MULTIPHOTON IONIZATION BY A VERY SHORT PULSE

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Detachment of the bound electron by an electric field pulse of duration only few or even part of the optical cycle long, however long compared to $\hbar/I$ ($\hbar$ - Planck constant, I - binding energy), is studied theoretically. This is the model problem for ionisation of atoms by extremely short laser pulses. Because of strong nonlinearity it does not reduce to the sum of monochromatic harmonics contributions and depend crucially on details of pulseshape. General analisis is presented in terms of analitical properties of the pulseshape function and explicit formulae are given for typical pulseshapes such as solitonlike, gaussian, lorenzian, etc., one- or half optical cycle long. Intensity and pulselength dependency of the ionisation probability are of approximately universal tunneling type at high intensities. However at moderate intensities in multiphoton regime they are very different for different pulseshapes, always orders of magnitude exeeding ionization by monochromatic wave of the same intensity and mean frequency.

A remarkable progress in generating very high intensity laser fields was closely related to corresponding reduction of pulse duration [1,2]. Therefore really multiphoton processes, i.e. that requiring simultanious participation of many ($\gg 1$) photons, are observed typically in experiments with ultrashort pulses (USP). With this reduction continuing, pulselengths become comparable to the optical field cycle duration [3-8]. Under such conditions the usual concept of transition (ionization) probability per time unit makes no sense. The only meaningful quantity remains the total - after the whole pulse - transition probability. Moreover the frequency spectrum of the pulses under consideration is very broad and because of extreme nonlinearity of the process its probability does not reduce to the sum of independent harmonics contributions. The physical essence of ionization process in the high intensity USP case may be thought of as interaction and competition of many harmonics contributions, depending not only on the spectrum, but also phase relations of different harmonics, i.e. higher order field correlations. In other words this means that the result is very sensitive to the exact pulse shape. In this article some extreme particular cases are theoretically studied, corresponding to USP's few or even half optical cycle long.

Recently few groups have investigated both experimentally and theoretically [] in a sense even more extreme limiting case - ionization of atoms by a pulse much shorter as compared to characteristic electron times (inverse optical transition frequency between neighbouring energy levels). They realized this conditions experimentally [] with alkaly atoms, excited to very high Rydberg states, corresponding to quasiclassical electron motion and small interlevel distances. Field pulse acted in such a case as an (quasi)instantaneous kick, moving electron from one - bounded - Kepler orbit to another - unbounded. Unlike that the problem discussed below is essentially quantum one: ionization from the tightly bound state, e.g. ground state, by the pulse one or one-half optical cycle long, but much longer then "atomic cycle" - $\hbar/I$, $\hbar$ being Planck constant and I - ionization energy. This means that the average energy of photon in the pulse is small compared to ionization energy. For atomic electron this is slowly varying perturbation, and therefore the same adiabatic treatment can be applied to this problem, which was exploited earlier [9] for ionization by intense monochromatic wave. That is based on the observation that the final - free - state of electron in the process under consideration is much more sensitive to such type of perturbations than the initial one - strongly bound and localized. So the transition probabilty is calculated as that of first order transition from the unperturbed initial atomic state to the final "exact" state of free electron in the strong time-dependent electric field. The latter of these states accounts for the field action nonperturbatively and contains the main contribution to the transition amplitude. For the fields below atomic, i.e. intensities up to PW/cm^2 range the most (an only) important defect of this approach is neglecting electron - ion Coulomb interaction in the final state, i.e. Born-type approximation. The significance of such type approximate solutions may seem questionable now. This kind of quantum problems - single electron in an external field, including both atomic and electromagnetic - certainly are within the limits of modern computing abilities. During the last decade several algorithms were proposed and successfully applied to the problems of multiphoton ionization and some others, related, such as UV higher harmonics generation []. Still analitic so-

lutions, even semiquantitatively correct, also have their advantages.¡¡ Being not restricted by any definite set of parameter values¿¿ They may be useful in representing an overall view of the process and trends due to variation of parameters, or as a starting point in analizing more complicated, e.g. multi-electron, systems.
Let the spatially uniform time dependent electric field $\mathcal{F}(t)$ be given by

$$\mathcal{F}(t) = \mathcal{F} \cdot f'(\omega t) \tag{1}$$

Here $f'(x)$ - derivative of the function f over its argument x, ω - inverse characteristic timescale of the pulse. In such a field the wave function of the free electron is

$$\psi_{\mathbf{p}}(\mathbf{r}, t) = \exp\Big[\frac{i}{\hbar}\Big(\mathbf{p}(t)\mathbf{r} - \int_0^t \frac{p^2(t')}{2m} dt'\Big)\Big] \tag{2}$$

with

$$\mathbf{p}(t) = \mathbf{p} + \frac{e\mathcal{F}}{\omega} \cdot f(\omega t) \tag{3}$$

Following the usual first order perturbation theory, the transition probability from initial state $\psi_0(\mathbf{r}) \exp[(i/\hbar)It]$ to the final state $\psi_{\mathbf{p}}(\mathbf{r}, t)$ can be calculated as

$$w_{i\mathbf{p}} = \frac{e^2\mathcal{F}^2}{\hbar^2\omega^2}\Big| \int_{-\infty}^{\infty} dx \cdot R_{\parallel}\Big(\mathbf{p} + \frac{e\mathcal{F}}{\omega} \cdot f(x)\Big) \cdot \exp\Big[i\frac{I}{\hbar\omega} \cdot \Phi(x)\Big]\Big|^2 \tag{4}$$

with phase function $\Phi(x)$ defined as

$$\Phi(x) = \frac{1}{I} \int_0^x \Big[I + \frac{p^2(x')}{2m}\Big] dx' - i\frac{\hbar\omega}{I} \cdot \ln f'(x) \tag{5}$$

Here

$$R_{\parallel}(\mathbf{p}) = \int \exp\left[-(i/\hbar)\mathbf{p}\mathbf{r}\right] \cdot \mathbf{n}\mathbf{r} \cdot \psi_0(\mathbf{r}) d^3r$$

- transition matrix element of coordinate component parallel to the field $\mathcal{F}$, $\mathbf{n}$ - the unit vector in the field direction.
In order to make the following analisis more spectacular it is convenient to use the representation of all quantities involved in the natural "atomic" scale, i.e. to define

$$\Omega = \frac{\hbar\omega}{I} \qquad \mathbf{q} = \frac{\mathbf{p}}{\sqrt{2mI}} \qquad \mathcal{E} = \frac{e\hbar\mathcal{F}}{\sqrt{2mI^3}} \tag{6}$$

Certainly this means the corresponding transformation of coordinate and time scales. Then dimensionless matrix element should be defined as

$$M(\mathbf{q}) = \left[\frac{\sqrt{2mI}}{\hbar}\right]^{5/2} \cdot R_{\parallel}(\mathbf{p}) \tag{7}$$

According to the above claim the whole consideration in this article is for $\Omega \ll 1$

The crucial parameter of the theory is then the ratio of dimensionless field and frequency :

$$\lambda = \frac{|\mathcal{E}|}{\Omega} \tag{8}$$

which is exactly inverse to the parameter γ introduced in [9] (if ω considered as a characteristic frequency of the process).

The factor $1/\Omega$ being the large parameter of the theory, the integral in (4) can be calculated by the stationary phase method.The stationary phase point(s) in the complex variable x plane is to be found from equation

$$\left.\frac{\partial\Phi(x,\mathbf{q})}{\partial x}\right|_{x_s} = 1 + \Big(\mathbf{q} + \mathbf{n}\cdot\lambda\cdot f(x_s)\Big)^2 - i\Omega\frac{f''(x_s)}{f'(x_s)} = 0 \tag{9}$$

Then the transition probability

$$w_{i\mathbf{p}} = 2\pi\Omega\lambda^2\left|\sum_s \frac{M\Big(\mathbf{q}+\mathbf{n}\lambda f(x_s)\Big)}{\sqrt{|\Phi''(x_s,\mathbf{q})|}} \cdot \exp[-\frac{i}{\Omega}\cdot\Phi(x_s,\mathbf{q})]\right|^2 \tag{10}$$

with summ over all saddlepoints x_s. Contributions of different saddlepoints are exponentially different and only that dominating must be kept in (10). Generally there is one such dominating saddle point - that corresponding to the lowest value of positive imaginary part of $\Phi(x_s,\mathbf{q})$. However in many cases, due to some symmetry of the pulse function $f(x)$, there exist pairs or groups of equivalent saddle points with equal values of $\Im\Phi(x_s,\mathbf{q})$ but different phase factors - $\Re\Phi(x_s,\mathbf{q})$. Interference of their contribution results in oscillations of the ionisation probability as a function of pulse parameters λ and Ω.

Considered as a function of its argument $\mathbf{q}$ this probability is momentum distribution function of emmited electrons. Note however that $\mathbf{q}$ in this formulae is momentum at the time instant when $f(x) = 0$. So if $f(\infty) \neq 0$,as it is e.g. in examples 1 and 4 below, the momentum distribution of ejected electrons is distribution (10) but shifted by $\delta\mathbf{q} = \mathbf{n}\lambda f(\infty)$, as is done in formulae (17) and (43) for above mentioned examples.

Typically distribution is gaussian around some average momentum $\mathbf{q}_m$, to be defined from the condition of minimum of $\Im\Phi(x_s, \mathbf{q})$, which accounting for (9) reduces to

$$q_{\parallel m} = -\frac{\lambda}{x''_{sm}} \cdot \Im\left[\int_0^{x_{sm}} dx \cdot f(x)\right] \tag{11}$$

and $\mathbf{q}_\perp = 0$, with $q_\parallel$ and $\mathbf{q}_\perp$ - momentum components, parallel and perpendicular to the field direction; $x_{sm} \equiv x_s(\mathbf{q}_m)$; x''_s - imaginary part of x_s.

In the vicinity of this sharp maximum, taking into account (9) and (11), imaginary part of $\Phi(x_s, \mathbf{q})$ can be transformed to

$$\Im\Phi(x_s, \mathbf{q}) = x''_{sm} +$$

$$+\Im\{\int_0^{x_{sm}} [\lambda^2 f^2(x) - q^2_{\parallel m}]dx + x_{sm}q^2_\perp + [x_{sm} - i(\lambda f'(x_{sm}))^{-1}](q_\parallel - q_{\parallel m})^2\} \tag{12}$$

with halfwidths defined by the second derivatives of the exponent in (10) over components of momentum.

A comment should be done about the preexponential factor in (10). In deriving this formula the matrix element $M\Big(\mathbf{q} + \mathbf{n}\lambda f(x)\Big)$ was treated as a regular function, slowly varying in the vicinity of x_s: $M(\mathbf{q}) \simeq M_0 \equiv M(0)$. However typically $M(\mathbf{q})$ contains a pole - singularity of the type $M(\mathbf{q}) = M_0/(1 + q^2)$ - in the momentum complex plane [9]. In the whole range of nonlinear absorbtion $\lambda \gg \lambda_c$ with λ_c defined below by (46), terms in $\Phi(x_s)$ proportional to λ or λ^2 are much larger then the last term $\sim \Omega$. Then this pole gets very close to the position of the saddle point. This modifies slightly evaluation of the integral in (4): instead of saddle point contribution enters half of the residue in that point, which enhances the preexponent in (10) by the factor $\pi/(4\mathcal{E}|f'(x_s)|)$. If two (or few) equivalent saddle points (and poles of M) are present in (10), each contribution to the transition amplitude must be multiplied with $sign\Im[f(x_s)] \cdot \sqrt{\pi/(4\mathcal{E}|f'(x_s)|)}$. However strictly speaking this corrections to the preexponential factor (like also one discussed below and due to violation of the standart stationary phase method in the vicinity

of the singularity in the pulseshape function itself) must be neglected: the preexponential factor in (10) and some following formulae should be considered only by the order of magnitude correct because of the abovementioned Born-type approximation.
Results for a few particular but representative enough examples are shown below.

1. Solitonlike half-cycle pulse (HCP)

$$f(x) = \tanh x \tag{13}$$

which means for electric field strength

$$\mathcal{E}(t) = -\frac{\mathcal{E}}{(\cosh \omega t)^2} \tag{14}$$

Momentum-resolved ionization probability

$$w_i(\Omega, \lambda, \mathbf{q}) = \frac{\pi\Omega}{\sqrt{\Omega^2 + \lambda^2}} \cdot (\lambda^2 + \zeta^2)|M_0|^2 \cdot$$

$$\cdot \exp\left\{ -\frac{2}{\Omega}\left[(1+\lambda^2)\arctan\frac{\zeta}{\lambda} - \lambda\zeta + \arctan\frac{\zeta}{\lambda}\cdot(\mathbf{q} - \mathbf{n}\lambda)^2\right]\right\} \tag{15}$$

with parameter

$$\zeta = \frac{1}{\lambda}\left[\sqrt{\Omega^2 + \lambda^2} - \Omega\right] \tag{16}$$

Accounting for above mentioned pole in the transition matrix element this formula shoud be modified to

$$w_i(\Omega, \lambda, \mathbf{q}) = |\pi M_0|^2 \cdot \exp\left[-\frac{1}{\Omega}\left((1+\lambda^2)\pi + 2\arctan\frac{\zeta}{\lambda}\cdot(\mathbf{q} - \mathbf{n}\lambda)^2\right)\right]\cdot$$

$$\cdot \sinh^2\left[\frac{1}{\Omega}\left((1+\lambda^2)\arctan\lambda + \lambda\right)\right] \tag{17}$$

Moreover formulae (15)and (17) are derived by the stationary phase method applied to evaluate integral in (4). However, for $f(x) = \tanh x$ with field decreasing the saddle point approaches $i\pi/2$ - the singularity of $f(x)$ itself (not that of matrix element). In the linear absorption regime $\lambda < \Omega \ll 1$ this violates conditions of the stationary phase method applicability also modifying numerically preexponential factor. In the framework of general analisis below the exact (in Born approximation) formulae will be derived valid for a weak field limit also. Coincedeng with (15) and (17) in the nonlinear field range, for weak fields they contain correction factors S^{reg} for (15) and S^{sing} for (17) represented in formulae (52) and (54).

2. Solitonlike one-cycle pulse (OCP)

$$f(x) = -\frac{3\sqrt{3}}{4\cosh^2 x} \tag{18}$$

The numerical factor is introduced to normalize $|f'(x_m)|$ to unity at both extrema of the field strength.
Momentum-resolved ionization probability

$$w_i(\Omega, \lambda, \mathbf{q}) = 8|2\pi M_0|^2 \cdot \exp\left[-\frac{\pi}{\Omega}(1+q^2)\right] \cdot$$
$$\cdot \Big[1 - \cos\Big(\frac{2}{\Omega}\Re\Phi(x_s, \mathbf{q})\Big)\Big] \cdot \sinh^2\Big[-\frac{1}{\Omega} \cdot \Im\tilde{\Phi}(x_s, \mathbf{q})\Big] \tag{19}$$

with

$$\Im\tilde{\Phi}(x_s, \mathbf{q}) \equiv \Im\Phi(x_s, \mathbf{q}) - \frac{\pi}{2}(1+q^2) = (1+q^2)(x_s'' - \pi/2)$$
$$-\frac{1}{6}\Big[(5q_\| - 2\tilde{\lambda}) \cdot \eta + \sqrt{1+q_\perp^2} \cdot \xi\Big] \tag{20}$$

$$\Re\Phi(x_s, \mathbf{q}) = (1+q^2)x_s' - \frac{1}{6}\Big[(5q_\| - 2\tilde{\lambda}) \cdot \xi - \sqrt{1+q_\perp^2} \cdot \eta\Big] \tag{21}$$

saddle point x_s defined by

$$x_s'' \equiv \Im x_s = \frac{\pi}{2} - \frac{1}{2}\arccos\frac{\sqrt{1+q_\perp^2+(\tilde{\lambda}-q_\|)^2} - \tilde{\lambda}}{\sqrt{1+q^2}} \tag{22}$$

$$x_s' \equiv \Re x_s = \frac{1}{2} tanh^{-1}\left[\frac{\xi}{\tilde{\lambda} + \sqrt{1+q_\perp^2+(\tilde{\lambda}-q_\|)^2}}\right] \tag{23}$$

field parameter $\tilde\lambda = (3\sqrt{3}/4)\cdot\lambda$ and

$$\xi = \sqrt{2\tilde\lambda\Big[\sqrt{1+q_\perp^2+(\tilde\lambda-q_\|)^2}+\tilde\lambda-q_\|\Big]} \tag{24}$$

$$\eta = \sqrt{2\tilde\lambda\Big[\sqrt{1+q_\perp^2+(\tilde\lambda-q_\|)^2}-\tilde\lambda+q_\|\Big]} \tag{25}$$

Formula (19) is presented in the form corresponding to the singular matrix element $M(\mathbf{q})$ as described above. The function sinh in (19) is accounting for contributions of two pairs of poles (saddle points): one pair with $x_s'' < \pi/2$ and another symmetrically above $\pi/2$. Contribution of the latter pair is significant only at the weakest fields $\lambda \ll \Omega^2$. It makes formula (19) to describe correctly (up to numerical factor ~ 1) linear absorption. In the whole nonlinear range $\lambda \gg \Omega^2$ this contribution is negligible and the sinh does not differ from the half of the exponential function of the same argument. The momentum $q_{\|m}$,corresponding to distribution function maximum, should be found from the equation

$$q_{\|m}\cdot x_{sm}''\Big|_{\mathbf{q}_\perp=0} = \eta \tag{26}$$

and substituted into (19) - (25). For small $\lambda \ll 1$ it is approximately $q_{\|m} \approx \sqrt{2\tilde\lambda}/\pi$. For large fields $\lambda \gg 1$ its value approaches $2\tilde\lambda/3$. Numerical data for $(3q_{\|m})/(2\tilde\lambda)$ are plotted in the Fig.1.

Oscillations in field and momentum dependences in (19) arise because of contributions interference due to pair of saddle points, symmetrical relative to imaginary axis. In the total - momentum integrated - ionization probability their amplitude decreases with field increasing as a result of distructive interference of different momenta contributions.

$$W_i(\Omega,\lambda) = \sqrt{\frac{2\pi\Omega}{u}}\cdot\frac{\Omega}{x_{sm}''}\cdot|M_0|^2\cdot\exp\Big[-\frac{\pi}{\Omega}(1+q_m^2)\Big]\cdot$$

$$\cdot\Big[1-\exp\Big(-\frac{2\tilde\lambda}{\pi\Omega}\Big)\cdot\cos\Big(\frac{4\sqrt{2\tilde\lambda}}{3\Omega}\Big)\Big]\cdot\sinh^2\Big[-\frac{1}{\Omega}\cdot\Im\tilde\Phi(x_{sm},\mathbf{q}_m)\Big] \tag{27}$$

with

$$u = \Big[x_s'' + \frac{1}{4}\frac{q\xi+\eta}{(1+q^2)\sqrt{1+(\tilde\lambda-q)^2}}\Big]_{\mathbf{q}=\mathbf{q}_m}$$

The oscillating term is written here in the form valid only for $\lambda \ll 1$ as for larger fields this term becomes negligible.

3. Gaussian one-cycle pulse (OCP)

$$f(x) = \exp{[\frac{1-x^2}{2}]} \tag{28}$$

Corresponding field pulse shape

$$\mathcal{E}(t) = -\mathcal{E}x \cdot \exp{[\frac{1-x^2}{2}]} \tag{29}$$

Then

$$\Phi(x_s, \mathbf{q}) = (1+q^2) \cdot x_s + 2\tilde{\lambda} q_{\|} Erf(x_s/\sqrt{2}) + \tilde{\lambda}^2 \cdot Erf(x_s) \tag{30}$$

Here $\tilde{\lambda} \equiv \sqrt{e} \cdot \lambda$; $Erf(x)$ -is error integral

$$Erf(x) = \int_0^x \exp{(-x^2)} \cdot dx$$

and

$$x_s(\mathbf{q}) = \sqrt{\ln \frac{\tilde{\lambda}^2}{1+q^2} \mp 2i \cdot \arccos \frac{-q_{\|}}{1+q^2}} \tag{31}$$

Only saddlepoints in the upper halfplane of x are relevant. So signs of roots must be chosen with positive imaginary parts. Therefore signs of real part are different for two saddlepoins. This is just an example of two equivalent saddlepoints interference.

$$x_s'' \equiv \Im x_s = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(\ln \frac{\tilde{\lambda}^2}{1+q^2})^2 + 4(\arccos \frac{-q_{\|}}{\sqrt{1+q^2}})^2} - \ln \frac{\tilde{\lambda}^2}{1+q^2}} \tag{32}$$

and

$$x_s' \equiv \Re x_s = \mp \frac{1}{\sqrt{2}} \sqrt{\sqrt{(\ln \frac{\tilde{\lambda}^2}{1+q^2})^2 + 4(\arccos \frac{-q_{\|}}{\sqrt{1+q^2}})^2} + \ln \frac{\tilde{\lambda}^2}{1+q^2}} \tag{33}$$

The equation for $\mathbf{q}_m$ in this case looks like

$$\mathbf{q}_m \cdot \int_0^{x''_{sm}} \left[1 - e^{-x''_{sm}u + u^2/2} \cdot \cos{(x'_{sm}u)} \right] du$$

$$= - \int_0^{x''_{sm}} e^{-x''_{sm}u + u^2/2} \cdot \sin{(x'_{sm}u)} du \qquad (34)$$

As x_{sm} itself is the function of $\mathbf{q}_m$, this equation together with (31) are the system of two coupled equations defining both x_{sm} and $\mathbf{q}_m$.

Imaginary parts of Erf functions in (30) also can be represented by integrals similar to those in (34). Taking into account (31)

$$\tilde{\lambda}^2 \Im Erf(x_s) = - \int_0^{x''_s} e^{-2x''_{sm}u + u^2} \cdot \left[(1 - q^2_{\parallel m}) \cdot \cos{(2x'_{sm}u)} + 2q_{\parallel m} \cdot \sin{(2|x|'_{sm}u)} \right] du \qquad (35)$$

Unlike general form of (31) - (33), equations (34) and (35) are written for $\mathbf{q} = \mathbf{q}_m$.

All this formulae become essentialy simplified and more transparent in the "multiphoton" ($\lambda \ll 1$) and "tunneling" ($\lambda \gg 1$) parameter ranges. For moderate intensities ($\lambda \ll 1$) momentum-resolved ionization probability

$$w_{i\mathbf{q}} \approx 4\pi\Omega \frac{\lambda^2}{\lambda^2 + \lambda_c^2} x''_{sm} |M(0)|^2 \cdot$$

$$\left[1 + \cos{\frac{\pi - 4q_\parallel}{x''_{sm}\Omega}} \right] \cdot \exp\left[- \frac{2}{\Omega} \cdot \left((1 + q^2)x''_{sm} - 1/(2x''_{sm}) \right) \right] \qquad (36)$$

with x''_{sm} given by

$$x''_{sm} = \sqrt{\ln \frac{1}{\tilde{\lambda}^2 + \lambda_c^2}} \gg 1 \qquad (37)$$

and

$$q_{\parallel m} \approx - \frac{\pi}{2x''^2_{sm}(x''^2_{sm} - 1)} \ll 1 \qquad (38)$$

Strictly speaking formula (36) is correct for low fields ($\lambda \ll \lambda_c \equiv \exp{(-1/(2\Omega^2)}$, linear absorption) and moderate fields ($\lambda_c \ll \lambda \ll 1$). In the intermediate range ($\lambda \sim \lambda_c$) it seems to be a reasonable interpolation. Oscillations of transition probability to any particular momentum due to interference of two saddlepoints contributions are very strong - up to complete cancellation. However in the total (momentum integrated) ionization probability they are gradually damped with the field increasing because of the momentum dependence of their phases.

$$W_i = 8\frac{\lambda^2 x''^2_{sm}}{\lambda^2+\lambda_c^2}\Big(\frac{\pi\Omega}{2x''_{sm}}\Big)^{5/2}|M(0)|^2\cdot$$

$$\cdot\Big[1+e^{-2/(\Omega x''^3_{sm})}\cdot\cos\frac{\pi}{x''_{sm}\Omega}\Big]\cdot\exp\left[-\frac{2}{\Omega}\cdot\left(x''_{sm}-1/(2x''_{sm})\right)\right] \qquad (39)$$

As to the strong field tunneling regime ($\lambda \gg 1$), formulae (48) - (50) are universal for any pulse shape the only difference being in the particular value of the parameter a - curvature at the pulse top. For gaussian pulse $a = 2$.

4. Lorenzian half-cycle pulse (HCP)

$$f(x) = \arctan x \qquad (40)$$

Corresponding field pulse shape

$$\mathcal{E}(t) = \frac{\mathcal{E}}{1+x^2} \qquad (41)$$

Then the saddle point is

$$x_s = i\tanh\left[(\sqrt{1+q_\perp^2}+iq_\parallel)/\lambda\right] \qquad (42)$$

and momentum distribution of ionization probability

$$w_i(\Omega,\lambda,\mathbf{q}) = |\pi M_0|^2\cdot\exp\left[-\frac{2}{\Omega}\left(|x_{sm}|-\lambda^2\cdot\varphi(|x_{sm}|)\right)\right]\cdot$$

$$\cdot\exp\left[-\frac{2}{\Omega}\left(q_\perp^2\cdot|x_{sm}|+(q_\parallel-\pi\lambda/2)^2\cdot\left(|x_{sm}|+\frac{2}{\lambda}(1-|x_{sm}|^2)\right)\right)\right] \qquad (43)$$

with

$$|x_{sm}| = \tanh\frac{1}{\lambda} \qquad (44)$$

and

$$\varphi(x) = \frac{1}{4}\int_0^x \ln^2\frac{1+x}{1-x}\cdot dx \qquad (45)$$

The numerical solution of (9) and (11) is strightforward for any reasonable pulseshape. However the general qualitative analisis is also possible and may be illuminating. There exist three essentially different areas in the plane of parameters (Ω, λ):

1. Weak fields and linear absorbtion for $\lambda \ll \lambda_c(\Omega)$ with λ_c - effective nonlinearity threshold, essentially dependent on the pulseshape and to be specified below for some typical pulseshapes. The general definition is

$$\lambda_c \cdot |f(x_{s0})| = 1 \tag{46}$$

with x_{s0} being the root of equation (9) corresponding to $\lambda = 0$. Terms proportional to λ and λ^2 in the righthandside of (9) can be neglected. The exponential factor in (10) reduces to exponentially small amplitude of high frequency harmonics, corresponding to the abovethreshold quantum energy $\hbar\omega > I$, always present in the Fourier spectrum of broadband signal.

2. Nonlinear regime: $\lambda > \lambda_c$. The last term in (9) can be ommited. Then

$$x_s = f^{-1}\Big(\frac{-q_{\|} \pm i\sqrt{1+q_{\perp}^2}}{\lambda}\Big) \tag{47}$$

Here $f^{-1}(y)$ - function, inverse to $f(x)$. The sign of the imaginary part of its argument must be fixed as to correspond to $x'' > 0$

2a. High fields - $\Omega^{-1} \gg \lambda \gg 1$.

Without any loss of generality one can always choose the point $x = 0$ to be the absolute maximum of $f'(x)$, i.e. field strengh, and $f'(0) = 1$. This last condition just fixes the exact value of λ. If there are few equivalent maxima each of them can be treated separately. In the range of interest around this point $f(x)$ can be approximated by cubic parabola

$$f(x) \approx f_0 + x - \frac{1}{6}ax^3 \tag{48}$$

$0 < a \sim 1$. Then after simple calculations

$$\Im\Phi(x_s, \mathbf{q}) = \frac{2}{3\lambda} \cdot \Big[1 + 4a\frac{(\mathbf{q} - \mathbf{q}_m)^2}{\lambda^2}\Big] \tag{49}$$

and

$$q_{\|m} \approx \lambda \cdot [f(\infty) - f_0] \tag{50}$$

This corresponds [9] to quasistatic tunneling during a short $\delta x \sim \sqrt{|\mathcal{E}|}$ time interval around field maximum. Momentum distribution of photoelectrons is gaussian with halfwidth $\Delta q_{\|} = \frac{\lambda}{4}\sqrt{3|\mathcal{E}|/a}$.

2b. $1 \gg \lambda \gg \lambda_c(\Omega)$ **- moderate fields** . Compared to the weak and strong field cases in this one the λ -dependence of $\Im\Phi(x_s, \mathbf{q})$ is more diverse depending on details of pulseshape, particularely singularities of the function $f(x)$ in the upper halfplane of the complex variable x. The gaussian shape $f(x) \sim \exp(-x^2/2)$ is the particular case with the only singularity of $f(x)$ being the essential one at the ∞. However most typically pulselike function $f(x)$ has singularities (poles, branching points) in the complex plane of variable x at some x_{pol} with the imaginary part $x''_{pol} \sim 1$. Then for weak fields just the $\exp(-2x''_{pol}/\Omega)$ defines the amplitude of high frequency Fourier component responsible for single quantum ionization. With the field increasing the saddle point x_s moves from x_{pol} to the real axis. Let the closest to the real axis singularity be the k-th order pole, i.e.

$$f(x) \approx \frac{A}{(x - x_{pol})^k}$$

for $|x - x_{pol}| \ll 1$. Then as will be shown below, the ionization amplitude in the whole domain $\lambda \ll 1$, including both weak and moderate field ranges, beside the weak field factor $\exp(-2x''_{pol}/\Omega)$ is dependent only on a single parameter

$$z = \frac{(\lambda A)^{1/k}}{\Omega} \tag{51}$$

and the moderate field range starts at $|z| \sim 1$ i.e. $\lambda_c \sim \Omega^k$. Note that the first two of above described examples are dominated by such type singularities: the first one corresponding to $k = 1$ and the second - to $k = 2$. The saddle points (and coincedent with them possible poles of matrix element $M\Big(\mathbf{q} + \mathbf{n}\lambda f(x)\Big)$ in preexponential factor) are

$$x_s(\mathbf{q}) = x_{pol} + \Big[\frac{A\lambda}{1+q^2}(\pm i\sqrt{1+q_\perp^2} - q_\|)\Big]^{1/k} \tag{52}$$

with both signs in the argument relevant, as all of these $2k$ points are in the close vicinity of x_{pol} which itself is in the upper halfplane. However in the moderate field strength range $|z| > 1$ only one of them is dominating - that with minimal value of x''_s. Or one pair of such points, if, depending on pole order k and $\chi \equiv \arg A$, there exist in the whole set (52) such a pair of mirror symmetric relative to the imaginary axis elements, with minimal value of imaginary part. The second example above with $k = 2$ and $\chi = 0$ corresponds just to such a case $(x_s - x_{pol})_{\mathbf{q=0}} = \sqrt{\lambda/2} \cdot (\pm 1 - i)$. In a general case

$$\Im\Phi(x_s, \mathbf{q}) = x''_{pol} - \frac{2k}{2k-1}\gamma \cdot (\lambda|A|)^{1/k} + q_\perp^2/(\Delta q_\perp)^2 + (q_\parallel - q_{\parallel m})^2/(\Delta q_\parallel)^2 \quad (53)$$

with

$$\Delta q_\parallel^2 \approx \Delta q_\perp^2 \approx x''_{pol} + o(\lambda^{1/k})$$

$$q_{\parallel m} = \frac{\sqrt{1-\gamma^2}}{(2k-1)x''_{pol}} \cdot (\lambda|A|)^{1/k}$$

and

$$\gamma = max_s \frac{(x_{pol} - x_s)''}{|x_s - x_{pol}|} \quad (54)$$

with index $s = 1, 2...2k$ marking different elements of the set (52). Thus in the moderate field strength range ionization probability increases as $\exp\left[4k\gamma|z|/(2k-1)\right]$ and the average momentum as $\lambda^{1/k}$. If there is only one dominating saddle point

$$W_i(\Omega, \lambda) = \sqrt{\pi}\Big(\frac{\Omega}{2x''_{pol}}\Big)^{3/2} \cdot |M_0|^2 \cdot \exp\Big[-\frac{2}{\Omega} \cdot x''_{pol} + \frac{4k}{2k-1}\gamma|z|\Big] \quad (55)$$

and

$$W_i(\Omega, \lambda) = \sqrt{\frac{\pi}{2}}\Big(\frac{\Omega}{x''_{pol}}\Big)^{3/2} \cdot |M_0|^2 \cdot \exp\Big[-\frac{2}{\Omega} \cdot x''_{pol} + \frac{4k}{2k-1}\gamma|z|\Big] \cdot$$

$$\cdot \Bigg[1 - \exp\Big[-\frac{2}{x''_{pol}\Omega} \cdot \Big(\frac{\gamma|z|}{2k-1}\Big)^2\Big] \cdot \cos\Big(\frac{4k}{2k-1}\sqrt{1-\gamma^2} \cdot |z|\Big)\Bigg] \quad (56)$$

if a pair of symmetric saddle points contribute. It should be noted that in all arguments of exponential and trigonometric functions only leading terms

in $\lambda \ll 1$ are shown in these formulae.
Derived by stationary phase assimptotic evaluation of integral in (4) formulae (55) and (56) are valid for moderate fields range $\Omega^k \ll \lambda \ll 1$. Their inapplicability for weak fields is clearly seen from the fact that they do not follow usual $\sim \mathcal{E}^2$ dependence as $\lambda \to 0$. The reason for that was already mentioned above in discussing the second example: at $z < 1$ contributions of all $2k$ saddle points, surrounding x_{pol}, become of the same order. It is easy to account for all of them, which would restore the correct $\sim \mathcal{E}^2$ behaviour in weak fields. Still it is not the whole story. The numerical coefficient appears to be wrong. The reason is that beside all these poles and saddle points the point x_{pol} itself is the essentiall singularity of integrand in (4) of the type $\exp\left[i(\lambda A)^2(x - x_{pol})^{-2k+1}/(2k-1)\right]$. Evaluation of integral in (4) accounting for the whole this structure in the complex plane, valid in the whole domain $|z| \ll 1$ i. e. weak and moderate field ranges, is possible in terms of fast converging power series in z. The result again is slightly different depending on the presense or absense of the pole in matrix element. If matrix element is regular (no pole) and slowly varying $M(\mathbf{q}) \approx M_0$

$$w_i(\Omega, \lambda, \mathbf{q}) = \lambda^2 |M_0|^2 \cdot \exp\left[-\frac{2x''_{pol}}{\Omega}(1+q^2)\right] \cdot \left| S_k^{reg}(z^k/\sqrt{2k-1}) \right|^2 \qquad (57)$$

with

$$S_k^{reg}(y) = 2\pi\sqrt{2k-1} \cdot y \sum_{n=0}^{\infty} \frac{(-1)^{(k+1)n} y^{2n}}{n! \cdot [(2k-1)+k]!} \qquad (58)$$

and for the case of singular matrix element

$$w_i(\Omega, \lambda, \mathbf{q}) = \lambda^2 |M_0|^2 \cdot \exp\left[-\frac{2x''_{pol}}{\Omega}(1+q^2)\right] \cdot |S_k^{sing}(z^k)|^2 \qquad (59)$$

with

$$S_k^{sing}(y) = 2\pi k \cdot y \sum_{n=0}^{\infty} (-1)^{(k+1)n} \cdot a_n \cdot y^{2n} \qquad (60)$$

and coefficients a_n defined as

$$a_n = \sum_{m=0}^{n} \frac{(2k-1)^{-m}}{m! \cdot [(2n+1)k - m)]!} \qquad (61)$$

Assimptotics of functions S_k^{reg} and S_k^{sing} at $|z| \gg 1$ coincede exactly with results of stationary phase calculations in the moderate field regime

$$S_k^{reg}(z^k/\sqrt{2k-1}) \approx \sqrt{\pi k/|z|} \cdot \exp\left(\frac{2k}{2k-1}\gamma|z|\right) \qquad (62)$$

$$S_k^{sing}(z^k) \approx \pi \exp\Big(\frac{2k}{2k-1}\gamma|z|\Big) \tag{63}$$

and their first terms substituted in formula (51) and (53) give exact result for weak field regime. Thus for pulseshape with pole type of singularity formulae (9)-(12) and (57)-(61) together describe completely ionization probability for any field strength, restricted only from above by atomic field, i.e. $\mathcal{E} \ll 1$. However weak field regime seems to be of more academic interest: for such short pulses effect is hardly experimentally observable.

The last of above examples corresponds to another type of pulseshape function singularity - logarithmic branching pount ("zeroth order pole").

For long pulses and approximately monochromatic fields the frequency dependence of true multiphoton process probability is very steep. As the whole consideration above shows, for very short pulses - HCP, OCP and, probably, few ($< 1/\Omega$) cycles long pulses - it is much slower, though still pretty steep. Qualitatively this slowing down can be explained as an increase, with the field increasing, of an average effective number n of photons absorbed per single ionization event. Because of a broad frequency spectrum of the pulse the process is single-photon one at a weak field and its multiplicity increases gradually to $n \sim \lambda^3$ [9] in tunneling regime $\lambda \gg 1$, while in monochromatic field it is restricted from below - $n > I/(\hbar\omega)$.

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