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Spinning relativistic particles in external fields

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Contents

Abstract. The motion of spinning relativistic particles in external electromagnetic and gravitational fields is considered. The self-consistent equations of motion are built with the noncovariant description of spin and with the usual, 'naive' definition of the coordinate of a relativistic particle. A simple derivation of the gravitational interaction of first order in spin is presented for a relativistic particle. The approach developed allows one to consider effects of higher order in spin. Concrete calculations are performed for the second order. The gravimagnetic moment is discussed, a special spin effect in general relativity. We also consider the contributions of the spin interactions of first and second order to the gravitational radiation of compact binary stars.

1. Introduction

The problem of the motion of a particle with internal angular momentum (spin) in an external field consists of two parts: the description of the spin precession and the account of the spin influence on the trajectory of motion. To the lowest nonvanishing order in c^{-2}, the complete solution for the case of an external electromagnetic field was given more than 70 years ago [1]. Gyroscope precession in a centrally symmetric gravitational field was considered to the same approximation even earlier [2]. Then, much later, spin precession was investigated in the case of the gravitational spin – spin interaction [3]. The fully relativistic problem of spin precession in an external electromagnetic field was also solved more than 70 years ago [4], and then in a more convenient formalism, using the covariant vector of spin, in [5].

The situation is different with the second part of the problem, which refers to the spin influence on the trajectory. Covariant equations of motion for a relativistic spinning particle in an electromagnetic field were written in the same paper [4], and for the case of a gravitational field in [6]. These equations have been discussed repeatedly from various points of view in numerous papers (see, e.g., [7 – 18]). The problem of the influence of the spin on the trajectory of a particle in an external field is not only of purely theoretical interest. In particular, it attracts attention because it is related to the description of the motion of relativistic particles in accelerators [19] (see also the recent review [20]).

In fact, it is far from being obvious whether one can observe in practice the discussed spin corrections to the equations of motion of elementary particles, for instance, an electron or proton. According to the well-known argument by Bohr (see [21]), the additional Lorentz force due to the finite size of the wave packet of a charged particle and to the uncertainty relation exceeds the corresponding component of the Stern – Gerlach force. However, this argument by itself does not exclude in principle the possibility of observing a common Stern – Gerlach effect, even a small one, in the presence of a larger background due to the uncertainty relation. In particular, in a recent paper [22] this possibility was claimed to be supported by numerical calculations. Moreover, spin-dependent correlations certainly exist in differential cross sections of scattering processes. So, it was

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proposed long ago to separate charged particles of different polarizations through the spin interaction with external fields in a storage ring [23]. Though this proposal is being discussed rather actively (see review [20]), it is not yet clear whether it is feasible technically.

There are however macroscopic objects for which internal rotation certainly influences their trajectories. We mean the motion of Kerr black holes in external gravitational fields. This problem is of importance in particular for the calculation of the gravitational radiation of binary stars. In this connection it was considered in [24–27]. However, when turning to these calculations, we found [28] that the equations of motion used in these papers taking account of spin to the lowest nonvanishing order in c^{-2} differ from those corresponding to the well-known gravitational spin–orbit interaction even in the simpler case of an external field. As we will see, the reason for the disagreement consists in different definitions of the center-of-mass coordinate. Moreover, it turned out that the widely used Papapetrou equations in the same c^{-2} approximation [6] also fail to reproduce the result for the gravitational spin–orbit interaction found in the classical work [2]. This discrepancy was pointed out long ago in [29]; however the explanation suggested in [29] does not appear satisfactory (see Section 3.2).

The present review is essentially based on recent works [28, 30, 31] where the equations of motion of a relativistic particle were derived with a noncovariant description of spin. These equations agree with well-known limiting cases. Though for an external electromagnetic field such equations (in the linear-in-spin approximation) have been obtained previously [19] (see also [20]), we would like to start with comments related to this approximation in electrodynamics.

2. Covariant and noncovariant equations of motion of a spinning particle in an electromagnetic field

2.1 Problems with covariant equations of motion

The interaction of spin with an external electromagnetic field is described, up to terms on the order of c^{-2}, by the well-known Hamiltonian (see, e.g., [32])

$$H = -\frac{eg}{2m}\,\mathbf{sB} + \frac{e(g-1)}{2m^2}\,\mathbf{s}[\mathbf{p}\times\mathbf{E}]\,. \tag{1}$$

Here $\mathbf{B}$ and $\mathbf{E}$ are external magnetic and electric fields; e, m, $\mathbf{s}$, and $\mathbf{p}$ are the particle charge, mass, spin, and momentum, respectively; and g is its gyromagnetic ratio. Let us emphasize that the structure of the second (Thomas) term in this expression has not only been firmly established theoretically, but has also been confirmed with high accuracy experimentally, at any rate in atomic physics. To avoid misunderstandings, let us note that, generally speaking, the last term in formula (1) should be rewritten in a Hermitian form (see, e.g., [33]):

$$[\mathbf{p}\times\mathbf{E}] \to \frac{1}{2}\,(\mathbf{p}\times\mathbf{E} - \mathbf{E}\times\mathbf{p}) = \mathbf{p}\times\mathbf{E} + \frac{i}{2}\,\nabla\times\mathbf{E}\,.$$

We will be mainly interested, however, in the semiclassical approximation, when field derivatives in the interaction linear in spin, are neglected. (Besides, the correction with $\nabla\times\mathbf{E}$ vanishes in the case of a potential electric field considered in [32].)

Let us try to construct a covariant equation of motion accounting for spin, which would reproduce in the same approximation the force

$$f_m = \frac{eg}{2m}\,\mathbf{sB}_{,m} + \frac{e(g-1)}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,[\mathbf{E}\times\mathbf{s}]_m - \mathbf{s}[\mathbf{v}\times\mathbf{E}_{,m}\,]\right) \tag{2}$$

corresponding to the Hamiltonian (1) (here and below a comma with a subscript denotes a partial derivative). A covariant correction f^{μ} to the Lorentz force $eF^{\mu\nu}u_{\nu}$ should be linear in the tensor of spin $S_{\mu\nu}$ and in the gradient of the tensor of electromagnetic field $F_{\mu\nu,\lambda}$; it may also depend on the 4-velocity u^{μ}. Since $u^{\mu}u_{\mu}=1$, this correction must satisfy the condition $u_{\mu}f^{\mu}=0$.

From the above-mentioned tensors, one can construct only two independent structures meeting the last condition. The first,

$$\eta^{\mu\varkappa}F_{\nu\lambda,\varkappa}S^{\nu\lambda} - F_{\lambda\nu,\varkappa}\,u^{\varkappa}S^{\lambda\nu}u^{\mu}\,, \tag{3}$$

reduces in the c^{-2} approximation to

$$2\mathbf{s}(\mathbf{B}_{,m} - \mathbf{v}\times\mathbf{E}_{,m}\,)\,, \tag{4}$$

and the second,

$$u^{\lambda}F_{\lambda\nu,\varkappa}u^{\varkappa}S^{\nu\mu}\,, \tag{5}$$

reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{s}\times\mathbf{E}]_{\,m}\,. \tag{6}$$

Mote that possible structures with the contraction $F_{\nu\varkappa,\lambda}S^{\varkappa\lambda}$ reduce to (3) and (5), due to the Maxwell equations and the antisymmetry of $S_{\varkappa\lambda}$.

Obviously, no linear combination of (4) and (6) can reproduce the correct expression (2) for the spin-dependent force. In a somewhat less general way this was shown in [28].

But why is it that the correct (in the c^{-2} approximation) formula (2) cannot be obtained from a covariant expression for the force? Obviously, one can easily reproduce by a linear combination of (4) and (6) those terms in (2) that are proportional to g. In other words, there is no problem to present in a covariant form the terms that describe, so to say, direct interaction of a magnetic moment with external fields. It is the terms in (2) independent of g and corresponding to the Thomas precession that cannot be written covariantly. Certainly, the Thomas precession can be described beyond the c^{-2} approximation, for arbitrary velocities. We mean as a noncovariance the nontensor form of the transformation law. Of course, the noncovariance of the equations does not mean that the physical observables have the wrong transformation properties. It is sufficient to recall electrodynamics in the Coulomb gage.

2.2 Relation between different definitions of the coordinate of a spinning particle

It was noted in [28] that the covariant formalism can be reconciled with the correct results if the coordinate $\mathbf{x}$ entering into the covariant equation is related to the usual coordinate $\mathbf{r}$ in the c^{-2} approximation as follows:

$$\mathbf{x} = \mathbf{r} + \frac{1}{2m}\,\mathbf{s}\times\mathbf{v}\,. \tag{7}$$

The generalization of this substitution to the case of arbitrary velocities

$$\mathbf{x} = \mathbf{r} + \frac{\gamma}{m(\gamma+1)}\,\mathbf{s}\times\mathbf{v}\,, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \tag{8}$$

was introduced in [20]. Obviously, after this velocity-dependent substitution, the Lagrangian depends explicitly on the acceleration, which does not allow the standard Hamiltonian approach to be applied. An essential advantage of using the coordinate **r** is, in our opinion, the possibility of applying the Hamiltonian formalism.

But why can the spin precession itself (as distinct from the spin influence on the trajectory) be described covariantly [4, 5] without any concern for the coordinate definition? First of all, the covariant equations of spin precession

$$\frac{\mathrm{d}S_\mu}{\mathrm{d}\tau} = \frac{e}{2m}\left[gF_{\mu\nu}S^\nu - (g-2)\,u_\mu F_{\lambda\nu}u^\lambda S^\nu\right] \tag{9}$$

(here S_μ is the covariant 4-vector of spin) are written in the semiclassical approximation, i.e., the coordinate dependence of external fields is completely neglected. Second, equations (9) are linear and homogeneous in spin. So, even if one went beyond the semiclassical approach here, but stayed within the approximation linear in spin, the use of the usual coordinate **r**, which differs from **x** in terms proportional to **s** only, would be completely legitimate.

Since relations (7), (8) are valid for a free particle as well, their origin can be elucidated with a simple example of a free particle with a spin 1/2. Here, instead of the Dirac representation with the Hamiltonian of the standard form

$$H_{\mathrm{D}} = \boldsymbol{\alpha}\mathbf{p} + \beta m\,,$$

it is convenient to use the Foldy–Wouthuysen representation [34]. Then the Hamiltonian is

$$H_{\mathrm{FW}} = \beta\,\epsilon_{\mathbf{p}}\,, \quad \epsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}\,,$$

and the 4-component wave functions $\psi_\pm$ of the states of positive and negative energies reduce in fact to the 2-component spinors $\phi_\pm$:

$$\psi_+ = \begin{pmatrix}\phi_+\\ 0\end{pmatrix}, \quad \psi_- = \begin{pmatrix}0\\ \phi_-\end{pmatrix}.$$

Obviously, in the Foldy–Wouthuysen representation the operator of coordinate $\hat{\mathbf{r}}$ defined by the usual relation

$$\hat{\mathbf{r}}\,\psi(\mathbf{r}) = \mathbf{r}\,\psi(\mathbf{r}) \tag{10}$$

is just **r**.

The transition from the exact Dirac equation in an external field to its approximate form containing only the first-order correction in c^{-2} is performed just by means of the Foldy–Wouthuysen transformation. Thus, in the resulting c^{-2} Hamiltonian the coordinate of a spinning electron is the same **r** as in the completely nonrelativistic case. Nobody makes substitution (7) in the Coulomb potential when treating the spin–orbit interaction in the hydrogen atom.

One more limiting case, which is of a special interest to us, is a classical spinning particle. Such a particle is in fact a well-localized wave packet constructed from positive-energy states, i.e., it is naturally described in the Foldy–Wouthuysen representation. Therefore, it is **r** that is natural to consider as the coordinate of a relativistic spinning particle.

A certain subtlety here is that in the Dirac representation the operator $\hat{\mathbf{r}}$ is nondiagonal. However, the operator equations of motion certainly have the same form both in the Dirac and Foldy–Wouthuysen representations. Correspondingly, the semiclassical approximation to both is the same. In particular, the time derivatives on the left-hand side of the classical equations of motion are taken of the same coordinate **r** that serves as an argument of the fields on the right-hand side of these equations.

As to the covariant operator $\hat{\mathbf{x}}$, it has the simplest form in the Dirac representation:

$$\hat{\mathbf{x}}_{\mathrm{D}} = \sqrt{\frac{\epsilon}{m}}\,\beta\,\hat{\mathbf{r}}_{\mathrm{D}}\sqrt{\frac{\epsilon}{m}}\,, \tag{11}$$

where $\hat{\mathbf{r}}_{\mathrm{D}}$ is the operator acting on the wave function in the Dirac representation according to rule (10). The covariance of the matrix element $\psi^\dagger\hat{\mathbf{x}}\,\psi$ is obvious: the matrix β transforms $\psi^\dagger$ into $\bar{\psi}$, and the factors $\sqrt{\epsilon/m}$ are needed for the covariant normalization of the wave functions.

Let us rewrite the operator $\hat{\mathbf{x}}$ in the Foldy–Wouthuysen representation. The matrix U of the Foldy–Wouthuysen transformation is

$$U = \frac{m+\epsilon-\beta\boldsymbol{\alpha}\mathbf{p}}{\sqrt{2\epsilon(m+\epsilon)}}\,. \tag{12}$$

The calculation, which is conveniently performed in the momentum representation where $\mathbf{r}_{\mathrm{D}} = i\nabla_{\mathbf{p}}$, results in the following expression:

$$\hat{\mathbf{x}}_{\mathrm{FW}} = U^\dagger\hat{\mathbf{x}}_{\mathrm{D}}U = \beta\left(\mathbf{r} + \frac{1}{m(m+\epsilon)}\,\mathbf{s}\times\mathbf{p}\right) - \frac{1}{2m}\left[(\boldsymbol{\alpha}\mathbf{p})\hat{\mathbf{r}} + \hat{\mathbf{r}}(\boldsymbol{\alpha}\mathbf{p})\right]. \tag{13}$$

Here

$$\mathbf{s} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \tag{14}$$

is the relativistic operator of spin. Note that the different components of the relativistic coordinate operator (13) do not commute. If we confine ourselves to the space of the positive-energy states, then we can put $\beta = 1$ in (13) and drop the terms with α. In this way we arrive at expression (8).

However, we wish to attract attention to the problems arising in the covariant formulation of the equations of motion of a spinning relativistic particle. One of them is that the constraints $u^\mu S_\mu = 0$ (or $u^\mu S_{\mu\nu} = 0$) should be taken into account. This problem is of a technical character, and is quite solvable (see, e.g., [14]).

Another one is much more serious. The covariant equations of motion contain a third time-derivative. For instance, the well-known Papapetrou equation [6] for a particle in an external gravitational field is

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\left(mu_\mu - S_{\mu\nu}\frac{\mathrm{D}u^\nu}{\mathrm{D}\tau}\right) = -\frac{1}{2}\,R_{\mu\nu\rho\sigma}u^\nu S^{\rho\sigma}\,. \tag{15}$$

Here, τ is the proper time, $u^\nu = \mathrm{D}x^\nu/\mathrm{D}\tau$, and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. As long as the term

$$-\frac{\mathrm{D}}{\mathrm{D}\tau}\left(S_{\mu\nu}\frac{\mathrm{D}u^\nu}{\mathrm{D}\tau}\right)$$

with the third time-derivative is treated perturbatively, no special troubles arise with it. However, the inherent shortcoming of Eqn (15) is that beyond the perturbation theory it evidently has spurious, nonphysical solutions.

2.3 Noncovariant formalism

Correct equations of motion in an electromagnetic field including spin to first order have been known for a fairly long time [19]. Though being fully relativistic, they are noncovariant and based on the initial physical definition of spin. According to this definition, spin is a 3-dimensional vector $\mathbf{s}$ (or 3-dimensional antisymmetric tensor s_{mn}) of the internal angular momentum defined in the rest frame of the particle. The covariant vector of spin S_μ (or the covariant antisymmetric tensor $S_{\mu\nu}$) is obtained from $\mathbf{s}$ (or s_{mn}) by the Lorentz transformation. By the way, an advantage of this approach is that the constraints $u^\mu S_\mu = 0$ and $u^\mu S_{\mu\nu} = 0$ hold identically. The precession frequency for a spin $\mathbf{s}$ at an arbitrary velocity is well-known (see, for instance, [32]):

$$\boldsymbol{\Omega} = \frac{e}{2m}\left\{(g-2)\left[\mathbf{B} - \frac{\gamma}{\gamma+1}\,\mathbf{v}(\mathbf{vB}) - \mathbf{v}\times\mathbf{E}\right] + 2\left[\frac{1}{\gamma}\,\mathbf{B} - \frac{1}{\gamma+1}\,\mathbf{v}\times\mathbf{E}\right]\right\}. \tag{16}$$

Naturally, the corresponding interaction Lagrangian (here the Lagrangian formulation is somewhat more convenient than the Hamiltonian one) equals

$$L_{\mathrm{el}} = \boldsymbol{\Omega}\mathbf{s} = \frac{e}{2m}\,\mathbf{s}\left\{(g-2)\left[\mathbf{B} - \frac{\gamma}{\gamma+1}\,\mathbf{v}(\mathbf{vB}) - \mathbf{v}\times\mathbf{E}\right] + 2\left[\frac{1}{\gamma}\,\mathbf{B} - \frac{1}{\gamma+1}\,\mathbf{v}\times\mathbf{E}\right]\right\}. \tag{17}$$

The equation of motion for the position has the usual form

$$\left(\boldsymbol{\nabla} - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nabla}_{\mathbf{v}}\right)L_{\mathrm{tot}} = 0\,, \tag{18}$$

where L_{tot} is the total Lagrangian of the system. The equation of motion for the spin in the general form is

$$\dot{\mathbf{s}} = -\{L_{\mathrm{tot}}, \mathbf{s}\}\,, \tag{19}$$

where $\{\ldots, \ldots\}$ is the Poisson bracket. For spin components, it is written as

$$\{s_i, s_j\} = -\epsilon_{ijk}\,s_k\,,$$

which is quite natural, according, for instance, to the well-known correspondence between the Poisson bracket and commutator. Correspondingly, in the quantum problem,

$$\dot{\mathbf{s}} = -i\,[L_{\mathrm{tot}}, \mathbf{s}]\,. \tag{20}$$

3. Spin precession in a gravitational field

In this section we present a simple and general derivation of the equations of spin precession in a gravitational field (restricting ourselves to first order in spin), based on a remarkable analogy between gravitational and electromagnetic fields. Due to this correspondence, the formulae of the previous section are naturally adapted for the case of an external gravitational field. In this way, we easily reproduce and generalize the known results for gravitational spin effects.

3.1 General relations

It follows from the angular momentum conservation in flat space-time taken together with the equivalence principle that the 4-vector of spin S^μ is transported parallel to the particle world line. The parallel transport of a vector along a geodesic $x^\mu(\tau)$ means that its covariant derivative vanishes:

$$\frac{\mathrm{D}S^\mu}{\mathrm{D}\tau} = 0\,. \tag{21}$$

We will use the tetrad formalism natural for the description of spin. In view of relation (21), the equation for the tetrad components of spin $S^a = S^\mu e^a_\mu$ is

$$\frac{\mathrm{D}S^a}{\mathrm{D}\tau} = \frac{\mathrm{d}S^a}{\mathrm{d}\tau} = S^\mu e^a_{\mu;\nu}u^\nu = \eta^{ab}\gamma_{bcd}\,u^d S^c\,. \tag{22}$$

Here,

$$\gamma_{abc} = e_{a\mu;\nu}\,e^\mu_b e^\nu_c = -\gamma_{bac} \tag{23}$$

are the Ricci rotation coefficients [35, § 98]. Certainly, the equation for the tetrad 4-velocity components is exactly the same:

$$\frac{\mathrm{d}u^a}{\mathrm{d}\tau} = \eta^{ab}\gamma_{bcd}\,u^d u^c\,. \tag{24}$$

The meaning of Eqns (22), (24) is clear: the tetrad components of both vectors vary in the same way, due only to the rotation of the local Lorentz vierbein.

In exactly the same way, the 4-dimensional spin and velocity of a charged particle with gyromagnetic ratio $g = 2$ precess with the same angular velocity in an external electromagnetic field [by virtue of equation (9) at $g = 2$ and the Lorentz equation]:

$$\frac{\mathrm{d}S_a}{\mathrm{d}t} = \frac{e}{m}\,F_{ab}S^b\,, \quad \frac{\mathrm{d}u_a}{\mathrm{d}t} = \frac{e}{m}\,F_{ab}u^b\,.$$

Thus, the correspondence

$$\frac{e}{m}\,F_{ab} \leftrightarrow \gamma_{abc}u^c \tag{25}$$

becomes obvious. The correspondence (25) makes it possible to obtain the precession frequency $\boldsymbol{\omega}$ of the 3-dimensional vector of spin $\mathbf{s}$ in the external gravitational field from expression (16) via the simple substitution

$$\frac{e}{m}\,B_i \to -\frac{1}{2}\,\epsilon_{ikl}\gamma_{klc}u^c\,, \quad \frac{e}{m}\,E_i \to \gamma_{0ic}u^c\,. \tag{26}$$

This frequency is

$$\omega_i = -\epsilon_{ikl}\left(\frac{1}{2}\gamma_{klc} + \frac{u^k}{u^0+1}\gamma_{0lc}\right)\frac{u^c}{u^0_{\mathrm{w}}}\,. \tag{27}$$

The factor $1/u^0_{\mathrm{w}}$ in (27) is related to the transition in the left-hand side of Eqn (22) to the differentiation with respect to the world time t:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\frac{\mathrm{d}}{\mathrm{d}t} = u^0_{\mathrm{w}}\,\frac{\mathrm{d}}{\mathrm{d}t}\,.$$

A subscript w is attached to the quantity u^0_{w} to emphasize that u^0_{w} is a world, but not a tetrad, component of 4-velocity. All other indices in expression (27) are tetrad ones, $c = 0, 1, 2, 3$; $i, k, l = 1, 2, 3$. The corresponding spin-dependent correction to the Lagrangian is

$$L_{sg} = \boldsymbol{\omega}\mathbf{s}\,. \tag{28}$$

However, in some respect, the first-order spin interaction with a gravitational field differs essentially from that with an electromagnetic field. In the electromagnetic case, the interaction depends, generally speaking, on a free phenomenological parameter, g factor. Moreover, if one allows for the violation of both P and T invariances, one more parameter arises here, the value of the electric dipole moment of the particle. The point is that both magnetic and electric dipole moments interact with the electromagnetic field strength, thus this interaction is gage-invariant for any value of these moments. Only the spin-independent interaction with the electromagnetic vector potential is fixed by the charge conservation and gage invariance.

Meanwhile, the Ricci rotation coefficients γ_{abc} entering into the gravitational first-order spin interaction (28), as distinct from the Riemann tensor, are not covariant. This interaction is fixed in a unique way by the angular momentum conservation in flat space-time taken together with the equivalence principle, and it has no free parameters [36, 37]. On the other hand, it is no surprise that the precession frequency $\boldsymbol{\omega}$ depends not on the Riemann tensor, but on the rotation coefficients. Indeed, this frequency should not have tensor properties: let us recall that a spin that is at rest in an inertial reference frame precesses in a rotating one.

This approach is applied below to the problems of spin–orbit and spin–spin interactions, as well as to spin precession in a plane gravitational wave.[1] We mostly consider the weak-field approximation. However, as distinct from the standard approaches, all three problems can easily be solved now for arbitrary particle velocities. The combination of a high velocity for a spinning particle with a weak gravitational field obviously refers to a scattering problem. Another possible application is to a spinning particle bound by other forces, for instance, by electromagnetic ones, when we are looking for the correction to the precession frequency due to the gravitational interaction.

Let us recall that in the weak-field approximation, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1,$$

there is no difference between the tetrad and world indices in $e_{a\mu}$ and the tetrad looks as follows:

$$e_{\mu\nu} = \eta_{\mu\nu} + \tilde{e}_{\mu\nu}, \quad |\tilde{e}_{\mu\nu}| \ll 1.$$

The well-known relation between the tetrads and the metric

$$e_{a\mu} e_{b\nu} \eta^{ab} = g_{\mu\nu}$$

in the weak-field approximation reduces to

$$\tilde{e}_{\mu\nu} + \tilde{e}_{\nu\mu} = h_{\mu\nu}.$$

The demand that tetrads should be expressed via the metric only, results in the so-called symmetric gage for the tetrads, where

$$\tilde{e}_{\mu\nu} = \frac{1}{2} h_{\mu\nu}.$$

Then, in the weak-field approximation the Ricci coefficients are

$$\gamma_{abc} = \frac{1}{2}\left(h_{bc,a} - h_{ac,b}\right). \tag{29}$$

[1] We are grateful to T Vargas for attracting our attention to the last problem.

3.2 Spin–orbit interaction. Weak field

In the centrally symmetric field created by a mass M, the metric is

$$h_{00} = -\frac{r_g}{r} = -\frac{2kM}{r}, \quad h_{mn} = -\frac{r_g}{r}\,\delta_{mn} = -\frac{2kM}{r}\,\delta_{mn}. \tag{30}$$

Here, the nonvanishing Ricci coefficients are

$$\gamma_{ijk} = \frac{kM}{r^3}\left(\delta_{jk} r_i - \delta_{ik} r_j\right), \quad \gamma_{0i0} = -\frac{kM}{r^3}\, r_i. \tag{31}$$

Substituting these expressions into formula (27) yields the following result for the precession frequency:

$$\boldsymbol{\omega}_{ls} = \frac{2\gamma+1}{\gamma+1}\,\frac{kM}{r^3}\,\mathbf{v}\times\mathbf{r}. \tag{32}$$

In the limit of low velocities, $\gamma \to 1$, the answer goes over into the classical result of [2]. The corresponding equation of motion derived from Lagrangian (28) is:

$$\ddot{\mathbf{r}} = -\frac{kM}{r^3}\,\mathbf{r} - 3\,\frac{kM}{mr^3}\left[\mathbf{v}\times\mathbf{s} - \frac{3}{2}\,(\mathbf{nv})\,\mathbf{n}\times\mathbf{s} - \frac{3}{2}\,\mathbf{n}\big(\mathbf{n}[\mathbf{v}\times\mathbf{s}]\big)\right],$$
$$\mathbf{n} = \frac{\mathbf{r}}{r}. \tag{33}$$

In the covariant approach the motion of a spinning particle is described by the Papapetrou equation (15), which in the linear-in-spin approximation reduces to

$$\frac{\mathrm{D}u_\mu}{\mathrm{D}\tau} = -\frac{1}{2m}\,R_{\mu\nu\rho\sigma}\,u^\nu S^{\rho\sigma}. \tag{34}$$

Note that the right-hand side of equation (34) is the only covariant structure possible here (up to a numerical factor).

For the nonrelativistic motion in the gravitational field created by a mass M, Eqn (34) gives

$$\ddot{\mathbf{x}} = -\frac{kM}{x^3}\,\mathbf{x} - 3\,\frac{kM}{mx^3}\left[\mathbf{v}\times\mathbf{s} - (\mathbf{nv})\,\mathbf{n}\times\mathbf{s} - 2\mathbf{n}\big(\mathbf{n}[\mathbf{v}\times\mathbf{s}]\big)\right],$$
$$\mathbf{n} = \frac{\mathbf{x}}{x}. \tag{35}$$

The reason for the disagreement between Eqns (33) and (35) is that they refer to the coordinates $\mathbf{r}$ and $\mathbf{x}$ defined in different ways [see (7)].

This discrepancy was pointed out long ago in [29] where the classical result [see (32) at $\gamma \to 1$] was derived from the scattering amplitude for a Dirac particle in a gravitational field. The explanation suggested in [29] for the disagreement is: "The quantum field theory of the spin-1/2 particle from which the classical result was derived does not have any spin supplementary condition.[2] This is because field theories deal with point particles and not with extended bodies."

However, first of all, spin in the Dirac theory of course satisfies the above-mentioned constraint (in the sense of expectation values). On the other hand, is a proton in a gravitational field a point particle or an extended body? A deuteron? A uranium nucleus? Obviously, an extended body can be treated as a point particle as long as we do not go into the details of its structure and as long as we do not consider its internal excitations. We dwell here on this point since one sometimes hears utterances similar to that above on "point

[2] The supplementary conditions $u^\mu S_{\mu\nu} = 0$ or $u^\mu S_\mu = 0$ are meant.

particles and extended bodies" even from some well-known theorists.

3.3 Spin – orbit interaction. Schwarzschild field

In the present subsection we treat the spin precession in the Schwarzschild field beyond the weak-field approximation (though neglecting the spin influence on the trajectory). The 3-dimensional components of the Schwarzschild metric can be conveniently written as

$$g_{mn} = -\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) - \frac{r_m r_n}{r^2}\,\frac{1}{1 - r_g/r} = -\delta^{\perp}_{mn} - n_m n_n\,\frac{1}{1 - r_g/r}\,. \qquad (36)$$

Nonvanishing tetrads are chosen as follows:

$$e_0^{(0)} = \sqrt{1 - r_g/r}\,, \quad e_m^{(k)} = \delta^{\perp}_{km} + n_k n_m\,\frac{1}{\sqrt{1 - r_g/r}} \qquad (37)$$

(in this subsection, the tetrad indices are singled out by brackets). Now, the nonvanishing Ricci coefficients (here, their last indices are world ones) are

$$\gamma_{(0)(i)0} = -\frac{kM}{r^3}\,r_i\,, \quad \gamma_{(i)(j)k} = \frac{1 - \sqrt{1 - r_g/r}}{r^2}\,(\delta_{jk} r_i - \delta_{ik} r_j)\,. \qquad (38)$$

At last, the precession frequency in this case is

$$\boldsymbol{\omega} = -\mathbf{L}\,\frac{r_g}{2mr^3}\left\{\frac{2}{u^0 + u^0\sqrt{1 - r_g/r}} + \frac{1}{1 + u^0\sqrt{1 - r_g/r}}\right\}. \qquad (39)$$

Here, m and $\mathbf{L}$ are the particle mass and the orbital angular momentum, respectively;

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left\{1 - r_g/r - (\mathbf{nv})^2\,\frac{1}{1 - r_g/r} - (\mathbf{v}^{\perp})^2\right\}^{-1/2}.$$

The rather cumbersome general expression (39) simplified for a circular orbit. Here,

$$u^0 = \left(1 - \frac{3kM}{r}\right)^{-1/2}, \qquad L = mr\left(\frac{kM}{r}\right)^{1/2}\left(1 - \frac{3kM}{r}\right)^{-1/2},$$

so that

$$\omega = \frac{(kM)^{1/2}}{r^{3/2}}\left[1 - \left(1 - \frac{3kM}{r}\right)^{1/2}\right]. \qquad (40)$$

The general case of spin precession in the Schwarzschild field was considered previously in [38]. Our expression (40) agrees with the corresponding result of [38] (the precession is considered there with respect to the proper time τ, but not with respect to t).

3.4 Spin – spin interaction

Let the spin of the central body be $\mathbf{s}_0$. The components of the metric linear in $\mathbf{s}_0$, which are responsible for the spin – spin interaction, are

$$h_{0i} = 2k\,\frac{[\mathbf{s}_0 \times \mathbf{r}]_i}{r^3}\,.$$

Here, the nonvanishing Ricci coefficients are

$$\gamma_{ij0} = k\left(\nabla_i\,\frac{[\mathbf{s}_0 \times \mathbf{r}]_j}{r^3} - \nabla_j\,\frac{[\mathbf{s}_0 \times \mathbf{r}]_i}{r^3}\right), \quad \gamma_{0ij} = -k\nabla_i\,\frac{[\mathbf{s}_0 \times \mathbf{r}]_j}{r^3}\,. \qquad (41)$$

The frequency of the spin – spin precession is

$$\boldsymbol{\omega}_{ss} = -k\left(2 - \frac{1}{\gamma}\right)(\mathbf{s}_0\nabla)\nabla\frac{1}{r} + k\,\frac{\gamma}{\gamma+1}\left[\mathbf{v}(\mathbf{s}_0\nabla) - \mathbf{s}_0(\mathbf{v}\nabla) + (\mathbf{v}\mathbf{s}_0)\nabla\right](\mathbf{v}\nabla)\,\frac{1}{r}\,. \qquad (42)$$

In the low-velocity limit this formula also goes over into the corresponding classical result [3].

3.5 Spin precession in a plane gravitational wave

Let a weak gravitational wave propagate along the axis 3. It is well known (see, for instance, [35, § 107]) that here coordinate conditions can be chosen in such a way that the only nonvanishing components of $h_{\mu\nu}$ are

$$h_{11} = -h_{22} = f_1(t - z)\,, \quad h_{12} = h_{21} = f_2(t - z)\,.$$

A straightforward (though rather tedious) calculation with formulae (27), (29) results in the following expressions for the components of the angular velocity:

$$\omega_{w1} = \frac{1}{2}\left(1 - \frac{\gamma}{\gamma+1}\,v_3\right)(\dot{f}_1 v_2 - \dot{f}_2 v_1)\,,$$
$$\omega_{w2} = \frac{1}{2}\left(1 - \frac{\gamma}{\gamma+1}\,v_3\right)(\dot{f}_1 v_1 + \dot{f}_2 v_2)\,,$$
$$\omega_{w3} = \frac{\gamma}{\gamma+1}\left[\dot{f}_1 v_1 v_2 - \frac{1}{2}\,\dot{f}_2\,(v_1^2 - v_2^2)\right]. \qquad (43)$$

The equations of motion in a plane gravitational wave follow from the corresponding Lagrangian $L = \boldsymbol{\omega}_w \mathbf{s}$. In the covariant approach this problem was considered in [39, 40].

4. Effects of higher order in spin

4.1 The idea of general formalism

The above rather simple considerations were quite sufficient for the description of the effects linear in spin. However, at least in the motion of rotating black holes (and possibly in some subtle spin effects for polarized nuclei of high spin in storage rings), an interaction of second order in spin may manifest itself. Anyway, going beyond the linear approximation in spin is of a certain theoretical interest. To study this general problem, a more sophisticated approach [30, 31] is needed. It is based on the following physically obvious argument already mentioned in Section 3.2: as long as we do not consider excitations of internal degrees of freedom of a body moving in an external field, this body (even if it is a macroscopic one!) can be treated as an elementary particle with spin. Thus, the Lagrangian of the spin interaction with an external field can be derived from the amplitude of elastic scattering of a particle with spin s by an external field. In this way we can describe the interaction of a relativistic particle to first order in the external field, but to arbitrary order in the spin. Explicit closed formulae were obtained in [30, 31] for the interaction of second order in spin. According to the arguments presented in Section 1, the discussion of the effects nonlinear in spin may be physically meaningful first of all in the classical limit $s \gg 1$. This limit is certainly adequate for rotating black holes. However, having in mind the above-mentioned problem of polarized nuclei, as well as some theoretical questions, the results were derived in [31] for arbitrary spins.

Below, we briefly present the technique for the investigation of effects of higher order in spin (a more detailed presentation is contained in [30, 31]) and the results obtained in this way. The details of the calculations are not necessary for understanding the results, and may be omitted when reading for the first time.

4.2 Equations of motion of a spinning particle in an electromagnetic field. Second-order effects

The Lagrangian of the spin interaction with an external field can be derived from the amplitude

$$-eJ^{\mu}A_{\mu} \tag{44}$$

of elastic scattering of a particle with spin s by a vector potential A_{μ} [30]. The matrix element J_{μ} of the electromagnetic current operator between states with momenta k and k' can be written (under P and T invariance) as follows (see [30, 31, 41]):

$$J_{\mu} = \frac{1}{\sqrt{\epsilon_k \epsilon_{k'}}}\,\bar{\psi}(k')\left\{p_{\mu}F_{\mathrm{e}} + \frac{1}{2}\Sigma_{\mu\nu}q^{\nu}F_{\mathrm{m}}\right\}\psi(k)\,. \tag{45}$$

Here, $p_{\mu} = (k'+k)_{\mu}/2$, and $q_{\mu} = (k'-k)_{\mu}$.

The wave function of a particle with an arbitrary spin ψ can be written (see, for instance, Ref. [32, § 31]) as

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi\\ \eta\end{pmatrix}. \tag{46}$$

Both spinors,

$$\xi = \left\{\xi^{\alpha_1\alpha_2\ldots\alpha_p}_{\dot\beta_1\dot\beta_2\ldots\dot\beta_q}\right\},$$

and

$$\eta = \left\{\eta^{\beta_1\beta_2\ldots\beta_q}_{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_p}\right\}$$

are symmetric in the dotted and undotted indices separately, and

$$p+q = 2s\,.$$

For a particle of half-integer spin, one can choose

$$p = s+\frac{1}{2}\,,\qquad q = s-\frac{1}{2}\,.$$

In the case of integer spin, it is convenient to use

$$p = q = s\,.$$

The spinors ξ and η are chosen in such a way that under reflection they go over into each other (up to a phase). When $p \neq q$, they are different objects that belong to different representations of the Lorentz group. If $p = q$, these two spinors coincide. Nevertheless, we will use the same expression (46) for the wave function of any spin, i.e., we will also formally introduce the object η for an integer spin, keeping in mind that it is expressed in terms of ξ. This will allow us to perform calculations in the same way for the integer and half-integer spins.

In the rest frame, both ξ and η coincide with a nonrelativistic spinor ξ_0, which is symmetric in all indices; in this frame, there is no difference between dotted and undotted indices. The spinors ξ and η are obtained from ξ_0 through the Lorentz transformation:

$$\xi = \exp\left\{\boldsymbol{\Sigma}\,\frac{\boldsymbol{\phi}}{2}\right\}\xi_0\,,\quad \eta = \exp\left\{-\boldsymbol{\Sigma}\,\frac{\boldsymbol{\phi}}{2}\right\}\xi_0\,. \tag{47}$$

Here, the vector $\boldsymbol{\phi}$ is directed along the velocity, $\tanh\phi = v$,

$$\boldsymbol{\Sigma} = \sum_{i=1}^{p}\boldsymbol{\sigma}_i - \sum_{i=p+1}^{p+q}\boldsymbol{\sigma}_i\,,$$

and $\boldsymbol{\sigma}_i$ acts on the ith index of the spinor ξ_0 as follows:

$$\boldsymbol{\sigma}_i\,\xi_0 = (\boldsymbol{\sigma}_i)_{\alpha_i\beta_i}\,(\xi_0)_{\ldots\beta_i\ldots}\,. \tag{48}$$

In the Lorentz transformation (47) for ξ, after the action of the operator $\boldsymbol{\Sigma}$ on ξ_0 the first p indices are identified with the upper undotted indices and the next q indices are identified with the lower dotted indices. The inverse situation takes place for η.

Then,

$$\bar{\psi} = \psi^{\dagger}\gamma_0 = \psi^{\dagger}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

where I is the product of unit 2×2 matrices acting on all indices of the spinors ξ and η. The components of the matrix $\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}$ are

$$\Sigma_{0n} = \begin{pmatrix}-\Sigma_n & 0\\ 0 & \Sigma_n\end{pmatrix}, \tag{49}$$

$$\Sigma_{mn} = -2i\epsilon_{mnk}\begin{pmatrix}s_k & 0\\ 0 & s_k\end{pmatrix}, \tag{50}$$

$$\mathbf{s} = \frac{1}{2}\sum_{i=1}^{2s}\boldsymbol{\sigma}_i\,.$$

The scalar operators $F_{\mathrm{e,m}}$ in (45) depend on two invariants, $t = q^2$ and $\tau = (S^{\mu}q_{\mu})^2$, where $S^{\mu} = (i/4)\epsilon^{\mu\nu\varkappa\lambda}\Sigma_{\varkappa\lambda}p_{\nu}/m$ is the spin 4-vector. In the expansion in the electric multipoles

$$F_{\mathrm{e}}(t,\tau) = \sum_{n=0}^{N_{\mathrm{e}}} f_{\mathrm{e},2n}(t)\,\tau^n\,,$$

the highest power N_{e} obviously equals s and $s-1/2$ for the integer and half-integer spins, respectively. In the magnetic multipole expansion

$$F_{\mathrm{m}}(t,\tau) = \sum_{n=0}^{N_{\mathrm{m}}} f_{\mathrm{m},2n}(t)\,\tau^n\,,$$

the highest power N_{m} constitutes $s-1$ and $s-1/2$ for integer and half-integer spins, respectively. It can be easily seen that $f_{\mathrm{e},0}(0) = 1, f_{\mathrm{m},0}(0) = g/2$.

Of course, the terms linear in spin in the amplitude (44) reproduce the well-known result (17). As to the interaction of second order in spins, respectively which we are interested in here, even the final formula for it is lengthy:

$$L_{e2} = \frac{Q}{2s(2s-1)}\left[(\mathbf{s}\nabla) - \frac{\gamma}{\gamma+1}(\mathbf{vs})(\mathbf{v}\nabla)\right]$$
$$\times \left[(\mathbf{s}\,\mathbf{E}) - \frac{\gamma}{\gamma+1}(\mathbf{sv})(\mathbf{vE}) + \left(\mathbf{s}[\mathbf{v}\times\mathbf{B}]\right)\right]$$
$$+ \frac{e}{2m^2}\frac{\gamma}{\gamma+1}\left(\mathbf{s}\,[\mathbf{v}\times\nabla]\right)\left[\left(g-1+\frac{1}{\gamma}\right)(\mathbf{s}\,\mathbf{B})\right.$$
$$\left. - (g-1)\frac{\gamma}{\gamma+1}(\mathbf{sv})(\mathbf{vB}) - \left(g - \frac{\gamma}{\gamma+1}\right)\left(\mathbf{s}\,[\mathbf{v}\times\mathbf{E}]\right)\right]. \tag{51}$$

Here, the particle quadrupole moment Q is defined as usual: $Q = Q_{zz}|_{s_z=s}$.

It is well-known that the electromagnetic interaction of the convection current and magnetic moment also contributes in the nonrelativistic limit to the quadrupole interaction. The value of this induced contribution to the quadrupole moment [already included into Q in formula (51)] is [41]

$$\Delta Q = -e\,(g-1)\left(\frac{\hbar}{mc}\right)^2 \begin{cases} s\,, & \text{integer spin}\,, \\ s-1/2\,, & \text{half-integer spin}\,. \end{cases} \tag{52}$$

In this formula, we have explicitly singled out the Planck constant $\hbar$ to demonstrate that the induced quadrupole moment ΔQ vanishes in the classical limit $\hbar \to 0,\ s \to \infty,\ \hbar s \to \text{const}$. Therefore, the contribution proportional to ΔQ does not in fact influence the equations of motion of a classical particle (though it plays a role in atomic spectroscopy [41]).

On the other hand, the electromagnetic interactions of the convection current and spin current also induce an interaction of second order in spin, which has a classical limit and is described by the last two lines of formula (51). This Q-independent part of the interaction (51) tends to zero in the nonrelativistic limit. Besides, it is reducible in spin; in other words, the structure $s_i s_j$ in it cannot be rewritten as an irreducible tensor $s_i s_j - (1/3)\delta_{ij}\mathbf{s}^2$. The Q-independent part of the interaction (51) does not have a quadrupole structure at all.

Of great interest is the asymptotic behavior of the interaction (51) as $\gamma \to \infty$. Surprisingly, though both Q-dependent and Q-independent parts of the interaction (51) by themselves increase with energy, there is a particular value of the quadrupole moment for which this interaction as a whole drops as $\gamma \to \infty$.

The situation resembles that for the interaction linear in spin. It is well-known (see, e.g., [11, 42, 43]) that there is a special value of the g factor, $g = 2$, at which the electromagnetic interaction linear in spin decreases with increasing energy. This follows immediately from formula (16) for $\gamma \to \infty$. Thus, the choice $g = 2$ for the bare magnetic moment is a necessary (but insufficient!) condition of unitarity and renormalizability in quantum electrodynamics. It holds not only for the electron, but also for the charged vector boson in the renormalizable electroweak theory. Other arguments in favor of $g = 2$ are given in [44–48].

The same situation holds for the second-order spin interaction in electrodynamics. There is a special value of the quadrupole moment Q at which this interaction also decreases with increasing energy. If we also assume $g = 2$, this value is

$$Q = -s(2s-1)\,\frac{e}{m^2}\,. \tag{53}$$

The same preferred value of the quadrupole moment was derived also otherwise, using the supersymmetric sum rules [45, 47, 48]. Again, (53) is a necessary condition of unitarity and renormalizability. And indeed, this is the value of the quadrupole moment of the charged vector boson in the renormalizable electroweak theory. For it,

$$g = 2, \qquad s = 1, \qquad Q = -\frac{e}{m^2}\,.$$

4.3 Second-order spin effects in a gravitational field

For a binary star, effects of second-order in spin are of the same order of magnitude as the spin – spin interaction when the spins of the components of the system are comparable [28]. The influence of the latter on the characteristics of the gravitational radiation becomes noticeable for a system of two extreme black holes [25]. Correspondingly, second-order spin effects in the equations of motion become substantial if at least one component of a binary is close to an extreme black hole [28]. Therefore, the investigation of these effects is not only of purely theoretical interest. In principle they may be observed with the gravitational wave detectors under construction.

The equations of motion in an external gravitational field to any order in spin can be obtained by means of a simple substitution from the corresponding equations of motion in an electromagnetic field.

The elastic scattering amplitude in a weak external gravitational field $h_{\mu\nu}$ is

$$-\frac{1}{2}\,T_{\mu\nu}h^{\mu\nu} \tag{54}$$

(in due time, we will go over to a generally covariant form). The matrix element $T_{\mu\nu}$ of the energy-momentum tensor between the states of momenta k and k' can be written as [30]

$$T_{\mu\nu} = \frac{1}{4\sqrt{\epsilon_k \epsilon_{k'}}}\,\bar\psi(k')\big\{4\,p_\mu p_\nu F_1$$
$$+ (p_\mu \Sigma_{\nu\lambda} + p_\nu \Sigma_{\mu\lambda})\,q^\lambda F_2 + (\eta_{\mu\nu}q^2 - q_\mu q_\nu)\,F_3$$
$$+ \left[S_\mu S_\nu q^2 - (S_\mu q_\nu + S_\nu q_\mu)(Sq) + \eta_{\mu\nu}(Sq)^2\right] F_4\big\}\,\psi(k)\,. \tag{55}$$

The scalar operators F_i in this expression are also expanded in powers of $\tau = (Sq)^2$:

$$F_i(t,\tau) = \sum_{n=0}^{N_i} f_{i,2n}(t)\,\tau^n\,. \tag{56}$$

Since we are interested in the equations of motion in empty space, the terms proportional to F_3 and F_4 in expansion (55) will be omitted, because when rewritten in the covariant form, they are proportional to the scalar curvature and Ricci tensor, respectively. Thus, the amplitude (54) can be presented in the following form:

$$-\frac{1}{2\sqrt{\epsilon_k \epsilon_{k'}}}\,\bar\psi(k')\left\{p_\mu F_1 + \frac{1}{2}\Sigma_{\mu\lambda}\,q^\lambda F_2\right\}\psi(k)\,h^{\mu\nu}p_\nu\,. \tag{57}$$

Clearly, (57) differs from (44), (45) by the following substitution only:

$$e\,A_\mu \to \frac{1}{2}\,h_{\mu\nu}\,p^\nu\,. \tag{58}$$

With this substitution in (51), one can obtain the second-order spin interaction of a particle with a gravitational field.

After this substitution, further calculations are performed in the same way as in the case of an external electromagnetic field. The only thing left is to get rid of the weak-field approximation by rewriting the expressions dependent on $h_{\mu\nu}$ via the Ricci coefficients in the terms of first order in spin, and via the Riemann tensor in the terms of higher orders.

The terms of second order in spin arising in this way can also be obtained otherwise. There is an instructive short-cut that allows one to derive without lengthy calculations the so-called gravimagnetic interaction [11], a gravitational analogue of the Q-dependent terms in formula (51). It was already mentioned that the analogy between first-order spin interactions in electrodynamics and gravity is incomplete. The electromagnetic interaction depends on the field strength, which is gage-invariant. However, the gravitational one depends not on the Riemann tensor, which is generally covariant, but on the Ricci rotation coefficients, which are not. In this respect, the second-order spin interaction discussed below, the gravimagnetic one, which depends on the Riemann tensor, is the gravitational analogue of the first-order spin interactions in electrodynamics.

The starting point of the derivation is the observation that the canonical momentum p_μ enters into a relativistic wave equation for a particle in external electromagnetic and gravitational fields through the combination

$$\Pi_\mu = p_\mu - eA_\mu - \frac{1}{2}\,\Sigma^{ab}\gamma_{ab\mu}\,.$$

Here and below, Σ^{ab} are the generators of the Lorentz group (which differ from our previous definition of Σ^{ab} by the factor $i/2$), and $\gamma_{ab\mu} = e^c_\mu\gamma_{abc}$. The commutation relation

$$[\Pi_\mu, \Pi_\nu] = -ieF_{\mu\nu} + \frac{i}{2}\,\Sigma^{ab}R_{ab\mu\nu} \tag{59}$$

demonstrates the remarkable correspondence

$$eF_{\mu\nu} \leftrightarrow -\frac{1}{2}\,\Sigma^{ab}R_{ab\mu\nu}\,. \tag{60}$$

The squared form of the Dirac equation in an external electromagnetic field

$$(-g^{\mu\nu}\Pi_\mu\Pi_\nu + m^2 + e\,\Sigma^{ab}F_{ab})\,\psi = 0$$

prompts that for an arbitrary spin s the Lagrangian

$$-\frac{e}{2m}\,\Sigma^{ab}F_{ab}$$

describes the magnetic moment interaction for $g = 2$; the factor $1/(2m)$ in this Lagrangian, additional to the above wave equation, becomes obvious from the comparison with the nonrelativistic limit.

Clearly, for an arbitrary g factor this covariant magnetic moment interaction is

$$\mathcal{L}_{\rm el} = -\frac{eg}{4m}\,F_{ab}\Sigma^{ab}\,. \tag{61}$$

This is in fact a covariant form of g-dependent terms in Lagrangian (17). As to the g-independent, Thomas terms in (17), it has already been pointed out that they cannot be presented in a covariant form with the usual, physical definition of the coordinate $\mathbf{r}$. In analogy with the magnetic moment

$$\frac{eg}{2m}\,\Sigma^{ab}\,,$$

it is natural to define the gravimagnetic moment

$$-\frac{\varkappa}{2m}\,\Sigma^{ab}\Sigma^{cd}\,.$$

Now, the correspondence (60) prompts the following gravitational analogue of the Lagrangian (61):

$$\mathcal{L}_{\rm gm} = \frac{\varkappa}{8m}\,\Sigma^{ab}\Sigma^{cd}R_{abcd}\,. \tag{62}$$

This is what we call the gravimagnetic interaction. Note that in the classical limit we have $\Sigma^{ab} \to S^{ab} = \varepsilon^{abcd}S_c u_d$.

The gravimagnetic ratio $\varkappa$, like the gyromagnetic ratio g in electrodynamics, may in general have any value. Still, it is natural that in gravity the value $\varkappa = 1$ is singled out as $g = 2$ in electrodynamics. Indeed, the analysis of the complete Lagrangian for the gravitational interaction of second order in spin, including of course $\varkappa$-independent terms, which correspond to the Q-independent terms in (51), demonstrate that just for $\varkappa = 1$ this total interaction asymptotically tends to zero with increasing energy [11, 30, 31]. The same conclusion is made in [49–52]. Unfortunately, the gravitational interaction for any spin is not renormalizable even at $\varkappa = 1$.

In any case, for $g = 2$ and $\varkappa = 1$ the equations of motion have the simplest form. Moreover, it has been shown in [11] that just this value of the gravimagnetic ratio, $\varkappa = 1$, follows from the wave equations in the Feynman gage both for the photon and graviton in an external gravitational field, as well as from the Rarita–Schwinger equation for $s = 3/2$ in a gravitational field.

The situation for spin 1/2 is worthy of a separate discussion. Obviously, no second-order spin interaction is possible here. Indeed, for spin 1/2 the properties of the spin matrices $\Sigma^{ab} = (i/4)(\gamma^a\gamma^b - \gamma^b\gamma^a)$ are such that $\Sigma^{ab}\Sigma^{cd}R_{abcd}$ degenerates into the scalar curvature R (times 1/2) without any spin dependence at all. So, our arguments in favor of $\varkappa = 1$ do not apply for spin 1/2. And indeed, the squared Dirac equation contains $R/4$, but with $\varkappa = 1$ one obtains here $R/8$. Nevertheless, we cannot see any real physical meaning in the recent proposal [53] to ascribe to the electron (which in fact has no gravimagnetic interaction at all) the gravimagnetic ratio $\varkappa = 2$.

Wave equations for particles of arbitrary spins in an external gravitational field were previously considered in [54]. The equation for integer spins proposed in [54] also corresponds to the gravimagnetic ratio $\varkappa = 1$. However, the value of $\varkappa$ prescribed in [54] for half-integer spins is different. Even in the classical limit $s \to \infty$ it does not tend to unity. This obviously does not comply with the correspondence principle, according to which at least in this classical limit there should be no difference between integer and half-integer spins.

5. Multipoles of black holes

Let us come back from elementary particles to macroscopic bodies. For a classical object, the values of both parameters g and $\varkappa$ depend in general on the various properties of the body.

However, for black holes the situation is different. It has been shown in [55] from an analysis of the Kerr – Newman solution that the gyromagnetic ratio of a charged rotating black hole is universal (and equal to that of the electron!): $g = 2$.

We will show that for the Kerr black hole the gravimagnetic ratio is $\varkappa = 1$. This value follows in fact from the analysis of the motion of the spin of a black hole in an external field in [24] (though this statement was not explicitly formulated there). We present here an independent and, in our opinion, simpler derivation of this important result.

At large distances from a Kerr hole, the hole can be considered as a point source of a weak gravitational field. To a linear approximation in the field of a hole at rest, the Lagrangian density corresponding to the interaction (62) can be written as

$$L = \frac{\varkappa}{4m} (\mathbf{s}\nabla)^2 h_{00}\, \delta(\mathbf{r}) \,. \tag{63}$$

The correction to the energy-momentum tensor induced by this interaction has a single component:

$$\delta T_{00} = -\frac{\varkappa}{2m} (\mathbf{s}\nabla)^2 \delta(\mathbf{r}) \,. \tag{64}$$

In the gage

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \,, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}_{\alpha} \,, \tag{65}$$

the static Einstein equation for the corresponding correction h_{00} to the 00-component of the metric is

$$\Delta h_{00} = 8\pi k T_{00} \,.$$

The correction itself is

$$h_{00} = \varkappa \frac{k}{m} (\mathbf{s}\nabla)^2 \frac{1}{r} \,. \tag{66}$$

Let us compare h_{00} with the corresponding contribution to the Kerr metric. In the Boyer – Lindquist coordinates, this metric is

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{r_{\mathrm{g}} r}{\Sigma}\right) \mathrm{d}t^2 - \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 - \Sigma\,\mathrm{d}\theta^2 \\ & - \left(r^2 + a^2 + a^2 \frac{r_{\mathrm{g}} r}{\Sigma} \sin^2\theta\right) \sin^2\theta\,\mathrm{d}\phi^2 + 2a \frac{r_{\mathrm{g}} r}{\Sigma} \sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}t \,, \end{aligned} \tag{67}$$

where $\Delta = r^2 - r_{\mathrm{g}} r + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$, and $\mathbf{a} = \mathbf{s}/m$. At $r_{\mathrm{g}} = 0$, the metric (67) describes a flat space in spheroidal coordinates [35].

Meanwhile, it is Cartesian coordinates that correspond in the flat space to the gage (65). The transition from the spheroidal coordinates to Cartesian ones is carried out with the required accuracy by the substitution

$$\mathbf{r} \to \mathbf{r} + \frac{\mathbf{a}(\mathbf{ar}) - \mathbf{r}a^2}{2r^2} \,.$$

In the Cartesian coordinates, the spin-dependent part of the 00-component of the metric

$$g_{00} = 1 - \frac{r_{\mathrm{g}}}{r} + \frac{r_{\mathrm{g}} a^2}{2r^3} (3\cos^2\theta - 1)$$

obviously coincides with h_{00} from formula (66) at $\varkappa = 1$. A somewhat more tedious consideration of the space components of the Kerr metric leads to the same result, $\varkappa = 1$.

Note that the motion of the Kerr black hole in an external gravitational field is not described by the Papapetrou equation even if one leaves aside the problem of the spin – orbit interaction, linear in spin. The point is that this equation refers to the case $\varkappa = 0$ [14].

It is proven in the same way that for a charged Kerr hole the gravimagnetic ratio $\varkappa$ is also unity. Moreover, the electric quadrupole moment of a charged Kerr hole also equals

$$Q = -2\,\frac{es^2}{m^2} \,, \tag{68}$$

the value at which the interaction quadratic in spin decreases with energy (this is the obvious limit of the general formula (53) as $s \to \infty$).

It can be demonstrated [56] that other, higher multipoles of a charged Kerr hole, both electromagnetic and gravitational, also possess just those values that guarantee that the interaction of any order in spin (but of course, linear in an external field) asymptotically decreases with increasing energy.

6. Gravitational interaction of spinning bodies, and radiation of compact binary stars

It is expected that in a few years the gravitational radiation from coalescing binary stars will be observed by laser interferometer systems. Its successful detection depends crucially on the accurate theoretical prediction of the exact form of the signal. In this way the observed effect becomes sensitive to the relativistic corrections of the orders c^{-2}, c^{-3} and c^{-4} to the motion of a binary system and to the radiation intensity. In particular, the spin – orbit interaction becomes essential. Moreover, effects of second order in spin may be observed in the gravitational radiation, in the case of two extreme Kerr black holes [25].

6.1 Spin interactions in a two-body problem

The spin interactions in a two-body problem can easily be obtained from the well-known results for the limiting case when one of the bodies (say, 2) is very heavy. In this limit we have the usual spin – orbit interaction with the frequency $\boldsymbol{\omega}_{ls}$ given in fact by formula (32) (the limit $\gamma \to 1$ is sufficient here):

$$V^1_{1ls} = -\boldsymbol{\omega}_{ls}\,\mathbf{s_1} = \frac{3}{2}\,\frac{k}{r^3}\,\frac{m_2}{m_1}\,\mathbf{ls_1} \,. \tag{69}$$

Here and below, $\mathbf{s_1}$ and m_1 are the spin and mass of the first body, $\mathbf{s_2}$ and m_2 are the spin and the mass of the second one, and $\mathbf{r}$ is the radius vector connecting the two bodies. Then, there is a so-called Lense – Thirring interaction of the orbital angular momentum $\mathbf{l}$ with the spin $\mathbf{s_2}$ of the central body [57]

$$V^1_{2ls} = 2\frac{k}{r^3}\,\mathbf{ls_2} \,. \tag{70}$$

Simple arguments of symmetry on particle permutation now dictate the form of the total spin – orbit interaction for the two-body problem:

$$V_{ls} = \frac{k}{r^3}\,\mathbf{l}\left[\frac{3}{2}\left(\frac{m_2}{m_1}\,\mathbf{s_1} + \frac{m_1}{m_2}\,\mathbf{s_2}\right) + 2(\mathbf{s_1} + \mathbf{s_2})\right] . \tag{71}$$

As to the spin – spin interaction, it is of the usual form, with $\boldsymbol{\omega}_{ss}$ given by formula (42) (again the lowest nonvanishing order in c^{-1} is implied):

$$V_{ss} = \frac{k}{r^3}\left[3(\mathbf{s}_1\mathbf{n})(\mathbf{s}_2\mathbf{n}) - \mathbf{s}_1\mathbf{s}_2\right], \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{72}$$

Of course, both expressions (71) and (72) can be derived directly, following, for instance, the approach of [35, § 106, Problem 4].

Let us now go over to the gravimagnetic interaction. This interaction (62) for particle 1 with the field created by a heavy mass m_2 reduces in lowest, first order in c^{-2} to the quadrupole form:

$$V_s^1 = \frac{3}{2}\,\frac{k}{r^3} m_2\, Q^s_{1mn} n_m n_n\,, \tag{73}$$

where the effective gravitational quadrupole moment of particle 1 is

$$Q^s_{1mn} = \frac{\varkappa_1}{m_1}\left(s_{1m}s_{1n} - \frac{1}{3}\,\delta_{mn}\mathbf{s}_1^2\right).$$

For the two-body problem under discussion, expression (73) generalizes to the following self-interaction of spins:

$$V_s = \frac{3}{2}\,\frac{k}{r^3}\left(\varkappa_1\,\frac{m_2}{m_1}\,s_{1m}s_{1n} + \varkappa_2\,\frac{m_1}{m_2}\,s_{2m}s_{2n}\right) \times \left(n_m n_n - \frac{1}{3}\,\delta_{mn}\right), \tag{74}$$

resembling the usual spin – spin interaction (72).

At $\varkappa_{1,2} \sim 1$, the effective quadrupole interaction (74) is of the same order of magnitude as the spin – spin one (72). Even in the most favorable case when they can become important, i.e., that of two extreme Kerr black holes, both interactions are of the c^{-4} order. The star rotation velocity is here $\sim c$, but the star radius is close to the gravitational one $r_g \sim c^{-2}$, so that each spin $s \sim c^{-1}$ [25]. The same argument demonstrates that the spin – orbit interaction is of the c^{-3} order [25].

6.2 Contribution of spin interactions to gravitational radiation

The spin interactions contribute in various ways to the gravitational radiation: through spin-dependent corrections to the orbit radius r and to the equations of motion used to evaluate the time derivatives, which enter into the usual expression for the gravitational quadrupole radiation; through the corrections to the 00-component of the energy-momentum tensor of the particles; through the gravitational analogue of the magnetic quadrupole radiation in electrodynamics; and through retardation effects.

In all our discussions of gravitational radiation, we restrict ourselves to the case of circular orbits, which is the most interesting one from the physical point of view [25]. Besides, the assumption of circular orbits essentially simplifies the calculations. Still the calculations remain tedious, so only the final results are presented here.

The relative correction to the radiation intensity generated by the spin – orbit interaction (71) is [28]

$$\frac{I_{ls}}{I_q} = -\frac{\mathbf{l}(73\mathbf{s} + 45\boldsymbol{\xi})}{12\,m_1 m_2 r^2}\,. \tag{75}$$

Here,

$$I_q = \frac{32k^4 m_1^2 m_2^2\,(m_1 + m_2)}{5r^5}$$

is the unperturbed quadrupole intensity;

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2\,, \quad \boldsymbol{\xi} = \frac{m_2}{m_1}\,\mathbf{s}_1 + \frac{m_1}{m_2}\,\mathbf{s}_2\,.$$

It can be easily checked that the corresponding result of [25, 26] would be reduced to agreement with (75) for the proper definition of the center-of-mass coordinate.

The correction due to the spin – spin interaction (72) is [25, 26]

$$\frac{I_{ss}}{I_q} = \frac{1}{48\,m_1 m_2 r^2}\,(649 s_{1t}s_{2t} - 223\mathbf{s}_1\mathbf{s}_2)\,. \tag{76}$$

The expressions for I_{ss}, as well as that for I_s below, have been averaged over the period of rotation. That is why both of them contain the spin components s_t orthogonal to the orbit plane.

And at last, the spin-self-interaction correction, generated by the gravimagnetic interaction (74) and by the above-mentioned gravitational analogue of the magnetic quadrupole radiation in electrodynamics, is [28]

$$\frac{I_s}{I_q} = \frac{1}{4m_1 m_2 r^2}\left[\left(27\varkappa_1 - \frac{1}{24}\right)\frac{m_2}{m_1}\,s_{1t}^2 + \left(27\varkappa_2 - \frac{1}{24}\right)\frac{m_1}{m_2}\,s_{2t}^2 - \left(9\varkappa_1 - \frac{7}{24}\right)\frac{m_2}{m_1}\,\mathbf{s}_1^2 - \left(9\varkappa_2 - \frac{7}{24}\right)\frac{m_1}{m_2}\,\mathbf{s}_2^2\right]. \tag{77}$$

This correction is also discussed in [58].

* * *

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