 + \eta = 1 . \tag{15}$$

Comparing Eqns (15) and (4), and having regard to $\eta > 0$ [14], we see that

$$\eta = \varphi , \tag{16}$$

and the efficiencies of the direct and the inverse Carnot cycles, given that they are equal, produce the golden section.

Substituting Eqn (16) into (10), and taking Eqn (4) into account, we arrive at

$$\tau = \varphi^2 . \tag{17}$$

This means that the efficiencies of the Carnot cycles are equal as long as the ratio of temperatures of a cooler and a heater of the ideal heat engine is equal to the square of golden section.

From Eqns (16) and (17) we get

$$\frac{\tau}{\eta} = \varphi , \tag{18}$$

which together with the equality

$$\tau + \eta = 1 \tag{19}$$

that follows from Eqn (10) proves that if the efficiencies of heat engine and refrigerator contained in the ideal heat engine are equal, then a unit segment is divided in the proportion of the golden section.

In this way, the establishment of equality (i.e. incorporation of symmetry) between the two branches of the dual structure of the Carnot cycle ('direct cycle – inverse cycle', or 'heat engine – refrigerator') leads to the occurrence of proportions of the golden section (that is, harmony). All in all, there are three numerical laws of harmony — the golden section, symmetry and asymmetry (or broken symmetry), all three being closely related [2]. One may expect therefore that, along with the symmetrical behavior [see Eqn (14)] of the two branches of the dual structure of the Carnot cycle, the golden section must also derive from their some asymmetrical behavior.

And indeed, if the relationship

$$\lambda = \eta^{-1/2} \tag{20}$$

holds, then equation (13) becomes

$$\eta + \eta^{1/2} = 1 . \tag{21}$$

Comparison between Eqns (21) and (4) gives

$$\eta = \varphi^2 . \tag{22}$$

Substituting Eqn (22) into (20) and (10), and taking Eqn (4) into account, we arrive at

$$\lambda = \varphi^{-1} , \tag{23}$$

$$\tau = \varphi . \tag{24}$$

Finally, from Eqns (22) and (24) we get

$$\frac{\eta}{\tau} = \varphi \,. \tag{25}$$

So we see that in the case of asymmetrical behavior of the two branches of the dual structure of the ideal heat engine ('heat engine – refrigerator') as expressed by efficiency relation (20), the Carnot cycle once again displays the proportions of the golden section (22) – (25).

In this way, we have found the linkage between the golden section and the Carnot cycle which is one of the fundamental principles of the science of thermodynamics. The golden section that had been observed previously in most diverse fields of science, art and human activity has been found in thermodynamics, one of the key chapters of theoretical physics.

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