Radiation of electromagnetic waves on instantaneous change of the state of the radiating system

B. M. Bolotovskiĭ, V. A. Davydov, and V. E. Rok

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR
Usp. Fiz. Nauk 126, 311–321 (October 1978)

The paper considers electromagnetic radiation arising when an instantaneous change with time of the parameters of the radiating system occurs. It is shown that under real conditions when the change in the parameters occurs in a finite time T, this quantity under certain conditions can be neglected and the radiation treated as the result of an instantaneous transition. Conditions are found for which this is true. As illustrations we determine the radiation fields arising on instantaneous change in magnitude and direction of a stationary point dipole moment, and also the field arising on instantaneous jump of the velocity of a point charged particle in a refracting medium.

PACS numbers: 41.10.Hv

CONTENTS

1. INTRODUCTION

We shall discuss a physical system which produces a static electromagnetic field. This may be a system of stationary electric charges or electric currents which are constant in time. Generally speaking, a change in the parameters of such a system is accompanied by radiation of electromagnetic waves. Suppose that the parameters of the system remain constant up to some moment of time and then in the course of a time interval T the system is rearranged so that its parameters change in a definite way, and at the end of the time interval T the rearrangement is completed and the parameters of the system take on new, final values which do not subsequently change. As an example we can consider a system possessing an electric dipole moment. If the dipole moment does not change, then the field of this system is electrostatic and there is no radiation. Now let us assume that the dipole moment of the system changes in the course of some finite time interval T from an initial value $\mathbf{p}_1$ to a final value $\mathbf{p}_2$. We can then state that before the beginning of the change there exists over all space an electrostatic field corresponding to the dipole with moment $\mathbf{p}_1$, and the change of the dipole moment from $\mathbf{p}_1$ to $\mathbf{p}_2$ is accompanied by radiation of electromagnetic waves; after the final value of the dipole moment $\mathbf{p}_2$ is established and all radiated waves have traveled to infinity, an electrostatic field corresponding to the dipole moment $\mathbf{p}_2$ is established over all space.

We pose the following question: Is it meaningful to discuss the case in which the change of the dipole moment occurs instantaneously? It is clear that any physical process, including the change of a dipole moment, requires a finite time to be accomplished. If the electric dipole is based on two charges of different signs, it is possible to change the value of the dipole moment by changing the location of these charges. To move the charges it is necessary to create a field, which in itself cannot be done instantaneously. In addition, the charges possess finite mass and therefore under the action of a finite force cannot change their location instantaneously. In analyzing any specific experimental arrangement, we reach the conclusion that an instantaneous change of the parameters of the system is impossible. However, this fact in itself does not mean that we cannot discuss an instantaneous change of a dipole moment.

Physicists often replace a real physical object by some idealized scheme. As an example we can cite a mathematical pendulum. How is a mathematical pendulum defined? It is a material point suspended on an ideal weightless hook and unstretchable fiber. None of the things entering into the definition exist in nature. Nevertheless the idea of a mathematical pendulum is an extraordinarily productive one which permits us to understand the most important features of the oscillations of a real pendulum. It turns out that the size of the body hung on the fiber is unimportant over wide limits and this justifies the idea of a material point in this case. If the mass of the fiber is much less than the mass of the suspended weight, we can consider the fiber weightless, and so forth.

Therefore we should not pose the question as follows: Is the instantaneous change of dipole moment possible? The answer to this question is clear, and it is a negative answer. However, this fact in no way prevents an idealized representation in terms of an instantaneous change from being useful. We shall show below when such an idealization is permissible.

The representation of an instantaneous change in the parameters describing a radiating source has a quite definite region of applicability. This region is defined as follows. Let us consider some stationary source of small size, and assume that radiation occurs as the result of change of some parameter (dipole moment, velocity, etc). Let this parameter change during some time interval T from a specified initial value to a specified final value. We expand the radiation field in monochromatic waves and take a wave with frequency ω. If the radiation source is stationary and the inequality $\omega T \ll 1$ is satisfied, then the radiation of the wave with frequency ω is determined only by the initial and final values of the parameter (dipole moment) and does not depend on the time T during which this parameter changes. The rearrangement time T drops out of the formulas which determine the radiation and we can consider the rearrangement instantaneous. If $\omega T \gtrsim 1$, and *a fortiori* if $\omega T \gg 1$, we cannot consider the rearrangement time equal to zero.

If the radiating system is moving, the criterion of instantaneous rearrangement changes. Let the velocity of the radiating system be V and the time of change of the parameter responsible for the radiation be T. In the rest frame of the source let the rearrangement of the parameter occur in a time T_1 and let a frequency ω_1 be radiated. Then in the rest system of the source we will have the criterion indicated above for instantaneous rearrangement:

$$\omega_1 T_1 \ll 1. \tag{1.1}$$

In order to obtain the criterion for prompt rearrangement in the laboratory system, it is necessary to express ω_1 and T_1 in terms of ω and T (the frequency and rearrangement time in the laboratory system) by means of a Lorentz transformation and to substitute these expressions into (1.1). Substituting into (1.1)

$$T_1 = T\sqrt{1-\frac{v^2}{c^2}}, \quad \omega_1 = \frac{\omega\,[1-(v/c)\cos\theta]}{\sqrt{1-(v^2/c^2)}}, \tag{1.2}$$

we obtain

$$\omega T \ll \frac{1}{1-(v/c)\cos\theta}, \tag{1.3}$$

where θ is the angle between the direction of the velocity and the direction of observation.

Thus, if the frequency of the wave satisfies the inequality (1.3), the rearrangement time does not enter into the expression for the intensity of radiation at frequency ω and we can speak of an instantaneous change of the parameter.

From Eq. (1.3) it is evident, in particular, that if the velocity of the radiation source is close to the velocity of light and we are discussing radiation forward, the criterion of instantaneous rearrangement is satisfied also for those frequencies for which $\omega T \gg 1$ (there is a region of frequencies which satisfies simultaneously the inequality (1.3) and the inequality $\omega T \gg 1$). Since for high frequencies radiated forward ($\theta = 0$) the instantaneous rearrangement criterion is satisfied and the principal loss to radiation is just at high frequencies, then in this case practically the entire loss to radiation (or in any case the principal part of it) can be obtained on the assumption of an instantaneous rearrangement of the source parameters.

It follows from Eq. (1.3) that taking into account the motion of the radiating system substantially extends the region of frequencies for which the instantaneous rearrangement criterion is satisfied. In fact, for a stationary system ($v=0$) we obtain from Eq. (1.3) $\omega \ll T^{-1}$. And if the system is moving, we obtain for the case of forward radiation ($\theta=0$)

$$\omega \ll \frac{1}{T}\,\frac{1}{1-(v/c)}.$$

At velocities close to the velocity of light the factor multiplying $1/T$ in the last inequality is proportional to the square of the total energy of the radiating system, and the instantaneous rearrangement criterion can be satisfied for frequencies which exceed the reciprocal of the radiation time by many orders of magnitude.

We emphasize this fact, because it does not always receive due attention. In particular, in the first editions of *Theory of Fields* by Landau and Lifshits[1] the radiation frequencies for which the rearrangement can be considered instantaneous were defined by the inequality $\omega\tau \ll 1$ (τ is the rearrangement time), and only in the last (sixth) edition is a criterion of the form (1.3) given.

This criterion can be rewritten in another form, multiplying both sides of the inequality (1.3) by the radiating system velocity v and dividing by the radiation frequency ω. We obtain

$$l = vT \ll \mathcal{L} = \frac{v}{\omega}\,\frac{1}{1-(v/c)\cos\theta}; \tag{1.4}$$

here l is the path traveled by the radiating system during the rearrangement time and $\mathcal{L}$ is a quantity of the dimensions of length, the so-called radiation formation length. This quantity plays an important role in the theory of transition processes of the type considered, particularly in the theory of transition radiation.[5] It can be seen from Eq. (1.4) that if the path of the system during the time of rearrangement is much less than the radiation formation length, the rearrangement can be considered instantaneous.

The criterion (1.4) qualitatively explains the features of transition radiation at a diffuse separation boundary.[6] It is necessary only to take into account that in a refracting medium it is necessary in the inequality (1.4) to write instead of the velocity of light c the value c/n, where n is the refractive index.

2. RADIATION ON INSTANTANEOUS CHANGE OF A DIPOLE MOMENT

Consider a system possessing a time-dependent dipole moment $\mathbf{p}(t)$. The size of the system will be assumed to be small and in what follows we shall neglect it. Then the density of dipole moment can be written in the form

$$\mathbf{p}(\mathbf{r}, t) = \mathbf{p}(t)\,\delta(\mathbf{r}) \tag{2.1}$$

(we assume that the dipole moment is located at the origin of coordinates). Let the dipole moment vector $\mathbf{p}(t)$ initially be $\mathbf{p}_1$ and then in the time from $-T$ to $+T$ change to $\mathbf{p}_2$ and then remain constant. Consider the radiation arising at frequency ω on such a change of the dipole moment. The Fourier component $\mathbf{A}_\omega$ of the vector potential is given by the formula

$$\mathbf{A}_\omega=\frac{1}{c}\int \mathbf{j}_\omega(\mathbf{r}')\frac{\exp\left(i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|\right)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}', \tag{2.2}$$

where $\mathbf{j}_\omega(\mathbf{r}')$ is the Fourier component of the current $\mathbf{j}$, which is related to the change of the dipole moment by the well known formula

$$\mathbf{j}=\frac{\partial \mathbf{p}}{\partial t}. \tag{2.3}$$

In our case

$$\mathbf{j}=\frac{d\mathbf{p}(t)}{dt}\delta(\mathbf{r}), \tag{2.4}$$

and this quantity is nonzero only at the origin and in the time interval from $-T$ to T. The current Fourier component $\mathbf{j}_\omega(\mathbf{r})$ is given by the formula

$$\mathbf{j}_\omega(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{j}e^{i\omega t}dt=\frac{\delta(\mathbf{r})}{2\pi}\int_{-\infty}^{\infty}\frac{d\mathbf{p}(t)}{dt}e^{i\omega t}dt. \tag{2.5}$$

If we take frequencies sufficiently low so that they satisfy the inequality

$$\omega T \ll 1, \tag{2.6}$$

we can set the exponential in the integral equal to unity and then we have

$$\mathbf{j}_\omega=\frac{\delta(\mathbf{r})}{2\pi}\int_{-\infty}^{\infty}\frac{d\mathbf{p}}{dt}dt=\frac{\delta(\mathbf{r})}{2\pi}\Delta\mathbf{p}, \tag{2.7}$$

where $\Delta\mathbf{p}=\mathbf{p}_2-\mathbf{p}_1$ is the change in dipole moment in a time $2T$. Exactly the same expression for $\mathbf{j}_\omega$ is obtained on the assumption that the dipole moment changes discontinuously from $\mathbf{p}_1$ to $\mathbf{p}_2$ and that this jump occurs at the moment of time $t=0$. Indeed, in this case

$$\mathbf{j}(t)=\Delta\mathbf{p}\delta(t)\delta(\mathbf{r}) \tag{2.8}$$

and the Fourier component of $\mathbf{j}(t)$ coincides with (2.7).

The inequality (2.6) signifies that the period $2\pi/\omega$ of the radiated wave is much greater than the rearrangement time T, and therefore we can neglect the rearrangement time.

Substitution of expression (2.7) into Eq. (2.2) gives

$$\mathbf{A}_\omega(\mathbf{r})=\frac{\Delta\mathbf{p}}{2\pi c}\frac{e^{ikr}}{r}\quad\left(k=\frac{\omega}{c}\right). \tag{2.9}$$

Carrying out the inverse Fourier transformation, we obtain an expression for the vector potential $\mathbf{A}$ as a function of the coordinates and time:

$$\mathbf{A}(\mathbf{r},t)=\frac{\Delta\mathbf{p}}{cr}\delta\left(t-\frac{r}{c}\right). \tag{2.10}$$

A dipole moment of the form (2.1) produces not only currents $\mathbf{j}$ but also a charge density ρ. The latter is determined by the formula

$$\rho=-\operatorname{div}\mathbf{p}(\mathbf{r},t). \tag{2.11}$$

Therefore the field is determined not only by the vector potential (2.10), but also by the scalar potential φ, which satisfies the wave equation

$$\Delta\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi(\mathbf{p}(t),\nabla)\delta(\mathbf{r}). \tag{2.12}$$

Equation (2.12) for the potential φ is obtained in the Lorentz gauge, where $\operatorname{div}\mathbf{A}=-(1/c)\partial\varphi/\partial t$. The solution of this equation has the form

$$\varphi=-\frac{1}{2}\left[1-\operatorname{sgn}\left(t-\frac{r}{c}\right)\right](\mathbf{p}_1\nabla)\frac{1}{r}-\frac{1}{2}\left[1+\operatorname{sgn}\left(t-\frac{r}{c}\right)\right](\mathbf{p}_2\nabla)\frac{1}{r}+\frac{\Delta\mathbf{p}\cdot\mathbf{r}}{cr^2}\delta\left(t-\frac{r}{c}\right); \tag{2.13}$$

here $\operatorname{sgn}x=x/|x|$ is the sign function.

Equation (2.10) for the vector potential $\mathbf{A}$ and Eq. (2.13) for the scalar potential φ completely determine the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$. The fields are related to the potentials by the well known expressions:

$$\mathbf{E}=-\operatorname{grad}\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\qquad \mathbf{H}=\operatorname{rot}\mathbf{A}. \tag{2.14}$$

Let us consider the structure of the fields in this problem. It follows from Eq. (2.10) that the vector potential is nonzero only on the spherical shell $r=ct$ which is expanding with the velocity of light from the point where the change in dipole moment occurred. Consequently, the magnetic field also is nonzero only at this shell. In constrast to the magnetic field, the electric field is nonzero over all space. The electric field pattern which follows from the formulas obtained can be represented as follows. Suppose that at a moment of time $t=0$ (which coincides with the jump in dipole moment from $\mathbf{p}_1$ to $\mathbf{p}_2$) there begins to expand from the point where the dipole is located a spherical shell whose equation is $r=ct$. Then at any moment of time the electric field inside this shell is the static electric field

$$\mathbf{E}_2=\frac{3\mathbf{r}(\mathbf{p}_2\mathbf{r})-\mathbf{p}_2r^2}{r^5}, \tag{2.15}$$

which is equal to the field of a static dipole with moment $\mathbf{p}_2$ located at the origin of coordinates. Outside the shell $r=ct$ the electric field also does not depend on time and is equal to

$$\mathbf{E}_1=\frac{3\mathbf{r}(\mathbf{p}_1\mathbf{r})-\mathbf{p}_1r^2}{r^5}. \tag{2.16}$$

This is the field of a static dipole with moment $\mathbf{p}_1$. An observer located outside the shell (at $r>ct$) does not yet know that the dipole moment at the origin of coordinates has changed and is already equal to $\mathbf{p}_2$, and not $\mathbf{p}_1$. Sooner or later the shell $r=ct$ expanding with the velocity of light will reach the observer, and at this moment the electric field at the point of observation will jump from $\mathbf{E}_1$ to $\mathbf{E}_2$. On the spherical surface $r=ct$ itself there is a variable electric field which depends on the coordinates and time. We have seen above that on this same expanding surface there is also a magnetic field. For the magnetic field it is easy to obtain the expression

$$\mathbf{H}=\operatorname{rot}\mathbf{A}=\frac{1}{cr^2}[\Delta\mathbf{p},\mathbf{r}]\left[\frac{1}{r}\delta\left(t-\frac{r}{c}\right)+\frac{1}{c}\delta'\left(t-\frac{r}{c}\right)\right]. \tag{2.17}$$

For the electric field at the surface $r=ct$ we find

$$\mathbf{E}=-\frac{1}{cr^3}[\mathbf{r}[\Delta\mathbf{p},\mathbf{r}]]\left[\frac{1}{r}\delta\left(t-\frac{r}{c}\right)+\frac{1}{c}\delta'\left(t-\frac{r}{c}\right)\right]=-\left[\frac{\mathbf{r}}{r},\mathbf{H}\right]. \tag{2.18}$$

It is evident that the fields at the surface of the sphere $r=ct$ are equal in magnitude and mutually perpendicular. In addition, at each point they are directed along a tangent to the sphere, i.e., they are perpendicular to the radius vector drawn from the origin of coordinates to the given point of the sphere. The three vectors **E**,**H**, and **r** form a right-handed set.

The fields (2.17) and (2.18) are a spherical electromagnetic wave propagating with the velocity of light from the origin of coordinates. At the points through which this wave passes, an instantaneous switching of the field occurs from one static value to the other. We take the vector $\Delta\mathbf{p}$ as the axis of the spherical surface $r=ct$, so that the poles of the sphere lie on the extension of the vector $\Delta\mathbf{p}$. Then the magnetic field of the wave (2.17) and (2.18) is everywhere directed along the tangent to the circles of latitude, and the electric field is directed along the tangent to the meridians. Let us further determine the intensity of radiation at frequency ω arising on instantaneous change of the dipole moment by an amount $\Delta\mathbf{p}$. The spectral density of radiation per frequency interval $d\omega$ and per solid angle $d\Omega$ is

$$dW_{\mathbf{n},\omega}=c\,|\mathbf{H}_\omega|^2 r^2\,d\Omega\,d\omega. \tag{2.19}$$

At large distances r from the origin the magnetic field $\mathbf{H}_\omega$ is simply expressed in terms of the vector potential $\mathbf{A}_\omega$ (2.9):

$$\mathbf{H}_\omega(\mathbf{r})=i\frac{\omega}{c}[\mathbf{n},\mathbf{A}_\omega], \tag{2.20}$$

where **n** is the unit vector in the direction of radiation. This gives

$$dW_{\mathbf{n},\omega}=\frac{(\Delta\mathbf{p})^2}{4\pi^2c^3}\sin^3\theta\,d\theta\,d\varphi\,\omega^2\,d\omega. \tag{2.21}$$

This formula can be obtained from the general formula for the spectral distribution in dipole radiation:

$$dW_{\mathbf{n},\omega}=\frac{|(\ddot{\mathbf{p}})_\omega|^2}{c^3}\sin^3\theta\,d\theta\,d\varphi, \tag{2.22}$$

where $(\ddot{\mathbf{p}})_\omega$ is the Fourier component of the second derivative of the dipole moment with respect to time. In the case discussed

$$(\ddot{\mathbf{p}})_\omega=\frac{i\omega}{2\pi}\Delta\mathbf{p}. \tag{2.23}$$

Substitution of this value into (2.22) immediately gives (2.21). Here θ is the angle between the vectors $\Delta\mathbf{p}$ and **n**, and φ is the azimuthal angle (θ and φ are the angles which determine the direction of radiation **n** in the spherical coordinate system in which the direction of $\Delta\mathbf{p}$ is taken as the axis).

As can be seen from Eq. (2.21), the radiation spectrum is proportional to the square of the frequency and therefore the total intensity of radiation diverges. Obviously the radiation spectrum must be cut off at frequencies which do not satisfy the inequality (2.6), this cutoff taking into account that the change of the dipole moment is not instantaneous but occupies a finite time T. Then for the total energy of radiation we obtain a finite value inversely proportional to T^3.

On taking into account the finite rearrangement time, the field pattern also changes. The magnetic field in this case will be nonzero not on a spherical surface, but in a spherical layer of thickness $2cT$ formed by two concentric spherical surfaces with a center at the location of the dipole. Both the inner and outer surfaces bounding the spherical layer are expanding from the origin with the velocity of light. Inside this layer there is also an electric field variable with time. In front of the layer and behind the layer there exist only static electric fields. The time of passage of the layer through any point of space is $2T$, and during this time the field changes from one static value corresponding to the dipole $\mathbf{p}_1$ to the other corresponding to the dipole $\mathbf{p}_2$.

3. INSTANTANEOUS STOP OR START OF A UNIFORMLY MOVING CHARGE IN VACUUM

Let a charge q previously moving uniformly with velocity **v** come to rest in a time T. Then, if the frequency of the waves radiated by it satisfy the inequality (1.3), the time T will not enter into the expression for the electromagnetic fields, radiation intensity, etc; in other words, for these frequencies the approximation of instantaneous stopping of the charge will be valid. The same can be said also of a starting charge. Let us consider first the case of instantaneous stopping of the charge. Let a charge q move uniformly along the z axis with velocity **v** and at the moment of time $t=0$ instantaneously stop at the origin. From the point of stopping of the charge there travels a wave of radiation located on the sphere $r=ct$. If the point at which the field is measured is located at a distance r from the origin, the wave of radiation will reach it at a moment of time r/c, and up to this time no signal of the stopping of the charge will reach the point r. Consequently, at this distance the potentials and fields will be described by the appropriate expressions for the potentials and fields of a charge moving uniformly with velocity **v** along the z axis. These expressions are easily obtained, for example, by using Coulomb's law and a Lorentz transformation,[1] namely:

$$\varphi=\frac{q}{R^*},\qquad \mathbf{A}=\frac{\mathbf{v}}{c}\frac{q}{R^*},$$
$$\mathbf{E}=\frac{q\mathbf{R}}{\gamma^2(R^*)^3}=\frac{q\mathbf{R}(1-\beta^2)}{R^3(1-\beta^2\sin^2\theta)^{3/2}},\qquad \mathbf{H}=\left[\frac{\mathbf{v}}{c},\mathbf{E}\right], \tag{3.1}$$

where φ is the scalar potential, **A** is the vector potential, **E** is the electric field, **H** is the magnetic field of a charge moving uniformly with velocity **v**, $\beta=v/c$, $\gamma=(1-\beta^2)^{-1/2}$,

$$R^*=\sqrt{(z-vt)^2+(1-\beta^2)(x^2+y^2)},$$

$(0,0,vt)$ are the coordinates of the moving charge, **R** is the radius vector from the charge to the point of observation at the moment t, and θ is the angle between **R** and **v**. Thus, it is evident that the electric field of a charge which has instantaneously stopped at the origin, measured at distances r greater than ct, is directed toward the point where the charge would be if it continued to move without stopping. This field is weakened in comparison with the Coulomb field in the region of angles θ close to 0 and π and enhanced in the region of angles θ close to $\pi/2$. Consequently, if this field is mapped by means of the lines of force, they should be crowded together on approach of θ to $\pi/2$.

On the other hand, at distances r from the origin less than ct, the Coulomb field of the charge q should be established:

$$\varphi=\frac{q}{r},\quad \mathbf{A}=0,\qquad \mathbf{E}=\frac{q\mathbf{r}}{r^3}\quad \mathbf{H}=0, \tag{3.2}$$

where r is the distance from the origin to the point of

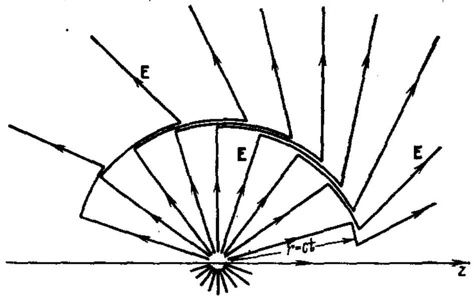

FIG. 1.

observation.

Thus, it is clear that the entire rearrangement of the field of a uniformly moving charge must occur on the sphere of radius ct (Fig. 1). Since the Coulomb field is isotropic, and the field of a uniformly moving charge has only axial symmetry, the lines of force of the electric field, while remaining continuous, must bend at the sphere $r=ct$. It is easy to calculate by means of Gauss's Law how the angle of inclination of the electrical lines of force to the z axis changes on passing through the spherical surface $r=ct$ (see Purcell[2]).

If φ_0 is the angle of inclination of an electric line of force to the z axis outside the sphere $r=ct$ and θ_0 is the angle of inclination of the same line of force inside the sphere $r=ct$, then at this sphere the line of force is bent in such a way that φ_0 and θ_0 are related as follows:

$$\operatorname{tg}\varphi_0=\gamma\operatorname{tg}\theta_0. \tag{3.3}$$

If we now assume that the axis of the sphere is the z axis, then the electric field at the sphere will be directed along the meridians and the magnetic field along the parallels. The electric and magnetic fields are singular at the sphere $r=ct$. Consequently, on instantaneous stopping of the charge an infinite energy is radiated. This should be expected, since we assume that the acceleration of the charge is infinite.

In a similar way we can discuss the question of the field of a charge which started at the moment $t=0$ from the origin with velocity v along the z axis (Fig. 2). In this case inside the sphere of radius $r=ct$ with center at the origin the field of a uniformly moving charge (2.1) is established. Outside this sphere, in the region where the signal of the starting of the charge has not yet arrived, the field will be purely Coulomb (3.2). On the sphere, as in the case of the stopping charge, a bending of the electric lines of force will occur. The electric field itself will be purely transverse at the sphere.

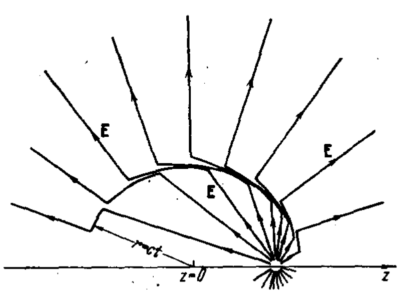

FIG. 2.

On the other hand, if the charge starts or stops not instantaneously but in some period of time T, then the sphere spreads into a spherical layer of thickness cT, inside which the electric lines of force of the uniformly moving charge will go over into the lines of force of the Coulomb field. Inside this transition region there will necessarily be present a transverse component of the electromagnetic field, the magnitude of which will be greater, the smaller is the value of T. Thus, all radiation of the starting or stopping of the charge will be concentrated inside this spherical layer of width cT.

4. FIELD TRANSFORMATION ON INSTANTANEOUS STOP OR START OF A ČERENKOV CHARGE

Tamm[3] investigated the influence of acceleration of a charge on Vavilov-Čerenkov radiation. He considered the radiation of a charged particle which is first at rest, then is instantaneously accelerated to a "superluminal" velocity $v>c/n$ (where $n=\sqrt{\varepsilon}$ is the refractive index of the medium), moves with this velocity for a finite time interval, and then instantaneously stops. Tamm investigated only the spectral characteristics of the resulting radiation, without discussing the spatial pattern of the fields. It is of interest to consider in more detail two problems more elementary than that posed by Tamm: the problem of instantaneous stopping of a superluminal charge and the problem of instantaneous acceleration of a stationary charge to superluminal velocity. An understanding of the structure of the fields arising in these situations will permit a clear and understandable representation of the radiation field in a broad class of problems, including the case considered by Tamm.

Let us first take the case of instantaneous stopping of a Čerenkov charge.[4]

Let a charge q move along the z axis with a velocity v greater than the phase velocity of light c/n in the medium, and at a moment of time $t=0$ instantaneously stop at a point $z=0$. Then the equations for the vector and scalar potentials $\mathbf{A}$ and φ have the form

$$\begin{aligned}\Delta\mathbf{A}-\frac{n^2}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}&=-\frac{4\pi}{c}\mathbf{j},\\ \Delta\varphi-\frac{n^2}{c^2}\frac{\partial^2\varphi}{\partial t^2}&=-\frac{4\pi}{n^2}\rho.\end{aligned} \tag{4.1}$$

Here the current density $\mathbf{j}$ and charge density ρ are given by the expressions

$$\begin{aligned}\mathbf{j}&=q\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t)\,\theta(-t),\\ \rho&=q\delta(\mathbf{r}-\mathbf{v}t)\,\theta(-t)+q\,\delta(\mathbf{r})\,\theta(t);\end{aligned} \tag{4.2}$$

the quantity $\theta(x)$ is the Heaviside step function,

$$\theta(x)=\frac{1}{2}\left(1+\frac{x}{|x|}\right).$$

Let us now expand the potentials $\mathbf{A}$ and φ in a Fourier integral of the form

$$\begin{aligned}\mathbf{A}(\mathbf{r},t)&=\int\mathbf{A}_{\mathbf{k},\omega}e^{i(\mathbf{kr}-\omega t)}\,d\mathbf{k}\,d\omega,\\ \varphi(\mathbf{r},t)&=\int\varphi_{\mathbf{k},\omega}e^{i(\mathbf{k},\mathbf{r}-\omega t)}\,d\mathbf{k}\,d\omega.\end{aligned} \tag{4.3}$$

Then the equations (4.1) for the Fourier components

$\mathbf{A}_{k,\omega}$ and $\varphi_{k,\omega}$ take the form

$$\mathbf{A}_{\mathbf{k},\omega}=\frac{4\pi}{c}\frac{\mathbf{j}_{\mathbf{k},\omega}}{k^2-n^2(\omega^2/c^2)},\qquad \varphi_{\mathbf{k},\omega}=\frac{4\pi}{n^2}\frac{\rho_{\mathbf{k},\omega}}{k^2-n^2(\omega^2/c^2)},\tag{4.4}$$

where $\mathbf{j}_{k,\omega}$ and $\rho_{k,\omega}$ are given by the expressions

$$j_{\mathbf{k},\omega}=\frac{q\mathbf{v}}{(2\pi)^3}\delta^-(\omega-\mathbf{kv}),\qquad \rho_{\mathbf{k},\omega}=\frac{q}{(2\pi)^3}[\delta^-(\omega-\mathbf{kv})+\delta^+(\omega)];\tag{4.5}$$

here

$$\delta^{\pm}(x)=\frac{1}{2\pi}\int_0^{\infty}e^{\pm itx}\,dt=\frac{i}{2\pi}\lim_{\gamma\to+0}\frac{1}{x\pm i\gamma}=\frac{i}{2\pi}\frac{1}{x\pm i0}.\tag{4.6}$$

It follows from Eqs. (4.3)–(4.6) that

$$\mathbf{A}(\mathbf{r},t)=i\frac{4\pi q\mathbf{v}}{c(2\pi)^4}\int d\mathbf{k}\,d\omega\frac{\exp[i(\mathbf{kr}-\omega t)]}{(\omega-\mathbf{kv}-i0)[k^2-n^2(\omega^2/c^2)]},\tag{4.7}$$

$$\varphi(\mathbf{r},t)=i\frac{4\pi q}{n^2(2\pi)^4}\int d\mathbf{k}\,d\omega\frac{\exp[i(\mathbf{kr}-\omega t)]}{k^2-n^2(\omega^2/c^2)}\left(\frac{1}{\omega-\mathbf{kv}-i0}+\frac{1}{\omega+i0}\right).\tag{4.8}$$

Examination of expressions (4.7) and (4.8) shows that the fields arising after the stopping of the charge divide space into three regions, whose boundaries shift with time. Up to the moment $t>0$ these regions and the fields in them have the following form (Fig. 3):

Region I:

$$\rho>\frac{vt-z}{\gamma},\quad r>\frac{ct}{n},\quad z<\frac{c^2t}{vn^2},$$

where $\rho=r\sin\theta=\sqrt{x^2+y^2}$ (here θ is the polar angle in coordinates fixed to the stopped charge and the axis Oz is parallel to $\mathbf{v}$). In this region the field coincides with the field inside the Čerenkov cone for a charge moving with velocity v at the point $z=vt$ (see Tamm[3]),

$$\varphi=\frac{2q}{n^2\sqrt{(vt-z)^2-\gamma^2\rho^2}},\quad A_z=\beta n^2\varphi,\quad A_x=A_y=0,\quad \beta=\frac{v}{c},\quad \gamma^2=n^2\beta^2-1.\tag{4.9}$$

The expressions given for the potentials of the electromagnetic field refer to points of space lying outside the sphere of radius $r=ct/n$ with center at the point of the stopped charge. Since the signal from the stopping of the charge, which propagates in the medium with a velocity c/n, has not yet reached region I, it is therefore natural that the field in this region remains the same as if the charge continued to move without stopping.

Region II: In this region the Coulomb field of the stationary charge q has already succeeded in forming itself:

$$\varphi=\frac{q}{n^2r},\quad A=0.\tag{4.10}$$

Finally, *region III* is the region left after removal from all space of regions I and II together with their boundaries. In this region there is no field.

Thus, we know the fields inside all three regions. It remains to find expressions for the fields at the boundaries. In view of the absence of dispersion and as a consequence of the instantaneous stopping of the charge, the boundaries of the regions are abrupt. The fields at the boundaries can be determined by means of Gauss's Law and simple geometrical considerations. Indeed, as a consequence of Gauss's Law the flux of electric induction through any surface containing inside itself the point $r=0$ with charge q must equal $4\pi q$. We shall take such a surface as follows (see Fig. 3). Consider a plane P perpendicular to the z axis and located behind the charge in regions I and III. We form a closed surface R from a portion of the plane P and some surface S lying entirely in region III. We calculate the flux of the electric induction vector through the surface R.

According to Gauss's Law

$$4\pi q=n^2\int_P\mathbf{E}\,d\sigma+n^2\int_S\mathbf{E}\,d\sigma\qquad(\mathbf{D}=\varepsilon\mathbf{E}=n^2\mathbf{E}).\tag{4.11}$$

The second integral vanishes as the result of the absence of field in region III. The first integral reduces to the sum of integrals over the internal region of the circle O (the part of the plane P belonging to region I) and over the circle O itself. The integral over the circle we designate by Φ_0. Thus,

$$4\pi q=n^2\int_{\rho<\rho_0}\mathbf{E}\,d\sigma+\Phi_0.\tag{4.12}$$

Using the fact that

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},$$

and also Eq. (4.9), it is easy to obtain

$$n^2\int_{\rho<\rho_0}\mathbf{E}\,d\sigma=n^2\left(-\nabla\int_{\rho<\rho_0}\varphi\,d\sigma-\frac{\mathbf{v}}{c}\frac{n^2}{c}\frac{\partial}{\partial t}\int_{\rho>\rho_0}\varphi\,d\sigma\right)=-4\pi q.\tag{4.13}$$

It follows from Eqs. (12) and (13) that

$$\Phi_0=8\pi q.\tag{4.14}$$

In other words the flux over the surface of the cone is $8\pi q$. It follows from the symmetry of the problem that the electric field must be directed along the generatrix of the cone and must be identical at all points of the circle O. From Eq. (14) and taking into account that

$$\rho_0=\frac{vt-z}{\gamma},$$

we find the magnitude of the field at the surface of the cone:

$$|\mathbf{E}|=\frac{4q\beta}{n(vt-z)}\delta\left(\rho-\frac{vt-z}{\gamma}\right).\tag{4.15}$$

Expressions (4.14) and (4.15) obviously are valid not only in the case of stopping of a Čerenkov charge, but also in the case of continuous motion of the charge in a medium without dispersion.

Applying Gauss's Law to closed surfaces encom-

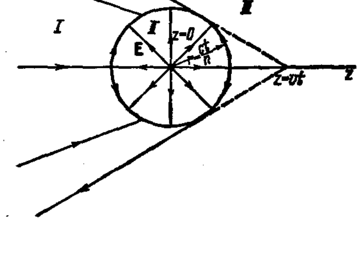

FIG. 3.

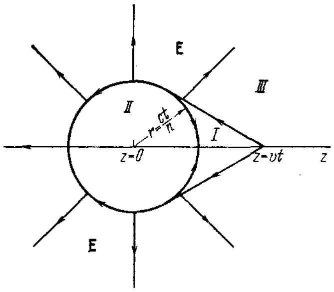

FIG. 4.

passing the elements of the boundaries separating regions I and II and regions II and III, we obtain similarly expressions for the fields at these boundaries:

$$\mathbf{E}_{\mathrm{I-II}} = -\mathbf{e}_\theta \frac{qn}{ct \sin\theta}\left[2\beta\left(\frac{\cos\theta - \beta n}{\beta n \cos\theta - 1} + 1\right) - \frac{1}{n^2}(1+\cos\theta)\right]\delta(r - r_0) \quad (4.16)$$

for

$$\pi \geqslant \theta > \arccos\frac{1}{\beta n}, \qquad r_0 = \frac{ct}{n}$$

and

$$\mathbf{E}_{\mathrm{II-III}} = \mathbf{e}_\theta \frac{q}{nct \sin\theta}(1 - \cos\theta)\,\delta(r - r_0) \quad (4.17)$$

for

$$0 \leqslant \theta < \arccos\frac{1}{\beta n};$$

here $\mathbf{e}_\theta$ is the polar unit vector of the spherical coordinate system with center at the point $z=0$.

In order to find the fields arising on instantaneous start of a Čerenkov charge at the moment of time $t=0$ from the point $z=0$, there is no necessity to solve Eq. (4.1) again. Imagine that one charge is at rest at the point $z=0$ and a second similar charge is moving with velocity $v > c/n$ along the z axis. At the moment when the second charge is at the point $z=0$, let it stop instantaneously and the first charge begin to move with velocity v along the z axis. It is clear that the pattern remains stationary, since the charges are indistinguishable. However, we can now obtain expressions for the fields of a starting charge, subtracting from the fields of the stationary pattern described above (i.e., from the sum of the Čerenkov and Coulomb fields) the fields just found for the stopping of the charge. It is easy to see that in the case of a start, space is also divided into three regions for $t > 0$ (Fig. 4).

In region I ($r > (c/n)t$, $\rho < (vt - z)/\gamma$, $z > c^2t/vn^2$) the field is equal to the sum of the field (4.9) and the Coulomb field (4.10).

In region II ($r < (c/n)t$) the potentials will be described by Eq. (4.9).

In region III, which consists of all space after sub-

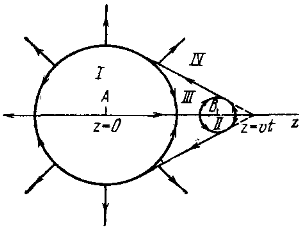

FIG. 5.

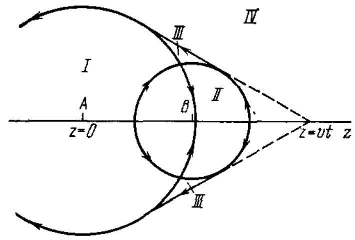

FIG. 6.

tracting regions I and II together with their boundaries, the field is pure Coulomb (4.10).

The results obtained are easily explained. Indeed, in the whole of region II it is already known that the charge started, and therefore the field of a uniformly moving Čerenkov charge is established there. In region I the signal of the starting of the charge lags behind the charge itself, and therefore in this region the field is the sum of the Coulomb and Čerenkov fields. In region III, where nothing is known of the starting of the charge, the field will remain pure Coulomb, as from a charge q continuing to remain at rest at the origin.

Knowing the field pattern for instantaneous start or instantaneous stop of a Čerenkov charge, we now can discuss the spatial pattern of the field in the problem posed by Tamm and mentioned above (motion of a Čerenkov charge in a finite segment of path).

Let a point electric charge up to the time $t=0$ be at rest, and at the moment of time $t=0$ instantaneously be accelerated to a velocity v greater than the phase velocity of light c and move with this velocity for a time interval T. At $t=T$ the charge instantaneously stops.

Consider the field pattern produced with this motion of the charge. It is shown in Fig. 5. In this figure the line $AB = vt$ is the path traveled by the charge. From point A as a center a spherical shell whose radius is $(c/n)t$ expands with a velocity c/n. Inside this shell (region I) the field coincides with field of a charge uniformly moving with velocity $v > c/n$ [see Eq. (4.9)]. From point B as a center a second spherical shell whose radius is $(c/n)(t-T)$ expands with velocity c/n. We consider the field pattern after stopping of the charge, so that $t > T$ and the radius of the second sphere is positive. Inside this sphere the field is equal to the field of a Coulomb center at rest at the point B (region II). Region III is part of the volume of the cone whose generatrices are the tangents to the spheres bounding regions I and II. In this region the field is equal to the sum of two fields: the field (4.9) produced by a uniformly moving Čerenkov charge and the Coulomb field produced by a charge at rest at point A. Finally, in region IV, which consists of all remaining space, the field is equal to the Coulomb field of a charge at rest at point A.

In the figure we have shown the field regions on the assumption that the spheres expanding from the beginning and end of the trajectory do not intersect. It is clear that sooner or later these spheres will partially overlap, i.e., and intersection of regions I and II will appear. In this portion of the space, which is common to regions I and II, the field will be equal to the sum of two fields: the Coulomb field of a charge at rest at point B and the Čerenkov field (4.9). The field pattern in this case is shown in Fig. 6.

The authors are grateful to V. L. Ginzburg for helpful remarks.

[1]L. D. Landau and E. M. Lifshits, Teoriya polya (Theory of Fields), Moscow, Nauka, 1973, p. 124.
[2]E. M. Purcell, Electricity and Magnetism, McGraw Hill, Russ. transl., Nauka, 1971, pp. 173–177.
[3]I. E. Tamm, Sobranie nauchnykh trudov (Collection of Scientific Works), Moscow, Nauka, 1976, p. 77.
[4]V. A. Davydov and V. E. Rok, in: Teoriya difraktsii i rasprostraneniya voln: VII Vsesoyuznyĭ simpozium po difraktsii i rasprostraneniyu voln (Theory of Diffraction and Propagation of Waves: Seventh All-Union Symposium on Diffraction and Propagation of Waves), Rostov-on-Don, 1977. Brief texts of reports, Moscow, 1977, Vol. 3, p. 333.
[5]I. M. Frank, Izv. AN SSSR, Ser. Fiz. 6, 3 (1942) [Bull. USSR Acad. Sci., Phys. Ser.].
[6]A. Ts. Amatuni and N. A. Korkhmazyan, Zh. Eksp. Teor. Fiz. 39, 1011 (1960) [Sov. Phys. JETP 12, 703 (1961)].

Translated by Clark S. Robinson