String theory: what is it?

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This is an attempt to describe the subject and the methodology of string theory as we understand them today, i.e., the entire set of problems which attract attention of theorists working in the field. The string model of Grand Unification of fundamental interactions is briefly discussed along with a broader string scenario of the unified field theory, a more mathematical concept, designed to facilitate understanding of the generic features of equivalence classes in different models of quantum field theory. A concise glossary of the most important notions unusual in physical literature but frequently used in papers on string theory is also included.

Having safely passed through the times of great enthusiasm, unrealized hopes, and unavoidable discouragement, string theory has entered at last the period of normal development and continues to attract attention of theorists. With the string program of unification of all fundamental interactions remaining at the top of the agenda, it is more and more widely understood to be rather an application than the content of the string theory. In fact, many, if not most, of the problems in this field have but an indirect relation to the physics of elementary particles. The progress of string theory is becoming more dependent on its own inner logic rather than on the need of any particular application. It is this inner logic rather than the difficulties of alternative approaches that directs further progress of the string scenario of unification, just as it should be with any rational theory. Moreover, following this logic, more and more various domains of physics and mathematics are getting involved in the framework of string theory, and this process creates a new basic construction in the edifice of modern natural science thus shedding new light on our ideas of the structure and interrelation of different sciences.

If one tries to characterize briefly the subject of string theory as it is currently interpreted, one will have to accept that it is no longer a specific theory or model but rather a *large set of methods and concepts* devised to provide a wide generalization of the standard formalism of quantum field theory and opening many new possibilities and applications for it. In this sense, string theory appears to be a branch of mathematical physics having its own value irrespective of any success of particular attempts to use it for constructing a model of some physical phenomenon. It is worthwhile to note that the most fruitful applications of mathematical formalism often occur where nobody expected them to arise when it was gaining momentum. Moreover, even the ideas which seem to have served as a source of this formalism can now appear irrelevant (suffice it to recall the concept of ether and its role in the discovery of Maxwell equations). For these reasons, we believe that the importance of particular physical ideas and scenarios that contributed to the development of string theory should be considered with caution to avoid overestimation: at best, they will still undergo numerous modifications; at worst, they will be replaced by new ones, absolutely unpredictable today. The actual significance of such scenarios is in pointing out a reasonable direction for further studies and providing criteria for evaluating the relative importance of various ideas. The existence of such criteria determines the value of theoretical physics from the viewpoint of pure mathematics. String theory is sufficiently productive in serving as a source of new mathematical problems and concomitantly indicating possible methods of solving them.

The purpose of these notes is to give a brief description of the content of modern string theory by way of illustrating the diversity of the concepts and methods involved. Each section of the present article deserves to be discussed at length in an extensive review (there are some already available in the literature) and appears to constitute a self-contained branch of knowledge.[1)] On the one hand, the scope of the present review is inevitably restricted by the fact that it deals with a field of knowledge which is still coming into existence and in which the importance and the relevance of different concepts and approaches are frequently revised, sometimes in the most radical way. This makes the choice of the material and its appraisal dependent not only on the author's taste but also on the time when the text is compiled. On the other hand, an attempt to embrace the entire scope of the theory requires all the details to be omitted; what follows is, in the end, somewhat reminiscent of the synopsis of a large book that can hardly be written by one author because it would require deep knowledge in too many disciplines. In order to make up more or less for the lack of such a book, we present references to the available publications (books and reviews as well as original articles) that focus on different aspects of string theory. Also included is a concise glossary of specific notions which are frequently used in papers on string theory. The present notes are intended to help those who are about to be involved in string studies to find their way in the ocean of papers and speculations that in some way pertain to this popular topic.

1. WHY IS STRING THEORY NEEDED?

String theory in the broad sense of the term has come into being since different fields of theoretical physics have been incessantly pushing into the foreground new problems

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that inevitably call for solution. These problems are arbitrarily divided into four classes.

A. Strong coupling theory and generic theory of nonlinear phenomena

"Synergetics" is a term recently coined to denote nonlinear and related phenomena.[1] In terms of stated objectives, synergetics is very close to the string theory, the difference being largely reduced to the use of more or less specific analytical apparatus by the latter which naturally precludes its universal application (although, nobody seems to know how serious this limitation may happen to be in the end). More specifically, nonlinear phenomena examined by string theory are conspicuous by a certain degree of symmetry, even if sometimes obscure. For example, an important breakthrough in string theory has been the interpretation of a number of complex nonlinear equations including Einstein and Yang–Mills equations as the symmetry principle of a certain quantum field theory (conformal symmetry of the two-dimensional sigma-model).[2]

In general, application of the string concept to nonlinear problems is based on the radical reformulation in terms of quantum field theory. Sometimes, such reformulation may even allow nonlinear equations to be *solved* (e.g., the "integrable" system theory[3–7]). More frequently, this approach results in a discovery of common traits inherent in seemingly different problems and the establishment of new criteria of "similarity" and "equivalence." Generally speaking, it may be expected that a major "output" of string theory in the future will be a theory of universality classes incorporating as its constituent components the "theory of catastrophes"[8–10] and the theory of phase transitions.[11] The latter theory or rather the problem of phase transition classification in two and three-dimensional systems may be regarded as one of the immediate sources of two important sections of string theory: the science of two-dimensional conformal models and the calculation of random surfaces.

Speaking of strings in the context of the strong coupling theory, it is worthwhile to mention the "naive" relationship between them. That is, true excitations ("quasiparticles") may assume the form of stretched one-dimensional filaments in the strong coupling regime. Moreover, such a situation is very common in our world largely due to the three-dimensional nature of space. The interaction between quantum chromodynamics[12] and the theory of strings appears to provide the most famous example of this kind. It seems appropriate to note that it was exactly this problem that gave rise to string theory.[13–17] These "naive" problems will be discussed in more detail below.

B. Theory of multiphase systems and interphase fluctuations

The scope of this problem is closely related to that of the previous one. In fact, the problem of strong coupling, at least if we are able to solve it or have an idea of how to attempt a solution, can be reduced to a choice of a viewpoint from which the examined system looks like a weakly interacting one. In physical terms, it means that a strongly interacting system can exhibit collective states (excitations, quasiparticles) which almost do not interact with one another. To put it in formal language, there may be a change of variables which turn the system of equations into a linear one ("integrable" case) or a weakly nonlinear one. In the latter case, the altered parameters may convert the system back to a highly nonlinear one. In other words, the interaction between quasiparticles may be strong again in a certain domain of parameter space. This calls for a search for other collective variables or other quasiparticles adequate to describe this region in parameter space. Such substitution of one set of quasiparticles by another makes up the substance of the science of phase states and interphase transitions. However, the classical field of theoretical physics devoted to this subject (statistical physics) confines itself to simple situations, that is it largely examines systems with a limited number of phase states and readily distinguishable interphase transitions. Recently, however, more complicated problems have acquired greater importance. In the first place, very interesting physical systems have been discovered having an infinitely large number of phases; moreover fluctuations between different phases are also large. It is assumed that reasonable universality classes for such systems are to be defined based on some new principles. The best known systems of this type are spin glasses and neural networks.[18–20,2)]

Another class of problems can be traced back to formal statistical physics that studies the correlation between micro and macrodescription. The notion of phases is certainly an essentially statistic, rough notion. One phase always undergoes either quantum or temperature fluctuations related to the formation of virtual rudiments of other phases. For this reason, any *accurately* measured parameter in one of the phases inevitably bears information about all other phases which can not be ignored unless it is for a certain degree of approximation. The importance of such information is a subject of endless discussions provoked from time to time on different occasions (the problem of scattering from a coherent state, the problem of "an exact renormgroup," and many other issues also lie within the scope of the theory of multiphase systems). Another situation (unification of interactions) subject to examination in the framework of this theory will be discussed in section C below.

An approach to a study of multiphase systems as implied by the string theory is based on the aforementioned reformulation of various nonlinear equations (e.g., state equations in different phases) in new terms smoothing such essential differences between phases and equations as the number of variables, the order and the number of equations, and even the dimension of the space in which they are defined. Such reformulation allows smooth interpolation between absolutely different types of equations, i.e. exactly what is necessary for the description of continuous transition of one phase to another. It should be indicated, however, that the use of practical implications of string theory is a matter of distant future and its applied value remains to be elucidated.

C. Unification of fundamental interactions

This problem deserves a special discussion, in the first place due to its crucial importance for natural science, and also because the development of a comprehensive theory of all fundamental interactions ("the theory of everything") appears to be the most ambitious project related to string studies.[21,22] Indeed, there are two projects, not one, which are complementary rather than mutually exclusive, each having its own value. Due to this, if one of the two scenarios fails in the end, the other is not necessarily doomed to the same fate.

The first scenario may be considered as a naive and straightforward application of string theory to unification of interactions. It treats strings as fundamental entities and considers one-dimensional extended objects rather than point particles to be the elementary units. From the point of view of the conventional theory of elementary particles, this amounts to the hypothesis of an infinitely large variety of particles with a regular mass spectrum, spins, and interaction patterns. Surprisingly, such a hypothesis is not in conflict with the available experimental data (the majority of additionally introduced particles have very high masses and are virtually unobservable). Nor does it deteriorate "the quality" of the theory as a quantum field theory despite the introduction of a new infinity (an infinite number of particles). Moreover, such an approach allows the original theory to be improved because the new infinity helps overcome long-standing ultraviolet divergences! Apart from the possibility to develop renormalized quantum gravity theory, this scenario makes it possible to reject the principle of renormalization which has been a real nuisance for many generations of scientists. In other words, there is a chance to suggest a *finite* fundamental theory. Also, it is gratifying to think that the above approach provides an opportunity for a further natural development of the Kaluza-Klein ideas[23-26] which allow the entire structure of the unification model (gauge symmetries, field content, coupling constants) to be encoded in geometric and even topologic properties of a certain manifold (the idea known as "compactification" formalism). A major drawback of the approach in question inherited from "pre-string" concepts of unification of interactions is the lack of selectivity: string unification models appear to be as numerous as conventional ones and impose practically no limitation on the selection of the gauge group, (string) field content, etc. On the other hand, the faith in the existence of a uniquely valid and truly fundamental "theory of everything" free from arbitrary interpretation makes up a major impetus to a search for Grand Unification[27-32] although this inference may be questioned by advocates of the anthropic principle.[33]

Most of the "model-builders" showing interest in the unification theory believe that many models are inherently discrepant, i.e. they may have anomalies,[34,35] be non-renormalizable or suffer from the "zero-charge" problem,[36] etc. Moreover, there was a time when it seemed absolutely impossible to devise a non-trivial consistent theory comprising both the standard model and quantum gravity. Anyway, one could hardly expect to have a broad selection of such theories. That was how matters stood on the eve of the 1984 "string boom" following the failure of the synthesis of the Kaluza–Klein approach with $D = 11$ supergravity which at that time was considered the most likely claimant to the role of the "theory of everything." At approximately the same time,[37,38] it was shown that a certain string model ("$E_8 \times E_8$ superstring"[38]) may play the role of a self-consistent unification model.[39] This model was distinguished from the variety of string theories known at that time by the criteria of finiteness and lack of anomalies. The further course of events demonstrated that what was initially regarded as the invincible pathology of alternative theories proved to be pathology of a specific formalism used for their analysis. (Today, theorists hardly need more evidence to confirm the feasibility of self-consistent theories with anomalies; *finite* models appear to be equally common in string theory). Moreover, the number of alternative string models including anomaly-free and finite ones has grown enormously. As a result, a great variety of models are now available instead of a unique self-consistent model—the naive string scenario has not become the last link in the vicious chain. Now, its advocates seem to have the only hope to cherish: that the analysis of non-perturbative behavior of strings may reveal, when completed, new unpredicted pathologies in the majority of the remaining models. However, this is but a delusive hope and, still worse, it is in conflict with that little which is already known about non-perturbative string theory. We shall see later that available evidence appears to support another, less-naive scenario.

The latter scenario is based on the ideas suggested at least simultaneously with the "naive" scenario. An obvious alternative to the identification of the "theory of everything" with the "unique self-consistent" model of the quantum field theory or string theory is its identification with a certain *unification* of all such models. In other words, different models can be interpreted as corresponding to different *phases* of a single theory. (Perhaps, it is worthwhile to mention the concept of "hadron democracy" or bootstrap as an analog of such an approach even if in a somewhat different context. Note that the formalism of "conformal boot-strap" suggested for more specific purposes may even prove to be a very effective calculation tool). It seems appropriate to observe that realization of such an idea of *a priori* equality of all conceivable models of field theory (or string theory for that matter) would impart literal meaning to the terms "theory of everything" and especially "unified field theory."

Practical implementation of such a scenario would require a uniform description of quite different models and their embedding into a certain unified "configuration" (or "phase") space of the "unified field theory." The next step is specification of dynamics in this space. Finally, such dynamics should distinguish individual points (phases) in this configuration space under certain conditions (e.g., at "low energies"). In a word, this scenario implies that the "theory of everything" has a complicated phase structure while known properties of the Universe are interpreted as a result of dynamic selection of one of the many *a priori* conceivable models of the quantum field theory. The string theory suggests at least a theoretical possibility to realize such a scenario[40,41] even though there is a large distance between such a possibility and practical implementation, not to mention the high probability of modification of the scenario in the course of the work. Remarkably, there is already indirect evidence of the adequacy of such ideas for the string theory formalism. A series of string models differing at the perturbative level has been shown to be naturally unified in the non-perturbative domain. (This is exactly what is stipulated by our scenario: given that different models are naturally unified into a whole, quantum fluctuations, possibly non-perturbative, must intermix them).

It would be a gross exaggeration to state that the second scenario (unlike the first one) draws much attention of theorists. Nevertheless, it does exist and, besides, it agrees better with the intrinsic nature of the string theory. It seems safe to prognosticate that the popularity of this scenario or one of its modifications will further increase among those interested in string theory.

D. Quantization of algebro-geometric structures

The importance of this problem in physics can be accounted for by the puzzle of quantum gravity. It still remains in the spotlight although there is no serious reason for combining the general theory of relativity with quantum mechanics. In most of the attempts to address this issue, the paradigm of quantum mechanics is given preference over the idea of space-time geometry.[3)]

This fully refers to the string scenario which treats space-time as no more than an effective object apparent under certain conditions. Such a property appears to be intrinsic in any theory designed for the desription of all conceivable features of quantum gravity including topology-changing fluctuations. It is remarkable that the relationship between geometry and dynamics becomes even closer in string theory, and that the Yang–Mills fields appear naturally and exhibit a deeply rooted connection with gravity.

The relationship between strings and gravity determined by the specific role of *two-dimensional* gravity in string theory is of a quite different aspect. Two-dimensional gravity is conspicuous for its totally quantum nature, and its investigation (indispensable in developing string theory) may throw light on the quantum gravity structure at large.

Finally, the mathematical slant of the problem should be mentioned. Many string models are remarkable for their algebro-geometric characteristics. In fact, almost any algebro-geometric object may be associated with a specific string model.[4)]

On the other hand, string theory being a quantum theory, it is no wonder that in many cases evaluation of string fluctuations and/or interactions reveals deformation of the initial algebro-geometric structure. This apparently leads to the natural description (or definition) of quantum groups, quantum spaces, and other objects of interest for modern mathematics. In this context, the specific position of string theory among other physical theories allows, on the one hand, various algebro-geometric objects to be easily associated with dynamics (a form of two-dimensional action) of string models and, on the other hand, string interactions to be accurately evaluated, most of the problems in string theory lending themselves to exhaustive solution within a finite period.

To summarize, we have tried to demonstrate who could possibly be interested in string theory, to identify reasons for such interest, and to define the sort of results to be expected. The next step is to provide even a more brief survey of the constituent components of the theory and indicate where the answers to selected questions should be sought. The following discussion is designed to illustrate the historic development of the theory in order to make obvious its inner logic. To a certain extent, understanding of this logic helps to divine further trends whereas past experience shows that nontrivial problems offered by the string program normally find solution sufficiently comprehensive to give rise to new ones which, being also solved, will raise... . We do not know how long and successful this chain of developments may happen to be nor can we predict if it breaks before the entire construction is close to completion. In the meanwhile, string theory has been fortunate...

2. STRINGS AS QUASIPARTICLES

To begin with, we enumerate situations in which strings make "spontaneous" appearance regardless of human wish and will. The feasibility of such a situation in itself requires string theory to be constructed and investigated. It is therefore natural that their description should precede both more speculative scenarious considering strings as fundamental objects and a description of mathematical formalism.

In the most naive sense of the word, a string is an extended one-dimensional object with tension which means that its energy $\mathscr{E}$ increases with increasing length L: $\partial\mathscr{E}/\partial L>0$. The string of musical instruments (a non-relativistic string) that gave the name to the entire theory is characterized by the "dispersion" law $\mathscr{E}=\text{const}+kL^2$ which is converted to linear $\Delta\mathscr{E}\sim\Delta L$ for small oscillations. Rewritten in terms of the amplitude A for small *transversal* oscillations, it again assumes the form of a quadratic expression $\Delta\mathscr{E}\sim A^2$. The theory of musical strings can hardly be expected to bring much surprise, but it had to be mentioned to make the picture complete. Polymers including protein molecules may be cited as another important example of non-relativistic strings.

More interestingly, strings can function as stable quasiparticles and have to be considered in studies of non-trivial phase states, e.g., in symmetry breakdown. Occurrence of strings in such situations is not at all infrequent, rather it is a rule, the most illustrious cases including vortices (tornados) in laminar flows, dislocation lines in a crystal lattice, Abrikosov lines in superconductors, Dirac lines attached to monopoles in gauge theories, "cosmic strings" in various models with a specific Higgs sector, etc. The high incidence of string-like entities in theories dealing with structural features of the world may be attributed to the *three*-dimensional nature of space. In order to answer the question about the structure of elementary topologically stable quasiparticles, it is necessary to know what should be eliminated from $\mathbb{R}^3$ to make it multiply connected. The obvious answer is *one*-dimensional lines. This means that stable *string*-like quasiparticles are sure to occur at least in situations with a characteristic ("order parameter") taking on values within a circle.[5)] Moreover, in such situations, the energy of quasiparticles is directly proportional to their length: $\mathscr{E}\sim L$, as follows from the equality of all fragments of the line, i.e., constant energy density. This dispersion law is characteristic of "relativistic" strings, and we see that relativistic strings originate naturally in non-relativistic systems. The *dynamics* of strings, as that of any other mechanical object, is determined not so much by the energy but by the action, i.e. an integral characteristic of the *two*-dimensional world surface swept over by a moving *one-dimensional* string. Integration of energy over time may provide an insight into what the action looks like. Having this in mind, it is not difficult to understand that the dispersion law $\mathscr{E}\sim L$ implies that the action is proportional to the world surface area $\mathscr{A}\sim S$. This formula certainly reflects equality of two dimensions, time and space, on the world sheet. Hence the term "relativistic string."

As a matter of fact, string theory begins with a study of quantum relativistic strings. Now that we have come to know the origin of "relativism," it is high time to address the problem of quantization. Certainly, the easiest way is to state that we are interested in the quantum mechanics of quasiparticles discussed in the previous paragraph. This would bring us to the problem of integration over their trajectories, i.e. (random) world surfaces taken, with weights $e^{i\mathscr{A}/\hbar} = e^{i\beta S}$. True, in the above examples this problem is of purely academic interest because quantum effects in such situations are insignificant. Other (temperature) fluctuations acquire far greater importance although there is practically no formal difference between quantum and temperature fluctuations: it is sufficient to substitute a real exponent for the imaginary one in the functional integral.

Now, take a step back to note that the solution of the "equation" "$\mathbb{R}^3 - ? =$ multiply connected" is as important as that of the equation "$\mathbb{R}^3 - ? =$ non-connected." This problem may be considered in terms of phase separation. Of course, the solution is "? = two-dimensional surface" because in the three-dimensional world the phases are separated by surfaces. In a *two*-dimensional world, $\mathbb{R}^2$-phases would be separated by lines. A study of these lines is useful because, for instance, in systems with phase transitions of the second kind the energy densities of the different phases coincide and the total free energy of the system turns out to be associated with phase-separating lines. Moreover, in a locally interacting system, energy is concentrated on the separation line and is in fact proportional to its length which brings us back to the dispersion law $\xi \sim L$ familiar from the above discussion. Further calculation of the partition function of the theory consists in summation over arbitrary (random) distribution of phase separation lines with weights $e^{-\beta L}$ (line intersection may be allowed or forbidden depending on the model under examination). A classical example of such problem is the Ising model, one of the most popular ones among those examined by string theory on different occasions. Coming back to the *three*-dimensional world $\mathbb{R}^3$, one obtains separation *surfaces*, free energies proportional to the area, and sums over random surfaces with weights $e^{-\beta S}$.

To summarize, the problem of calculating $\Sigma e^{-\beta L}$ may need solution in studies on (i) thermodynamics of $(2+1)$-dimensional multiphase systems and (ii) quantum mechanics (thermodynamics) of (relativistic) point particles. Due to (ii), these problems are included for examination in the theory of particles to be solved by standard methods of the conventional quantum field theory (QFT). Similarly, the problem of $\Sigma e^{-\beta S}$ computation pertains to studies of (i) thermodynamics of $(3+1)$-dimensional multiphase systems and (ii) quantum mechanics (thermodynamics) of relativistic strings. Already this aspect of string theory) appears to be a generalization of the theory of particles and the conventional quantum field theory (QFT).

This hierarchy is open for further elaboration. It seems appropriate to design a theory of membranes, $(2+1)$-dimensional objects, and a general theory of p-branes, $(p+1)$-dimensional systems. Within this scheme, the usual way of cognition of the world by means of analysis of $(3+1)$-dimensional field theories falls under (ii) for 3-branes. However, string theory has the advantage of additional options, i.e. some problems of higher hierarchical levels can be solved at lower levels. Studies of the most general $(p+1)$-dimensional QFT models in terms of *strings* rather than p-branes will be discussed later. As a matter of fact, such a possibility is not unexpected since any local QFT may be described, apart from other modes, in terms of point particles. Such a description is referred to as first quantization formalism (or, to use the equivalent term, the Feynman diagram technique). However, its efficiency is low excepting some specific (and very important!) cases when the analysis may be confined to the perturbative regime. It is remarkable how rapidly the efficacy of this formalism grows with the switch-over from particles to strings. Even more remarkably, this formalism can still be applied to a study of conventional local QFT. It is understandable that the transition from strings to p-branes with $p > 1$ would result in a further complication which makes strings a sort of "the golden mean": their formalism is sufficiently rich and sophisticated to describe adequately the important features of the examined models but remains relatively simple to ensure practicability.

Before advancing to new topics, it is worthwhile to address a few more problems involving either strings or other important elements of string theory. To begin with, it should be emphasized that the above discussion of string functions in the capacity of quasiparticles was largely confined to the "generic situation" when their appearance was due to reliable topologic factors.

At the same time, there is room for more complicated (and more interesting, for that matter) motifs, e.g. dynamic ones. The most important example of this kind is the confinement phase of $(3+1)$-dimensional non-abelian gauge theories including QCD. In this phase, the force lines of the gauge field due to mutual attraction form narrow tubes with a practically constant linear energy density. Therefore, in this phase, relativistic strings function as quasiparticles (hadrons in terms of QFT). In more than one aspect, they are reminiscent of Abrikosov lines in superconducters but differ from them in that they result from a purely dynamic process. Unlike the superconducting (Higgs) phase, the confinement phase involves an intermediate stage, i.e. formation of condensates which first cause a spontaneous "breakdown" of gauge symmetry and thereafter induce, for topologic reasons, the appearance of string-like quasiparticles. Due to their dynamic nature, strings in the confinement phase are, first, essentially quantum, and second, their "thickness" that controls the accuracy of the description in terms of string theory is not an independent parameter.

The latter case constitutes one of the most difficult problems in application of string theory to the confinement phase: only sufficiently long (excited) strings or hadrons with high spins and masses are easy to describe. The problem of hadron strings in QCD is not merely an example illustrating the application of string theory to the description of the strong coupling regime (the confinement phase in the Yang–Mills formulation of QCD). This is the problem that gave rise to string theory: an attempt to formulate QCD (a strong interaction theory) in terms of string theory (the so-called dual resonance models[13-17]) preceded by several years its formulation in terms of Yang–Mills fields.

In this section, we have already mentioned statistic systems, their different phases, and interphase transitions with special reference to situations with phase transitions of the second kind. There is one more unexpected aspect of the relationship between phase transitions of the second kind and string theory. The correlation radius tends to infinity during such transitions and at points of phase transitions of the second kind any system acquires an additional symmetry (conformal).

Conformal symmetry is rich (infinite-dimensional) only in two (and one)-dimensional situations, and this class of conformal theories is readily distinguishable among various two-dimensional QFT models. More than that, conformal statistic systems *a priori* defined on lattices rather than in the $\mathbb{R}^2$ continuum are very easy to describe in terms of *local* QFT due to the infinite correlation radius. The problem of phase transitions of the second kind in $(2+1)$-dimensional systems that can otherwise be referred to as the problem of two-dimensional conformal theories is a classical problem of modern theoretical physics that has been an object of numerous studies. On the other hand two-dimensional conformal systems are considered to be a key subject in string theory specifying the form of the string action on the world sheet (in the simplest situation, i.e., for the so-called bosonic strings, the action is merely a surface area).

As regards two-dimensional statistic systems (i.e. physics of thin films), the theory of the quantum Hall effect and the closely related theory of anyons are worth special consideration. At least the former theory warrants description in terms of multiple phases, with interphase transitions (deconfinement transitions) usually being associated with restoration of conformal invariance. It is both interesting and difficult to go beyond the scope of static approximation to be described by the $(2+1)$-dimensional Chern–Simons model, yet another important participant in string theory. This range of problems draws much attention due to striking qualitative effects such as quantization of Hall conductivity[42–44] and anyonic superconductivity.[45]

There are several other fields of theoretical physics awaiting reformulation in terms of string theory. The feasibility of such reformulation is beyond any doubt. In the first place, this concerns the theory of polymers and biological membranes. Theoretically, string theory appears to be specially designed for such application because it traditionally populates lines and surfaces with various auxiliary objects and examines the resulting effects. However, there have been few serious attempts in this direction. Another field is chaos theory.[46–48] Efforts to construct a similar theory based on the science of fractals[49] have been numerous because it is closely related to the quantum gravity doctrine and hence to string theory. A further survey of this correlation will be developing in parallel to the investigation of the relationship between chaos and quantum theory which is in fact one of the main objectives of chaos studies. Multiphase systems such as spin glasses and neural networks[18–20] are also related to the problem in question. Although their direct links with string theory at present remain unclear, the few available items of evidence of parallelism (e.g., the p-adic formalism[18,50,51]) are really impressive.

Moreover the string theory may prove a useful tool for reformulation of various problems of discrete mathematics in terms of continuum (analytic) mathematics and vice versa. The first success was marked by progress in quantum mechanics which admits two equivalent formulations, matrix (discrete) and functional (continuous) and is best reflected in the Wiener-Dirac-Feynman concept of path integral.[52] From this point of view, a major contribution of string theory consists in the suggestion to examine a broader class of path integrals, i.e., random surface (not only line) integrals, in order to substantially extend the range of problems amenable to formulation in these terms. This sole prospect may keep up interest in string theory because the importance (and the difficulty) of developing effective discrete mathematics for further progress of natural science especially in the fields of biology and artificial intelligence can hardly be overestimated.

This is the end of the story about "spontaneous" strings in problems that *a priori* have nothing to do with them. We could see that such problems are not few. Let us now turn to speculative issues and discuss the problem of *fundamental* strings. These strings, unlike those examined earlier in this review, are purely hypothetical, the basis for the hypothesis of their existence being purely speculative. More than that, no experimental evidence can be anticipated in the foreseeable future. Indirect items of evidence, if any, would hardly constitute proof, but even they are unlikely to be obtained. No matter how disappointing this inference may seem, there is every reason to arrive at it. It is at present still beyond the power of human civilization to attain a really interesting (Planck) energy level at which quantum properties of gravity could be fully manifested (conventional approach would require increasing the energy of modern accelerators by 16 orders of magnitude!). It may be consoling to think that we can hardly imagine what would befall mankind if this task were accomplished. It should not be forgotten that the above conditions are reminiscent of those in which the Universe came into existence, and nobody can be expected to be so bold as to predict the consequences of reproducing them.[6)]

It is therefore not only the lack of material means but also common sense that favors reasoning rather than experimentation in this field in an attempt to perceive the fundamentals of the Universe within at least the next few decades. In the meanwhile, the fundamental string hypothesis remains the best product of the human mind for this purpose. At this point, it appears most appropriate to pass to the description of this hypothesis.

3. FUNDAMENTAL STRINGS AS A MODEL OF FUNDAMENTAL INTERACTIONS

Development of string scenarios for unification of fundamental interactions (electromagnetic, weak, strong, and gravitational) is motivated by disadvantages of more traditional methods. It is worthwhile to recall what is known about fundamental interactions to enable the reader to understand some difficulties inherent in the conventional approach. The first three interactions in the above list (excepting gravitational) can be comprehensively described by the so-called standard model[12,27–32] which is in fact a combination of the Glashow–Weinberg–Salam (GWS) model and quantum chromodynamics (QCD).[7)]

The only thing we know for sure about gravity is that *classical* gravity does exist and is adequately described by the general theory of relativity with the Einstein–Hilbert action $M_{\rm Pl}^2\int\sqrt{G}\mathscr{R}\,d^4x$. The limited capacity of our experimental tools does not allow observation of any effect of quantum gravity or a correction, if any, to the action such as $\int\sqrt{G}\mathscr{R}^2 d^4x$. In other words, there is perfect harmony if viewed from the standpoint of conventional physical methodology: the theory available agrees with experimental observations. Pending new experiments, one may hope that the results will not in the end conform to the theory which would require amendments and create a new job for theorists. However, there is actually no need to wait for new experiments to be conducted—it is time now to think of which amendments *may* possibly be needed. There is a great deal of confidence that all corrections will be restricted to minor changes in the gauge group and field content, at most to introduction of a new hierarchy in the degree of symmetry breakdown.

It is less safe to conjecture that at a more fundamental level, the standard model will be replaced by some kind of a supersymmetric (?) model of Grand Unification. There is no reason to reject an alternative variant that all or most of the fields in the standard model will resemble hadrons constituted by the fields of a different and more fundamental model with confinement. Also, it is conceivable that there is a hierarchy of such models because the lack of information is fraught with a great variety of permitted variants. However, it should be emphasized that none of these suggested modifications bring entirely new ideas or lead us beyond the limits of *renormalizable gauge* models. This is not at all accidental because renormalizability is the highest demand to be made of the *local* $(3 + 1)$-dimensional theory, but even this is unattainable without gauge symmetry due to the presence of vector bosons.[12,27;8]

Certainly the property of renormalizability in the absence of finiteness in the ultraviolet domain does not give much cause for enthusiasm with respect to a true fundamental theory designed to describe the physical world at arbitrarily small distances. However, instead of lamentation, one could do better by inquiring why the standard model *must* be a renormalizable and a gauge theory. Dismissing the notorious answer that man is unable to think of anything better (such an answer would suggest narrow-mindedness and inadequacy of the employed formalism rather than indicate properties of the Universe), there is the only explanation available to the effect that the standard model is an *effective* low-energy theory. Such an answer indeed implies a lot, in particular it emphasizes that all effective theories are usually renormalizable and always possess specific symmetries, gauge symmetry being one of them. Phonons in solids provide a typical example of an effective theory. Specific structure of the crystal lattice and its interaction with electrons can be infinitely complex, but its trace remaining in the phonon sector is merely a small number of effective interaction constants. Events that occur at small distances (of the order of the length of the crystal lattice are immaterial for phonon physics. This is an immediate analog of the renormalizability principle. Of crucial importance for such universality is isolation of the phonon sector from all the others: in the long-wave (low-energy) approximation, gapless (massless) excitations, phonons, are split off from all the others (massive ones). It remains, however, to be explained why complicated interactions at the micro—level do not influence the very existence of phonons, that is, why are they not mixed with other excitations and converted to massive objects. The explanation is that there is a *symmetry* that maintains masslessness. With special reference to phonons, this is translational invariance spontaneously broken by the presence of crystal lattice. In other words, phonons are Goldstone particles. Other symmetries, besides the spontaneously broken global one, that can ensure masslessness and therefore the very existence of the low-energy sector are gauge and chiral symmetries, and also supersymmetry. Evidently, the number of such symmetries is limited, and the occurrence of any of them may be considered as an indication that we are dealing with an effective theory. Moreover, such a view of the standard model allows one more of its remarkable properties to be accounted for, that is it explains why all its equations are second order differential equations. This is a really intriguing puzzle for everyone who had once completed a course of general physics, but it is very easy to solve if one assumes that it is merely a property of a low-energy approximation rather than a *fundamental* natural law which is actually nonexistent.[9]

To sum up the results of this speculative analysis, the key properties of the standard model (such as those of any other model of Grand Unification) are themselves most reliable indicators of its nonfundamentality. The "fundamental theory of everything" should be sought elsewhere. It is an analysis of gravity problems that may possibly indicate the direction of further search if it proceeds from the assumption of the absolute value of the quantum paradigm, i.e. aims at the construction of quantum gravity. Such striving could be justified already by the fact that there is no known way to combine quantum matter with classical gravity, at least in the case of matter capable of emitting and absorbing gravitational waves. However, unlike classical gravity, quantum gravity has serious intrinsic problems. One of them is the unlimited growth of interaction between virtual gravitons with increasing energy which makes gravity nonrenormalizable. This phenomenon which distinguishes gravity from other $(3 + 2)$-dimensional gauge models can be attributed to the fact that the function of the charge is performed by the energy momentum tensor and this results in additional amplification of interactions with an increase in energy. Due to nonrenormalizability, gravity can not be adequately interpreted as an effective low-energy theory. In fact, it is devoid of the ability to "forget" details of its own organization at ultra-short distances. Nevertheless, the structure of gravity needs to be properly understood if the existence of massless gauge gravity is to be explained. This implies knowledge of the fundamental theory itself.

There is no need to mention that a somehow quantized GTR can not assume the role of such a theory due to irreparable ultraviolet divergences. The in-depth reason for ultraviolet divergences is known to originate from a desire to have a *local* quantum field theory, i.e., a theory dealing with point particles. The most natural radical approach to combatting divergences (and seemingly the only acceptable one in the case of gravity) implies renunciation of the locality concept and assumption of nonlocal models as a substitute for fundamental theory. Vague criteria of causality and unitarity and, more important, the lack of effective methodology are major difficulties encountered in dealing with such theories. Currently, string models are the sole class of basic concepts available for analysis that may actually have a claim to becoming a nonlocal but causal fundamental theory. String models, of all known theories, contain fields subject to interpretation as gravitons and also ensure self-consistent cut-off of divergences near the Planck energy. It should be emphasized once again that without such cut-off quantum gravity would remain a strong coupling theory which, even if not pathologic, is unlikely to allow existence of massless excitations (gravitons). Such a theory appears to realize a topologic rather than a Goldstone phase.[10]

Another important problem of quantum gravity is the occurrence of fluctuations and transitions with altered topology that is difficult to describe in terms of gravitons, i.e. in terms of the naive field approach. Alteration of topology as well as any other change in background geometry means in the graviton language formation of a specific graviton condensate. Field theory encounters no difficulty in describing isolated condensates, but it is not so easy to discuss many of them taking into account their numerous reciprocal fluctuations (think of the aforementioned multiphase systems!). This problem resembles attempts to describe a single condensate in the formalism of first (but not second) quantization for particles: such a description is theoretically feasible but hardly makes any sense. This analogy suggests expediency of transition from the second quantized theory to the "third quantized" theory. Such is one of the characteristic methods in string theory illustrated by an attempt of its direct application to topologic fluctuations in four-dimensional quantum gravity that resulted in the theory of "baby universes."[11]

One of the conceptual implications of this theory is an inference of the unavoidably effective nature of both fundamental interactions per se and their parameters (charges, masses). This confirms apprehension aroused in the course of the foregoing discussion, i.e., before the analysis of quantum-gravity effects. Certainly, the sources of these fears are different, but there is every reason to believe that the way out in either case is the same while the most important criterion of fundamental nature of the theory is its ability to overcome difficulties inherent in quantum gravity.

Let us now set aside for a time the problem of topologic fluctuations to return to the subject of divergences and modes of their elimination. This implies examination of nonlocal models which may specifically originate from string theory. There are several questions to answer. First, how and under what conditions can the structure of the local quantum field be recovered? Second, how is the conflict between nonlocality and causality resolved? Third, how is the quantum string theory organized and what are its potential pros and cons? The latter issue will be discussed at length in the following sections. But to begin with, it is necessary to answer the first two questions.

String tension that restricts their length is crucial for the existence of a *local* low-energy limit. Although the total energy of *relativistic* strings is believed to be due to their tension (which implies that the absolute energy minimum corresponds to point strings), they may have a certain characteristic length due to quantum fluctuations (uncertainty principle) which is defined in the unification models by the Planck scale $M_{\rm Pl}^{-1}$.[12]

Examined from large distances (compared to $M_{\rm Pl}^{-1}$) that are the only ones available with current experimental techniques, such strings are indistinguishable from points and their extended structure is practically unobservable. There is reason to think that elementary particles generally assumed to be points are actually extended strings. The proof of such an inner structure would be the possibility to excite internal oscillations of the strings, say, if the energy conveyed by a probe particle scattered by the string might be transformed into the energy of such an excitation instead of altering the motion of the string as a whole or creating new string-particles. Fortunately, the actual energy spectrum of an internal string excitation is discrete which allows (in agreement with the general logic of the theory of elementary particles) each excitation to be regarded as a new "elementary" particle and the scattering process with ensuing internal string excitation to be interpreted as mere conversion of one sort of particles into another, different types of internal oscillations being associated with specific particle varieties. String oscillations differ by the number of the harmonic ("number of nodes"), polarization patterns ("direction"), and amplitude. The number of the harmonic and the (quantized) amplitude are related to the oscillation energy. Since the latter is in fact the energy of *internal* string oscillations, it is responsible for the particle rest-mass: different harmonics correspond to different masses.

Polarization in its turn appears to be related to the particle spin or, more precisely, direction of spin depends on polarization whereas its magnitude is a function of the number of the harmonic, that is the zero (unexcited) level has spin 0, polarization of the first harmonic is defined by the vector (direction of oscillations) and corresponds to spin 1, polarization of the second harmonic is defined by the second rank tensor (spin 2), etc. Half-integer spins are encountered in more complicated string models in which one-dimensional lines bear additional structures. To conclude, from the standpoint of the theory of elementary particles, a noninteracting string is organized as a collection of particles with different spins and masses. The mass spectrum m in the majority of string models is defined (before the interactions are included) by the simple formula

$$m^2 \sim M^2(\alpha_0 + N),$$

where N are nonnegative integers. However, this formula must not be considered to offer an exhaustive characteristic of the spectrum because the same "tree" approximation shows a higher degree of degeneracy and different particles may have identical masses. The entire spectrum can be conveniently characterized by the generating function

$$Z(t) \equiv \sum e^{-2\pi t(m/M)^2},$$

where the sum is over all string excitations. This function containing information about mass degeneracy can also be used as a source of spin information (especially pertaining to low level spins). The following formulas for the simplest classes of string models, i.e., open and closed bosonic strings, give an idea of the shape of the generating functions:

$$\begin{aligned} Z_{\rm op}(t) &= [q^{1/24}\prod_{n=1}^{\infty}(1-q)^n)]^{-(D-2)} Z_{\mathscr{L}}^{(D)}(t), \\ Z_{\rm cl}(t) &= \int_{-1/2}^{+1/2} |Z_{\rm op}(t-is)|^2 ds, \end{aligned} \tag{1}$$

where D is the space-time dimension and $q \equiv e^{-2\pi t}$. Factor $Z_{\mathscr{L}}$ is responsible for corrections associated with the two-dimensional Liouville field $x^0(\xi)$ that describes the *time* coordinate of the string in space-time. The main difference of this factor from other two-dimensional fields $x^i(\xi)$ $(i = 1...D-1)$ describing space coordinates lies in the sign which it has in the formula for two-dimensional action on the world surface:

$$\mathscr{A} = M^2 \int G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\alpha x^\nu d^2\xi + ..., \tag{2}$$

the tensor $G_{\mu\nu}(x)$ (the space-time metric) has the signature $(-,+,+,...,+)$.[13] Formula (1) suggests that $G_{ij} = \delta_{ij}$. One of the corrections to the naive picture of string particles is that the factor $Z \neq 1$ whenever the D-dimensional Lorentz invariance is broken due to additional terms which are not explicitely shown in Eq. (2). Traditionally, models with $Z_{\mathscr{L}} = 1$ (more precisely, free from *any* quantum anomaly as two-dimensional field theories) are referred to as "critical" string models. The above example indicates that critical strings have more "naive" and predictable properties. It will be shown below that the property of criticality is closely associated with the interesting low-energy limit in string models which predetermines the specific role of critical strings in the unification theory.

It ensues from formula (1) that the particle spectrum in the open string model contains tachions, particles with negative m^2 at $D>2$, while massless ($m^2=0$) particles occur only at $D=2, 26,...$, having zero-spin at $D=26$ and being a $(24 = D-2)$-component vector at $D=26$. Of primary importance for the theory of unification is the strictly *massless* sector of the string model whereas all other string-particles have masses of the Planck scale and can not be produced in modern accelerators. This approach implies that the diminutive (in this scale) masses of leptons, quarks, and Higgs bosons are *beyond* the bounds of the tree approximation or are even more likely due to *non-perturbative* effects of string interaction. For this reason only the case $D=26$ of all open string models with tree characteristics defined by formula (1) may be of phenomenological interest for the purpose of unification (analysis of string models with $D>26$ can not be conducted by currently available methods). It has been shown that in this case, G_{0i} in (2) may be assumed to be equal to 0 which makes $Z_{\mathscr{L}} = 1$ in Eq. (1). Therefore, the above inference that the massless sector is represented by the $(D-2=24)$—component vector is not in conflict with Lorentz invariance. Existence of the $(D-2)$-component vector-particle is compatible with D-dimensional Lorentz symmetry only in the case of this particle being represented by a *gauge* massless boson. Thus, having started from the assumption of massless particles in the tree approximation as a condition for the existence of a nontrivial low-energy sector, we come to the conclusion that the string must be critical. This implies that the massless sector has an additional gauge symmetry which is in turn capable of maintaining masslessness of the particles even after interactions are included.

The open "superstring" model contains additional two-dimensional fields on the world surface and may be defined by an action principle more complicated than (2). Also, the action has an apparent two-dimensional supersymmetry. The principle of two-dimensional supersymmetry alone defines fermionic strings whereas an additional selection of states, the so-called GSO-projection, is necessary to obtain a "superstring." The generating function of the tree spectrum for the open "superstring" has the form:

$$\begin{aligned} Z_{\mathrm{op,\,su}}(t) = {} & (D-2)[\prod_{n=1}^{\infty}(1+q^n)(1-q^n)^{-1}]^{D-2} Z_{\mathscr{L},1}^{(D)}(t) \\ & + \frac{1}{2}\{[q^{-1/16}\prod_{n=1}^{\infty}(1+q^{n-(1/2)})(1-q^n)^{-1}]^{D-2} \\ & - [q^{-1/16}\prod_{n=1}^{\infty}(1-q^{n-(1/2)})(1-q^n)^{-1}]^{D-2}\} Z_{\mathscr{L},2}^{(D)}(t). \end{aligned} \tag{3a}$$

The model becomes critical at $D=10$. In such a case, the first line in Eq. (3a) that describes the *fermionic* sector of the model (Ramond sector) coincides with the second line describing the *bosonic* sector (Neveu–Schwarz sector).[14] Then, the generating function may be reduced to

$$Z_{\mathrm{op,\,su}}(t) = 16[\prod_{n=1}^{\infty}(1+q^n)(1-q^n)^{-1}]^8. \tag{3b}$$

Co-incidence of two constituents of the generating function is one of the manifestations of space-time supersymmetry of the critical superstring. Unlike two-dimensional supersymmetry, space-time supersymmetry occurs only in the critical dimension of $D=10$ and only after GSO projection (here, we come across a common feature of *critical* strings: peculiar duality of properties on the world sheet and in space-time; in the examined case, such a property is represented by supersymmetry). It follows from Eq. (3b) that this model does not include tachions while the massless sector contains 8-component gauge bosons and 8-component gauginos, their gauge partners. The model would have every prospect to play the role of the fundamental theory if it were not for some quantum anomalies, the condition $D=10$ being necessary but not sufficient. The true *critical* superstring contains a large number of fields with non-abelian gauge symmetry (SO(32) in one of the simplest cases) at the massless level and tensor particles with properties identical to those of gravitons (the latter feature is common for all *critical* models of *closed* strings). We shall discuss different string models after a brief review of string *interaction*.

Interaction between strings is of primary importance for explanation of casuality of the theory. We have already agreed that noninteracting strings can be regarded as a collection of different sorts of particles (Lorentz symmetry may be broken in non-critical models). Hitherto, the sole evidence of non-locality of the theory is an infinite number of particle varieties. It is essential to stop at this point to have *interactions* represented as interactions between point particles without any *a priori* action at a distance. This would remove all difficulties associated with causality as is the case with the conventional local QFT. String theory has chosen the simplest way to attain this goal because it permits only *local* string interaction in a single point. This means that a string can break into two only in a certain point rather than split lengthwise along its entire length or along a certain portion. This considerably restricts the choice of interaction. An immediate analog of such a limitation in the theory of particles is that splitting of particles is forbidden, in fact that there is an absolute ban on *any* interaction. In this context, *interacting* strings are directly analogous to *free* particles which accounts for marked simplification and (not infrequently) for solvability of problems in string theory.

From a more formal point of view, locality of interactions (that warrants causality and readily distinguishes strings from other non-local QFT models) means prohibited branching of world surfaces and implies that the world sheet for interacting strings is a *smooth* two-dimensional surface. Conversely, string interaction is reminiscent of branching of their world lines that breaks smoothness. This may be considered to provide another formulation of analogy with *free* particles. It is conceivable that from the mathematical point of view smooth surfaces are much easier and more *efficient* to work with than singular ones. This accounts for successful application of the mathematical formalism of string theory. Also, it seems appropriate to note that strings hold a unique position in the p-bran series. For mathematical purposes, it is quite natural to define p-bran interaction with the same limitation and confine it to smooth $(p+1)$-dimensional world hypersurfaces. However, it is insufficient to ensure locality of interactions and hence a physical interpretation of the theory at $p>1$ when they are concentrated on sub-manifolds of codimension 2 (i.e. on $(p-1)$ rather than 0-dimensional ones).

Speaking of locality of interactions, one can not but emphasize its role in eliminating ultraviolet divergences. Locality of interaction along with string extent ensures that interaction occurs only between string fragments with a characteristic length $1/E$, after the transferred momentum and energy E have exceeded the characteristic value M (inverse string size). For example, only the fraction M/E of the total string mass interacts with a graviton. This serves to assure additional supression of interactions with increasing energy and, in the end, finiteness of the theory.

Description of a specific interaction, e.g. emission of a particular string excitation (a particle), is of paramount importance in the string theory formalism. To this effect, an additional term with a "source" is introduced into the action specifying weights in the functional integral (in this case, two-dimensional action that defines weights in the sum over surfaces) as is usually done in QFT. As regards strings, such a term is the integral of an operator (the so-calles vertex operator) while the "source current" may be interpreted as the D-dimensional field of the corresponding particle. For example, the emission of a massless graviton (if such a one is present among the string excitations) corresponds to the vertex operator $\partial_a x^\mu \partial_a x^\nu$, with the appropriate term having the form

$$\int h_{\mu\nu}(x)\partial_a x^\mu \partial_a x^\nu \mathrm{d}^2\xi.$$

Bearing in mind that the graviton field in gravity is connected with D-dimensional metrics via the relation

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

one arrives at the conclusion that the entire action (2) may be considered to describe a string in the external graviton field defined by the D-dimensional metrics $G_{\mu\nu}(x)$.

4. STRING TOMOGRAPHY OR STRING THEORY AS A RADON TRANSFORMATION IN THE SPACE OF QFT

The last remark is very important for string theory, especially for its further development and application. Also, it allows very broad generalization: for each conventional field theory is supposed to have its string counterpart. To demonstrate this, distribution of a *probe* string should be examined in relation to the fields of a given theory regarded as external (background) fields. In this case some integral background characteristics are encoded by string propagation patterns, i.e. in the properties of a two-dimensional theory on its world surface. Such an approach is widely applied to different fields of natural science and is often referred to as tomography (the use of the scattering features of rays that pass through the body to characterize its inner structure) or as a Radon transformation (integral transformation of a generic type; its simplest variant-Fourier transform, "probes" the function by an averaged "interaction" with an imaginary exponent).

The success of the method is wholly dependent on the efficacy of mapping *interesting* properties of the object under study into *simple* properties of the probe. From this viewpoint, string theory may be very useful for the study of QFT. In principle, there is every reason to analyze the field theory using probe *particle* propagation, rather than that of strings. However, such an approach does not appear to provide the necessary information if the dynamics of a field theory is ranked among its interesting properties. Suppose, that the external fields satisfy (D-dimensional) equations of motion, i.e. a certain differential (or integro-differential, finite difference or any other) equation. If the solution is in the form of a plane wave (i.e., the equation of motion is *linear*), resonance phenomena will be apparent in propagation of the probe particle and interesting information (e.g., wave length) will be obtained. Conversely, if the original equation is *non-linear*, this will be hardly possible because the probe particle (which does not interact with anything but the external field) is unable to feel all the constraints imposed on the external field at different points which by no means line up.

A string is quite a different entity. Apart from being extended, which means that it is sensitive to both the external field gradient and its alterations at *finite* distances (this obliterates the distinction between the solutions of differential and finite difference equations), the string is able to split into many strings, that is, it can interact! This allows information about nonlinear relations to be conserved. If, suppose, the external field configuration is described by the decay of the wave into two, followed by a reconstitution from the fragments, the probe string may prove to be able to follow this evolution. Theoretically, the same is attainable with "probe" particles allowed to interact with one another. However, this would expose quite a different aspect of the problem—how to define the transformation itself, i.e., how to define the interaction between interacting particles and the external field. A simple geometric approach valid for free particles (as worded in terms of smooth line geometry or trajectories in a space with defined external fields) is irrelevant in the case of interacting particles because it is unclear how to estimate the effect of the external fields on the interaction vertex. Actually, the only way to use interacting probe particles is to take them from the theory under study. Such an approach unifying external field and first quantization techniques may be useful, but it is *not universal* since it requires introduction of specific probe particles for each theory. Strings are quite a different matter. String interaction does not interfere with smoothness and therefore does not give rise to any special problems with respect to assessment of interactions with external fields or, to put it otherwise, excitation spectrum for the entire variety of string models is sufficiently rich to include practically all conceivable QFT.

Be that as it may, translation into the string language is really useful. It allows many equations of motion for external fields to be converted to symmetry principles! Conformal symmetry is the most common one involved. Let us come back to our initial example, a string in a gravitational field.

The external field $G_{\mu\nu}(x)$ is associated with the string model with the action (2). The condition of conformal invariance (vanishing of the beta-function) in a two-dimensional theory has the form of

$$\mathscr{R}_{\mu\nu} = o\mathscr{R}/M^2$$

($\mathscr{R}_{\lambda\mu\nu\rho}$ is the curvature tensor built from the metric $G_{\mu\nu}$) and can be reduced in the longwave approximation to Einstein equation $\mathscr{R}_{\mu\nu} = 0$ for the external field $G_{\mu\nu}$. Yang–Mills, Laplace, and other equations can be interpreted in a similar manner.[15]

An advantage of any tomography technique is the lack of specificity because it allows results of the study to be represented in an identical form due to the use of the same apparatus for examining different subjects, individual organs, etc. The situation is very similar as far as strings are concerned. In the low-energy limit gravitational and Yang–Mills fields, abelian and non-abelian gauge groups, bosons and fermions may exist; supersymmetry may emerge and undergo breakdown; space-time, if any, may be 4 or 15-dimensional and show a simple or a complicated topology; in a word, low-energy theories may belong to different classes in the absence of any apparent way to smoothly shift from one of them to another, but putting them into the string language yields two-dimensional field theories which appear to be much easier to classify and interpolate. For this reason, string theory has a good prospect to become a valuable tool not only for the "naive" scenario of unification of fundamental interactions, but also for a more ambitious scenario of "the unified field theory" mentioned in the introductory part of this article (and for applications to multiphase systems at large).

An approach to this scenario based on string theory at large may be described as follows. First, all possible models of the field theory (phases in multiphase systems) are substituted by corresponding models of *two-dimensional* QFT (in this case, probe string propagation is used to define a map from the set of all theories into a set of two-dimensional ones). The latter set has at least one advantage of being free from serious restrictions pertaining to renormalizability because there is a great variety of well-defined two-dimensional QFT models. Strictly speaking, an optimal selection of the *entire* space of two-dimensional theories as a universal configuration (phase) space is hardly possible due to the fact that poorly-structured *generic* two-dimensional models do not possess any remarkable algebro-geometric properties. Doubtless, various two-dimensional conformal models (shown above to be associated with a large number of differential and other equations) must be included as points of the phase space. This is a truly remarkable set with a number of unexpectedly deep properties and numerous structures. However, it is not even connected in any reasonable topology. This certainly reflects the nonconnectedness of the original space of various equations, field theories, etc. Meanwhile, our primary objective is to search for mapping into a set of two-dimensional models in order to obtain reasonable interpolations, i.e., a *connected* space.

Connectedness can be achieved by mere addition of all integrable models to all two-dimensional conformal ones. The resultant space of integrable two-dimensional theories is by no means less remarkable than its subset comprising only conformal theories. It is already connected but rather resembles a net than a normal space. Specifically, its fundamental group appears to be very complicated in reasonable topologies. This space is sure to need further extension without the loss of its algebro-geometric structures. Reasonable addition can be expected to lead beyond two-dimensional theories. In all likelihood, it will be determined by the algebra of (double?) loops, but at this point opinions of different authors begin to diverge, and it would be premature to discuss possible options till concensus is achieved (a putative variant is examined in Refs. 40, 41). It should only be noted that definition of a configuration space is the first but not the last step.

It needs to be understood what dynamic principles can be naturally defined on this space and which dynamic structures may result from them. Of course, this problem can not be separated from that of the space itself because geometry and especially algebra does not only characterize space but also determine all the rest (this makes up the basic idea of geometric quantization). For the time being, all these issues are also a matter of conjecture. In the meanwhile, the space of integrable theories has become one of the most important areas of current studies. Knowledge of its structure is expected to facilitate greatly further generalization.

5. CONFORMAL FIELD THEORY

Our next purpose is to discuss two-dimensional conformal models the role of which in string theory and its applications was analyzed in a previous section. We shall start with fundamentals of the general theory and thereafter consider the most important classes of conformal models. Two-dimensional space is assumed to be Euclidian which is not in conflict with the existence of a Minkowsky signature in the real D-dimensional space-time. A special analysis of the dependence on the two-dimensional metrics g_{ab} is beyond the scope of this section. This relation must in the first place be taken into account in switching over from conformal models to string ones, a matter of further discussion.

Major characteristics of conformal symmetry in two dimensions are that it is infinite-dimensional and is interpreted in *complex-analytic* terms. The former characteristic means that the condition of conformal invariance is very limiting and hence interesting. The latter property provides a good prospect for the development of efficient mathematical apparatus (everyone who has studied the theory of complex variables knows that complex analysis is far more powerful than the real one). The first step to formalization is to introduce complex co-ordinates $(z,\bar{z})$ on a two-dimensional surface. It is convinient to interpret $\bar{z}$ as "time" in two-dimensions to avoid terminological complication. Conformal invariance implies that the energy-momentum tensor is traceless,

$$T^a_a = T_{z\bar{z}} = 0.$$

Then, the conservation law $\partial_a T^{ab} = 0$ (which usually implies only conservation of total energy and momentum $(\partial + \bar{\partial})\oint T_{0b} = 0$) turns into the holomorphicity conditions

$$\bar{\partial}T = 0, \quad \partial\bar{T} = 0 \quad (T \equiv T_{zz}, \bar{T} \equiv \bar{T}_{\bar{z}\bar{z}}),$$

which implies conservation ($\bar{\partial}L[f] = 0$) of an infinite set of variables:

$$L[f] = \oint f(z)T(z)dz$$

with any holomorphic vector fields $f(z)$. $L[f]$ form a set of generators of infinite-dimensional Lie algebra (conformal symmetry of the model in question). A reasonable choice of the basis in the space of holomorphic vector fields depends on the global properties of the two-dimensional space. Specifically, it is appropriate to take $f_n^0(z) = z^{n+1}/dz$, where n is any integer or a non-negative integer if the space has the topology of a Riemann sphere with two and one punctures respectively. Corresponding generators

$$L_n \equiv L[f_n^0] = \int z^{n+1}T(z)dz$$

form a Virasoro algebra:[15a)]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}cn(n^2-1)\delta_{n+m,0}.$$

This commutation relation may be represented in terms of the operators $T(z)$ as the operator expansion:

$$T(z)T(z') \sim \frac{c}{2(z-z')^4} + \frac{T(z+z')/2)}{(z-z')^2} + \text{reg},$$

where the parameter c is referred to as the central charge of the Virasoro algebra and constitutes an important characteristic of the conformal model. Due to the requirement that the vector fields that define the conservation laws be holomorphic, conformal models are considered to be intermediates between generic two-dimensional field theories (where only a finite number of Killing vectors are allowed to perform this role) and topologic models (where *any* vector field has a conservation law). String models are topologic and derived from conformal ones by the introduction of a Liouville field and reparametrization ghosts (which does not interfere with conformal invariance and allows the total central charge $c + c_{\text{ghost}} + c_{\mathcal{L}}$ to be converted to zero) and subsequent factorization with respect to action of the conformal group (altogether, these operations may be interpreted as integration over two-dimensional metrics). The conformal models themselves most of all resemble integrable models as far as the number of conservation laws (one-parametric set) is concerned but are distinguishable from them due to the very simple structure of conserved quantities that can be chosen linearly dependent on the energy-momentum tensor and directly related to the complex structure of the two-dimensional space-time. In the case of a generic two-dimensional integrable model the same role is played by the complex structure of an *auxiliary* (spectral) two-dimensional surface.

All the states of the conformal model may be divided into representations of the Virasoro algebra (the corresponding operators creating these states from "vacuum" are frequently referred to as "vertex operators" for reasons obvious from the previous section). It should be borne in mind that the Virasoro algebra associated with a holomorphic $T(z)$ affects only the chiral components of the vertex operators dependent on holomorphic fields. Each vertex operator is a bilinear combination of chiral and anti-chiral operators. It is assumed in the further discussion that classification by representations of the Virasoro algebra is most reasonable for chiral operators. Highest weight representations associated with so-called Verma modules have been most thoroughly investigated. A Verma module is built up by the primary field which satisfies the relation

$$T(z)V(0) \sim \frac{\Delta}{z^2}V(0) + \frac{1}{z}\partial V(0) + \text{reg}.$$

Similar to the operator expansion of $T(z)T(z')$ that defines the Virasoro algebra, this formula reflects the physical sense of the energy-momentum tensor as a generator of infinitesimal coordinate transformations. Moreover, this formula suggests that

$$[L_n, V] = 0$$

for all $n > 0$ which makes the *primary* field actually of highest weight with respect to the Virasoro algebra. (Note that the energy-momentum tensor itself is not the primary field in the case of a non-vanishing central charge). Furthermore,

$$[L_0, V] = \Delta V;$$

and the parameter Δ is called the *dimension* of V. The dimension of a chiral operator coincides with its spin, the spin of the total vertex operator being the difference between "left" and "right" (chiral and anti-chiral) dimensions. A Verma module is an orbit of a universal envelope of the Virasoro algebra which passes through V, i.e. the set of all the fields of the form $[L_{-n1}, [L_{-n2}, ... [L_{-nk}, V]...]]$ with dimensions $\Delta + n_1 + n_2 + ... + n_k$ and their linear combinations, the so-called "Virasoro descendants." The integer $n_1 + n_2 + ... + n_k$ is referred to as the *level* of the operator in the module. By definition, a Verma module is the highest weight representation but, possibly, reducible. It is really reducible if the module has a new *primary* field at a certain non-vanishing level which is termed a nil-vector. A Verma module associated with a nil-vector is a submodule of the original one and must be eliminated in order to construct an *irreducible* representation (in other words, it is necessary to find a factor of the module with respect to equivalence "nil-vector ~ 0," hence the term "nil-vector;" the norm of this field is nil). Actually, there are three possibilities: (i) no nil-vector, (ii) each consecutive submodule is embedded into the previous one, (iii) all nil-vectors of each module belong to one of the two intersecting submodules.[16)]

Case (i) is a most general one. Case (ii) is also common while case (iii) for Virasoro algebra can be realized only at a specific central charge value

$$c = 1 - [6(p-q)^2/pq],$$

with integer mutually—simple $p,q \geqslant 3$ (the so-called minimal representation series).

The simplest classification of conformal models is according to the set of representations of Virasoro algebra formed by vertex operators. The set of vertex operators is restricted by the requirement of closure of operator algebra and in practice can not be as small as a single representation

FIG. 1

of Virasoro algebra though there are cases when operator expansion closes with a *finite* number of irreducible representations. The corresponding conformal models are termed rational. The simplest example of rational representations are the (p,q)-minimal models[58-61] containing a finite number of representations from the above minimal series. All these minimal models have $c<1$, but there are many rational theories with $c>1$ in which the values of c are not necessarily rational. Another important feature of the conformal model is two-dimensional unitarity, i.e., the positiveness of norms of all vertex operators. Elimination of nil-vectors (having vanishing norms), i.e., restriction to irreducible representations is a necessary but not a sufficient condition of unitarity: many fields from a Verma module may have negative norms and *not* be excluded by eliminating nil-vectors. Specifically, only the one series $(p,q)=(p,p+1)$, of all the minimal models, is unitary. The simplest representative of this series is $(p,q)=(3,4)$, the Ising conformal model (i.e. the Ising model at a phase transition point). The significance of two-dimensional unitarity for *string* models remains unclear because unitarity can be restored by integration over metrics.

Operator expansion of products of vertex operators defines intertwining operators of different representations.[62] The intertwining operator in a conformal model is a trilinear relation between triplets of holomorphic vertex operators, a specific case of more general (multilinear) operators referred to as conformal blocks. A conformal block is a holomorphic analog of a correlation function. Correlation functions of full (non-chiral) vertex operators in the conformal theory are assumed to be bilinear combinations of conformal blocks and their complex conjugates. Unlike correlators (real and single-valued functions of vertex operator coordinates), conformal blocks are holomorphic but not necessarily single-valued, that is they can have both poles and branch points. There are no essential singularities in conformal blocks of rational theories, and they are totally determined by orders of the poles and by monodromy properties.

Although the Virasoro algebra is common for all conformal models, special classes of theories with broader infinite-dimensional symmetries are also worth considering. We require, as before, that symmetry be associated with certain conserved, i.e. holomorphic, currents. Such symmetry is usually referred to as a chiral algebra of the model due to its being holomorphic. Thus, a chiral algebra always contains the Virasoro algebra (rather its universal envelope), but it also may be broader. An example of a very rich chiral algebra is the Kac-Moody algebra.[17]

The Kac-Moody algebra is a quantum analog of a loop algebra with values in the Lie algebra G, and is itself a Lie algebra (not to be confused with quantum *deformation*—the so-called quantum group which is not a Lie algebra). The generators $J^{\alpha}(z)$ carry, besides the coordinate z, the indices α labelling the generators of the algebra G, and commutation relations are expressed via its structure constants $f^{\alpha\beta}_{\gamma}$ and the Killing metric $h^{\alpha\beta}$:

$$J^{\alpha}(z)J^{\beta}(z') \sim \frac{kh^{\alpha\beta}}{(z-z')^2} + f^{\alpha\beta}_{\gamma}\frac{J^{\gamma}(z')}{(z-z')} + \text{reg.}$$

The parameter k is called the central charge of the Kac-Moody algebra $\widehat{G}_k$. In a model with such symmetry, a chiral algebra is the universal envelope $\mathscr{U}\widehat{G}_k$ of the Kac-Moody algebra. Specifically, the energy-momentum tensor which generates the Virasoro algebra becomes a function of the currents J^{α}. In this case, the simplest choice is:

$$T_G(z) = \frac{1}{2(k+C_G)}\operatorname{Tr} J^2(z)$$

The resulting conformal model is referred to as the Wess-Zumino-Novikov-Witten model (MWZNW) while the aforesaid embedding of the Virasoro algebra into the envelope of the Kac-Moody algebra is termed the Sugawara formula with $c = k \dim G/(k+C_G)$ where the parameter C_G is a dual Coxter number of algebra G, e.g., $C_{\mathrm{sl}(n)} = n$. Along with a quadratic trace, it is useful to consider in the universal envelope other operators of a similar type:

$$\widehat{W}^{(n)}_G(z) = \operatorname{Tr} J^n(z).$$

With respect to the operator $T = W^{(2)}$, the Kac-Moody currents have spin 1, and the $W^{(n)}$ operators have spin n. It is useful to classify MWZNW states by representations of the Kac-Moody algebra which automatically serve as (reducible) representations of the Sugawara-like Virasoro algebra.

This approach should be applied to any other chiral algebra. The structure of highest weight representations of all chiral algebras is reminiscent of that described above for the Virasoro algebra: there always are Verma modules, nil-vectors, minimal representation series, and rational and minimal conformal models many of which are non-unitary. Similar to the Virasoro algebra, the model needs to be neither rational nor, especially, minimal. The advantage of minimal models is that they most adequately reflect the structure of a chiral algebra and therefore are very useful for mathematical purposes, e.g., in structural studies of the algebra and its representations. The minimal models for the Kac-Moody algebras associated with *simple* finite-dimensional Lie algebras occur at positive integer k values; they are rational and unitary, and allow a simple Lagrangian formulation with action

$$\mathscr{A}_{\mathrm{WZNW}} \sim k\Big[\int \operatorname{Tr}(g^{-1}\partial_a g g^{-1}\partial_a g)\mathrm{d}^2\xi + i\varepsilon^{abc}\int \operatorname{Tr}(g^{-1}\partial_a g g^{-1}\partial_b g g^{-1}\partial_c g)\mathrm{d}^3\xi\Big].$$

The field $g(\xi) = g(z,\bar{z})$ defines the mapping of a two-dimensional surface into the group G. The second item (the Wess-Zumino term) is represented in the action as an integral over the three-dimensional manifold the boundary of which is formed by the two-dimensional surface in question. Due to the arbitrary choice of the three-dimensional extension, the action is defined up to an integer multiple of $2\pi ik$ which does not affect the functional integral $\int \mathscr{D}g e^{-\mathscr{A}}$ (for integer k). The equations of motion for this non-local action are local: $\bar{\partial}J = 0$, where $J = g^{-1}\partial g$ while equal-time commutation relations ensure that the currents $J(z)$ form the Kac-Moody algebra $\widehat{G}_k$. The inverse transition from the Kac-Moody algebra to the action $\mathscr{A}_{\mathrm{WZNW}}$ is also possible leading to another notable interpretation:[63] this action is related (by the operator d^{-1}) to the Kirillov-Costant form of the Kac-Moody algebra. The latter is unambiguously defined by any Lie algebra and its coadjoined orbit (in this sense the WZNW model is totally *algebraic*: it is sufficient to mention "the Kac-Moody algebra," and the canonical procedure will lead to the functional integral:

$$\int \mathscr{D} g \exp(-\mathscr{A}_{\mathrm{WZNW}}).$$

Conformal blocks of the WZNW model are closely related to representations of quantum groups. It has already been indicated that the operator expansion which defines conformal blocks may be interpreted in terms of intertwining operators, i.e., it defines variables such as $3j$-symbols and other Racah coefficients. Strictly speaking, in order to obtain anything related to the characteristics of the original algebra G, it is necessary to eliminate all the descendant fields from the operator expansion and leave only the primary fields. A perfectly justified procedure of such a type leads to *string* models (see below). In the framework of the conformal model, it is necessary to use something of the type of a projection of an operator algebra (i.e., actually to eliminate the contribution of the descendants). The pertinent methodology is known as fusion rules[62] and, due to non-zero central charges, characterizes the deformation of algebra G, i.e., a quantum group, rather than the algebra G itself because projection breaks the structure of the Lie algebra.

Another line of reasoning leading from the WZNW model, develops the idea of the Wess–Zumino term in the action. It appears that the entire action, not only its part, can be represented by a Wess–Zumino term: the three-dimensional Chern–Simons integral:

$$k\int \mathrm{Tr}(AdA + \frac{2}{3}A^3)\mathrm{d}^3\xi.$$

The way in which the functional integral that defines this three-dimensional topological theory depends on boundary conditions at the two-dimensional boundary of the three-dimensional space is again given by the expression $\exp[-\mathscr{A}_{\mathrm{WZNW}}]$, whereas the functional integral of MWZNW is the integral over boundary conditions.[64] This directly leads to topological theories in general[65] and to the problem of the topology of three-dimensional manifolds (knot theory) in particular.[66] On the other hand, the Chern–Simons action is closely related to even more remarkable four-dimensional topological theory $\int \mathrm{Tr}\, F\tilde{F}\, \mathrm{d}^4\xi$. Finally, the Chern–Simons term plays a role in the theory of quantum anomalies and physics of $(2+1)$-dimensional systems (e.g. fractional quantum Hall effect, theory of anyons). All this makes studies of the three-dimensional interpretation of the WZNW model very popular.

Basically, it is conceivable that any d-dimensional theory must be associated with a certain $(d+1)$-dimensional topological model dependent solely on boundary conditions in codimension one. The case of the WZNW model indicates that at $d=2$, *conformal* models are related in this way to *local* three-dimensional topological models. A set of such theories, if they must be formulated in terms of a Lie algebra, is evidently confined to Chern–Simons-like models which, in their turn, are related to MWZNW. True, this is not a direct argument but rather the simplest of the existent justifications of the faith that many two-dimensional conformal models must be related to WZNW model, while universal envelopes of the Kac–Moody algebras must contain all conceivable chiral Lie algebras.[18)]

A more specific formulation of the statement concerning the "relation" of a conformal theory to the WZNW model should mention that the former is a reduction of the latter, i.e., is produced by selection of a certain subset in the space of states invariant with respect to dynamic flows (equations of motion). To make such a selection possible, it is necessary to change the dynamics (i.e., the energy-momentum tensor) in such a way that allows the existence of an invariant system of constraints which could be imposed on the states. To begin with the bilinear relation between $T(z)$ and the currents $J(z)$ may be preserved (although this is not obligatory):

$$T(z) = \iint C_{ab}(z, z', z'')J^a(z')J^b(z'')\mathrm{d}z'\mathrm{d}z'' + \int d_a(z, z')J^a(z')\mathrm{d}z'.$$

Restrictive constraints are imposed on the coefficients C_{ab} and d_a in order to ensure compatibility of the Kac–Moody and the Virasoro commutation relations for $J(z)$ and $T(z)$ respectively. However, even in the case of the simplest "local" anzatz

$$C_{ab}(z, z', z'') = C_{ab}/(z-z')(z-z''),$$

$$d_a(z, z') = d_a/(z-z')^2$$

with constant C_{ab} and d_a (Refs. 67, 68), the set of embeddings is not limited to the Sugawara one. There is a large number of possible reductions, but the Goddard-Kent-Olive (GKO) and the Drinfel'd-Sokolov (DS) reductions constructed according to subalgebras $H\subset G$ are among the most interesting of them (Refs. 67–69).[19)] In order to describe reduction, one ought to define a non-Sugawara tensor $T(z)$, determine compatible constraints (conditions of reduction), and find primary fields with respect to $T(z)$ although in the original Kac–Moody algebra they need not necessarily be primary. In case these fields include operators from the universal envelope $\mathscr{U}\widehat{G}_k$ that form a closed (not necessarily linear) subalgebra, the latter should be regarded as a *chiral algebra* of the reduced theory and the corresponding symmetry should apparently be subjected to gauging on transition to a *string* model. Certainly, one may start by imposing an arbitrary system of constraints, but this will require checking that conformal symmetry remains intact, that is that the part of the chiral algebra commuting with the constraints still contains the Virasoro algebra generated by a "deformed" energy-momentum tensor.

The simplest type of a MWZNW reduction is the coset-models or the GKO-construction.[72] The starting point is specification of the subalgebra

$$\widehat{H}_{k'} \subset \widehat{G}_k.$$

We shall discuss the simplest variant:

$$H\subset G; \quad k' = k.$$

There are many interesting reductions when G is a semi-simple, and not necessarily a simple, algebra, $\widehat{G} = \Pi_i G_{i,k_i}$, and $k' = \Sigma_i k_i$. The energy-momentum tensor of the GKO model is

$$T_{G/H} = T_G - T_H.$$

The central charge is

$$c_{G/H} = c_G - c_H.$$

The currents $J^{\alpha}\|(z)$ with indices $\alpha_{\|}$ belonging to $\hat{H}$ have similar commutators with T_G and T_H (they are primary fields of dimension one with respect to both these tensors) and therefore commute with $T_{G/H}$. Thus, the GKO reduction may be defined by conditions $J^{\alpha}_{+}\|(z)=0$. As usual, only half of the constraints can be imposed due to the presence of the nonvanishing central charge k: this fact is indicated by the subscript " + ."

The primary GKO fields can be picked out from among the primary fields of the original MWZNW which automatically makes the GKO model both rational and unitary for integer values of k when the MWZNW to be reduced has these properties. The most effective description of the GKO reduction is achieved in the formalism of free massless fields (see below). In this formalism reduction means only the rejection of a part of the noninteracting fields. Besides, in it there is no difference between the GKO and other reductions which is important for the development of a *general* string theory even if it is less convenient for the analysis of specific, especially rational and unitary coset-models. GKO-models form quite a broad and most widely studied class of conformal theories. Many of these models contain "accidental" global and even local symmetries the origin of which in most cases can be easily traced to "accidentally unbroken" generators of the original chiral algebra. This class includes the already familiar *unitary series* $(p,p+1)$ of the Virasoro minimal $(p,p+1)$ models $(\mathrm{su}(2)_1\times \mathrm{su}(2)_{p-2}/\mathrm{su}(2)_{p-1})$, and also the two-dimensional supersymmetric *unitary* minimal series $\mathrm{su}(2)_2\times \mathrm{su}(2)_{p-2}/\mathrm{su}(2)_p$; it is minimal for the $N=1$ Virasoro superalgebra regarded as a chiral algebra), as well as *unitary* minimal $W_{\mathrm{sl}(n)}$-models $\mathrm{su}(n)_1\times \mathrm{su}(n)_{p-2}/\mathrm{su}(n)_{p-1}$; [here the chiral algebra is Zamolodchikov's $W_{\mathrm{sl}(n)}$ algebra (Refs. 73–76)], and many others (including nonminimal, nonrational, and even nonunitary models, the latter arising when WZNW with noninteger k is reduced).

Drinfeld–Sokolov (DS) reductions[77-79] provide a slightly more complicated and instructive example. In this case, a maximal nilpotent subalgebra is selected as H (upper triangle matrices for $\mathrm{sl}(n)$ in the common matrix representation) although not all of its generators are supposed to vanish: first-level generators (they correspond to *simple* roots in the case of $\mathrm{sl}(n)$ and stand at the first diagonal above the principal one) are made equal to one. In general, a nonzero value may be attributed only to a time-independent operator that commutes with the energy-momentum tensor. However, this restriction may be weakened because the nonvanishing value is *constant*: it is sufficient that the operator be a primary field of vanishing dimension. In this case the commutator retains only the term with a derivative which equals zero. This allows the DS reduction to preserve conformal symmetry although it is the slightly modified Virasoro algebra, not the Sugawara algebra, that remains unbroken. (With respect to T_G, *all* currents are of dimension one rather than zero; the adequate tensor is $T_{\mathrm{DS}}=T_G-\partial H_\rho$ or, more precisely, is a restriction of the right-hand part onto the reduced Hilbert space; $H_\rho(z)$ is the current which corresponds to the Carten element of algebra G associated with the vector ρ on the Carten plane; ρ is the half-sum of *all* positive roots). Moreover, even a *richer* fragment of $\mathscr{U}\hat{G}$ e.g. adequately modified operators $\hat{W}_G^{(n)}$, survives in this situation. They are associated with Cazimir operators of finite-dimensional algebra G (but not $\hat{G}$!), and like them are not all independent because of the restricted number of degrees of freedom. However, the relations between them are not linear. Following the DS reduction, the number of algebraically independent W-operators appears to be equal to the rank of G; moreover this finite set of operators is closed not only with respect to Poisson brackets but also to quantum commutators (which is far from being obvious). In other words, quantum (i.e., adequately modified) W-operators form a closed *operator* algebra or the Zamolodchikov algebra W_G (Refs. 73–76, 80, 81). This algebra becomes a Lie algebra only in the (naively taken) limit rank $G\to\infty$ (Refs. 82–85); in the case of finite ranks, it is at best quadratic (there is such a basis). W-algebras provide a highly non-trivial example of chiral algebras, but the problem of its localization (gauging), i.e., construction of corresponding string models (the so-called W-strings[86-93]) remains obscure.

The theory of *DS* reduction is rather broad and needs to be substantially improved. It is worthwhile to note that the *DS* reduction gives rise to the Toda conformal models[94] (Liouville theory being the simplest of them) which, along with their integrable analogs, constitute a classical object of investigation in mathematical physics (the sine-Gordon model[4] is considered to be the simplest integrable Toda model).

6. TOPOLOGICAL AND STRING MODELS

The foregoing discussion did not address *gauging* the algebra of constraints (the algebra $\hat{H}_k$, in the case of GKO) as another possibility to describe any reduction. As usual, this requires supplementation of the original action (e.g. $\mathscr{A}_{\mathrm{WZNW}}$) with constraints multiplied by gauge fields and addition of terms taking into consideration the non-abelian structure of the algebra of constraints. The central charge defines the coefficient in front of the gauge field action, moreover, the quantum measure includes the functional integral over the ghost fields containing the interaction of ghosts with a gauge field in case of a non-abelian algebra and with fields of the original model when this algebra is not a Lie algebra. (By definition, the latter case is ruled out in the GKO model and in Yang–Mills theory and is not therefore discussed in the usual textbooks on elementary-particle physics, although it is possible in other situations). The states of the reduced theory are denoted as classes of BRST cohomology (this condition distinguishes them from a variety of functions dependent on all the fields of the original model, gauge fields and ghosts). The BRST formalism (95–98) is an important technical tool of the string theory since transition from conformal models to *string models* requires gauging the chiral algebra.

Methodological aspects of reduction as applied to the WZNW model have been elaborated in Ref. 99. The coset G/G, i.e., the case of $H=G$, occupies a special place among these reductions. Although, by definition, almost all degrees of freedom in this model are eliminated, something may survive. This is particularly easy to observe in the BRST formulation because it uses gauge fields and thus allows the existence of something inbetween a degree of freedom and the lack of it; configurations are feasible having vanishing field *tension* that are *not* purely gauge ones. The very existence, the number, and the properties of such configurations depend on the *global* structure of the space where the theory is defined (by boundary conditions). Therefore, *effects of this kind*[20)] are referred to as topological (or cohomological, to use a more correct term).

The WZNW model with completely gauged Kac–Moody symmetry is a typical example of topological theory in which all the observables (correlation functions) can be described in terms of BRST cohomologies. In fact, they may be reduced to cohomologies of the module space of flat G-connections on two-dimensional surfaces and provide an example of the description of sophisticated topologies in terms of functional integrals. The efficacy of such an approach to topological problems is associated with the possibility of appealing to physical intuition and using more specific methods for the analysis of functional integrals (e.g., quasiclassical calculations). Moreover, this approach allows a gradual transition from topological investigations to the analysis of more interesting algebro-geometric structures on the same space (merely by *incomplete* reduction of the original model).

It is known that the BRST formalism is very close to supersymmetry but contains half the number of generators as compared with the ordinary Gol'fand-Likhtman supersymmetry[102] (in fact, only one in the two-dimensional situation in question) and has nothing to do with shift generators, i.e. with the energy-momentum tensor. In ordinary supersymmetry theories, one can ignore the energy-momentum tensor in examining vacuum states and describe the vacuum sector in terms of topological models. This sector can be rather complicated in supersymmetry models and often contains "valleys" and "plateaus" due to cancellation of quantum corrections which eliminate vacuum degeneracy in other situations.[21)] Instead of the advantages provided by the algebraic structure in case of the WZNW model, construction of topological models based on supersymmetric ones provides an opportunity to vary the form of the action, usually the potential. So far, the most striking example of using this possibility is the establishment of a relation between topological models (their classification) and catastrophe theory.[115,116] Also, this appears to be the simplest way to explain their relation (already known) with integrable equations.

As a matter of fact, the term "topological" ought to be applied to the field theory models that are independent of the space-time metric and coordinates. There are two possibilities to build up such a theory (reminiscent, by the way, of the two scenarios of interaction unification discussed in a previous section). First, it is possible to construct a model in which even the classical action or at least the equations of motion do not depend on either the metric or the coordinates (e.g. a model having Weyl invariance with respect to metric rescaling, $g \to hg$) and further require cancellation of anomalies in order to avoid development of such dependence due to quantum corrections. (Also, it is not forbidden to demand that the corrections be cancelled between different orders of perturbation theory). Such a possibility is realized in the above examples of Hamiltonian reductions and supersymmetric models and also in more complicated cases, such as those of the Chern–Simons topological models,[117,118] when the number of dimensions D is odd (with Lagrangian $d^{-1}\mathrm{Tr}\, F^{(D+1)/2}$, e.g. $A\,\mathrm{d}A + (2/3)A^3$ for $D = 3$), or topological θ-models for even D (named after the θ-term; the Lagrangian is $\mathrm{Tr}\, F^{D/2}$ e.g. $\mathrm{Tr}\, F\tilde{F}$ for $D = 4$). Another possibility is to choose a theory containing a metric-dependence and integrate over the metrics (over two-dimensional metrics when the models are two-dimensional). Association with the previous variant is apparent at least if the original model is conformal.[22)] It has already been said that integration over two-dimensional metrics may be described as (i) an introduction of new fields (Liouville field and reparametrization ghosts) leading to the modification of the model which leaves it conformal but makes the central charge vanishing, followed by (ii) gauging the Virasoro algebra. Such interpretation results, at the last stage, in a more or less routine reduction procedure and yields a topological field theory. However, this theory differs from those discussed previously in one aspect. This difference is manifested in Ward identities which remember either the original symmetry of the model or its operator algebra (i.e., they are "projections" of the latter on the topological sector; see Refs. 120, 121 for examples of such recursion relations). The difference is supposed to be due to the fact that the integral over the metrics (hence, over Liouville field) is not ordinary (101): the metrics is restricted by the condition of *positive definiteness*; therefore, the integration domain in metric space has a boundary which can make a contribution to the integral. In fact the *topology* of the two-dimensional space is altered and this gives rise to recursion relations connecting correlators on surfaces of different topology. We shall return to this question in the last section. Now, it is time to recall from where the integral over metrics came to string theory and how it can be calculated.

7. PERTURBATION THEORY OF STRINGS OR MASSLESS FREE FIELDS ON RIEMANN SURFACES

Coming a few steps back, there is a problem of summation over surfaces with weight $e^{-M^2 S}$, where S is the surface area measured in the external metric $G_{ij}(x)$ of $(D-1)$-dimensional space:

$$S = \int (\det_{(ab)} G_{ij}(x) \partial_a x^i \partial_b x^j)^{1/2} \mathrm{d}^2\xi. \tag{4}$$

The sum over surfaces implies summing over all the smooth embeddings including those of different topology. In the picture of strings-particles, it means that an interaction is introduced and "unitarization" performed. Conformity between contributions of different topologies at this stage may be achieved if a constant of string interactions (such a parameter is a priori intrinsic in string models, although *a posteriori* it frequently has no special interest due to such phenomena as dimensional transmutation) is introduced as the coefficient in front of the Euler characteristic in two-dimensional action (moreover, it is necessary to co-ordinate "volumes" of integration for different topologies, a separate problem to be solved within the framework of the theory of a universal module space). It is natural to consider the sum over surfaces as a functional integral in two-dimensional field theory. It was already indicated that the action in this integral depends only on embedding of the surface into $(D-1)$-dimensional space and is independent of the choice of coordinates ξ and metric $g_{ab}(\xi)$ on the two-dimensional world surface. Reparametrization invariance of the action (a sort of gauge symmetry) is usually not difficult to preserve in functional integration, but things are quite different as regards metric-dependence—Weyl invariance is as a rule broken. This alone precludes the use of Eq. (4) as a reasonable *effective* action of the two-dimensional theory since it must depend on g_{ab}. But such a dependence in its turn, suggests that the sum over surfaces must include summing not only over embeddings but also over two-dimensional metrics. As far as the action is concerned, formula (4) is equally unsuitable and because of strong nonlinearity it would anyway be impossible to work with such action. Polyakov formalism[56] in string theory implies the necessity to take

$$\int G_{ij}(x)\partial_a x^i \partial_b x^j g^{ab}(\det g)^{1/2}d^2\xi. \tag{5}$$

as a substitute for the action of Eq. (4). At least in principle, the measure in the integral over the metrics must be defined by norm:[56]

$$\|\delta g\|^2 = \int \delta g_{ab}\delta g_{a'b'} g^{aa'}g^{bb'}(\det g)^{1/2}d^2\xi,$$

although in practice, simpler measures are employed (Refs. 122–125).[23)] In order to calculate functional integral one has to fix the reparametrization invariance and introduce corresponding ghost fields (b,c) with spins 2 or -1 respectively.[56] The metric retains the only true degree of freedom, the Liouville field $x_0(\xi)$:

$$g_{ab}(\xi) = g^{(0)}_{ab}(\xi)e^{Mx_0(\xi)},$$

and, as has been mentioned already, the x_0-dependence after integration over the fields x_i, b, c emerges in the effective quantum action (actually, due to x_0-dependence of the determinants and Green's functions of Laplace operators). The correct quantum action, stable with respect to quantum corrections has the form of Eq. (2), with the Minkowsky signature appearing automatically (if $D \leqslant 26$).[24)]

In the case of a bosonic string examined here as the simplest example, it is necessary to replace dots by ghost action and to make corrections related to topology and interfering, generally speaking, with D-dimensional Lorentz-invariance:

$$\begin{aligned}\mathscr{A}_{\text{bos}} \sim \int (M^2 G_{\mu\nu}(x)\partial_a x^\mu \partial_b x^\nu g^{ab}_{(0)} + \alpha M x^0 \mathscr{R}_{(0)} \\ + b^{ab}\partial_a c_b)(\det \bar{g}_{(0)})^{1/2}d^2\xi + \dots .\end{aligned} \tag{6}$$

There is no need to integrate over the metric $g^{(0)}_{ab}(\xi)$: this is a reference metric with curvature $R_{(0)}$. $g^{(0)}_{ab} = \delta_{ab}$ might be suggested as the simplest choice, but in this case $R_{(0)} = 0$ which makes it clear that such a simple option is inapplicable to most of the cases: according to the Gauss–Bonnet theorem, the integral $\int \mathscr{R}_{(0)}\sqrt{\det g_{(0)}}\, d^2\xi$ is proportional to the Euler characteristic of the surface and is not zero. The parameter α in Eq. (6) varies with the model type (in the given case it depends on D: $\alpha \sim D - 26$). The dots stand for corrections. In the first place, they take into account the contribution of the boundaries of the two-dimensional surface (if there are any), and secondly, they restore conformal invariance. In other words, formula (6) is accurate if the two-dimensional surface is closed and if the *space-time* metric $G_{\mu\nu}(x)$ is chosen to ensure the vanishing of the β-function, e.g. if $G_{\mu\nu}(x) = \delta_{\mu\nu}$.

Similar reasoning appears to be equally relevant in more complicated situations than the bosonic string model of Eq. (4). The original two-dimensional action may be more sophisticated than simply the surface area, it may even from the beginning depend on a two-dimensional metric although in this case the determination of the correct effective quantum action (two-dimensional conformal theory, an analog of Eq. (6)) becomes a separate problem.

The metric retains discrete (topological) degrees of freedom, besides Liouville-like ones. Their existence can be attributed to the impossibility to reduce, by means of surface reparametrization, all metrics to $g_{ab} = g^{(0)}_{ab}e^{Mx}0$ with *the same* $g^{(0)}_{ab}(\xi)$ because all equivalence classes of the metrics differ not only in the choice of $x_0(\xi)$ but of $g^{(0)}_{ab}(\xi)$ as well. However, while the former arbitrariness is *infinite-dimensional*: (x_0 may be any function of two-dimensional co-ordinates), almost all the $g^{(0)}_{ab}(\xi)$ are reparametrizationally equivalent: there is only one finite-parametric family of reference metrics $g^{(0)}_{ab}[y](\xi)$. The parameters y are related to the modules of the complex structure of the two-dimensional surface and form a module space, their number (module space dimension) being a function of topology. Integration over all metrics implies not only the integral over the Liouville field but also a *finite-dimensional* integral over the module space. In fact, introduction of the Liouville field and reparametrized ghosts brings us back to the situation without metrics but with a number of corrections: first, the proper approach was used; second, the action of Eq. (6) contains extra fields x_0 which were absent in Eq. (2); third, it does not contain a square root; fourth, the metric left an additional trace, i.e., dependence on the modules of complex structure on the surface.

The latter finding implies that in string model studies, we actually have to deal with Riemann surfaces; moreover it emphasizes the role of conformal models which were shown to be associated with the complex structure (see above). Thus, for a given topology, the integral over the metrics is identical to the integral over the module space of a certain variable (correlator) in the conformal theory. It is clear that one can not integrate any correlator—it must be independent of the co-ordinates on the surface. In practice, this requirement means that integration should be confined either to correlators of zero-dimensional vertex operators or to integrals of unit-dimensional operators (conventionally referred to as "observables." From the more formal point of view, the observable must be an element of BRST-cohomology which brings us back to the idea that the string model emerges from the conformal one when the Virasoro algebra is gauged. In connection with this, it is appropriate to recall that any chiral algebra is subject to gauging, and the "observables" in the corresponding "string model" can be represented in the form of integrals over the module space. Sometimes such an approach does not encounter serious difficulty (for example, in the case of the module space of flat connections associated with gauging the Kac-Moody symmetry), but certain cases, e.g. modules associated with W-symmetry or problems of W-gravity[86–93] remain to be clarified.

The module space of closed oriented Riemann surfaces with punctures is in itself a complex *orbifold*, having discrete cone-like singularities at the points associated with additional discrete symmetry. Moreover, the module space is *a priori* non-compact and its boundaries correspond to surface degeneration (pinching handles, colliding punctures, etc.). It has a highly complicated topology: specifically, there are modular equivalent domains on the Teichmuller covering space.[35,125] Modular transformations allow collision of the punctures to be considered as an equivalent of a particular handle pinching and module space to be compactified by addition of only singular surfaces with pinched handles (in the absence of coinciding punctures). Following such a Deligne–Mumford compactification, the topology of module space becomes equivalent to the space module topology of fat graphs and easy to investigate.[128,129] The complex dimension of the module space $\mathcal{M}_{p,n}$ is $3p-3+n$, where p is the number of handles (genus) and n is the number of punctures (exceptions from the rule are $\dim_{\mathbb{C}}\mathcal{M}_{0,n}=0$ for $n=0,1,2,3$ and $\dim_{\mathbb{C}}\mathcal{M}_{1,n}=1$ for $n=0,1$, and can be attributed to the existence of *conformal Killing vectors*, overall holomorphic vector fields with zeroes in punctures).

An objective of the perturbation theory of strings is to devise expressions for "multi-loop amplitudes" in various string models. In practice, it implies defining correlators of vertex operators in arbitrary conformal models on arbitrary Riemann surfaces. This problem being solved, it remains to select a specific class of observables, i.e., vertex operators with correlators structured like *measures* on corresponding module spaces with punctures designating the positions of the vertexes. A somewhat excessive solution of the problem (identification of *all* correlators) is a specific feature of the *perturbation theory of strings* which, on the one hand, facilitates its broader application, but on the other hand, lessens its adequacy for the specific problem of *string* model studies as compared with certain alternative methods (see the section on "non-perturbative" approaches below).

The perturbation theory of strings is based on the formalism of free massless fields (Refs. 125, 130, 131), i.e. the Gauss field theory with the action

$$\int \partial_a\phi\partial_b\phi g^{ab}(\det g)^{1/2}d^2\xi.$$

The field ϕ is free in that it is lacking self-action (vertexes of the ϕ^3, ϕ^4, and other types). However, this theory is not altogether trivial due to the interaction with the background metric g^{ab}. Its basic objects are the determinants and the Green's functions of the Laplace operators $\Delta=g^{ab}\partial_a\partial_b$ on the Riemann surface. They all depend on both the Liouville field and the modules of complex surface structure; moreover, one can easily find objects which holomorphically depend on the modules (i.e., the dependence is compatible with the complex structure on the module space). It seems appropriate to examine the Laplace operators Δ_j defined for the fields of any spin j (they are sometimes referred to as *j—differentials*). The determinant formula, for example, looks like this:

$$\operatorname{Det}\Delta_j=\exp(c_j\mathcal{A}_{\mathcal{L}})\det N_j\det N_{1-j}|\operatorname{Det}\bar\partial_j|^2, \tag{7}$$

where $c_j=2(6j^2-6j+1)$ is the central charge of the Virasoro algebra for j-differentials and

$$\mathcal{A}_{\mathcal{L}}\sim\int\partial_a x^0\partial_b x^0 g^{ab}_{(0)}(\det g_{(0)})^{1/2}d^2\xi\sim\int\mathcal{R}\frac{1}{\Delta}\mathcal{R}$$

is the Liouville action, while two finite-dimensional determinants $\det N$ take into account the contribution of zero-modes of the operators $\bar\partial_j$ and $\bar\partial_j^{+}$ (holomorphic j and $(1-j)$ differentials respectively). The main part of the formula is the chiral determinant $\operatorname{Det}\bar\partial_j$ which is holomorphically dependent on the complex coordinates on the module space, $\partial\operatorname{Det}\bar\partial_j/\partial y=0$, and is a section of the determinant bundle over the module space. The operator $\bar\partial_j$ acts from the space of the j-differential into the space dual to the space of $(1-j)$-differentials. If the fields (j-differentials) over which the integration is carried out in the functional integral are multi-valued on the Riemann surface, $\operatorname{Det}\bar\partial_j$ also becomes an analytic function of the boundary conditions (in fact, a section of the aforementioned bundle of flat connections over the module space) while the Laplace action is supplemented by an additional term, the so-called Quillen anomaly.[35,130,132] This dependence on boundary conditions plays a key role in the theory of integrable systems. Explicit formulas for $\operatorname{Det}\bar\partial_j$ express it in terms of special functions: Jacobi and Riemann theta-functions.[133,134] Jacobi theta-functions are defined as holomorphic functions on g-dimensional complex tori the phases of which are shifted after going around non-contractable contours by a linear function of co-ordinates.[25)]

Tori differ in the choice of lattices defined by symmetric complex nondegenerate matrices T_{ij} ($i,j=1...g$) with positively-defined imaginary parts; a set of such matrices forms an upper Siegel half-space. Each closed Riemann surface of genus g can be embedded using a Jacobi map into a g-dimensional complex torus (called its Jacobian) the corresponding T_{ij} being holomorphically dependent on modules of the surface and referred to as period matrices. There is more than one embedding of the same surface possible, different embeddings being related to modular transformations

$$T\to\frac{AT+B}{CT+D},$$

which form the $\mathrm{Sp}(g,\mathbb{Z})$ group. Jacobi theta-functions obtained in this way, with the argument being a function of a point on the *surface* and T its period matrix, are referred to as Riemann theta-functions. It should be borne in mind that *not all* Siegel matrices are period matrices (for $g>3$). The problem of identification of a subset of period matrices in Siegel space and, hence, the problem of independent identification of Riemann theta-functions, is known as the Shottky problem. It has a simple solution in the form of a transcendental equation for $g=4$ which is of importance in the superstring model,[135,136] but an effective general solution remains to be found. An implicit solution is known in the form of the Novikov hypothesis[137,139] (which has now been proved) according to which the *Riemann* theta-functions are characterized by the fact that they satisfy a system of nonlinear equations, actually the Kadomtsev–Petviashvili (KP) hierarchy.[3,5] Riemann theta-functions together with information about their zeroes are practically all that is needed to construct Green's functions or any other variables in free field theory.

Formula (7) possesses the property of holomorphic factorization with which we have already become familiar when discussing conformal theories: correlators are representable as bilinear combinations of holomorphic conformal blocks. Although an integral (over module space) is also an example of a bilinear combination, it is sometimes useful to talk about holomorphic factorization of correlators *before* integration over module space. For instance, bilinear combinations in the case of rational conformal models contain a *finite* number of conformal blocks before, not after, integration. The property of holomorphic factorization is also useful for identifying string models on open and nonoriented surfaces. Corresponding module spaces are *not* complex, but they can be embedded as half-dimensional real subspaces into complex module spaces of closed surfaces, using the technique of doubles. Moreover, measures on these subspaces are defined through chiral components of the full measure (see Refs. 140–142).

Measure on a module space is one of the simplest variables in string theory. It defines the partition function (vacuum amplitude) of a 26-dimensional boson string and is known as the Mumford measure, having been extensively investigated in various parametrizations of the module space.

Parametrizations deserve a special discussion since their different varieties are useful for different purposes.

(A). It is very convenient to use period matrices for the analysis of modular properties (the theory of modular forms being one of the key subjects of the classical theory of *elliptic* functions). Unfortunately, the Shottky problem makes it very difficult to use this approach for $g>4$ (although the theory for $g=2,3,4$ is also underdeveloped; see Refs. 143, 144 for string applications).

(B). A very useful approach is to consider Riemann surfaces as algebraic manifolds. The problem is constituted by the lack of a simple and single-valued parametrization of a fixed-genus module space. An important exception is genus 2 (and 1) when all the Riemann surfaces are hyperelliptic. The formalism of free fields for hyperelliptic surfaces (and abelian coverings of the Riemann sphere in general) is especially simple: everything is expressed in terms of hyperelliptic integrals. It is very often applied to the preliminary analysis of complex topology effects in the studies of new models. Moreover, certain special problems, e.g. the Ashkin–Teller model[145] or the theory of the Korteveg-de-Vries equation[3] can be reduced to the hyperelliptic formalism. Coming back to the *generic* problem of representing Riemann surfaces as algebraic ones, note that the problem of description of determinants and Green's functions in terms of defining equations has never been solved. There are two goals to strive for. First, such a representation must be convenient for a uniform interpretation of all the genera and, in the end, for summing over the genera (see for instance chapter 12 in the review of Ref. 130). Second, algebraic surfaces may be defined over arbitrary numeric fields; on this route, connections between string theory and important sections of algebraic geometry can be established, the closest of them being the Arakelov theory:[146] see Ref. 51 for the p-adic string theory dealing with these problems (although it remains to be further elaborated).

(C) Parametrization of module space may be achieved by choosing a surface, cutting out a neighborhood of a point (or points), "rotating" this neighborhood, and gluing it back. The formalism based on this idea is referred to as the Krichever construction[147,148] and is currently considered the most useful one. Its major tool is an analysis of holomorphic sections of different bundles extendable or *non*-extendable inside or outside the cut-out neighborhood. Module space is normally associated with sections extendable neither inside nor outside.[149] Moreover, changes of surface topology (e.g., handle gluing operators) can be described in similar terms.[150] The so-called string operator formalism[151–153] and representation of a universal module space[154–157] in the form of an infinite-dimensional Grassmanian[148] are also associated with the Krichever construction. This construction is equally important for an analysis of integrable problems; in fact, it was originally suggested for this purpose.[147]

Another constituent of the formalism of free massless fields is a reformulation of various conformal models in terms of a variety of free fields with quadratic actions. Traditionally, this procedure is sometimes called bosonization. Generally speaking, this problem is an example of a search for action angle variables. However, it is important that a transition to these variables in the case of two-dimensional conformal models is virtually local. Free field representation has been reported for the WZNW model[158] and for various types of its reduction.[79,159] It is especially convenient for the description of a Verma module; and as applied to this range of problems, it is often referred to as the Feǐgin-Fuks-Dotsenko–Fateev–Felder formalism.[160–163]

The Feigin–Fuks–Felder operators or screening operators are used to describe in these terms irreducible representations and, hence, rational conformal models. (See Ref. 158 for their Lagrangian interpretation). One more important class of conformal theories, $N=2$ supersymmetric sigma-models,[164,165] appears to reduce to free fields using the Nicolai transformation; however, very much remains to be elucidated concerning this class (see Ref. 166; for further examples see also Refs. 167, 131. On the whole, applicability of the formalism of free massless fields (i.e. the existence of a replacement of variables) ought to be regarded as a *definition* of a conformal theory.

There is a little bit more delicate relation between this formalism and two-dimensional integrable theories.[168] Dependence of the chiral determinant Det $\bar\partial$ on boundary conditions imposed on the fields on a surface has been mentioned earlier. Boundary conditions for a closed surface are parametrized by the points of a Jacobian (when the fields are sections of a linear bundle over the surface; in case of multidimensional bundles, the Jacobian is replaced by the space of maps of the surface fundamental group into the structure group of the bundle; the following reasoning is equally valid for this general situation). It is possible to define a set of commuting dynamic flows on the Jacobian, e.g., simply uniform rectilinear motion (windings) with different directions and velocities, and examine $\tau\{t\} = \text{Det}\,\bar\partial$ as a function of corresponding "times" $\{t\}$. This τ-function [Refs. 169–171] satisfies nonlocal bilinear Hirota identities the infinitesimal version of which is represented by an infinite set of compatible (due to commutativeness of the original flows) differential equations that form an integrable hierarchy (Kadomtsev–Petviashvili or the integrable hierararchy of the Toda chain in the case of linear bundles on arbitrary Riemann surfaces, and Korteveg–de Vries hierarchy in the case of hyperelliptic surfaces, etc.). Such a view of integrable theories (evidently, the most reasonable one) suggests the absence of space-time interpretation of their "times."

Sometimes, it is possible to consider lower equations of the hierarchy, e.g., the sine-Gordon equation, to be two-dimensional relativistic systems; such an interpretation may be useful in the construction of integrable interpolations between individual conformal models (we have already mentioned this idea in the context of inductive design of a "unified field theory" (see Refs. 172, 173 for examples of specific results concerning interpolations defined by the renorm-group flow). It should be emphasized that in such cases the Riemann surface used in the general construction ("spectral parameter surface") is not directly related to two-dimensional space-time: an adequate description of their interrelation is likely to be achieved in the double-loop theory (see Refs. 40, 41).

8. NONPERTURBATIVE STRING THEORY

The next objective of string theory following the development of calculations on Riemann surfaces (discussed under the heading of perturbation theory of strings) is to sum over all such surfaces including summation over topologies. A direct calculation of integrals over module spaces and a summing up of the resultant values are equally hopeless and senseless. What is really important is to reveal new *structures* supposed to come to light as a result of summing over modules and topologies. Those believing in the string program as presented in the introductory portion of the present article will, from the very beginning, understand that these new structures should allow a unified approach both to all the Riemann surfaces regardless of their topology and (which is equally important) to all the string models irrespective of the underlying conformal theories.

This second aspect of nonperturbative calculations acquires special significance in the context of the *comprehensive* string theory understood as the "unified field theory" or dynamic unification of all string models into a single whole. *Perturbative* analysis can not be expected to take into account such a whole, even if it exists, because perturbative expansions close to different extremums are defined only by their immediate neighborhoods and show no interrelation whatever. On the contrary, *nonperturbative* (exact!) results know everything: it means, if applied to our situation, that a nonperturbative calculation for a single string model must actually know about all the others! Of course, this is not the case with a rigorous deductive approach which may account for the absence of a well-defined method of nonperturbative calculations in quantum mechanics: perturbative series are normally asymptotic (diverging) and may be summed up with a given variable only after additional "nonperturbative" information (usually, about analytic properties of the potential and its behavior at infinity) is available.

All this indicates that additional information must be *postulated* in the form of some new principles taking into consideration the current state of string theory which is yet to be *invented*. However, these principles are expected to meet very rigid requirements, that is whatever may be declared to be the nonperturbative answer, its perturbation theory expansion must reproduce the sum over surfaces. Therefore, the safest way is to start with attempts at summation in search of new principles instead of trying to postulate directly. This process is far from being completed which makes accentuation even less reliable here than in other sections of this article.

The most straightforward ways to summation over modules and topologies originate from the problem of a unified description of all Riemann surfaces. The basis of this approach is the universal module space (UMS), a combination of all module spaces in which spaces $\mathscr{M}_{p-1,n+2}$ (pinching handle), $\mathscr{M}_{p_1,n_1+1} \times \mathscr{M}_{p-p_1,n-n_1+1}$ (pinching cycle homologous to zero), and $\mathscr{M}_{p,n-1}$ (coincidence of marked points) are attached to the boundaries of $\mathscr{M}_{p,n}$. Different definitions of UMS may exclude some of these gluings and determine specific gluing rules (see Ref. 154). One of the purposes of introduction of such a space is to fix relative norms of the measures on spaces of different topology: this is often referred to as factorization conditions (and their corollaries for correlators of observables, i.e., for *integrals* with these measures, are termed recursion relations). Construction of UMS and string measures on UMS is performed "from the top down", i.e., from complex topologies to simpler ones. Actually, the factorization condition defines the measure on $\mathscr{M}_{p,n}$ as a restriction of the measure on the module space with higher p and n, and the problem of summation of perturbation theory series consists in "guessing" what should be called the measure on $\mathscr{M}_{\infty,\infty}$.

To a certain extent, the inverse procedure exploits the idea of a handle gluing operator (HGO). To sum up a series means in this language to exponentiate the HGO integrated over handle size and location (in fact, a whole family of HGOs graduated by codimensions which is necessary for the assessment of interaction between handles). Perhaps, this problem is not so hopelessly unsolvable, especially if an effective description of surfaces as (non-abelian) coverings over the Riemann sphere should be found. As a matter of fact, the most effective technique in operations with UMS and HGO is associated with the Krichever construction (string operator formalism), and the UMS is usually represented as a factor of some version of an infinite-dimensional Grassmanian. This Grassmaninan is in turn explicitly related to algebras of the $\mathrm{Gl}(\infty)$, $\mathrm{Sl}(\infty)$ or $W(\infty)$ types and through them to the corresponding WZNW models and their reductions. The latter include a large number of conformal models, and for this reason, the Grassmanian appears to be involved in the problem of describing a set of all conformal models ("phase space") of the comprehensive string theory.

Considerable methodological advancement in the field of nonperturbative calculations that allowed more accurate formulation of our expectations with respect to this issue is associated with the formalism of matrix models.[174–176] This is a common tool of discrete mathematics that has for a long time been employed in statistical physics and has proved to be a most effective alternative to the Polyakov formalism as regards string (but not conformal) models. Application of this method aims at the construction of a Regge calculus for two-dimensional gravity. However, it differs from the standard Regge formalism in that it is confined to triangulations with fixed (and equal) lengths of the sides of triangles. Despite the fact that the cause of the equivalence of quantum gravity as *defined* above and Liouville theory remains obscure,[26)] answers obtained by either method invariably coincide.

From the methodological point of view, the value of such modification of the Regge technique is evident: in the absence of summation over side-lengths of triangles, any triangulation can be replaced by a dual graph with triple vertexes whereas the problem of enumerating such graphs is the problem of evaluating the integral

$$\int dm\, \exp(m^2 + tm^3).$$

It is certainly interesting to analyze *correlators*, i.e., to introduce excitations into triangulations. The simplest approach to the description of excitation is to replace some of the triple vertexes of the graph by quadruple ones, etc. If the contributions of triangulations of different topology are in addition to be distinguished, number-valued integration variables m should be replaced by square $N \times N$ matrices M using the t'Hooft method. In other words, it is necessary to consider the integral

$$\mathcal{L}_N\{t\} = \int dM \exp(\sum_k t_k \mathrm{Tr}\, M^k). \tag{8}$$

This integral defines the partition function of a discrete matrix model. In order to obtain the correlators in a string model, it is necessary to take the continuum limit, $N \to \infty$. Continuum limits may differ depending on the expected behavior of the coefficients $\{t_k\}$. It is possible to restore the expansion in topologies (in inverse powers of N) in one of such limits. A more interesting double-scaling limit[177-180] yields the sum over all topologies. It readily appears from formula (8) that taking limits implies certain analytical continuations (more than that, the continuum limit by its definition implies singular behavior of $\mathcal{L}\{t\}$, i.e., the divergence of integral (8). Therefore, it is necessary to define some reasonable properties of the partition function (8) which are preserved on taking the limit. Such properties may include *equations* (Ward identities) imposed on $\mathcal{L}\{t\}$. They have the form of Virasoro constraints[181,185]

$$l_n \mathcal{L}_N\{t\} = 0, \quad n \geq -1,$$

where, in the case of model (8).

$$l_n = \sum_k k t_k \partial/\partial t_{k+n} + \sum_a \partial^2/\partial t_a \partial t_{n-a},$$

$$[l_n, l_m] = (n-m) l_{n+m}.$$

Another mode of putting them down are the recursion relations mentioned above. These identities result in $\mathcal{L}_N\{t\}$ being a τ-function of the integrable hierarchy of the Toda-chain,[184] i.e., it satisfies the *bilinear* Hirota equations. The variable that satisfies the equations of Toda hierarchy themselves is

$$\varphi_N = \log(\mathcal{L}_{N+1}/\mathcal{L}_N).$$

It is convenient to take the continuum limit directly in these terms, trying to preserve the integrable structure. The double-scaling limit is the one in which the hierarchy of the Toda chain turns into the Korteveg-de Vries (KdV) hierarchy, and the nonperturbative partition function of the two-dimensional quantum gravity is a τ-function of the KdV hierarchy $\tau\{T\}$ (the times T are selected linear combinations of the times t and N) satisfying the system of Virasoro constraints (somewhat different from those described above). This τ-function may itself be represented as a matrix integral quite different from (8):

$$\mathcal{L}\{T\} \sim F(Q) \int dX \exp V(X) + V'(Q)X) \tag{9}$$

with $V(X) = X^3$, $T_n = n^{-1} \mathrm{Tr}\, Q^{-n}$, $F(Q)$ is a function constructed in a particular manner over $V(X)$. This formula is known as the Kontsevich model.[129,186] There is evidence[187-190] that the same τ-function has a purely topological meaning, i.e., it defines the generating functional for topological invariances: Chern classes of divisor bundles over module space. Different modes of definition of topological models (directly reflecting topological properties and arising from averaging over metrics) lead to identical results, as was to be expected.

Matrix models can also be constructed when quantum gravity occurs along with the initial conformal model. At present, the results are available only for the Virasoro (p,q) minimal models: they all have $c < 1$. Remarkably, nonperturbative partition functions for all models with $q = 2$ are defined by the same formulas (8) and (9) (Ref. 191): the nonperturbative answer for one model does indeed know about the existence of another! An analog of formula (8) for models with $c > 2$ is more complicated, being represented by multi-matrix models[181,192] with the number of matrices being $q - 1$; at the same time, there is still only a single matrix in formula (9): it is only necessary to replace the potential by $V(X) = X^{q+1}$. The KdV hierarchy is replaced by the q-reduction of the KP hierarchy and the Virasoro constraints make room for the $W_{\mathrm{sl}(q)}$-constraints. Therefore, the Kontsevich model (9) with an arbitrary $V(X)$ is close to the nonperturbative partition function for the entire class of string models,[186] i.e., they are all associated with the KP hierarchies and the "Toda lattice" or, what is the same, with *linear* bundles over Riemann surfaces.

From the space-time viewpoint, all these models may be interpreted as $D = 2$ string models (the Fateev–Dotsenko scalar field that arises from bosonization of minimal models plays the role of the x_1-coordinate while the Liouville field as usual performs the function of x_0). The restriction $c \leqslant 1$ is attributed to the requirement of the absence of tachion excitations, i.e., to the stability of the phase described by a given model (it can not be ruled out that other models of heterotic strings without tachions may give rise to new classes of nonperturbative models). Of special interest are the so-called "$c = 1$ models" in which the Gaussian theory of the x_1 field taking values in a circle or in a segment of length R stands for the original conformal theory. Interest in these models can be accounted for by: first, the presence of the free parameter R, second, the expected appearance of W_∞-conditions on the nonperturbative partition function, third, the presence of a *continuous* (rather than discrete, as in other cases) family of observables at $R = \infty$, and last, but not least, the necessity of improving the available technique of nonperturbative calculations for the description of the $c = 1$ models (reasonable matrix models determining the nonperturbative partition function and containing R-dependence have never been reported).

The moral to be drawn from these first nonperturbative results of string theory is that an integrable structure may be the natural structure that arises as a sequel of summation of perturbative series. Evaluation of a nonperturbative partition function requires that all perturbations contained in the theory be exponentiated. An effective action in such a situation is sure to be infinitely parametric. This suggests a rich symmetry associated with the freedom of arbitrary variable (field) substitutions in the functional integral (which is usually lacking due to the requirement of minimality motivated, for instance, by renormalizability). This symmetry (apparent as Virasoro constraints which were examined in the above examples of matrix models) reflects the invariance of the partition function with respect to a very broad class of substitutions of its own arguments ("times"). This freedom may be used to make an optimal choice of "times." It ensues from matrix models that such a choice may include partition functions that turn out to be τ-functions of integrable hierarchies.

It is worthwhile to mention one more viewpoint of the same results. The Virasoro constraints and their $W_{\mathrm{sl}(q)}$-generalizations (i.e. recursion relations) can be interpreted as symmetries induced by the operator algebra of vertex operators when restricted to the set of observables (primary fields of definite dimensions). This algebra of observables is a truly invariant characteristic of the model independent of the type of the surface, modules, and integration over metrics. Investigation of algebras of observables and reformulation of available information about string models in terms of these algebras are the most important current problems of string theory. (See Refs. 193, 194 for very interesting but preliminary results of the studies on $c = 1$ models, including those pertaining to four-dimensional geometry.) Gauging these symmetries ("third quantization" and construction of adequate "string field theory") will be a further development in string theory.

Setting apart all these details, the general scheme of designing a string theory using the inductive approach ("from the bottom up") as the most adequate one at the stage of theory invention may be described as follows:

(i) Take any conformal model and select a chiral (sub) algebra in its operator algebra.

(ii) Build up a corresponding string model by gauging the chiral algebra. The "remaining" operator algebra (its "projection" on the classes of the BRST-cohomologies associated with the chiral algebra or, simply, its "factor" with respect to the chiral algebra) makes up the algebra of observables. We note that the algebra of observables includes operators that affect topology. The string model may be described as a new conformal model (with extra Liouville and ghost fields; with the zero central charge of the chiral algebra modified with regard to these fields; and with permission to consider only a part of the vertex operators, i.e., the "observables") or as topological gravity (based on the fact that correlators of the observables are topological invariants). In the former case, the algebra of observables may be regarded merely as a fragment of the operator algebra of a new conformal model (with all the descendants with respect to the modified chiral algebra being ignored). In the latter case, algebra of observables has the form of recursion relations.

(iii) Design a "nonperturbative string model" by gauging the algebra of observables. Gauged fileds introduced in this way are associated with observables and interpreted as *physical fields* (in space-time, if any). From the two-dimensional viewpoint, these fields "perturb" the action of the string model, i.e., convert it to another string model, the nonperturbative partition function being the same for all models. It was already mentioned that stage (iii) has until now been examined only for a single class of conformal theories, the minimal Virasoro series. The "times" T_n in the theory of Eq. (9) play the role of *physical fields*. Investigation of this case reinforces belief in two major characteristics of a nonperturbative partition function: first, it is actually universal if regarded as a function of physical fields (one for the entire class of minimal models); second, it is integrable, being a τ-function. These two statements appear to be the main predictions of string theory for the theory of fundamental interactions. However, it will take many efforts to clarify the assertion that the exact effective action of the standard model as a function of Yang–Mills fields is a logarithm of a τ-function (i.e. satisfies special loop-equations associated with the Hirota bilinear relations).

As far as string theory itself is concerned, its problems naturally include a search for a self-contained *a priori* definition of a universal nonperturbative partition function. It is only after such a truly fundamental principle has been found that a *deductive* construction of string theory and the "correct" view of the entire range of problems known under this name will be possible.

9. INSTEAD OF A CONCLUSION, OR A COMMENTARY ON STRING MODEL BUILDING

Instead of appraising achievements of string theory at large, it appears more expedient to briefly outline developments in the field of science that gave rise to this theory. It may be useful with a view to evaluate more accurately the discrepancy between the expected and obtained results and reveal major trends in the divergence. Now, what have we got as STRING MODELS OF GRAND UNIFICATION? Almost by definition, there is a class of critical finite string models in the low-energy limit of which the standard model or any defined model of the Grand Unification might originate. This class appears to be boundless, as long as it is considered in terms of the more or less understood *perturbative* physics of strings.

The earliest and most famous model is the ten-dimensional heterotic string model with the $E_8 \times E_8$ gauge group compactified on the Calabi–Yau manifold with dynamically broken supersymmetry at low energies (see Refs. 38, 39, 195). Construction of this model suggests string compactification from 26 to 10, and thereafter to 4, dimensions (it is the requirement of the "elegant" intermediate ten-dimensional stage that accounts for the early illusion that the model is unique). A generalized procedure of this type has the aspect of a construction of a large family of so-called four-dimensional strings[196–199] many of which are equally acceptable from the phenomenological point of view.

All the difficulties in determining the "optimal" string unification model are the same ones as those inherent in the conventional scenario of the Grand Unification: the experimental information is utterly insufficient not only to determine the theory structure near the Planck scales (where the difference between the string and any other unification first becomes apparent), but even to choose between the Great Desert scenario and models designed within the framework of the technicolor concept (also, to elucidate the possibility of four-dimensional supersymmetry in the world of elementary particles at lower-than-Planck energies). These examples illustrate the *current* problems of experimental physics. (Investigation of the effects at the level of radiation corrections in the Glasow–Weinberg-Salam model is about to exclude the most meaningful of the technicolor models which strengthens the belief in the reality of the Grand Desert; however, recent measurements of three constants of fundamental interactions: $\alpha = e^2/\hbar c$, the Weinberg angle, and Λ_{QCD}, are most likely to support the supersymmetric scenario—introduction of superparticles facilitates the intersection of lines of renormgroup evolution of the *three* parameters in *one* point.[53]

Quantum gravity effects, the only ones specific for the string theory (they are *impossible* to describe within the framework of conventional Grand Unification), are supposed to be far beyond the scope of experimental possibilities, while the immediate aim is to discover at least *classical* gravitational waves of *high* amplitude. In this context, the objective of the string model building may be to obtain convincing evidence of the existence of string models giving rise to phenomenologically acceptable models of Grand Unification as their low-energy limit (i.e., at energies 10^{15} GeV) and, hopefully, to understand (for *purely theoretical reasons*) why only one of such models is distinguished. The former objective appears to have been accomplished: there is much evidence that practically *any* conceivable field theory at energies of the order of 10^{15} GeV may be considered as the low-energy limit of some string model. True, it is essential that in the course of string model studies many new concepts and ideas were generated in the theory of Grand Unification itself (e.g., those of dilaton vacuum condensate, Kahler structure of the action for scalar fields, extra Z-bosons, dynamic supersymmetry breakdown, natural appearance of hidden matter, etc.).

However, all these ideas bear but a historical relation to string unification (or rather to string theory as a field of mathematical physics or, even more precisely, to the way of thinking); they are neither necessary nor even "natural" in string models and may exist or not, be productive or fail *irrespective* of the validity of the fundamental string concept. As to the possibility to understand theoretically what the "correct" Grand Unification model should be, this appears to be a matter for the remote future. In fact, specification of this problem and a description of current approaches to its solution have been the major subjects of this article.

Discussions of phenomenologically acceptable unification string models are largely focused on two classes of models:

(A) Ten-dimensional heterotic strings compactified on Calabi–Yau spaces; (see Refs. 39, 195 for the original ideas concerning this approach and Refs. 115, 116 for the description and classification of Calabi–Yau spaces and current methods of investigation of such models using orbifolds, catastrophe theory, and $N=2$ supersymmetric sigma-models in general).

(B) Four-dimensional heterotic strings not infrequently referred to as four-dimensional strings and built up with the use of 22-dimensional lattices.

The most promising approaches appear to be those that distinguish four-dimensional space. They do not have to be specially looked for because studies on string theory inevitably suggest apt ideas: the string and the dimension $D=4$ are perfectly aware of each other regardless of our will. Their close relation at the most elementary level is manifested in that $D=4$ is the minimal space-time dimension at which the generic world surfaces still intersect. This phenomenon is described in the simplest terms by the hypothesis of "renormalization" of any other dimension into 4 due to the effects of quantum gravity, e.g., the effect of formation of a Hausdorf (fractal) dimension. Technically, this idea is evinced in the occurrence of conformal sigma-models at $D=4$ in which conformal invariance is maintained due to the presence of a topological term (intersection index of two-dimensional surfaces in four-dimensional space.[200] Another remarkable feature of string unification is the automatic appearance of the Minkowsky signature in space-time as a consequence of the two-dimensional Weyl anomaly.

There is one more fact closely associated with the former two: complexification of a two-dimensional space results in a four-dimensional one. The above nonperturbative calculations suggest a certain degree of distinction of $c=1$ string models that already have a sufficient number of degrees of freedom and symmetries which does not, however, lead to the loss of vacuum stability. Namely, the value $c=1$ corresponds to $D=2$ although for different reasons it may be expected that complexification will be needed. Two of these possibilities are sufficiently naive to be mentioned without many comments. In the first place, it should be noted that the scalar supermultiplet in models with two-dimensional $N=2$ supersymmetry contains a pair of scalar fields which, on the one hand, suggests a doubling of the variable D in the case of the sigma-model approach, while on the other hand, the appearance of $N=2$ supersymmetry on the world sheet looks quite natural in the light of the current concept of interaction between string and topological models. Secondly, it should be remembered that transition from open to closed strings may be considered as complexification (including complexification of vertex operators) and, hence, as a doubling of the dimension D (number of degrees of freedom). These inferences appear to be confirmed by the analysis of certain characteristics of the algebra of observables in $c=1$ models[194] which reveals a rudimentary twistor structure intrinsic in $D=4$. Today, such ideas just begin to offer themselves for in-depth examination, and it will take many efforts to assess their applicability to the theory of fundamental interactions. At present, it is enough to emphasize that these ideas are related to studies on string *dynamics* and comparative dynamics of different string models. Interestingly, the logic of theory development has brought the conclusion (or at least is about to bring it) that the number of space-time dimensions and characteristics of its signature will most likely turn out to be key issues in the search for dynamically distinguished models of string unification within the framework of the future string theory. In fact, these are exactly those problems which prompted our studies beyond the conventional local quantum field theory. These problems, besides being crucial for the above purposes, are supposed to have solutions (beginning to show in the light of string theory) that must satisfy the most strict taste, that is $D=4$ and the signature is $(-,+,+,+)$.

10. A FEW WORDS ABOUT THE LITERATURE

This is the end of a brief review of the fundamentals of a new scientific discipline—string theory. The mode in which it is presented here (few formulas, many words) can hardly be considered totally adequate to a discussion of problems pertaining to mathematical physics but it allowed a concise description of a broad range of problems. The following is a check-list of published works on different sections of string theory to make up for the lack of necessary details that alone make a theory interesting. The list is not simply incomplete—most of the original publications are deliberately not included. Instead, there is a selection of recent and most detailed papers in which the lacking references can be found.

Unfortunately there are no more or less comprehensive textbooks on string theory either in the English or the Russian languages. Until now, the methodology and the very essence of string theory are best of all presented in a book by A. Polyakov[200] although many results and even fields of this science remained beyond its scope. A large fragment of string theory, the simplest string models and their properties in the tree and one-loop approximation, is presented in the monograph of M. Green, J. Schwartz, and E. Witten.[21] The volume is actually a collection of original papers that date from a few years of the "golden era" of the string scenario of Grand Unification. The subject of a recent monograph by S. Ketov[201] is close to that of the previous one. Much more concise reviews of the same subject[202-205] were published in a special "string" issue of Uspekhi fizicheskikh nauk (translated in Soviet Physics Uspekhi) in 1986, i.e. in the prime of the "string boom" (see also Ref. 206). The position of string theory on the eve of this boom was depicted at length in reviews by J. Schwarz and M. Green which have not until now lost their value for those who are about to begin studying this discipline;[207] true, the major part of these materials are included in the book of Ref. 21. A brief account of the philosophy of string unification can be found in Ref. 22. All these publications contain practically no information about the most up-to-date developments in modern string theory starting with the formalism of Riemann surfaces ("multiloop calculations"). The best review of this subject was published by V. Knizhnik in Soviet Physics Uspekhi,[130] see also Refs. 125, 131, 208, and 209. A somewhat simplified account of the same problems can be found in Ref. 210 and also in a small book with very carefully selected materials.[25] Detailed reviews dealing with the most recent events in string theory (general theory of conformal models, nonperturbative methods, matrix models) are lacking in the literature. Therefore, we shall not only cite references but also make brief commentaries about selected notions and methods. Certain mathematical issues considered in modern quantum field theory were covered a few years ago in a small "glossary" published by M. Olshanetskiĭ in Soviet Physics Uspekhi.[211] An encyclopaedic survey[212] and a capital review[213] are much more comprehensive and reciprocally additive "glossaries:" almost all the algebro-geometric issues dealt with in these volumes are relevant to different sections of string theory.

11. GLOSSARY

ALGEBRA—this is first of all a large section of mathematics the scope of which is difficult to delineate (see Ref. 213). Text-books in classical algebra are Refs. 214, 215, those on algebraic geometry are Refs. 216, 217.

Besides this broad meaning, the term algebra has a narrow one. Specifically, it is used to designate a set with binary operation: mapping $A \times A \to A$ (most frequently such a set is a module over a commutative ring, the latter being usually represented by integer, rational, real, or complex numbers). Standard university courses normally concentrate on LINEAR ALGEBRA (actually, algebra of matrices). Very important in physics are Lie algebras, i.e., algebras in which bilinear operation has properties of a commutator: $[a,a] = 0$, and which conform to the Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. Refs. 218–220 are the most readily available text-books of Lie algebras. Lie algebras occur in physics as algebras of infinitesimal symmetry, but there is every reason to consider *non*-Lie-like symmetries as well, e.g., those associated with W-algebras. The structure of a Lie algebra does not suggest a possibility to *multiply* elements (generators) of the algebra. Therefore, it is not unusual that a Lie algebra has to be examined along with its infinite-dimensional universal envelope produced by all kinds of formal generator products and their linear combinations. Other constructions related to Lie algebras that most frequently occur in string theory are the "Tanaka–Kreĭn construction" (Ref. 220, chapter 12) or the description of the Lie groups in terms of their representations, along with its constituents of the type of the theory of Racah coefficients: $3j$-, $6j$-symbols, etc.;[62] quantum deformations of Lie algebras (quantum groups); root and weight systems; orbits of joint representations (see Ref. 220 for them and their role in the general theory of representations).

VIRASORO ALGEBRA—central extension of vector field algebra on a circle (circle diffeomorphisms, Diff S^1/S^1); it is also possible to consider algebras of holomorphic vector fields on Riemann surfaces with punctures (the Krichever–Novikov algebra,[57] see also Ref. 149). The Virasoro algebra occurs in string theory in two major aspects: as an algebra of conformal symmetry generators[58-61] and as a (subalgebra) algebra of observables in some (all?) string models. Implementation of the latter function is related to the so-called Virasoro constraints in the theory of matrix models;[181-185] in a narrower class of models, additional more rigid W-constraints are imposed on the functional integral. See Refs. 126, 221, and 222, for geometric quantization of the Virasoro algebra.

DOUBLE-LOOP ALGEBRAS—these are deformations of two-loop algebras, i.e., holomorphic maps of two-dimensional complex manifolds into conventional (finite-dimensional) Lie algebras G (see Refs. 40, 41 about their fundamental role in string theory). The *Mojal–Baker–Fairlie algebra*[223-225] is a special case corresponding to $G = U(1)$; this is a deformation of the algebra of Hamiltonian vector fields on Riemann surfaces (the algebra of area-preserving deformations). Deformation includes a central expansion ($2g$-parametric for a surface of genus g) and a quantum deformation (which is practically isomorphic to Weyl symbol algebra in quantum mechanics[226]). This algebra has already appeared in several contexts in string theory: as an analog of the Virasoro algebra in membrane theory, as one of the interpretations of the Lie W_∞-algebra,[82-85] and as the algebra of observables (rather its "half") in the important string model "$c = 1$" (Refs. 193, 194). One of the remarkable features of the Mojal–Baker–Fairlie algebra is manifested in that quantum deformation *does not* affect the Lie algebra structure as is the case with finite-dimensional algebras and the Kac–Moody algebras (0- and 1-loop algebras).

KAC–MOODY ALGEBRAS—specific infinite-dimensional algebras of finite growth, that are central extensions of one-loop algebras. The book of V. Kac[227] is the best manual of "simple" algebras of this type. Of similar value *with respect to their applications as one-loop algebras is Ref.* 228. Kac–Moody algebras occur in string theory largely in the role of chiral algebras. The universal envelope of Kac–Moody algebras includes other important chiral algebras:

the Virasoro and W—algebras. The Kac–Moody algebras are immediately related to the WZNW conformal model which can be considered as the product of geometrical quantization of such algebras (the MWZNW action equals d^{-1} of the Kirillov–Costant form[63,220] of the Kac–Moody algebra). Distinction of "simple" Kac–Moody algebras as chiral algebras in current field theory is most likely tentative and can be attributed to their being relatively easy to investigate. From this point of view, hyperbolic algebras and especially double-loop (possibly, also D, $D-1$ or $D/2$-loop) algebras are as interesting as Kac–Moody algebras. A more fundamental significance of Kac–Moody algebras for string application follows from that they are one-loop algebras: note that closed string is virtually a loop in space-time. See Ref. 229 about attempts to use the loop algebra formalism for reformulation of the results of perturbative string theory (such an activity is referred to as "STRING FIELD THEORY").

ALGEBRA OF OBSERVABLES in topological and string models is an analog of the operator algebra in conformal theory. Unlike the conformal case, observables are independent of the point on the surface which makes their algebra more likely to become a genuine algebra (single-valued bilinear operation). Its associativity is directly dependent on the completeness of the model (unitarity is sufficient but not necessary). The algebra may include both a commutative ring (ground ring) and a Lie algebra (symmetry algebra). The algebra of observables is readily identifiable in topological models defined in the form of a factor of a conformal model, i.e., when the central charge is zero and the observables are represented by vertex operators of zero dimension or by integrals of unit-dimensional operators (currents). In this case, the role of the algebra of observables is assumed by a factor of the full operator algebra (such an approach to the string model suggests a simple elimination of all the Virasoro descendants in the right-hand parts of the operator expansions). Identification of the algebra of observables by three-point functions (note, however, that it is not *a priori* clear whether a genuine *algebra* may appear in this way) requires examination of "chiral correlators", i.e., analogs of conformal blocks. Full real correlators provide information only about the symmetric part of the algebra of observables, i.e., about the commutative ring. The importance of the algebra of observables is that it is the most crucial invariant characteristic of the topological model independent both of its representation in the form of the conformsl measure and of the genus of the surface. It is this algebra that must be regarded as the chiral (i.e., gauged) algebra in the transition from the string model to the (string) field theory in space-time. Investigation of the algebras of observables is just beginning: examples of the simplest topological models are presented in Ref. 65, those of string models in Ref. 193, 194. Specificity of string models should be attributed to the presence among the generators of the algebra of operators that alter surface topology. In case of representation of a string model in the form of some kind of topological gravity, the role of the algebra of observables is played by the so-called **recursion relations**.[120,121]

ANYONS—are particles in $(2+1)$-dimensional field theory with special statistics differing from boson, fermion, and parafermion statistics. The statistics is characterized by properties of monodromy of a multiparticle wave function at a fixed point of time (i.e., by the transformation rules under permutation of arguments—the particle coordinates). Monodromy in D-dimensional theories of *point* particles with $D>3$ is usually *almost* independent of the *manner* of argument permutation. For instance, all trajectories of two moving particles that finally lead to their permutation are homotopically equivalent if the space dimension is $D-1>2$. This example already shows that the case of $D-1=2$ is distinguished: the trajectories may get entangled and there are $\mathbb{Z}$ classes of homotopical equivalence. In case of a system of several particles, the situation looks a little more complicated also for $D>3$: it is sufficient to take a set of particles that form a frame of reference in space in order to understand that there may be nonequivalent trajectories since $\pi_1(SO(D-1))\neq 0$. This fundamental group for $D>3$ is equal to $\mathbb{Z}_2$, and it can be demonstrated that a further increase in the particle number does not produce a more complicated structure of trajectory equivalence classes. All this results in the existence of a *single nontrivial (fermion) statistics*[27)] (and a bit earlier was denoted by the word "almost"); note also that parafermions may appear only in the case of a non-single-valued wave function, i.e. when it is a section of a non-trivial bundle.

For $D=3$, not only is $\pi_1(SO(2))=\mathbb{Z}$ (which suggests the existence of an infinite number of nonequivalent statistics) but also monodromies of multi-particle wave functions are in no sense factored into monodromies of two- or three-particle functions. Nonequivalent statistics are different representations of the *braid group*.[212] Such nontrivial statistics are called anyonic. Taking into consideration the collective nature of anyon statistics (i.e., their marked dependence on the particle number), there is no hope to have anyons described in a formalism similar to Grassman variables for fermions (at least nothing of the kind has so far been invented). Field formalism appears to be more adequate because it describes anyon statistics as a non-local one. It is for this description that the additional U(1) gauge field is introduced with which the original particles interact, the action for this field being the abelian Chern–Simons integral $\theta\int A\,dA$. The parameter θ distinguishes between nonequivalent statistics. In such a formalism, the existence of a specific anyon phase becomes apparent (in which it is possible to integrate over original fields, with quanta of field A being quasiparticles; sometimes, they are referred to as anyons). In real physical systems where manifestation of the effects related to anyon statistics is anticipated, field A may be composite (i.e., associated with a condensate phase, etc.). Such physical systems include two-dimensional films showing fractional quantum Hall effect.[43] Another object (for the time being, purely theoretical) is systems with anyon superconductivity.[45] For reviews on the physics of anyons see Ref. 230.

WESS–ZUMINO TERMS in actions of gauge quantum-field models are non-explicitly invariant contributions to the Lagrangian that change under gauge transformations by total derivatives (so that the action remains invariant). The simplest example are the Chern–Simons terms in odd-dimensional Yang–Mills theories; also, they frequently occur in odd-dimensional *sigma-models* on homogeneous spaces (it is quite understandable since such sigma-models can be described in terms of gauge fields). Similar expressions can be obtained from antisymmetric tensors of a higher rank instead of vector fields. Among important examples of this kind is the topological model of A. Schwartz introduced by him[231] to describe the Rey–Singer torsion and the Wess–Zumino term in the action of ($D = 11$)-supergravity. Theories in which the *entire* action is a Wess–Zumino term of one of the above types are usually topological. Specific properties of more general models with non-vanishing kinetic terms include renorminvariance of the Wess–Zumino terms in all orders of perturbation theory (exceptions from this theorem are caused by anomalies and are easy to take into account).[34,232] The problem of nonperturbative renormalization is very interesting (and is directly related to the problems of *fractional* quantum Hall effect and anyon superconductivity, see Ref. 45). There are examples of non-Chern–Simon–Wess–Zumino terms (e.g., σ-models in even dimensions and especially on non-homogeneous manifolds). True, a relation with Chern–Simons terms is sometimes apparent even though it is intricate. For instance, the Wess-Zumino term $\int \mathrm{Tr}(g^{-1}dg)^3$ in the two-dimensional WZNW model is related to the three-dimensional Chern–Simons integral.

GEOMETRIC QUANTIZATION–is a large section of mathematical physics dealing with dynamic systems on geometric manifolds (most frequently, on group and homogeneous spaces). The classical theory of dynamic systems on homogeneous spaces is discussed in Refs. 233, 234. Application of geometric quantization techniques to a very broad range of problems is illustrated in Ref. 235, and the role of geometric quantization on coadjoined orbits in representation theory is described in Refs. 63, 220, and 236.

HETEROTIC STRING–is a critical string model in which the right and the left conformal blocks originate from different conformal theories. The most restrictive requirements are imposed on such models by modular invariance conditions. The first example of a heterotic string was described in the paper of Ref. 38 where the left-hand constituent was a model of a ten-dimensional superstring while the right-hand one arose from compactification of a 26-dimensional boson string on a 16-dimensional torus. Modular invariance was readily obtained if the torus was made from an *even self-dual* lattice such lattices occur only in dimensions divisible by 8 (Ref. 237) and include Γ_8 and Γ_{16}—root lattices of algebras F_8 and SO_{32} as well as the *Leech lattice* Γ_{24} (Refs. 238–241). *String* compactification on root (and weight) lattices of algebra G in space-time results in gauge symmetry with group G which may be either $E_8 \times E_8$ or SO_{32} for the ten-dimensional heterotic string. The majority of the four-dimensional string models discussed in the context of the string scenario of interaction unification are also heterotic (Refs. 196–199). The word "heterotic" originates from a biological term for designation of highly viable hybrids.

HYPERELLIPTIC SURFACES—abelian double coverings over a sphere, i.e., Riemann surfaces of genus g which can be defined by algebraic equations of the form:

$$y^2 = \prod_{i=1}^{2g+2}(x-\lambda_i).$$

It is possible to fix three of the $2g+2$ parameters λ_1 at points 0, 1, ∞ using transformations from the group $\mathrm{Sl}(2)$ which acts by fractionally linear transformations of the coordinates x on a Riemann sphere ("substrate"). The remaining $2g-1$ parameters are modules of the complex structure of hyperelliptic surfaces. The finite group of permutations of parameters λ_i plays the role of the modular group. With the exception of cases $g = 0,1,2$, hyperelliptic surfaces are only subspaces of codimension $g-2$ in the module space (at $g = 3$, this is a divisor in the module space defined by the condition of vanishing of a certain theta-constant). Hyperelliptic parametrization is very convenient for explicit calculations since it provides simple formulas for various transcedental objects including the Riemann theta-functions and their zeroes; in many respects, hyperelliptic functions are almost as well-known as ordinary elliptic ones. Calculations of free fields on hyperelliptic surfaces and other abelian coverings of the Riemann sphere are discussed in Refs. 130, 145, 242–245). Application of these results to the superstring model is described in Refs. 131, 246, and 247. Unfortunately, there is as yet no effective generalization of such a formalism for non-abelian coverings.

HOLOMORPHIC FACTORIZATION—is a key property of two-dimensional conformal models determining the role played by complex geometry in string theory. Holomorphic factorization consists of the representability of correlation functions in the form of bilinear combinations of holomorphic sections of certain bundles over module spaces—CONFORMAL BLOCKS. The idea of holomorphic factorization has been formulated in the basic (for conformal field theory) article of Ref. 58 with special reference to rational models where the number of independent conformal blocks is finite. Often the term "holomorphic factorization" is used in a narrower sense—possibility of expansion into a *finite* bilinear combination of conformal blocks. In string models with complicated topology, such a restrictive requirement can be met only *before* integration over module space. One of the fundamentals of present-day string theory (holomorphic factorization of multi-loop amplitudes in a model of 26-dimensional boson strings and, hence, reduction of corresponding measures on the module space to the Mumford measure) is referred to as the *Belavin–Knizhnik* theorem (Refs. 125, 130, 135). The situation with holomorphic factorization in the superstring model[208] is also instructive.

GRASSMANNIAN—is the space of infinite matrices A_{nk} factorized with respect to the equivalence relation

$$\sum_{n,k} A_{nk}y^nz^k \sim \sum_{n,k} A_{nk}(y + \sum_{i>1} u_iy^i)^n(z + \sum_{j>1} v_jz^j)^k$$

with any vectors u_i, v_i, and is an infinite-dimensional analog of ordinary Grassmannians $U(2N)/U(N) \times U(N)$. Certain constraints may be imposed on the matrices A_{nk} (e.g., trace finiteness); depending on the rigidity of these constraints, Grassmannians are categorized into Segal–Wilson[148] and Sato[169,170] Grassmannians. The former occurs in the Krichever construction on finite-genus Riemann surfaces, the latter (of a more general sort) appears on examination of "infinite genera" and in describing the solutions of the Virasoro constraints in matrix models (Refs. 186, 248, 249). The Krichever construction makes it possible to use the Grassmannian for the description of the UNIVERSAL MODULE SPACE.[154–157] Analysis of matrix models supports the idea[40,41] of the feasibility of a "dual" situation when (a subset) of the Grassmannian can be treated as the space of all string models (see Ref. 186). The Grassmannian is a natural object in the theory of τ-functions. If a finite-dimensional Grassmannian is considered to be a set of hyperplanes in $2N$-dimensional space, one can speak of a projection of one hyperplane onto another and examine a deformation of the metric and the measure under this projection. A similar construction in an infinite-dimensional case allows the τ-function to be identified with determinants of projection operators for certain "hyperplanes."[148] The subspace in the Sato Grassmannian distinguished by *string equations* (see Ref. 186) appears to be of special interest, but its invariant description is at present unavailable.

INTEGRABLE MODELS—they form a large section of mathematical physics discussed at length in complementary manuals on the theory of integrable systems.[3-7,168] The key issue of the theory of integrability is the relationship between the infinite hierarchy of compatible equations of the form

$$\partial u/\partial t_k = F_k\{u, \{t\}\}$$

(as a rule, differential) and bundles over **Riemann surfaces.** The line of reasoning that leads from equations to bundles usually takes into account the *Lax representation* of the original equations (including the "*dressing*" *formalism*), the "auxiliary" linear *spectral* problem, the requirement that its *monodromy* is constant in the course of evolution, and the interpretation of this monodromy in terms of Riemann surfaces (having the *spectral parameter* as the coordinate on these surfaces). As a result, evolution defined by the integrable hierarchy is interpreted as a set of commuting (linear, of different directions and velocities) motions on the Jacobian of the surface (its modules are invariants of the motion). The inverse sequence starts from the definition of a system of *a priori* commuting motions in the module space of a bundle over the spectral surface (motions on the Jacobian are the simplest abelian case associated with linear, i.e., one-dimensional bundles) and determination of τ-functions as averages over free fields on this surface which are sections of the bundle. The next step is a system of bilinear *Hirota equations* on the τ-FUNCTION which are simple identities for the correlators on the Riemann surfaces (in fact, they reflect orthogonality of creation and annihilation operators). The integrable equations themselves can be obtained by the expansion of the non-local Hirota equation into an infinite series the terms of which are ordinary (as a rule, differential) equations. Examples of non-local integrable hierarchies are given in Ref. 250. Problems pertaining to the transfer of this construction to super-Riemann surfaces are discussed in Ref. 251. A major function of integrable systems in string theory (and, possibly, in a broader context) is to characterize the generating functions of nonperturbative correlators (nonperturbative partition functions) as τ-functions. In this sense, integrability should be considered as a property of effective actions not liable to further averaging. In this context, integrability is a *classical* property which is not subject to quantization. Nevertheless, there is an interesting problem of quantization of isolated integrable equations regarded as field theory models (usually two-dimensional ones, e.g., the sine-Gordon model, the non-linear Schrödinger equation, etc.). This problem is also important for understanding of the relationship between conformal and integrable systems, e.g., for investigation of interpolations between conformal models. See Refs. 4 and 172, 173 for such problems and their relation to quantum groups. Ref. 252 reports interesting preliminary results concerning the application of the free field formalism to the problem of quantization of integrable equations.

The Toda lattice system,[253] the Zakharov–Shabat hierarchy, the Kadomtsev–Petviashvili (KP) hierarchy, the Korteveg–de Vries hierarchy, and the Boussinesq hierarchy are the most thoroughly investigated integrable hierarchies. The latter two are the simplest of the so-called A_n reductions of the KP hierarchy associated with the simple Lie algebras A_1 and A_2. The "discrete" hierarchy of the Toda chain (TC) (with one of the "times" in it being discrete) is equally well-known. All these hierarchies have already appeared in the studies of matrix models and, consequently, in nonperturbative string theory.

QUANTUM ANOMALIES—the term has been coined to describe symmetry breakdown during transition from classical to quantum theory.[34] Most anomalies may be regarded as a property of regularized determinants of differential operators.[34] See Refs. 35, 125, 130 for more details about the best known case of two-dimensional determinants. In this case, there are "gravitational," "Weyl," "holomorphic," and "Quillen" anomalies associated with dependences of Det$2\bar\partial$ on the choice of coordinates of Det Δ on the conformal factor in the metric, with corrections to holomorphic factorization of Det Δ as a function of modules of complex structure of the Riemann surface and modules of linear bundles over it (i.e., its representation in the form of $|\mathrm{Det}\ \bar\partial|^2$), respectively. Theories with *internal* anomalies are currently interpreted as having extra degrees of freedom which decouple in the classical approximation but affect such characteristics as vacuum energy, central charge, etc. Specific two-dimensional anomalous models have been examined from this point of view in Refs. 254–256. An important example of an anomalous degree of freedom in string theory is the Liouville field; its interpretation with in the framework of the said scheme allows the Minkowsky signature to be naturally interpreted as an "anomalous effect;" more complicated signatures (several pluses and several minuses) are obtained in a similar way in W-string models.[86-93] Using such an approach, anomaly-free (CRITICAL) models are assumed to be those that do not undergo symmetry changes during transition to the classical limit. (For instance, a non-critical string has the $SO(D-1)$ double symmetry in flat D-dimensional space-time and the broader $SO(D-1,1)$ one in the classical approximation. The Lorentz-symmetry $SO(25,1)$ in critical dimension $D=26$ is retained even at the quantum level. The criticality condition is essential in that the model needs to have a nontrivial low-energy limit; it therefore plays an important role in building up string models of the Grand Unification.

QUANTUM GROUPS—they have been discovered by L. Faddeev and co-workers who studied integrable systems.[4] A well-known approach to quantum groups using their relationship with Hopf algebras has been suggested by V. Drinfeld.[257] A somewhat different view of quantum groups is illustrated by Ref. 258 where they are treated as symmetries of quantum spaces. The quantum group theory is one of the most rapidly developing branches of mathematics and its string interpretation involves the use of conformal and integrable systems. The relationship between quantum $3j$-symbols (Clebsh–Gordan coefficients) and monodromies of conformal blocks in the theory of free massless fields is of the simplest type. The theory of quantum groups is also closely related to knot theory. Long-term objectives of quantum group studies include the development of a comprehensive theory of "quantum" q-hypergeometric Gaussian functions (arising from finite-difference equations in the same manner as ordinary hypergeometric functions arise from the solution of differential equations). For reviews see Ref. 259.

CHIRAL ALGEBRAS—it would be natural to identify the operator algebra of *all* chiral vertex operators as the total chiral algebra of the conformal model. However, this object is very complicated and can not be called an algebra in the strict sense of the word. Normally the term "chiral algebra" refers to a subalgebra of the universal envelope of the Kac–Moody algebra which is adequate for the conformal model in question (see, for example, Ref. 62). The entire chiral algebra or a part of it is subject to gauging on transition from a conformal model to its string counterparts. It would be correct to apply the term "chiral algebra" to a symmetry which may be accurately gauged. This is not however a definition in *internal* terms of a conformal model and therefore is not used in the literature on conformal theory.

COMPACTIFICATION—this is an idea that can be traced back to the Kaluza–Klein scenario which implies that properties of fundamental interactions (gauge groups, field contents, coupling constants) may been coded in the geometric properties of a particular compact manifold. According to the literal interpretation of the idea, space has additional compact dimensions, and motion along these dimensions is impossible unless the particle energy exceeds the inverse compactification radius that defines the mass scale. In this case, the low-energy sector contains only the zero-modes of fields on a compact manifold; their characteristics can be defined by the geometrical, sometimes topological, properties of this manifold. The gauge group is associated with manifold isometry, the number of generations with the number of zero-modes, and the coupling constants with the overlap integrals of zero-modes, etc. Ideas of compactification are reviewed in Refs. 22–26. These ideas are used in string models. There are two features of string compactification worth mentioning. First, a change in the relative significance of the gauge group and isometry: the former is associated with the more intricate characteristics of a compact manifold and may exceed the isometry group, an important example being the appearance of the gauge group G during compactification on a torus associated with the weight lattice G or the so-called Frenkel–Kac mechanism[260] used in the simplest models of heterotic[38] and four-dimensional[196–198] strings. Second, equivalence of compactifications on various manifolds which precludes proper definition of the problem of "the existence" of *compact* dimensions in string compactification, their presence and number being dependent on the choice of one of the alternative equivalent models. Specifically, the aforementioned heterotic string models (including those of "four-dimensional strings") may be interpreted as compactificated theories although this is not the only possible interpretation.

THE KRICHEVER CONSTRUCTION—this is a mapping that associates a point of an infinite-dimensional Segal–Wilson **Grassmannian** with a set of data {Riemann surface, bundle over it, puncture, local co-ordinate system in its neighborhood, local bundle trivialization}. In order to obtain such a mapping, it suffices to expand a section of a bundle with a fixed singularity at the puncture into a Laurent series:

$$f(z) = (z - z_0)^{-n}(1 + \sum_{K>0} A_{nk}(z - z_0)^k.$$

In this case, the matrix A_{nk} meets the condition of trace convergence and defines the equivalence class under factorization with respect to co-ordinate changes holomorphic in the neighborhood of z_0, i.e., a point of the Grassmannian. Examination of sections with essential singularities $\sim\exp\Sigma_m t_m z^m$ (the Baker–Akhiezer "functions") reveals a sufficiently simple ("integrable") dependence of the Grassmannian point on the "times" t_m (bundle module or "boundary conditions"), see Refs. 147, 148 for details. Various characteristics of string models, from chiral algebras[57,149] to determinants and even string amplitudes, can be rewritten in terms of (equivalence classes) of the matrices A_{nk}. Such are the contents of STRING OPERATOR FORMALISM.[151–153] The Grassmannian may be used (at least theoretically) as a representation of the UNIVERSAL MODULE SPACE.[154–157] There is an interesting problem of the "closure" of the Grassmannian, i.e., of "infinite-genus surfaces;" specific convergence conditions defining the Segal-Wilson Grassmannian[148] appear to be excessively limiting, and it is more reasonable to deal with the more general Sato Grassmannian[169,170] especially in the context of matrix models.[186,248,249] The Grassmannian or a part of it may also be used in the capacity of a configuration space of the "unified field theory" for labelling various string models by the points of Grassmannian.[186]

CONFORMAL MODELS—they appear in studies of phase transitions in $(2+1)$-dimensional systems, in interpretation of multi-dimensional differential equations in terms of the symmetry properties of two-dimensional **sigma-models** and in designing string models after integration over metrics (in the latter case, a conformal model could also exist before integration). The formalism of conformal field theory which allows computation of correlation functions in conformal models originates from a classical pioneer work of Ref. 58 in which the basic concepts of holomorphic factorization, the Virasoro algebra, rational, unitary, and minimal conformal models were first introduced and a classification of these models (in fact, the classification of minimal closed operator algebras) was suggested; this established their still not entirely *understood* connection with the representation theory of the Virasoro algebra.[160] The next important step is the creation of the formalism of free massless fields[161] by analogy with the theory of the Verma modules.[160] An important role in this formalism belongs to screening operators[161] analogs of the Feĭgin–Fuchs operators in representation theory (see Ref. 163 for their interpretation in terms of the BRST-complex). Application of the formalism of free massless fields to the WZNW model and its reductions is discussed in Refs. 79, 131, 158, 159, 209, and 261. (Ref. 158 suggests an interpretation of the screening operators as being associated with a non-local change of co-ordinates in the functional integral). Attempts to develop a classification of rational conformal theories are reviewed in Ref. 62. A description of topological models and two-dimensional quantum gravity (including noncritical strings) in terms of conformal models is presented in Refs. 262–263 and 122–126 respectively.

MATRIX MODELS (random matrix theory)—is a self-contained science investigating multiple matrix integrals including functional integrals with matrix-valued fields (there is even a monthly journal publishing pertinent materials). Matrices may be other than square, and vector models play an important role in applications. Integrals with a Gaussian distribution are the most developed part of the theory although *non*-quadratic actions are equally interesting. In physics, matrix models occur in different contexts, viz. replica methods, spin glass theory, neural networks (Kirkpatrick vector model[18]), quantum gravity problem, Yang-Mills theory, etc. Matrix models are used in studying problems that involve fat graphs including topological studies of module spaces of bundles over Riemann surfaces.[120,129] As a rule, matrix models with a scalar action are examined (in the exponent which dictates the measure of the integration, the trace of matrices appears) and of correlators of only scalar variables are evaluated. Equations of motion and Ward identities in such matrix models are frequently referred to as LOOP-EQUATIONS. Matrix models with a sufficiently large number of *parameters*[28)] (coupling constants or background fields) are closely related to integrable systems.

Matrix integrals as functions of the parameters turn out to be τ-functions of integrable hierarchies. However, generic τ-functions do not arise in this way; instead, they are restricted by an additional constraint which is traditionally called the STRING EQUATION and depends on the model.[177–180] An invariant description of the subset in the Sato Grassmannian specified by the string equation remains an unsolved problem (see Refs. 248,249,264,286). One of the reasons for the importance of matrix models in string theory is the existence of the two-dimensional gravity formalism based on these models which is efficient in both perturbative and nonperturbative domains[177–180] and is closely related to topological gravity.[120,187–190] Actually, each string model (showing stability with respect to nonperturbative corrections) can be associated with a (multi)matrix one, or, more precisely, with its *double-scaling continuum limit*[177–180] that may again be represented in the form of another *matrix* model (the Kontsevich model) and is in a certain sense identical for all original models (see Ref. 186 for details). After finding a more invariant formulation that does not use an intermediate stage with a matrix integral but is directly expressed in terms of τ-functions or a Grassmannian, these results will be applied to the construction of a "unified field theory," i.e., a natural unification of all string models. The most important types of matrix models examined in connection with the problem of two-dimensional gravity are as follows.

Discrete one-dimensional matrix model

$$\int dM \exp \sum_k t_k \mathrm{Tr}\, M^k;$$

(see Ref. 184) for a most important case of Hermitean matrices M and Ref. 256 for a model with unitary M; Refs. 266–268 discuss a modified model for the case of complex square matrices M); models with orthogonal, real, and even vector M (i.e., matrices of the $1\times N$ size) are also discussed. The Hermitean *discrete* one-matrix model can also be represented in the form of the **Kontsevich model**[186]

$$F_V(Q)\int dX \exp \mathrm{Tr}(-V(X) + V'(Q)X)$$

with *potential*

$$V(X) = (X^2/2) - n \log X,$$

where n is the size of matrix M and $t_k = T_k + \delta_{k2}/2$, $T_k \equiv (1/k)\mathrm{Tr}\, Q^{-k}$. The double scaling continuum of this model is described by the Kontsevich model with $V(X) = X^3/3$ (Ref. 186); for details about this continuum when $n\to\infty$ under certain constraints with respect to the behavior of the coefficients t_{2k} and $t_{2k-1} = 0$ see Ref. 268. For $V(X) = X^{K+1}/(K+1)$, the Kontsevich model describes double scaling continuum of the K-matrix discrete model

$$\Big(\prod_a^K \int dM_a \exp\sum_k t_k \mathrm{Tr}\, M_a^k\Big)$$

$$\times\ \exp(M_1M_2 + M_2M_3 + \ldots + M_{K-1}M_K).$$

For any $V(X)$, the partition function of the Kontsevich model as a function of the times T_k is a τ-function of the *KP* hierarchy satisfying the string equation $L^V_{-1}\tau = 0$.

THE REPLICA METHOD is a specific modification of the diagram technique for the case when interactions without energy transfer take place. Also in such a case, there are restrictions on acceptable diagrams: following elimination of all lines associated with these interactions, they must remain connected (attachment of extra loops to the diagram with interaction lines of this type alone is prohibited). Interactions without energy transfer are not infrequent in physics, but the most typical example are the effective interactions of quasi-particles due to rescattering by random impurities in models of solid-state physics.[269] The most interesting up-to-date applications include the quantum Hall effect and spin glasses. The problem of an adequate modification of the diagram technique is in fact the problem of evaluating $\langle\!\langle \log Z\rangle\!\rangle$ instead of $\langle\!\langle Z\rangle\!\rangle$ where $\langle\!\langle \cdots\rangle\!\rangle$ means averaging over impurities. The replica method uses a representation of $\log Z$ as the limit of $Z^N - 1$ as $N\to 0$. The values of $\langle\!\langle Z^N\rangle\!\rangle$ at positive integer N can be found by examining a new system composed of N copies (replicas) of the original one. At the end of the calculations one must let N go to zero. The simplest qualitative result attainable with the use of the replica method is the explanation of the Anderson localization as confinement (associated with asymptotic freedom) in sigma-models which describe propagation of quasiparticles in the field of random impurities. There is no rigorous substantiation of the replica method. Moreover, it should be borne in mind that the most interesting effects using this method have been found in the case of its risky application, i.e. when the $U(N)$ or $SO(N)$ symmetry between replicas was broken or effects associated with asymptotic freedom which can disappear at the point $N = 0$ were employed. An alternative method is the supersymmetric formalism even though the range of its applications is currently not so wide. The use of the replica method in spin glass theory and the theory of the quantum Hall effect is reported in Refs. 18 and 270, 271 respectively. In the latter case, the problem consists in the analysis of the specific sigma-model on the Grassmannian $\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$ which may be either two-dimensional, with a topological term in the static situation, or three-dimensional, with a Wess-Zumino term in nonstatic situation (e.g., in case of the Coulomb interaction responsible for the *fractional* quantum Hall effect; a correct derivation of the three-dimensional model is not available in the literature). The interpretation of the quantum Hall effect itself can be reduced to an analysis of the renormalization properties of such σ-models (see Refs. 272–274 for the static case; a similar study in the nonstatic situation remains to be conducted).

TODA MODELS—they form an important class of two-dimensional models in quantum field theory. Toda models may be conformal and integrable (non-conformal). They are associated with simply-laced Lie algebras. The number of fields is equal to the rank of the algebra, they are represented in the form of vectors on the Cartan plane. Action in the conformal gauge for the metric has the form

$$\int\left[\frac{1}{2}|\partial\phi|^2+\sum_\alpha e^{i\vec{\alpha}\vec{\phi}}\right],$$

where α are all *simple* roots of the algebra in the case of the conformal Toda model; an integrable model can be obtained by the addition of one more "lowest" root (the same root which is added to create a Kac-Moody algebra from a simple Lie algebra). In the case of the sl(2) algebra, the conformal Toda model is known as the Liouville theory and the integrable one as the sine-Gordon model. Moreover, there are interesting models with the reversed sign in front of the kinetic term and without i in the exponents. The theory of the Toda models as classical dynamic systems is presented in Ref. 275. At the quantum level, the conformal Toda models may be considered as reduced WZNW models.[94] They (and their supersymmetric analogs) play an important role in the theory of conformal models (they are related to both the MWZNW and the conformal sigma-models described in terms of quasihomogeneous polynomials (see Refs. 115, 116) for such a description and Ref. 166 for its relation to the Toda models). They are equally important for the theory of W-gravity.[86–93]

MODEL—a concept from the theory of Lie groups.[29]

It denotes a set of group G representations taken with unit multiplicities. The concept and the early examples of models were first suggested in Ref. 276. The models originate spontaneously (as a set of W_G-primary fields) during the examination of string compactifications on root and weight lattices of simply-laced G algebras and also from the WZNW theory.[277–278]

THE WESS-ZUMINO-NOVIKOV-WITTEN MODEL (MWZNW)—this is a two-dimensional theory with equations of motion $\bar\partial J^\alpha=0$, where the $J^\alpha(z)$ form the Kac-Moody algebra with respect to equal-time commutation relations. Unlike the non-chiral equations $\partial_a J^\alpha_a=0$, analyticity conditions in the MWZNW can not be deduced from any local action. The action of MWZNW was introduced by E. Witten[275–281] and contains the specific multi-valued Wess-Zumino term; the first example of the model with such terms can be found in Ref. 282. Such Lagrangians (in fact, elements of cohomologies rather than ordinary measures) have been studied in the general context by S. Novikov.[283] This action also permits an interpretation as d^{-1} of the Kirillov-Costant formula for the Kac-Moody algebra.[63] The Wess-Zumino term is absent in the case of an abelian algebra.

The energy-momentum tensor in the MWZNW is defined by the HALPERN-SUGAWARA FORMULA.[284,285] Interesting MWZNW reductions are GKO-Coset-models[72] and the Drinfeld-Sokolov reductions.[77–79] The latter group of reductions naturally result in the appearance of W-algebras. The "generalized Sugawara construction"[67–70] is also important even though its algebro-geometric meaning remains unclear. The up-to-date theory of the WZNW model and its reductions is an important constituent component of string theory. Fundamental results concerning the quantum MWZNW are reported in Ref. 286 together with a derivation of the KNIZHNIK-ZAMOLODCHIKOV EQUATION (basic Ward identity for the MWZNW). See Refs. 131, 158, 209 for the formalism of free massless fields for this class of models. An important approach to the description of reductions as "gauged WZNW models" has been developed in Ref. 99. The relation of the MWZNW to quantum groups and the three-dimensional Chern-Simons model is discussed in Refs. 99 and 117, 118, 164, respectively.

THE KONTSEVICH MODEL—is an important matrix model defined by the integral

$$\mathcal{L}_V[Q]\sim\int dX\exp(-\mathrm{Tr}\,V(X)+\mathrm{Tr}\,V'(Q)X)$$

(the coefficient in front of the integral is specifically dependent on the "potential" V and the matrix Q). It simultaneously describes various models of topological gravity (including the simplest of them[120] for $V(X)\sim X^3$ (Ref. 129)) and the double scaling continuum of all multimatrix models, i.e. string models built up from all the Virasoro minimal conformal models (with $V(X)\sim X^{K+1}$) corresponding to the K-matrix model). Regardless of the choice of potential, $\mathcal{L}_V[Q]$ is τ-function of the KP-hierarchy as a function of the times

$$T_k=\frac{1}{k}\mathrm{Tr}\,Q^{-k}$$

and satisfies the string equation which, together with the Grassmannian point associated with the τ-function, is dependent on V. See Ref. 186 for details.

THE CHERN-SIMONS MODEL—is the quantum theory of the Yang-Mills gauge field A in space-time of $D=2n-1$ dimensions with the Lagrangian $d^{-1}(\mathrm{Tr}\,F^n)$. The action being independent of the metric, the model is virtually topological although the problem of topology-preserving regularization for the general case remains to be solved (the naive regularization using the kinetic term Tr $F^2_{\mu\nu}$ appears in this case to be unacceptable). The Chern-Simons model is especially popular at $D=3$ when the space of states is formed by the space of flat connections on a two-dimensional surface and there is a close relation with two-dimensional conformal theories (especially with the WZNW models and their reductions), quantum groups and knot theory (Refs. 64, 99, 117, 118). See Ref. 287 for a five-dimensional Chern-Simons model. Chern-Simons models also occur in the theory of quantum anomalies[34] and in numerous physical applications. The latter are interrelated because the Wess-Zumino terms including the Chern-Simons term arise spontaneously in effective actions of odd-dimensional theories (e.g., as anomalies of fermion determinants).[34,288-292] Their appearance leads, first, to the breakdown of global symmetry (space parity) and, second, to nontrivial dynamic effects especially at $D = 3$; the Chern–Simons term is also associated with massive electrodynamics,[293,294] anyon statistics,[230] etc.

MODULES—are parameters describing continuous families of equivalence classes of algebraic geometry objects. These classes include Riemann surfaces (equivalence class with respect to holomorphic changes of coordinates) and bundles over them, and also multi-dimensional complex spaces. (Sometimes, it is possible to build up *conformal* sigma-models using such spaces; example: Calabi–Yau spaces. In view of this, space modules are often referred to as modules of corresponding conformal models). An exact definition of module space taking into account delicate details always arising in a description of equivalence classes is given in terms of *schemes* (the most concise description can be found in an article entitled "Theory of Modules" (in Ref. 295). String model studies have most frequently to deal with the following construction related to chiral algebras and defining the *local* structure of the corresponding module space including its dimension. The field $\varepsilon(z)$ of spin $1 - j$ is associated with the chiral algebra generator $K(z)$ of spin j in such a way as to generate an infinitesimal "symmetry" transformation by the operator

$$\oint \langle \varepsilon K \rangle$$

(the angular brackets indicate a scalar product in the space where K and ε take on values, e.g., in a fibre of the bundle). Almost any field $\varepsilon(z)$ defined on a small (contractible) circle is holomorphically extendable inside or outside the disc. Modules turn out to be associated with the number (usually finite) of fields $\varepsilon(z)$ which are extendable neither inside nor outside; their number for linear bundles on closed surfaces is $(2j-1)(g-1)+n$, where g is the surface genus and n is the number of punctures where $\varepsilon(z)$ may have simple poles. The simplest case: modules of complex structure of the surface are associated with the Virasoro algebra in the described manner.[149]

MODULE—this is an algebraic concept synonymous with vector space, i.e., a set with addition and multiplication by elements of a certain ring with distributivity conditions. The VERMA MODULE is a representation of highest weight obtained as a linear envelope of formal products of any number of descending generators of a Lie algebra by the highest weight. The Verma module is a reducible representation if it contains vanishing norms, the so-called NIL VECTORS. Unlike some irreducible representations, the infinite-dimensional (as a vector space) Verma module has a simple structure. In the case of the Kac–Moody, Virasoro, and W-algebras, the Verma modules are easy to construct from free massless fields. For details about the Kac–Moody and Virasoro cases see Refs. 227 and 160 respectively.

OPERATOR ALGEBRA, OPERATOR EXPANSION—this is a key concept in quantum field theory. In the case of two-dimensional conformal models, the operator algebra is drastically simplified due to holomorphic factorization.[58-61] It associates an infinite series in powers of $(z_1 - z_2)$ and, possibly, logarithms with operator-valued coefficients with each pair of operators $a(z_1)a(z_2)$. The singular terms of the expansion (there is usually a finite number of them) may be interpreted as equal-time commutation relations; Jacobi identities are valid in case a complete set of operators is examined. In such a case the operator algebra is associative. This requirement is *not* fulfilled automatically if the operator algebra is originally defined by its "three-point functions", i.e., by mapping of the tensor cube of the field space into complex numbers (evidently, such a formulation carrying information about the nonsingular terms of the operator expansion implies that the operator algebra differs from algebra in the narrow meaning of the term: the third field is not uniquely defined by the other two). Under certain additional conditions, the operator algebra can be extended to mapping tensor degrees of field spaces into sections of bundles over module spaces of the Riemann surfaces with punctures (these sections are called CONFORMAL BLOCKS). Examination or *rational* conformal models results in some simplifications (see Ref. 62 for a review of the appropriate theory). An attempt to suggest a reasonable formal definition of operator algebra is described in Ref. 296. Of special interest is the operator algebra in a certain sense factorized over a chiral algebra (e.g. over the Virasoro algebra), that is the operator algebra is restricted to a set of primary fields. Straightforward methods of such a "projection" lead to such concepts as FUSION RULES[62] and VERLINDE ALGEBRA.[297] *Gauging* a chiral algebra, i.e., a transition to the corresponding string (topological) model, is even more interesting. Monodromy matrices play the role of conformal blocks in the description of the resulting algebra of observables, the principal invariant characteristic of the string model (moreover, in many cases this is a $\mathbb{Z}_2$-monodromy which means that in going around a closed circuit, the phase can change only by an even or odd multiple of $2\pi i$). Incidentally, the resultant algebra of observables may prove to be a true algebra where the third observable can be *uniquely* defined by the two others. But even in this case, it need not necessarily be an anticommutative Lie algebra; it often contains a commutative subring (ground ring); see Refs. 194, 298–300 for the most important examples.

OPEN AND NON-ORIENTABLE STRINGS—are string models defined on open (i.e., having boundaries) and/or non-orientable surfaces. Such surfaces having any number of handles, open string models inevitably include closed strings. Open and non-orientable surfaces may be considered as factors of specific closed surfaces, the so-called doubles[301] with respect to $\mathbb{Z}_2$-symmetry. Module spaces of doubles form half-dimensional real subspaces in ordinary complex module spaces while measures on these subspaces defining correlators in the open string theory merely coincide with holomorphic constituents of ordinary string measures (for closed strings); see Refs. 40–42. Punctures may be regarded as a specific type of a surface boundary; the same refers to the neighborhood of a puncture arising in the Krichever construction—this accounts for parallelism between the open string theory and the string operator formalism.[40] In the p-adic case, the analogs of the open string theory are much simpler than the analogs of closed strings: the former are described in terms of ordinary p-adic numbers from the $\mathbb{Q}_p$ field while the latter are defined over the Ω_p field which is an enlargement of the algebraic closure of $\mathbb{Q}_p$ and is a very sophisticated analog of complexification, i.e. a transition from $\mathbb{R}$ to $\mathbb{C}$.

RECURSION RELATIONS—are a form of representation of algebra of observables in string models especially in topological gravity. These are relations between different correlators including those on the surfaces of different genera and with a different number of punctures. *E*. Witten was the first to introduce the term "recursion relations" and to examine this phenomenon thoroughly. See also Ref. 121.

RENORMALIZATION PROPERTIES OF MODELS WITH TOPOLOGICAL AND WESS–ZUMINO TERMS—this is a classical problem of the modern quantum field theory unyielding to easy analysis. The most relible assertions: renorminvariance of topological and Wess–Zumino terms (up to the readily recognizable anomalies of quantum determinants) in all orders of perturbation theory. Some models (e.g., sigma-models on homogeneous spaces) may be predicted to contain a nontrivial fixed point (zero of the β-function, conformal model) if the coefficient in front of the topological term is $\theta = (2n+1)\pi$, provided "nonperturbative effects" are taken into account. Nonperturbative effects also result in renormalization of the topological term,[272] the general picture of renormgroup evolution in a double-charge theory is shown in Fig. 2 left below. This result can be readily-obtained[272] in the instanton gas approximation;[302,303] see Ref. 304 for an example of an *accurate* calculation in the one-dimensional lattice model. These findings are experimentally confirmed by integer-valued (static) quantum Hall effect where the above picture is observed experimentally[305] and the theory is defined by the σ-model with a topological charge.[270–274] An important application of this result is illustrated by the possibility to distinguish string models in four-*dimensional* space-time with a specific topological charge: intersection index of (world) surfaces.[200] Reliable reports on nonperturbative renormalization of Wess–Zumino terms are unavailable; Ref. 306 illustrates difficulties inherent even in the instanton approximation. Experimental findings concerning the fractional (non-static) quantum Hall effect may possibly be interpreted as indicating that the renormgroup behavior of such models must be even more interesting (see Fig. 2 right): θ is the coefficient in front of the Wess–Zumino term and there must be *many conformal points*[30)] ("the ladder" in the figure prompts numerous analogies including those with p-adic Bruhat–Tits trees and spin glasses or Feĭgenbaum bifurcation pictures, etc.). If applied to the Hall effect, θ is in both cases the Hall conductivity σ_{xy}, g^{-2} is the conventional conductivity σ_{xx}, and the arrows indicate the direction of renormgroup evolution with the growth of the effective sample size; the really observed values are those at the arrow "ends" ("dressed" parameters), at the origin there are the bare parameters which depend on the microscopic properties of the substance and are not universal. The difference between the two situations lies in the possibility and impossibility to ignore electron interactions in the specimen both with one another and with the impurities and the background fields. Vanishing of the "dressed" value σ_{xx} is known as "localization" or the phenomenon of asymptotic freedom (unlimited growth of g with increase in size). If $\sigma_{xy} = (2n+1)\pi$, "delocalization" occurs; the lines have conformal points indicated by crosses.

REPARAMETRIZATION GHOSTS—arise when the gauge is fixed in the course of integration over two-dimensional metrics. They are Grassmann vector fields on Riemann surfaces (a change of coordinates is always a vector field), that is they have spin -1 (while a conjugate ghost field has spin $+2$). Discovery of these ghosts by Polyakov[56] provided an explanation for the origin of the magic $D = 26$ dimension in the boson string model ($26 = c_2$ is exactly the central charge for the Virasoro algebra generated by energy-momentum tensor of ghosts)

$$T_{\mathrm{ghost}} = -b\partial c + 2(\partial b)c)$$

and stimulated development of the modern string theory based on the first quantization approach, i.e. on the analysis of two-dimensional conformal models and two-dimensional quantum gravity. The role of reparametrization ghosts is especially important in the BRST-formalism which interprets all observables of the *string* model as elements of cohomologies of the nil-potent BRST-operator $Q_{\mathrm{BRST}} = cT + bc\partial c$.

RIEMANN SURFACES—are two-dimensional surfaces with a given complex structure. There are closed (having no boundaries) orientable surfaces of different genera g (g is the number of handles) and open (with boundaries) non-orientable surfaces. It is sufficient to investigate only the closed orientable surfaces since the information obtained may be transferred to all other cases using the technique of doubles.[301] The following objects of the theory of Riemann surfaces are of primary importance for string theory: bundles of j-differentials, Jacobi maps, period matrices, modules of complex structure, module spaces, and special functions:—Jacobi and Riemann theta-functions together with information about their zeroes. Readers can update their knowledge in this field using the following books (Refs. 133, 134, 216, 307, 308).

SIGMA-MODELS—These are models of d-dimensional QFT with D-fields $x^{\mu}(\xi)$ and action

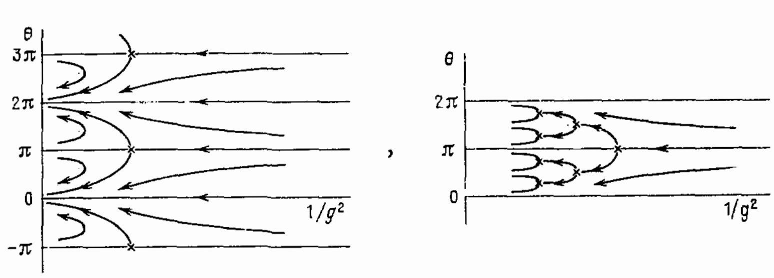

FIG. 2

$$\int G_{\mu\nu}(x)\partial_a x^\mu \partial_a x^\nu d^d\xi.$$

The fields may be interpreted as defining a map onto a D-dimensional manifold (target space) with the metric $G_{\mu\nu}(x)$. Sigma-models are most suitable for the translation of geometric information into the QFT language and allow the QFT to be applied to the solution of geometric problems. There are sigma-models with supersymmetry including extended supersymmetry $N = 1,2,3,4$. A major advantage of the σ-model is that the manifold geometry is manifested in its simple properties as quantum field theory. For example, the homotopical nontriviality of the manifold is reflected in the existence of the topological (if $\pi_D \neq 0$) and/or the Wess–Zumino (if $\pi_{D+1} \neq 0$) terms of the action. Sigma-models on homogeneous manifolds G/H may be regarded as Yang–Mills models with the gauge group H (and the matter in the adjoint representation of the global group G); in this case, say, the Wess–Zumino term can be frequently represented in the form of a Chern-Simons term. Supersymmetrization of the model is virtually the introduction of vectors from the tangent space of the manifold as fermion degrees of freedom. In other words, supersymmetric sigma-models reflect the properties of tangent bundles; extended supersymmetries are related to Kähler and hyper-Kähler structures on the manifold, etc. String models that can be interpreted in terms of D-dimensional space-time are described as two-dimensional conformal sigma-models. Conformal symmetry is achieved either by imposing differential equations (of the Einstein type) on G or by adjusting (frequently *dynamic* due to nonperturbative renormalization) the topological terms. The latter variant is also used in the theory of the quantum Hall effect[270–274] and in one of the models of four-dimensional strings.[200] A review of the general properties of sigma-models can be found in Refs. 309, 310, the sigma-model approach to the low-energy limit of string models (Fradkin–Tseĭtlin formalism) has been discussed in Ref. 2.

PARTITION FUNCTION (STATISTICAL SUM)—this term is used in discussions of string models in two, somewhat different, meanings. Firstly, it designates string amplitudes without external lines (vertex operators). In this case, g-loop partition functions are considered to be a contribution of genus-g Riemann surfaces to the "vacuum energy" in a given string model. Such partition functions can be determined both for string and conformal models. The term "statistical sum" was coined in connection with the analogy of this problem for $g = 1$ to the problem of computation of free energy at finite temperature. One-loop partition functions resemble integrals over t of spectral generating functions $Z(t)$ which already appeared in the main text, but differ from them in numerical factors (sometimes, even infinite ones). Formally, this can be accounted for by the requirement of modular invariance or duality (in terms of dual-resonance models). Less formally, the implication of this difference is that very short strings may be regarded as high excitations of strings of a moderate length $\sim 1/M$. This is important to understand when working, say, on the formalism of the "string field theory."[311]

Secondly, the term "nonperturbative partition function" is used to denote the generating function for *all exact* correlation functions of a string model, i.e., a functional containing comprehensive information about the given model (in fact, even about a class of models). Such wording is warranted because examination of nonperturbative effects calls for "exponentiation" of all vertex operators (both naive observables and "handle gluing operators"), that is for the extension of the action by introducing all kinds of perturbations with arbitrary coefficients or "times" (the term was derived from the theory of integrable hierarchies closely related to the problem in question in the case of an adequate choice of the perturbation basis). The "vacuum amplitude" in the theory with such a perturbed action (summed over all orders of perturbation theory) forms the "nonperturbative partition function." The nonperturbative partition function characterizes a class of models rather than an individual model since exponentiation of certain perturbations may be interpreted as an alteration of the original model, e.g., a breakdown of its symmetry. Theoretically, such a class must include *all* string models if a sufficiently large number of perturbations is considered—this is one of the basic ideas in building up a "unified field theory" or a natural unification of all string models. The integral defining the Kontsevich matrix model is a good example of a nonperturbative partition function (in this case, *the class* of *models* is all string models associated with minimal Virasoro models).

STRING MODEL–is obtained by gauging the Virasoro algebra in a conformal model with zero central charge of the Virasoro algebra (if the original central charge differs from zero, it must be associated with a specific gauge field the dynamics of which makes the effective central charge become zero). *Any* conformal model can serve as an example of such a string model provided it is supplemented with a system of reparametrization ghosts and a Liouville field with characteristics dependent on the central charge of the original model. In this case, transition to the string model may be considered as averaging (integration) over two-dimensional metrics. Observables of the resulting theory are in a certain relation with primary fields of the original theory (rather, its extension, e.g., through topology-altering operators) whereas the operator algebra is replaced by the algebra of observables which shows no dependence on the location of points on the Riemann surface. The properties of the algebra of observables are an invariant characteristic of the string model. It is also useful to consider models obtained by gauging broader chiral algebras, e.g., the Kac–Moody or W_G-algebras. Gauging the entire operator algebra, (indeed its "third quantization," see Ref. 312 for the interpretation of this concept in a more specific context) may be expected to yield an effective field theory in space-time (which should be called "string field theory" or "second–quantized string theory"). The hypothesis of string model equivalence to topological gravity models appears to be a promising approach for string model studies in the nonperturbative domain.

STRING FIELD THEORY–is an analog of a second-quantized theory of particles (i.e., a model of a usual local field theory) for a given string model. The following example can illustrate the difference between the first and second-quantized formulations: while the dynamics of scalar relativistic particles in the first-quantized approach is defined by a sum over lines with weight e^{-ML} and a specific line-branching rule, say, only triple triple branchings are allowed and each enters with weight λ, then in the second-quantized formalism *the same* model is described by an integral over the fields $\phi(x)$ with the action

$$\int (\phi(\nabla_\mu \partial^\mu + M^2)\phi + \lambda\phi^3)\det G)^{1/21/2} d^D x.$$

Both formulations immediately lead to Feynman diagrams (the former—directly to expressions in α-parametrization). The string model is originally defined in terms analogous to the first—quantized formulation which is normally sufficient for the development of the diagram technique. This is exactly what is done in the FORMALISM OF FREE MASSLESS FIELDS on Riemann surfaces when integrals over the parameters α in ordinary Feynman diagrams (i.e., over the line lengths (modules) are actually integrals over modules of Riemann surfaces, whereas the integrands are given by powers of Riemann theta-functions rather than by those of rational functions. However, it is not so easy to represent the same diagrams in the form which could be interpreted as second-quantized, i.e., in the form of a functional integral over string fields $\Phi\{C(t)\}$—functionals of contours $C\{t\}$ in space-time-with the action $\int(\Phi\Delta\Phi + \lambda\Phi^3)$ (with selected operations $\int$, Δ, and multiplication of fields $\Phi\times\Phi\to\Phi$). The formal difficulties are caused by the necessity to take into account additional numerical factors such as the (inverse) modular group volumes in front of expressions for string amplitudes. Less formally, the problem consists in the existence of duality (e.g., the equivalence of the contributions of t and s-channel diagrams to the 4-string amplitude). These problems are not so critical in the case of open string models where it is possible to suggest without much difficulty artificial rules for defining the string action that are likely to ensure correct results.[229] The progress is not so striking as regards closed strings. For various ideas pertaining to the construction of second-quantized formulation of string models (largely the simplest ones: boson strings and superstrings) see Refs. 313–315. There is also the problem of reformulation of "nonperturbative calculations" in the same language, that is the definition of effective string action taking into consideration various deformations of the original string model and summing all orders of perturbation theory. We are not aware of any serious attempt to reproduce using this approach even a well-known result (nonperturbative partition function for the Virasoro minimal models).

SUPERSTRING—this is a specific string model derived from a fermion string by means of GSO projection. The fermion string is a two-dimensional supersymmetric generalization of the boson-string model, a natural synthesis of the Neveu-Schwarz and Ramond (NSR) models.[32] In the critical dimension $D = 10$, a superstring has no tachionic excitations and exhibits space-time supersymmetry. Also, it is possible to describe the critical superstring in the Green–Schwarz formalism in which this supersymmetry is manifestly realized in the absence of two-dimensional fields with half-integer spin on the world sheet. The model was first suggested in Ref. 55, the evidence of cancellation of one-loop divergences and anomalies for the SO(32) gauge group is presented in Ref. 37. Ref. 38 formulates a related HETEROTIC STRING model with the $E_8\times E_8$ gauge group which is also free from anomalies. Different approaches and results relevant to super- and heterotic strings are described in Ref. 21. Also, see Ref. 131 for superstrings on Riemann surfaces in both the NSR and Green–Schwarz formalisms and the problem of demonstration of *finiteness* in all orders of perturbation theory (only incomplete evidence is currently available).

THE BELAVIN-KNIZHNIK THEOREM—in the broad sense, this is an assertion of the precise meaning of holomorphic factorization of measures on module spaces which define, after integration, correlators in the string model. It is a broad generalization of the remark about holomorphic factorization made in Ref. 58 in the context of the definition of rational conformal models on a Riemann sphere. Ref. 135 showing for the first time the fundamental role of complex geometry in string theory, is actually a consideration of the model of the 26-dimensional boson string. It provides evidence that the measure of the module space of Riemann surfaces of genus g which defines the g-loop contribution to a partition function (vacuum diagram) is equal to the squared module of a holomorphic section of the Mumford bundle (Mumford measure) divided (in the case of an aptly selected section) by $\det(\mathrm{Im}T)^{13}$ (for $g\geqslant 2$), where T is the period matrix of the surface and 13 is the number of *noncompact* space-time dimensions divided by two. Poles of the measure at the boundaries of module spaces are identified as being associated with tachionic and dilatonic excitations.

THE DUISTERMAAT–HECKMAN THEOREM—is an assertion of "the accuracy of the quasi-classical approximation," i.e., the possibility to replace the integral with the action S by a sum over solutions of the equations of motion $\delta S = 0$ if two requirements are fulfilled: global action of a compact group-on the integration space is defined, and the dynamic principle and symplectic structure are compatible with this action. The theorem has been proved for finite-dimensional integrals in Refs. 104–106. In the infinite-dimensional situation, it turns into a statement of supersymmetric quantum theories of a special form.[107–108] In the formulation of quantum mechanics in terms of loop space, the global action of the compact group U(1) may be defined as contour reparametrization. In this manner, it is possible to have a symplectic interpretation of supersymmetric models and the NICOLAI TRANSFORMATION.[103] The Duistermaat–Heckman theorem is closely associated with equivariant cohomologies, index theorems, and other topical problems.

THETA-FUNCTIONS–are special functions necessary to make calculations involving Riemann surfaces. Elliptic functions for genus $g = 1$ are known better than others; they may be defined in terms of elliptic integrals or in the form of infinite series. The two options reflect the possibility to describe a surface of genus $g = 1$ either as a complex elliptic curve or as a flat torus. These descriptions do not coincide at $g\geqslant 2$: generalization of the former is realized through definition of the surface as an algebraic manifold (this being an ineffective or at least a poorly developed approach excepting certain cases such as hyperelliptic curves) whereas the generalization of the latter description defines a g-dimensional torus (JACOBIAN), rather than the surface itself, into which the surface is holomorphically embedded by the so-called JACOBI MAP. Given a Jacobian, the surface is unambiguously reconstituted. The analog of elliptic theta-functions for a jacobian is readily constructed in the form of p-fold series and referred to as Jacobi theta-functions. Their inverse image on the surface is known as the Riemann theta-functions and are rather sophisticated objects. Even the description of a set of g-dimensional tori that are jacobians of certain surfaces is a problem (known as the SHOTTKY PROBLEM). The Riemann theorem concerning zeroes that describes the image of the surface in its jacobian as a transcedental equation in terms of Jacobi theta-functions is a key one in the theory of Riemann theta-functions. According to the theorem, it is possible to choose points $R_1,..,R_{g-1}$ on a surface of genus g in such a way that for *any* other points $z_1,...z_{g-1}$

$$\theta(\mathbf{z}_1+...+\mathbf{z}_{g-1}-\mathbf{R}_1-...-\mathbf{R}_{g-1})\equiv 0.$$

Since as many as $g-1$ points are arbitrary, the above equation defines a subset of codimension $(g-1)$ in the g-dimensional jacobian i.e. a one dimensional complex space which is actually the surface image. Applicability of the theorem depends on information about the points $R_1,...,R_{p-1}$ which are holomorphic functions on the module space (alluded to in the main text above as "information about zeroes" of the theta-function). Moreover, it is also useful to know the Jacobi map

$$z\to \mathbf{z}=\int^z \boldsymbol{\omega},$$

itself, i.e., the formula for canonical 1-differentials $\omega_i(z)$ on the surface. The formalism of free massless fields expresses multiloop correlators in terms of the entire set of special functions θ, ω, R. This is not an exhaustive solution of the problem since these objects are not independent. Some comfort may be found in the fact that all these variables are easy to determine and substitute into the general formulas derived in this formalism if explicit parametrization of the surface is defined (for an example of this procedure in the simplest case of hyperelliptic surfaces see Ref. 244). General information about theta-functions can be found in Refs. 133, 134.

TOPOLOGICAL GRAVITATION–this term was suggested by E. Witten for a specific approach to the definition of string models based on a postulate of a generating function of topological invariants of module spaces of bundles over Riemann surfaces as a total nonperturbative partition function of the model. Such a definition may be compatible with factorization conditions connecting contributions of different topologies to string models because topological invariants are inevitably sensitive to the arrangement of module space boundaries. The role of these conditions and, concomitantly, of the algebra of observables is played by a set of the so-called recursion relations. The simplest example: two-dimensional Witten's topological gravity[120] which deals with the generating function of the Chern classes of divisors on the conventional module space of surfaces with punctures. This function can be rewritten in the form of the Kontsevich matrix model with potential $V(X)=X^3$ (Ref. 129) and coincides with the double scaling limit of the conventional matrix model, i.e. with the total nonperturbative partition function of two-dimensional quantum gravity. In this case, recursion relations can be presented in the form of Virasoro constraints imposed on the total partition function which is in itself a τ-function of the Korteveg-de Vries hierarchy. There seems to be no doubt as regards the validity of this scheme in general, but comparabale exhaustive results on the specific relationship between topological characteristics of module space with nonperturbative partition functions of string models and τ-functions of integrable hierarchies are still lacking in other situations. Different aspects of this problem are discussed in Refs. 186–190, 316.

TOPOLOGICAL MODELS–they are quantum-mechanical or quantum-field theories in which all correlation functions are independent of the choice of co-ordinates and metric in space-time or in other spaces relevant to theory definition. This allows the correlation functions to be used as topological invariants of these spaces. A common approach to introduction and investigation of a large class of topological models is based on the use of a functional integral with classical action independent of coordinates and metrics. An obligatory requirement in such a theory also is the invariance of the measure in the functional integral, specifically, the absence of quantum anomalies. A. Schwarz was the first to analyze a topological model (theory of antisymmetric tensor fields) in connection with an evaluation of the Ray-Singer torsion.[231] E. Witten proposed a general formulation of the concept of topological models.[65,120] An important example: Yang–Mills topological theories and topological sigma-models. As a rule, even-dimensional theories of this type use topological charges (e.g. $\mathrm{Tr}\int d^4x F\tilde{F}$) as the action whereas the so-called Wess-Zumino terms play this role in odd-dimensional theories (e.g., the Chern–Simons action $\mathrm{Tr}\int d^3x(A\,dA+(2/3)A^3)$). The three-dimensional CHERN–SIMONS MODEL[117,118] appears to be the most popular one due to its relation to other important problems: topological classification of three-dimensional spaces (knot theories),[66] two-dimensional conformal theories, and quantum groups. First results on a five-dimensional analog of the Chern–Simons model were reported in Ref. 287. The theory of the BRST—cohomologies (which are virtually identical to equivariant cohomologies) constitutes the mathematical apparatus of topological models. The possibility of describing generic topological theories (in which the dependence on metric characteristics is present in the classical approximation but disappears after final evaluation of the functional integral) in the same terms remains unexplored. The examples are quantum gravity theories including (in two dimensions) string models.

An alternative approach to the definition of topological models consists in the analysis of generating functions of topological invariants of different spaces (the model depends on the choice of both the space and the class of invariants). The problem of conditions in which the generating function may be considered as a partition function of a field theory (including the choice of an adequate and "complete" basis in the space of topological invariants) awaits clarification. The situation in which the examined space bears a relation to the metrics space is called TOPOLOGICAL GRAVITY. In a two-dimensional situation, models of topological gravity describe topology of module spaces of Riemann surfaces and bundles over them and are closely associated with string models. A most conspicuous finding obtained in this field is Witten's hypothesis of the relationship between topology and integrability (its simplest version has already been proved). According to this hypothesis, partition functions of (some ?) models of topological gravity are τ-functions of integrable hierarchies.

FORMALISM OF FREE MASSLESS FIELDS–it is composed of two major portions. The first is the theory of free fields per se on Riemann surfaces. It expresses all correlators in terms of Riemann theta-functions and their zeroes (more precisely, in terms of a set of points $R_1...,R_{p-1}$ on a surface, such that

$$\theta(\mathbf{z}_1 + \dots + \mathbf{z}_{p-1} - \mathbf{R}_1 - \dots - \mathbf{R}_{p-1}) \equiv 0.$$

for any $z_1,\dots,z_{p-1}$; a Jacobi transformation). There are four closely related but different systems of free massless fields: scalar with the action

$$\int (\partial\phi\,\bar\partial\phi + \alpha\mathscr{R}\phi),$$

the Grassmann b,c-system with one field of spin j and another one of spin $1-j$ and with the action $\int b\bar\partial c$, a similar *boson* β,γ-system; and a scalar analog but with a restricted domain ϕ of variation of (a field with values in a segment or a ray which are the simplest examples of orbifolds). The first three field types are described in Refs. 158, 125, for multiloop calculations of the latter field type (based on the Prime manifold theory,[308] see Refs. 317, 318. It is also useful to analyse multicomponent free scalars with boundary conditions intermixing the components, i.e. free field theories taking on values in tori and toric orbifolds. A torus case is easily reduced to a one-dimensional situation, the case of orbifolds has not been analyzed in full details (with the exception of the $g=1$ case discussed in Ref. 319).

The second constituent of formalism of free fields is reduction of various conformal models to Gaussian integrals (in this context, such a formalism is sometimes termed bosonization or Coulomb gas representation). Reduction of the Virasoro minimal models was performed by V. Dotsenko and V. Fateev.[161] Transition to free fields in the WZNW model and its reductions is described in Refs. 158, 261 and 131, 209 respectively. Also, see Ref. 131 for the Green-Schwarz superstring model and Refs. 122–126 for a similar approach to two-dimensional gravity.

P-ADIC STRINGS—they are analogs of string models defined on *P*-adic curves. The field of p-adic numbers $\mathbb{Q}_P$ is an analog of the field of real numbers $\mathbb{R}$ and can be obtained by enlarging the set of

$$|x|_P \equiv P^{-\mathrm{ord}_P x},$$

rational with respect to the *non-Archimedian* norm, where $\mathrm{ord}_p x$ is the power in which the *simple* number P enters the expansion of the *rational* number x into simple factors (the norm is non-Archimedian in the sence that $|x+y|_P \leqslant \max |x|_P, |y|_P$). All the fields $\mathbb{Q}_P$ are nonequivalent. $\mathbb{Q}_P$ is composed of formal semi-infinite series

$$a \equiv \sum_{i \geq -m} a_i P^i,$$

where all the a_i are the elements of the finite field F_P of residues modulo P, (that is numbers from the set $0,1,\dots,P-1$). The fields $\mathbb{Q}_P$ are not *algebraically closed*, which means that some algebraic equations with coefficients from $\mathbb{Q}_P$ may have no solution in $\mathbb{Q}_P$. The analog of an algebraically closed and full field of complex numbers $\mathbb{C}$ is the enlargement of the algebraic closure of $\mathbb{Q}_P$; it is denoted by Ω_P. See Refs. 146, 237, 320 for details about determination of p-adic numbers. It is convenient to define $\mathbb{Q}_\infty \equiv \mathbb{R}$. A set of all *simple* numbers P supplemented with the point ∞ coincides with the "spectral ring" Spec$\mathbb{Z}$ of all the simple ideals of the integer ring $\mathbb{Z}$. The relation between the real and p-adic structures is defined by the *product formula* or the *expansion* of *unity*: for any rational x (i.e., x belonging to any $\mathbb{Q}_P$), $\Pi_{\mathrm{Spec}\mathbb{Z}} |x|_P = 1$. This statement may also be formulated in terms of adels.[33]

Riemann surfaces may be considered as algebraic curves over a field of complex numbers, i.e., as one-dimensional complex manifolds defined by algebraic equations, modules of complex structure being related to the coefficients of the equations. Each equation can be examined over the complex number field and over other fields including $\mathbb{Q}_p$; in the latter case the manifolds are referred to as *arithmetic*. Such a change of viewpoint may turn nondegenerate curves into degenerate ones (For example, the elliptic curve (torus) defined by the equation $y^2 = x(x-1)(x-\lambda)$ and nondegenerate over $\mathbb{R}$ or $\mathbb{C}$ for $\lambda \neq 0,1,\infty$ is singular if considered as a curve over $\mathbb{Q}_p$ whenever $\lambda = 0,\ 1,\ \infty \bmod P^n$. The exponent n characterizes the degree of degeneracy of the arithmetic curve over a given point P in the ring Spec$\mathbb{Z}$. Different variables defined on algebraic curves including Green's functions and determinants of Laplace operators may also be considered on arithmetic curves. Given $P \neq \infty$, many such objects are constant (and equal to unity) on a module space excepting the points where the curve becomes degenerate. (For the same example of an elliptic curve, Det $\Delta \sim |\lambda(1-\lambda)|^{1/6}$, and its analog at the point P of the spectral ring is $\mathrm{Det}_P \Lambda \sim |\lambda(1-\lambda)|_P^{1/6} \neq 1$ only for $\lambda \neq 0,\ 1,\ \infty \bmod P$). In this sense, the product formula decomposes these variables into products of elementary p-adic constituents which are entirely defined by the singularities of the module space. These ideas were further developed in the Arakelov–Faltings theory for divisors and heights (Green's functions) on arithmetic curves examined over Spec $\mathbb{Z}$.[146] (The same book describes the application of these ideas to the proof of the Mordell theorem, a weak form of the Great Fermat theorem, and gives a definition of **Mumford measure**). The determinants and the Green's functions of Laplace operators on the curves over $\mathbb{Q}_p$ can be defined by Gaussian functional integrals with fields "living" on Bruhat–Tits trees;[50] see Ref. 51. This formalism is directly analogous to the Polyakov formalism for conventional open strings. The p-adic analog of closed strings requires an adequate description of curves over Ω_P (not only over $\mathbb{Q}_P$) and remains to be found.

One more remarkable fact about the p-adic string theory is (the unexpected!) validity of the product formula for (some?) string *amplitudes*—integrals over module spaces. It is nontrivial that the integrals for p-adic amplitudes are specifically defined as integrals over p-adic numbers and in this sense, the expansion of unity in formulas for string amplitudes in a way commutes with integration. The simplest example of this phenomenon is the expansion of the Veneziano formula for a 4-point function on the sphere;[322] no equally elegant cases have been reported. The reasons for the development of such an "adelic" property in integrals (which ones?) over module spaces remain to be elucidated; see also Ref. 50. Other concepts of p-adic strings are discussed in Ref. 323.

τ-FUNCTION—in the simplest case of $U(1)$-bundles over Riemann surfaces and KP-hierarchy, this is a correlator of free fields (most convenient—of spinors $\tilde\psi,\psi$ with action $\int \tilde\psi\bar\partial\psi d^2z$) on a Riemann sphere with two punctures of the form

$$\tau_G\{T_k\} = \langle\langle e^H \rangle\rangle_G = \langle 0|e^H G|0\rangle,$$

where

$$H = \sum_k T_k J_k, \quad \text{U(1)-current } J(z) \equiv \tilde{\psi}\psi(z) = \sum_k J_k z^{k-1},$$

and

$$G = \exp \sum_{k,l} A_{mn} \tilde{\psi}_m \psi_n.$$

Since the "correction" to the action, $H + \log G$, is quadratic over fermion fields, it is possible to use the Wick theorem to calculate the correlators $\langle\langle e^H \rangle\rangle_G$. Following the Miwa transformation

$$T_k = \frac{1}{k}\sum_a (\lambda_a^k - \bar{\lambda}_a^k) e^H$$

may be presented as

$$e^H = :\prod_a \tilde{\psi}(\bar{\lambda}_a)\psi(\lambda_a):,$$

and

$$\tau_G\{T_k\} = \frac{\prod_{a,b}(\bar{\lambda}_a - \lambda_b)}{\prod_{a<b}(\lambda_a - \lambda_b)(\bar{\lambda}_a - \bar{\lambda}_b)} \det{}_{ab} \langle\langle \tilde{\psi}(\bar{\lambda}_a)\psi(\lambda_b)\rangle\rangle_G.$$

The operator G is defined by a point of the Sato Grassmannian,[169,170] which is in turn defined by the matrix A_{mn}. If this point simultaneously belongs to the Segal–Wilson Grassmannian,[148] the τ-function may be interpreted as a correlator on the corresponding Riemann surface and the operator G as an adequate combination of handle-gluing operators for the Riemann sphere. As a result of an obvious identity we have

$$\oint (\psi(z)|0\rangle \times \psi(z)|0\rangle) = \sum_n \psi_n|0\rangle \times \psi_{-n}|0\rangle \equiv 0$$

(an annihilation operator affects either the one or the other vacuum) and it is possible to perform "conjugation" with the help of the operator G, (actually, to replace both the $|0\rangle$ in the identity by $G|0\rangle$). The τ-function satisfies the system of the bilinear Hirota equations of the type

$$\oint \frac{dz}{z} \tau_G(T_k' + z^k k^{-1}) \tau_G(T_k'' - z^k k^{-1}) = 0$$

for any sets of times $\{T_k'\}$, $\{T_k''\}$. Expansion of the Hirota equations into a series in powers of $(T_k' - T_k'')$ yields a system of integrable equations: the KP hierarchy. Additional restrictions on the form of G result in a reduction of the KP-hierarchy, including that of KdV-(sl(2)), Bussinesq-(sl(3)), and other sl(n)-(or merely n-) reductions. (Note also that the condition for the sl(2)-reduction for G from the Segal–Wilson Grassmannian implies that the Riemann surface must be hyperelliptic). Conversely, if the fields $\tilde{\psi},\psi$ of the original construction took on values in multi-dimensional (rather than linear) bundles over the Riemann surface, there emerge τ-functions of more general hierarchies. In the analysis of string models, τ-functions arise, first, from the Miwa parametrization—merely as correlators of free fields on Riemann surfaces, and second, as nonperturbative partition functions (this time, at a nontrivial site). These two roles of τ-function appear to reflect a more general fact of occurrence of Riemann surfaces (or rather points of the Sato Grassmannian) in string theory in two different aspects: as string world surfaces and as parameters on a manifold of string models. An in-depth investigation (and, thereafter, application) of this fact is an immediate objective of string theory; see Refs. 40,41. Due to the equivalence of some string models and models of topological gravity, the emergence of τ-functions in the role of nonperturbative partition functions may indicate that they also have a topological implication.[120,129,186] In connection with this, it should be noted that in the Kontsevich model, the operator G is expressed in terms of the potential $V(X)$, see Ref. 186; in the general case of analysis of nonperturbative partition functions, the operator G is defined by the "string equation." The relation between integrability and topology as well as topological and/or algebro-geometric implications of the string equation remain obscure.

W_G-ALGEBRAS—they were introduced by A. Zamolodchikov[73] as closed associative *finitely*-generated operator algebras which contain operators of integer spin exceeding 2. The general proof of the existence of classical W-algebras associated with simple Lie algebras G is presented in Ref. 74; see also Refs. 324,325 for an updated concept. The structure of the corresponding module space (an analog of the complex structure module for the Virasoro $W_{\mathrm{sl}(2)}$-algebra) remains to be established. No deep *a priori* reasons are known for the existence of closed quantum W-algebras (it is not trivial that they become closed on a finite number of generators) only a clumsy explicit construction in terms of free fields[76,80] is available. W_G-algebras for $G = A_{N-1} = \mathrm{sl}(N)$ are frequently called W_N-algebras. For $G \neq \mathrm{sl}(2)$, W_G-algebras are not Lie algebras. The algebra of generators is quadratic in the case of a properly chosen basis. *Non*-Lie-like corrections formally disappear in the $N \to \infty$ limit; W_∞-algebra is a Lie algebra at least in the case of a naive definition.[82–85] W-algebras may be considered as describing symmetries of some models of statistical physics and conformal field theory; in the latter case they are usually included in the chiral algebra and it is worthwhile to be gauge them in transition to string models. Corresponding models of W-STRINGS are described in Refs. 86–93, but the theory of W-strings is still in embryo. Similar to generators of the Virasoro algebra, generators of W-algebras come into being in the form of equations for non-perturbative partition functions, e.g., in the double-scaling limit of multimatrix models.[192,181] See Ref. 71 for specific $\tilde{W}$-algebras associated with discrete multimatrix models.

[1)] Moreover, experts in many of these fields prefer to avoid the term "string theory". This, however, can not obscure the fact that there is a standpoint which allows all these issues to be combined to form a novel system of concepts. As usual, such unification has both its own value and outcomes useful for professional researchers in various scientific disciplines.

[2)] As a matter of fact, these systems include any amorphous states. Evolution of the amorphous medium may be considered to proceed via a sequence of almost identical metastable phases not separated by potential barriers, different symmetries or any other qualitative characteristics. It should be emphasized that "amorphous phase" and "spin glass phase" are sometimes referred to as entities even though they are virtually composed of an infinite number of phases differing in configuration of the background fields.

[3)] This specific idea is considered less valuable than the general philosophic principle of Einstein according to which the validity of a basic theory is largely determined by its mathematical beauty. The idea can certainly be expected to show algebro-geometric elegance, but its refinement must not be necessarily expressed in terms of four-dimensional Riemann geometry.

[4)] Manifolds with given metrics or topology give rise to sigma-model and topological model respectively. Something in between (e.g. a complex structure) is reflected in the patterns of instanton fluctuations on the world sheet. Similarly, Lie algebra may be associated with string models for which it plays the role of global symmetry, chiral algebra or *full* symmetry algebra, etc.

5) It is most often a phase of fermion field or an element of the U(1) group. Not all of the above examples appear to explain readily how quasiparticles can possibly be included in this scheme (e.g. vortices in fluids, etc.). Variables satisfying the equation $\mathrm{curl} A = 0$ may seem to be a more universal criterion. However, the practical solution of this equation is $\mathbf{A} = G^{-1}\,\mathrm{grad}\, G$, where G is an element of a compact U(1).

6) The risk of such experiments has been repeatedly emphasized by L. B. Okun in connection with discussions of the problem of catalysis of false vacuum decay.

7) Discussion of the difficulties and obscurity of the standard model is beyond the scope of the present review. Suffice it to mention their three types. First, there is the problem of confinement in QCD. In this context the word "problem" does not imply a real "difficulty." There is a complicated problem of describing the strong coupling regime, but the lack of its solution (or its partial solution) by no means raises doubts as to the adequacy of QCD as the theory of strong interactions. Second, the GWS model has some problems concerning the Higgs sector. It is unclear which of the many possible variants of its arrangement should be given preference. We are not prone to regard this as a serious problem in that future experiments may facilitate selection of one or more of these predictable (and even more or less accurately *calculated*) options. We should hardly be surprised at any particular choice because there is no reliable theoretical reason to prefer any one of the many alternative possibilities (fundamental or composite Higgs boson or bosons; in the case of a fundamental boson, the question arises whether it forms a supermultiplet with one of the known fermions; in the case of a composite boson, the question is whether it is composed of strongly interacting W and Z bosons or new elementary particles, etc.). Perhaps even more important, we can not hope to understand which option is more preferable until we manage to go beyond the framework of ideas assumed as the basis in formulation of the standard model. Third, GWS suffers from the problem of zero-charge. Strictly speaking, it is also the problem of strong coupling, but unlike the case of confinement, we are not confident that it can be solved taking into account nonperturbative effects. A less expensive (and almost certainly more correct) option would be to assume the Grand Unification hypothesis, i.e. interpretation of the standard model as a result of spontaneous breakdown of the broader gauge symmetry with some simple gauge group, SU(5), SO(10) and even $E_8 \times E_8$. True, the concept of Grand Unification brings about a new "problem," the so-called hierarchy problem,[22,23] which in turn can be solved by introduction of supersymmetry. The scenario of Grand *Super*-unification appears to be free from such problems. More than that, there is indirect experimental evidence testifying to its validity (e.g. the "fortunate" correlation between the measured values of the constants of the three interactions[53] and high proton stability).

8) At present, there is only one example of a model of a local QFT that appears to have advantage over renormalizable theories, being absolutely free from ultraviolet divergences. This is the $N = 4$ Yang–Mills theory.[54] However, it can hardly be used for the construction of anything compatible with the standard model. Symbolically, this $N = 4$ model remarkable in more than one aspect is in fact a by-product of a superstring model study of Ref. 55.

9) For example, Maxwell's equation is a second order equation because action has the form of $\int F^2_{\mu}$. Actually, it may be presented as

$$\int \left(F^2 + \frac{a}{M^2} F^3 + \frac{b}{M^4} F^4 + \ldots\right);$$

due to the fact that all corrections to equations of motion with three, four, etc. derivatives are insignificant when energy and momentum are low compared with M. Note that such an approach also offers no difficulty with respect to the number of initial conditions which is normally dependent on the order of the equation: only a part of these conditions (exactly as many as is necessary) are compatible with the criterion of the absence of massive excitations.

10) Gauge invariance can ensure that fields with certain nontrivial (!) transformation properties are massless only provided that there are some such otherwise symmetry can not be expected to yield any useful information. It is sufficient to recall by way of example the confinement phase, a close analog of the topologic phase, in which the QCD gauge symmetry in no way restricts the mass of colorless hadrons.

11) In order to avoid misunderstanding, it should be emphasized that this is a phenomenological theory, i.e. it confines itself to fluctuation studies and ignores other properties and problems of quantum gravity including the divergence problem. For this reason, all inferences deduced from the theory of baby universes are purely qualitative and unreliable. The value of such theories is in that they help to predict events that may occur in a given situation. The final conclusion can be arrived at only in the framework of a comprehensive theory of quantum gravity, and the string theory rather than one of its technical methods may be destined to play this role.

12) In order to avoid misunderstanding, it is worthwhile to note that the fundamental parameter $M = M_{\mathrm{Pl}}$ that defines the coefficient in front of the action

$$M^2 \int G_{\mu\nu}(x)\partial_a x^{\mu} \partial_b x^{\nu} \quad (\mu, \nu = 1, \ldots, D)$$

on the world surface (which is in fact the subject in question) must not necessarily coincide with the standard value

$$\hat{M}_{\mathrm{Pl}} \sim (\hbar c/\gamma)^{1/2} \sim 10^{19}\,\mathrm{GeV}$$

For example, scenarios with the compactification mechanism include one more parameter, R_{comp}, and

$$\hat{M}^2_{\mathrm{Pl}} \sim M^{D-2} R^{D-4}_{\mathrm{comp}}$$

However, the high stability of fundamental interaction parameters in time and a dynamic analysis of string compactification make significant deviation of R_{comp} from $1/M$ unlikely even though such a scenario with one or more intermediate scales is not completely ruled out. We do not emphasize differences between M_{Pl} and $\hat{M}_{\mathrm{Pl}}$ each time we come to mention these parameters in order to avoid minute discussion.

13) The Liouville field in Polyakov formalism is interpreted as determining the conformal factor of the two-dimensional metric

$$g_{ab}(\xi) = e^{\chi(\xi)} \delta_{ab}$$

(its sole *non*-gauge degree of freedom). At first sight, the naive action for a bosonic string,

$$\int G_{ij}(x)\partial_a x^i \partial_b x^j g^{ab} (\det g)^{1/2} d^2\xi$$

with positively denoted G_{ij}, is independent of x_0, but this is mere illusion: due to quantum anomalies[56] the correct action looks like Eq. (2) with a nonvanishing component G_{00}, or in a broader context, G_{0i}. Moreover, G_{00} is negative at least till $D \leqslant 26$. This automatic appearance of the Minkowsky signature in an *a priori* Euclidian theory is one of the most pretty effects in string theory from the point of view of fundamental interactions: the existence of the Minkowsky signature in this world is interpreted as a *quantum anomaly*! In order to count the number of dynamic two-dimensional degrees of freedom, it is necessary to subtract 2 from the total number $D - 1$ of fields x^i (due to invariance with respect to reparametrization of the two-dimensional world surface) and to add 1 (Liouville field). The resulting number $D - 2$ occurs in formulas for $Z(t)$. Note that in formulas (1) and (3a), $Z_{\mathcal{L}}$ include only those effects which are associated with the difference between the x_0 field and all the other x_i fields but unrelated to its mere existence. The overall contribution of the Liouville field to $Z(t)$ is defined by the product $\eta^{-1}(q) Z_{\mathcal{L}}(t)$.

14) This relation is in fact the Jacobi–Riemann identity for elliptic functions and may be known to some of the readers as

$$\vartheta^4_{10}(0, \tau) = \vartheta^4_{00}(0, \tau) - \vartheta^4_{01}(0, \tau).$$

15) It is understandable that the majority of string models should deal with low order, most frequently, second order equations. This is certainly an advantage as regards the philosophy of unification models. However, examination of abstract equations may require selection of string models with more intricate low-energy limits. Note also that string theory allows solution of the old problem concerning obtaining the equations from the symmetry principle. Despite the common delusion, gauge invariance does not solve this puzzle because it is impossible to account for elimination of tr F^3-like corrections to Yang–Mills action tr F^2 without appealing to the non-symmetry principle of minimality. It is natural to use the minimality principle in examining renormalizable and/or effective low-energy models, but it does not fit equally well in discussing a fundamental theory. String interpretation of an equation assumes it to reflect conformal symmetry of an entirely non-obvious object—a two-dimensional theory on the world sheet of the probe string. Of course, the choice of corrections to the equation remains arbitrary and transforms into an arbitrary choice of the string model, but in the case of a fixed model, the relation between the equation and symmetry is uniquely defined.

15a) The basis $f^{kn}_n(z)$ for surfaces with a more complicated topology depends on the choice of the complex structure and is defined by more sophisticated formulas, the requirement being that holomorphic field poles were located only at punctures. The corresponding generators $L\,[f^{kn}_n]$ form a modification of the Virasoro algebra known as the Krichever–Novikov algebra.[57] The operator expansion of $T(z)T(z')$ is local and naturally independent of the choice of basis in the space of vector fields. Therefore, the generators $L\,[f^{kn}_n]$ may be represented as *formal* linear combinations of L_n although certain cohomologies of algebra depend on the choice of the basis which suggests that they are non-isomorphic,—a common thing about the theory of infinite-dimensional algebras.

[16] Verma modules depicted as cones show the following patterns (See Fig. 1).

[17] The notion of a chiral algebra is vague. In the broadest sense of the word a (holomorphic) "symmetry" of a conformal model is the operator algebra of all (chiral) vertex operators (both primary ones with respect to $T(z)$ and their descendants). Identification of such a symmetry as a chiral algebra is possible (and even reasonable in *string* models where all the Virasoro descendants are excluded and indispensable in the full string theory), but such an interpretation is not yet universally accepted in the literature. The notion of a chiral algebra is normally used in one of two contexts: for specific reductions of the WZNW model (see below) when a chiral algebra is considered to be a member of an unbroken algebra in the universal envelope of the original Kac–Moody algebra or when it is meant subsequently to *gauge* a symmetry defined by a chiral algebra (Virasoro algebra in string models, W-algebra in W-string models, and total Kac–Moody algebra in constructing the simplest *topological* models). Note that such a narrow interpretation is in part a concession to generalization (a chiral algebra must not necessarily be a Lie algebra, e.g. W-algebras are quadratic). More essential, construction of the *complete* string theory suggests that the *entire* algebra of vertex operators must be gauged: the corresponding gauge fields are nothing else but D-dimensional fields that describe particle-strings from the space-time, but not two-dimensional, viewpoint. The general term is frequently used in the narrower sense as long as such a narrow problem is not on the agenda.

[18] It is not unusual that, besides generators of ordinary symmetry producing a chiral Lie algebra, a conformal model has a set of operators that form another distinct algebraic structure, a commutative ring (ground ring). If it is included into a chiral algebra, the latter ceases to be a Lie algebra. In a sense, this ring should be associated with "the topological sector" of the model in which case the Lie-like structure is to be regarded as its "symmetry" (and hence, its chiral algebra). Substantiation of such separation in conformal models of the generic type requires an in-depth analysis.

[19] By analogy with such a "generalized Sugawara construction," it is possible to examine "generalized W-operators" of the form

$$\oint dz_1 \ldots \oint dz_n C_{a_1 \ldots a_n}(z, z_1, \ldots, z_n) J^{a_1}(z_1) \ldots J^{a_n}(z_n) + \ldots .$$

For examples obtained using the "local anzatz" see Ref. 70; an example of the nonlocal type yielded by the analysis of matrix models is given in Ref. 71.

[20] Formally, the occurrence of such effects can be accounted for by the presence of derivatives in formulas for gauge transformations (e.g. in the abelian theory $\delta A = d\varepsilon$, but condition $dA = 0$ does not necessarily imply the existence of ε for, say, $A =$ const on the circumference). Certainly, such phenomena are largely applied to solve mathematical problems, e.g. calculation of cohomologies. At the same time, they are interesting physical effects breaking the narrowly-interpreted paradigm of local action: they look like long-range interactions unrelated to particle propagation. A famous example is the Aharonov–Bohm effect (and, as a matter of fact, the static Coulomb interaction). In an ordinary four-dimensional situation, the beauty of the effect is however overshadowed by the presence of a massless photon. Therefore, a more elegant example is the long-range action in a three-dimensional massive electrodynamics containing no massless *particles* capable of propagating over long distances.[100] See also the recent discussion of the "physical view" in connection with the study of the important "$c = 1$" string model.[101] Also note, that "cohomological" effects are commonplace in string theory—they occur at every step.

[21] Supersymmetric models have one more feature which makes their study very promising: they bear information about *symplectic* structures on loop spaces, at least in the case when the action is at most quadratic over fermionic fields.[103] In this sense, supersymmetric models may in the future play the same role in symplectic geometry studies as conformal models play in the analysis of complex geometry. Symplectic nature of quantum supersymmetric models is of great importance as regards the applicability of Duistermaat–Heckman-like theorems (about the exactness of quasiclassical approximation)[104–108] and, in the end, for their exact solvability. This range of problems also includes important cases of index theorems and their striking applications[109–112] as well as the theory of Nicolai transformations.[113,114]

[22] Equations of motion in a conformal theory are conditions of holomorphicity and their solutions would be only constants if it were not for nontrivial boundary conditions (including singularities). Therefore, there is not much to be done to convert a conformal model to a topological one: only eliminate sources of singularities and branch points, i.e. central charges and nonvanishing dimensions. It is exactly this variant that is discussed in the text. Speaking of a *non*-conformal model, the effective theory derived from it by averaging over the metrics might be nontrivial. However, the system of solutions of equations of motion in this effective theory must be invariant (the equations being covariant) with respect to *arbitrary* substitutions of coordinates. This assertion refers to more than two dimensions: such a property must also be possessed by quantum gravity in any number of dimensions. Examples of such effective covariant models are discussed in Ref. 119.

[23] Moreover, the actions of Eqs. (4) and (5) completely coincide only for equations of motion for the metric g_{ab}, the final proof that the integral over g_{ab} with the specified measure does convert (5) to (4) being available only for a similar one-dimensional problem (the theory of relativistic particles). The problem of *proving* that different variants of formalism produce results from one equivalence class is a weak point of string theory in general; on the other hand, more than plausible *hypotheses* of such coincidence usually suggest the existence of remarkable and (as yet) incomprehensible relations between quite different things. (The example of an equivalence of different measures in the integral over metrics: the one mentioned earlier,[56] that induced by the DS reduction,[122–126] and the simply free one,[123–125] may prove unconvincing even if important. See below and in Ref. 127 for a much more striking example: equivalence of the Polyakov formalism and the random lattices approach).

[24] In order to avoid possible misunderstanding, it should be noted that most of the old papers argue that it is possible to neglect the integral over the conformal factor in the case of $D = 26$ (and for other critical strings). However, such a viewpoint does not seem consistent; current ideas are discussed in the text. A peculiar feature of critical strings is that the Liouville field x_0 (except for the signature, which is sometimes also important!) is *indistinguishable* from the remaining fields x_i although this does not mean that it is altogether lacking!

[25] Reference books on special functions usually contain only *elliptic* Jacobi functions corresponding to $g = 1$. Theta-functions at other g-values are a broader class of functions that resemble elliptic ones in more than one aspect. Specifically, they are also representable as Gauss series (only multi-fold) and satisfy specific second order equations that relate dependences on the matrix T and a point on the torus). Another important class of special functions not included in the commonly available reference books is constituted by integrals of theta-functions. Their analog for $g = 0$ is hypergeometric functions (integrals of power functions) which are especially complicated for $g > 1$ since they are defined (or rather interesting) only for *Riemann* theta-functions.

[26] The common explanation is that the shape of triangles is immaterial when their size is very small in the continuum limit. The thing to worry about in the first place in case of a doubt as to the validity of the procedure is the absence of a continuum limit. However, the limit can be obtained by an explicit calculation and is actually well-defined. In this sense, such a definition of two-dimensional quantum gravity is as relevant as the standard definition suggested by Polyakov. Nevertheless, the problem remains as to why these definitions coincide (in fact, this is the problem of equivalence of quantum measures). This problem possibly pertains to an interesting question of algebraic points distribution in module space (see Ref. 127).

[27] This way of reasoning for the case of $D - 1 = 3$ may be illustrated by a well-known simple experiment: take two identical triangles, link corresponding vertices with three lines (loose), and turn one of the triangles through 360°. The lines will get tangled. Now, try to disentangle them without a change in the position of the triangles. It would not be difficult if there were only two lines (it is sufficient to carry one of the lines over the upper triangle), but it is impossible to do in case of three or more lines. However, the lines will easily become disentangled regardless of their number if the triangle is turned through 720° rather that 360°. Note incidentally that it is impossible to derive the existence of fermion statistics from a consideration of two-particle systems (although this is frequently attempted) since it follows from the above examples that the invariant statement refers only to multi-particle (at least $(D - 1)$-particle) systems.

[28] In a broad outline, the number of background parameters must coincide with the number of integration variables. In models of the type of

$$Z\{t_k\} = \int dM \exp \sum_k t_k \mathrm{Tr}\, M^k \quad \text{(Refs. 184,185)}$$

such variables are actually the eigenvalues M and for infinite-dimensional matrices there is obtained a one-parameter family, e.g. a set of the infinite number of "times" $\{t_k\}$. In the Kontsevich models

$$Z_K\{\Lambda\} = \int dX \exp\left(\sum_k t_k \mathrm{Tr}\, X^k\right) + \mathrm{Tr}\, \Lambda X \quad \text{(Ref. 186)}$$

the number of eigenvalues in the matrices X and Λ is identical. The origin of integrability of matrix models of this type consists in the possibility to reduce the integration to a change of external variables (the choice of gauge). More interestingly, the natural parametrization of the action in the matrix integral also turns out to be natural from the point of view of integrability, viz. $Z\{t_k\}$ is a τ-function of the Toda chain directly in the variables $\{t_k\}$.

29) It should be noted, to avoid misunderstanding, that the word "model" has been used throughout the article in its common meaning which has nothing to do with group theory.

30) I wish to thank D. Khmel'nitsky for the convincing explanation of this fact.

31) "Omitted by Author"

32) Curiously (and even symbolically), the Neveu-Schwarz *string* was the very first example of a supersymmetric system and gave an incentive to supersymmetry studies in physics (the algebra of supersymmetry as a mathematical object had formerly been suggested by Gol'fand and Likhtman).[102]

33) An adel ring consists of sequences $A = (a_{(\infty)}, a_{(2)}, a_{(3)}, ..., a_{(P)}, ...)$, where $a_{(P)} \in Q_P$ with a few additional restrictions and rational numbers that form a subgroup of *principal adels* comprising elements of the form $X = (x, x, x, ..., x...)$. See Ref. 321 for more details about adels and idels. The philosophy behind adels and the product formula may be stated as follows: all "natural" physical variables are trivial (equal to unity?) if considered over a ring of adels (or at least principal adels) rather than common numbers: $f(X) = \Pi_{\mathrm{Spec}\, Z} f_P(x) = 1$. Hence, the existence of non-trivial physical variables $f_\infty(x)$ can be interpreted as a result of "symmetry breakdown": for unknown reasons, we can see only one (real) constituent, its non-triviality being, however, totally compensated by a "latent" p-adic component. It will be shown below that such "natural" functions include not only the norm $f_P = |...|_P$, but also string measures determined from Bruhat–Tits trees and, possibly, certain string amplitudes (at least the Virasoro amplitude).

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