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STOCHASTIC INSTABILITY OF NON-LINEAR OSCILLATIONS

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1. INTRODUCTION

THE topic of this review is relatively new and little known, especially among physicists. Hence, we shall begin with a simple example. Let us imagine the motion of some non-linear oscillator, e.g., an ordinary pendulum, but without damping, when acted on by an external very weak periodic force. The equation of motion of such a system can be written in the form

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = \varepsilon F(t), \tag{1.1}$$

where φ is the angle of deviation of the pendulum from the equilibrium position, ω_0 is the frequency for small oscillations, and $\epsilon \to 0$.

Let the perturbation be harmonic initially:

$$F(t) = F_0 \cos\Omega t. \tag{1.2}$$

The solution of the latter problem is well known (see e.g.[2] and Sec. 2.2), and we shall describe qualitatively a case close to resonance in which the frequency Ω of the perturbation is approximately equal to the frequency $\omega(a)$ of unperturbed motion of the oscillator. In view of the non-linearity, the latter depends on the amplitude a of the oscillations ($\omega(0) = \omega_0$). In contrast to linear resonance, the frequency of a non-linear oscillator varies when a resonance perturbation acts on it because of the variation in amplitude of the oscillations. Consequently, in turn, the amplitude and frequency of the oscillator undergo oscillations (beats) over a certain range Δa, $\Delta\omega$.

Now let us suppose that the perturbation F(t) contains many harmonics:

$$F(t) = \sum_n F_n \cos n\Omega_0 t, \tag{1.3}$$

and that the frequency spacing (Ω_0) between them is much smaller than the beating interval $\Delta\omega$, so that several harmonics of the periodic perturbation of (1.3) simultaneously occur in resonance. How will the pendulum oscillate in this case? In spite of the smallness of the perturbation ($\epsilon \to 0$), no solution of this problem has yet been rigorously found. However, a semiqualitative theory (this review will be concerned with it to a considerable degree), as well as numerical experiments (Chap. 4), lead to a rather unexpected conclusion: the pendulum will oscillate as though a "random" force were acting on it. In particular, the energy of the oscillations will increase on the average in proportion to the time, in spite of the non-linearity of the system (in the absence of damping). We shall call this special type of motion of an oscillatory system stochasticity.

First of all, study of the stochasticity phenomenon is interesting from the purely physical standpoint, because it connects two seemingly rather different fields of physics: the theory of non-linear oscillations and statistical mechanics. It is in essence one of the possible mechanisms by which statistical laws can arise in a dynamic system. From this standpoint, elucidation of the fundamental features of stochastic motion is also of considerable importance in the general theory of non-

linear oscillations. Stochasticity is no less significant from the standpoint of applications, mainly those concerning the problem of stability of non-linear oscillations of conservative systems, which arises in the most varied fields of physics and technology.

Let us take up this problem in somewhat greater detail.

The most difficult point in the problem of stability of oscillations is precisely the case of a conservative system. In this case, only the so-called neutral stability can exist. Here, small perturbations of the starting conditions cause limited oscillations about the unperturbed solution that neither grow nor decay with time. In particular, this means that any unconsidered factor, however small, can transform such a neutral stability into instability. This is one of the reasons why we do not yet have a general theory of stability of non-linear oscillations of conservative systems, in spite of considerable efforts along this line, especially in the last decade. Another reason is that the fundamental field of application of the theory of non-linear oscillations has involved until recently self-oscillatory or self-regulating systems, which are not conservative. This biased orientation of the theory of oscillations is well noted in the monographs (see, e.g.[50]). It has had the result that even such flexible new methods as the Krylov-Bogolyubov-Mitropol'skiĭ asymptotic expansion (the KBM method[1,2]) have been used to a very limited extent, while the entire topic of the theory of non-linear oscillations has been artificially narrowed to the problem of existence and stability of periodic solutions. On the other hand, it has proved possible in celestial mechanics (however paradoxical this may be, the latter is not usually included in general courses of non-linear oscillations[50]) to restrict the treatment to constructing approximate solutions only for very short intervals of time (on the characteristic scale), by virtue of the shortness of human life. This fact is a highly specific feature of modern celestial mechanics, a feature that has led to rejection of Poincaré's original program of constructing a general, although qualitative, theory of motion of conservative systems.

However, new, important applications of the theory of non-linear oscillations have recently appeared. They involve the problem of prolonged retention of charged particles in a limited region of space by means of an electromagnetic (usually magnetic) field having a special configuration. These are the magnetic traps for a thermonuclear reactor,[55] charged-particle accelerators,[6] and especially, accumulators for proton and proton-antiproton counter beams.[56] We can also classify here the problem of non-linear interaction of waves in a plasma (see, e.g.[33]). These applications pose the problem of studying motion of a non-linear oscillatory system as a whole, i.e., over an unlimited time range and for arbitrary initial conditions. On the other hand, even the first experiments[9,57] and calculations[10] in the field of new applications have indicated that the problem of stability of non-linear oscillations is related to the problem of appearance of statistical laws in a dynamic system. Further, it has turned out that such a relation was also found by Hedlund and by Hopf[12] within the framework of the so-called ergodic theory (see Chap. 3). The latter is a special branch of mathematics that has arisen from attempts to provide a foundation for the laws of thermodynamics and statistical physics on a purely mechanical basis. The first application of this important result of Hedlund and Hopf to problems of physics, and until recently the only one, was made by Krylov,[13] who studied the statistical behavior of molecular systems. The recent mathematical and physical studies that are reviewed in this article should be considered to be a development, or more properly, a reactivation of this field. In particular, the ergodic theory has been developing vigorously recently, mainly in connection with the studies of Kolmogorov[31] and of Anosov and Sinaĭ[14-16] (Sec. 3.5), and it has left far behind the statistical mechanics that gave rise to it. However, the modern ergodic theory describes only the limiting case, which is often unattainable in principle, of maximum instability of motion. Hence, it does not suffice per se for solving the mentioned applied problems. On the other hand, a mathematical theory has recently arisen that has been stimulated in part by celestial mechanics. It treats precisely the opposite limiting case of maximum stability of non-linear oscillations of a Hamiltonian system. This very general theory was also started by Kolmogorov,[58] and has been developed by Arnol'd[59] and Moser[60] (the KAM theory). Evidently, we must combine both limiting theories in order to solve practical problems. Unfortunately, it is currently impossible to construct rigorously such a unifying theory, owing to great difficulties in principle. However, one can develop a semiqualitative theory, which we shall call the theory of stochasticity. This theory makes it possible to depict an overall pattern of instability of non-linear oscillations supplemented with estimates of orders of magnitude. Often this proves sufficient for applications, in any case, when combined with experiments, either numerical or "real."

The fundamental concept of the theory is that of nonlinear resonance (Sec. 2.2). This interesting phenomenon, an example of which is described above, was apparently studied first by Lagrange in celestial mechanics, where it acquired the name of librational motion of planets (see, e.g.[3]), and in more explicit form, in the theory of accelerators, in connection with the mechanism of autophasing discovered by Veksler and McMillan[4,5] (see also[6]). Below, we shall introduce and study nonlinear resonance with the example of relatively simple one-dimensional models, beginning with the rather general case of a non-linear oscillator acted on by an external periodic perturbation (Sec. 2.1), and ending with an "elementary" model (Sec. 3.1). The fundamental method of study will be analytical estimation based partially on the current ergodic theory (Chap. 3), and corroborated by numerical experiments with the "elementary" model (Chap. 4).

The theory of stochasticity makes it possible to picture the overall structure of the phase space of a nonlinear oscillatory system, which proves to be permeated everywhere by a dense system (Sec. 2.4) of non-linear resonances. Owing to the interaction of these resonances in the neighborhood of the separatrix of each of them, a so-called stochastic layer always arises. The latter is the nucleus of instability. This fact was known in general form even by Poincaré,[61] and has been studied in detail by Mel'nikov,[62] who was able to calculate the

so-called splitting of the separatrix, which characterizes the lower bound of the width of the stochastic layer. The upper bound was estimated in[20,8] by using the stochasticity theory, and it proved to agree in order of magnitude with the splitting of the separatrix. Recently, Alekseev[63] has rigorously proved the existence in the stochastic layer of quasi-random trajectories, thus justifying the name of the layer.

For a system having one degree of freedom, the rather thin stochastic layers of different resonances do not intersect in the phase plane (see, e.g., Fig. 1). Hence, the instability is localized within one stochastic layer, and is not dangerous from the practical standpoint. Rigorous treatment of this problem in the KAM theory[59] makes it possible to formulate and prove in this case the theorem of permanent ($t \to \infty$) stability of motion. The same situation also arises in an autonomous system having two degrees of freedom, since its motion is confined in phase space to a surface of constant energy. We shall call both these cases one-dimensional. In the multidimensional case, the stochastic layers of different resonances generally intersect one another to form an intricate system of "channels" along which diffusion can occur to great distances, i.e., real instability. The first example of such an instability was constructed by Arnol'd.[35] Hence it is usually called Arnol'd diffusion. The semiqualitative theory of this instability has been developed in[8].

The width of a non-linear resonance and its stochastic layer is determined by the size of the "perturbation" with respect to the "unperturbed" system, which we understand to be a so-called maximally stable system (Sec. 3.3), i.e., a system having separable variables. If the perturbation is great enough, adjacent resonances in phase space overlap (Sec. 2.3). Then the stochastic layers broaden to the size of their resonances to form a continuous stochastic region in phase space. Here a strong stochastic instability arises (including even the one-dimensional case). This is the most dangerous instability of non-linear oscillations. This review is mainly concerned with describing this instability in particular with the examples of simple models (Chaps. 2, 3) and some applications (Chaps. 5 and 6).

For the reader's convenience, we also summarize the fundamental ideas and concepts of the current ergodic theory (Chap. 3).

The stochasticity theory not only permits one to obtain practically-important criteria and characteristics of instability of non-linear oscillations, but also to trace in detail the transition from dynamic to statistical description of the motion of a mechanical system. The latter is essential for a deeper and more correct understanding of the statistical models. This problem is discussed briefly in Chap. 7, and also in Sec. 3.6. We refer readers who are interested in this problem in greater detail to the excellent monograph of Krylov,[13] whose studies have appeared to us to be the first step toward constructing a physical theory of stochasticity.

2. A ONE-DIMENSIONAL NON-LINEAR OSCILLATOR ACTED ON BY A PERIODIC PERTURBATION

The fundamental problem of this review is to demonstrate the phenomenon of stochastic instability of non-linear oscillations by means the simplest models that are still physical (i.e., bear a relation to actual mechanical systems). One such model is a one-dimensional nonlinear oscillations by means the simplest possible models that are still physical (i.e., bear a relation to actual mechanical systems). One such model is a one-dimensional nonlinear oscillator acted on by a periodic perturbation, as described in the Introduction. The pattern of motion of this model is very pictorial, but as we shall see, not at all trivial. Some important practical problems can be reduced to this model. Some of them, including also multidimensional models (in the first approximation), will be discussed below. Of course, this does not mean at all that one can solve all non-linear oscillation problems with the discussed model. However, as we shall see below, it permits one to study the fundamental features of non-linear oscillations, by means of which we can approach the solution of more complex problems. Among the latter, for example, are the so-called Arnol'd diffusion, a special form of stochastic instability of multidimensional systems. We shall not treat it in any detail, and shall refer interested readers to the special studies[35,8] (see also Chap. 6).

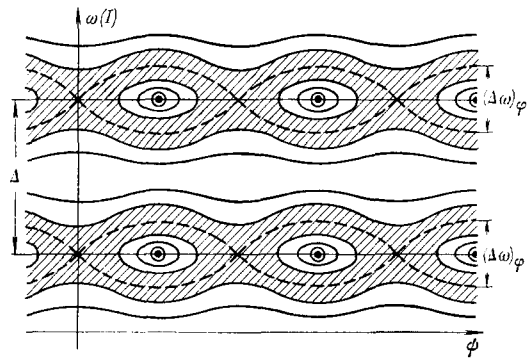

FIG. 1. Diagram of the phase trajectories in the neighborhood of two resonances with moderate non-linearity (2.6). $(\Delta\omega)_\phi$ is the width of the non-linear resonance; Δ is the frequency spacing between adjacent resonances. The dotted lines show the separatrices of the first-order approximation; they break down in subsequent approximations, and stochastic layers are formed in their place, which are shown by the cross-hatching (Sec. 6.1).

Henceforth we shall always assume the perturbation to be small in order to allow approximate methods. For this reason, the fundamental effect of the perturbation involves resonance. This statement is not so obvious for a non-linear as for a linear system, but it still proves to be correct (Sec. 2.2).

2.1. An Isolated Non-linear Resonance

The most convenient variables to describe the motion of a non-linear oscillator are the variables: action (I) and angle (θ). The action is related to the energy W of the system be the relation

$$\frac{dW(I)}{dI} = \omega(I), \quad \theta = \omega t + \varphi,$$

where $\omega(I)$ is the frequency of the non-linear oscillations. When there is a perturbation, the equations of motion of the oscillator in terms of the variables (I, θ) of the unperturbed motion have the form:

$$\left.\begin{aligned} \dot{I} &= -\varepsilon \frac{\partial V(I, \theta, \vartheta)}{\partial \theta}, \\ \dot{\theta} &= \omega(I) + \varepsilon \frac{\partial V(I, \theta, \vartheta)}{\partial I}, \quad \dot{\vartheta} = \Omega, \\ \mathcal{H}(I, \theta, \vartheta) &= \mathcal{H}_0(I) + \varepsilon V(I, \theta, \vartheta), \end{aligned}\right\} \tag{2.1}$$

Here $\mathcal{H}$ is the Hamiltonian, while the perturbation ϵV is small ($\sim \epsilon \ll 1$) and periodic in θ and n with a period of 2π. The phase ϑ characterizes the external perturbation having a period $T = 2\pi/\Omega$. We shall take the non-linearity of the oscillator to be its non-isochronicity, i.e., dependence of the oscillation frequency on the energy. The following dimensionless parameter serves to characterize the non-linearity:

$$\alpha = \left| \frac{I}{\omega} \frac{d\omega}{dI} \right| = \left| 2 \frac{\partial^2 \mathcal{H}_0}{\partial I^2} \middle| \frac{\partial \mathcal{H}_0}{\partial (I^2)} \right|. \tag{2.2}$$

Anharmonicity of the oscillations, i.e., presence of higher harmonics, does not necessarily involve non-linearity. Thus, revolution of a relativistic particle in a magnetic field gives an example of a non-linear, but harmonic oscillator. On the other hand, motion of an ultrarelativistic particle in a rectangular potential well is an anharmonic oscillation having a constant frequency.

In the general case, the perturbation ϵV is expanded in a double Fourier series

$$V(I, \theta, \vartheta) = \sum_{m, n} \frac{V_{mn}(I)}{2} e^{i(m\vartheta + n\theta)} + \text{c.c.} \tag{2.3}$$

A resonance (m, n) (the m are harmonics of oscillations of the oscillator with the nth harmonic of the external perturbation) is determined by the condition

$$m\Omega + n\omega \approx 0. \tag{2.4}$$

The required accuracy of this equality depends on the width of the resonance (see below in Sec. 2.2.).

The simplest case is that of a single resonance: m, n = +1 ($\omega \approx \Omega$). Then two (real) terms remain in the summation of (2.3). One of these, having the argument $\vartheta - \theta$ ($\omega, \Omega > 0$) is the resonance proper, while the other ($\vartheta + \theta$) is a high-frequency perturbation. We can neglect the latter term in the first approximation, e.g., on the basis of the KBM averaging method (Krylov-Bogolyubov-Mitropol'skiĭ[1,2]). More subtle effects of such a perturbation will be discussed below (see Chap. 6).

If the width of the resonances is small enough, then only one resonance can occur under the given initial conditions. We shall call it the real resonance, or simply the resonance of (2.4). Under these initial conditions, all other terms in the summation of (2.3) correspond to virtual resonances.* Naturally, virtual resonances affect the motion. However, in a non-linear system, this effect does not amount to a trivial superposition, and we shall call it the interaction of resonances.

Whenever the spacing between resonances is "large enough", a term that will be defined more precisely below (Sec. 2.3), the isolated-resonance approximation must still be satisfied, in which we can ignore interaction of resonances. This corresponds with the ordinary KBM method of averaging. If we retain only one resonance term in the summation of (2.3), and introduce the resonance phase $\psi_{mn} = m\vartheta + n\theta$, we get from (2.1) the equations of motion in the form

$$\left.\begin{aligned} \dot{I} &= \varepsilon n V_{mn} \sin \psi_{mn}, \\ \dot{\psi}_{mn} &= m\Omega + n\omega(I) + \varepsilon n V'_{mn} \cos \psi_{mn}, \end{aligned}\right\} \tag{2.5}$$

where the prime denotes the derivative with respect to the action I.

2.2. Phase Oscillations in the Approximation of Moderate Non-linearity

Let the parameter α in (2.2) satisfy the condition of moderate non-linearity:

$$\varepsilon \ll \alpha \ll 1/\varepsilon. \tag{2.6}$$

As we shall see below, the variation $\Delta I = I - I_r$ is always small in this case ($|\Delta I| \ll I_r$), where the value I_r corresponds to exact resonance $(m\Omega + n\omega(I_r) = 0$. Hence, independently of the form of the function $\omega(I)$, $V_{mn}(I)$ of the motion near a non-linear resonance of (2.5) is approximately described by the following (conservative) universal Hamiltonian:

$$\left.\begin{aligned} \mathcal{H}_u &= n\omega' \frac{(\Delta I)^2}{2} + \varepsilon n V_{mn}(I_r) \cos \psi_{mn}, \\ \omega' &= (d\omega/dI)_{I=I_r}. \end{aligned}\right\} \tag{2.7}$$

The system of (2.7) resembles a pendulum of mass $(n\omega')^{-1}$ in a gravitational field having an acceleration $g = \epsilon n^2 \omega' V_{mn}$. Figure 1 shows the pattern of trajectories in the phase plane I, ψ_{mn} for two non-linear resonances without taking account of the interaction between them. In the initial phase plane (I, θ) of the oscillator, the chain of resonances is closed, and it contains n regions. Each such region contains a stable equilibrium position, or an elliptical stationary point* ($\psi_{mn} = \pi$; $g > 0$), which is indicated in Fig. 1 by the symbol $\odot$. Adjacent regions are separated by positions of unstable equilibrium, or by a hyperbolic stationary point ($\psi_{mn} = 0$; $g > 0$, symbol $\times$ in Fig. 1). The trajectory connecting two consecutive hyperbolic points, which is indicated in Fig. 1 by a dotted line, is called the separatrix. It bounds the region of non-linear resonance within which phase oscillations occur, i.e., limited variation in the phase ψ_{mn}. The phase varies monotonically in time outside the separatrix. The maximum width of the non-linear resonance (dimensions of the separatrix) is determined from (2.7):

$$\left.\begin{aligned} (\Delta I)_\varphi &= 4\sqrt{\varepsilon V_{mn}/\omega'}, \\ (\Delta\omega)_\varphi &= \omega'(\Delta I)_\varphi = 4\sqrt{\varepsilon\omega' V_{mn}} = \frac{4\Omega_{mn}}{n}, \end{aligned}\right\} \tag{2.8}$$

where Ω_{mn} is the frequency of the small phase oscillations. With moderate non-linearity as in (2.6), the variation of both I and ω is small:

$$\frac{\Delta I}{I} \sim \sqrt{\frac{\varepsilon}{\alpha} \frac{V_{mn}}{\omega I}}, \quad \frac{\Delta\omega}{\omega} \sim \sqrt{\varepsilon\alpha V_{mn}/\omega I},$$

This justifies the universal description of a non-linear resonance using the Hamiltonian of (2.7).

We can explain the physical meaning of the phase oscillations as follows. Owing to the non-linearity, a change in the amplitude of the oscillations at a resonance causes the frequency of the oscillations to depart from the resonance value. Hence, it stops further change in the amplitude of the oscillations. However, the frequency shift that arises here leads to departure from the reson-

*We note that not every virtual resonance can become real, since the condition (2.4) is not generally satisfied by all m, n.

ance phase, so that the amplitude of the oscillations finally begins to vary in the opposite direction, the oscillation frequency again returns to the resonance value, etc.

Although non-linearity thus even stabilizes an isolated resonance, i.e., it restricts the variation ΔI, the latter proves to be substantially larger according to (2.8) than in the non-resonance case: $\Delta I \sim \epsilon I$. This justifies the distinguishing of resonances in a non-linear system. For the same reason, the real parameter in the expansion when $\alpha \sim 1$ is $\sqrt{\epsilon}$, rather than ϵ. In particular, in the first approximation in terms of $\sqrt{\epsilon}$, we can neglect the last term in Eq. (2.5) for ψ_{mn}, which is a linear correction term for the frequency.* We must take it into account only when $\epsilon \gtrsim \alpha$. In particular, the condition for stabilization of a resonance by non-linearity can be obtained approximately from the estimate†

$$\varepsilon n V'_{mn} \leqslant n (\Delta\omega)_\varphi,$$

or

$$\frac{\varepsilon}{\alpha} \leqslant \frac{\omega V_{mn}}{I (V'_{mn})^2}. \qquad (2.9)$$

Apparently, non-linear resonance was first studied by Lagrange in connection with the so-called librational motion of planets,[3] and in a more explicit form, in the theory of accelerators in connection with the mechanism of autophasing discovered by Veksler and McMillan[4,5] (see also[6]). We have adopted the fundamental terminology concerning non-linear resonance from the latter studies. A universal theory of non-linear resonance in the approximation of moderate non-linearity has been given in[7] (see also[8]).

2.3. Interaction of Non-linear Resonances

Interaction of non-linear resonances is the fundamental effect that we shall take up in this study. Its significance consists in the fact that the very dangerous stochastic instability of non-linear oscillations arises when the interaction is strong enough.

Evidently, interaction of resonances depends on the ratio of the width $(\Delta\omega)_\varphi$ (2.8) of the resonance to the distance to the nearest neighboring resonance $\Delta = |\omega_{i+1} - \omega_i|$ (see Fig. 1).

We shall call the quantity

$$s = \frac{(\Delta\omega)_\varphi}{\Delta} \qquad (2.10)$$

the coupling constant of the resonances.

The isolated-resonance approximation corresponds to the condition: $s \ll 1$. On the other hand, when $s \gtrsim 1$, which implies overlap of adjacent resonances, the pattern of motion must change substantially. In fact, as the universal Hamiltonian of (2.7) shows, an isolated non-linear resonance is equivalent to the motion of a system in a certain potential well. When there are several resonances, there are just as many wells (see Fig. 1). If $s \gtrsim 1$, the adjacent wells merge, so that the system can travel from one well to another.* Evidently, this implies the development of a certain instability. Our fundamental assertion consists in the idea that stochastic instability of non-linear oscillations sets in when the coupling constant $s \gtrsim 1$, i.e., the motion becomes highly irregular and seemingly random (see Sec. 3.6). Hence, we can also call the coupling constant s the stochasticity parameter.

We shall call the condition

$$s \sim 1 \qquad (2.11)$$

the stochasticity boundary.

Apparently, stochastic instability of non-linear oscillations was first observed in numerical experiments by Goward and Hine,[9] who also obtained criteria of instability like (2.11). Analytical estimates are given in[10], where arguments are also given in favor of the idea that this instability is stochastic. The latter viewpoint has subsequently been corroborated in a series of studies, which are reviewed in the monographs[8,11] (see also below). It turned out that stochastic instability is a special case of manifestation of statistical laws. The relation of the latter to instability was first noted in an especially constructed example by Hedlund and Hopf.[12] Krylov[13] has analyzed it in detail for the case of molecular collisions, and Anosov and Sinaĭ[14-16] have proved it rigorously under very general conditions.

We shall describe below the fundamental features, the mechanism, and nature of stochastic instability, using the example of simple models.

2.4. A Basic Model of Stochasticity of a Hamiltonian System

Let us examine the special case of a periodic perturbation of a non-linear oscillator in the form of very brief "pulses" that depend only on the phase θ: $V(I, \theta) \to TV(\theta) \sum_k \delta(t - kT)$ (2.1). Then we can write the equations of motion in the form of a canonical transformation $I, \theta \to \bar{I}, \bar{\theta}$:

$$\bar{I} = I - \varepsilon T \frac{\partial V}{\partial \theta} = I - \varepsilon I_0 f(\theta), \qquad \bar{\theta} = \theta + T\omega(\bar{I}) \qquad (2.12)$$

which has the generating function $F(\bar{I}, \theta) = \bar{I}\theta + T(\mathcal{H}_0(\bar{I}) + \epsilon V(\theta))$. This transformation describes the change in the quantities I and θ when acted on by a single "pulse", or as we shall say henceforth, in one step. Of course, when $T \to 0$, the transformation (2.12) goes over into the Hamiltonian equations (2.1), but without a linear correction term for the frequency, because of the condition that we have adopted: $\partial V/\partial I = 0$. The last expression for $\bar{I}$, where $f(\theta) \sim 1$ is a dimensionless function of the phase, while I_0 is a constant having the dimensions of action that has been introduced to make the relations more graphic.

Further, let us examine the special case in which $f(\theta) = \sin\theta$, i.e., the perturbation has only one harmonic. We can write the condition for resonance in the form $T\omega = 2\pi m$, where m is an integer. In terms of the differential equations (2.1), or as we shall say, of continuous

*This term remains even for a linear oscillator.

†Schoch [66] and Mel'nikov [62] have made detailed studies of resonance with weak non-linearity; Ford and Lunsford [85] have recently discovered a qualitatively new phenomenon: the persistence of a considerable region of stochasticity with an arbitrarily weak non-linearity.

*We note that several adjacent resonances then simultaneously become real.

time, this system of resonances corresponds to expansion of a periodic δ-function in a Fourier series:

$$\sum_k \delta(t-kT) = \sum_{m=-\infty}^{\infty} \frac{e^{im\Omega t}}{T},$$

Here the spacing between resonances is $\Delta = 2\pi/T = \Omega$, while $V_{m_1} = 2I_0/T$ (2.3), (2.12). The stochasticity parameter of (2.10) takes on the form

$$s^2 = \left| \frac{8}{\pi^2} \varepsilon\omega' I_0 T \right|. \qquad (2.13)$$

First of all, we shall show from simple physical considerations that when $s \gg 1$, the motion of the system of (2.12), which we shall call the basic model of stochasticity,* actually resembles randomness.[10] In order not to do this, let us call attention to the fact that the two variables I and θ play quite different roles in the motion of the system in (2.12). The phase space of the system is a cylinder that is closed in θ and infinite in I. The change in I per step is always small, since $\epsilon \ll 1$. We can easily see that the transformation of the phase here has fundamentally the nature of an dilatation, the coefficient of which we shall define to be

$$K(\theta) \equiv \frac{d\bar{\theta}}{d\theta} - 1 = -\varepsilon\omega' I_0 T f'(\theta). \qquad (2.14)$$

The latter expression is derived from (2.12). By comparison with (2.13), we find that $|K| \sim s^2$.

When $K \gg 1$, two essential processes occur. The first of these is local instability of motion, which means an exponential divergence of neighboring trajectories, at least in phase: $(\Delta\theta_n) \approx (\Delta\theta)_0 (\overline{K})^n$. Here $\overline{K}(\theta)$ is the geometric mean of the dilatation coefficient along the trajectory, and n is the serial number of the transformation step, or the discrete time. The second process, the so-called mixing, is in this case a distribution of trajectories emerging from a small region $\Delta\theta \lesssim 2\pi/|K|$ over the entire phase range of 2π even after a single step. It is easy to envision that this leads to disappearance of correlations in the sequence of phases $\theta_0, \theta_1, \ldots, \theta_n$. That is, the motion becomes seemingly "random" in terms of phase. Of course, these arguments are too greatly simplified. Nevertheless, as we see it, they are very pictorial. This is very important because the exact mathematical theory of motion of even such a simple model as (2.12) still faces insuperable obstacles. Still, it proves possible to deepen somewhat these pictorial ideas by using the modern ergodic theory (Chap. 3). Let us imagine for a moment that we have already proved the "randomness" of the motion in terms of phase. Then we arrive at a typical situation in modern statistical mechanics in deriving a kinetic equation in the random-phase approximation. One usually postulates that the angular variables are random quantities, and this makes it possible to write the kinetic equation for the distribution function of the action variables, which serve as the integrals of motion of the unperturbed system. We refer the reader to the special studies[17,18,27] for the details of deriving the kinetic equation (see also Chap. 7).

*The meaning of this term is explained below (Sec. 6.3), after we have seen which properties of stochasticity are described by this model, and which are not; the basic model was introduced in [10].

3. FUNDAMENTAL CONCEPTS OF ERGODIC THEORY

Ergodic theory is a branch of mathematics that studies discrete or continuous transformations having a preserved measure (integral invariant). As the name shows, ergodic theory arose from attempts to create a basis for statistical laws, i.e., derive them from dynamic equations of motion. Its currently most interesting object (at least for physicists) is the case of maximum instability of motion of a Hamiltonian system. The opposite case of maximum stability is treated by the KAM theory (see Sec. 6.1). According to the Liouville theorem, the integral invariant (and the measure of space) is the phase volume, the element of which we shall denote by $d\Gamma$. We shall set the total volume of the limited phase space of the system to be $\Gamma = 1$.

The aim of this chapter is to explain by simple examples the fundamental concepts and language of modern ergodic theory, and thus to facilitate reading the special studies in this field. We are convinced that familiarity with this theory is now an urgent necessity for all those who are interested in the problem of stability of nonlinear oscillations and allied problems.

Furthermore, we shall use the most recent results of ergodic theory for a more detailed analysis of the motion of the basic model of (2.12) (see Sec. 3.5).

3.1. An Elementary Model of Stochasticity

First of all, we should simplify even further the basic model of (2.12). This involves the fact that ergodic theory is applicable mainly to systems having a finite phase space, which is not true of the basic model. The motion of the latter actually resolves into two substantially different processes (when $s \gg 1$): dynamic motion within a limited region of phase space that generates a "random" succession of phases $\theta_0, \theta_1, \ldots, \theta_n, \ldots$; and unlimited diffusion in I. Only the former of these processes is related properly to ergodic theory, whereas the latter is described by the kinetic equation. We shall show how one can separate the two processes, using the example of the basic model. For convenience in writing down the transformation, we shall start by transforming to the new phase $\psi = \theta/2\pi$ having the period of unity. It is convenient to assign the latter by the operation of taking the fractional part: $\{x\} = x - [x]$, where [x] is the integral part. Now let us introduce instead of I the new variable $\varphi = \{T\omega(I)/2\pi\}$. The meaning of this is that we want to distinguish only those changes in I that lead to a change in the phase ψ, and thus to eliminate the diffusion process. In the new variables, the transformation takes on the form

$$\bar{\varphi} = \{\varphi + kf(\psi)\}, \quad \bar{\psi} = \{\psi + \bar{\varphi}\} \qquad (3.1)$$

and is also canonical. Here $k = \epsilon\omega' I_0 T/2\pi$. The phase space of the system of (3.1) is now restricted to a unit square having sides that are identical in pairs, or to a unit torus. We shall call the model of (3.1) the elementary model.

In this chapter, we shall use it along with other simple models to illustrate the fundamental concepts of modern ergodic theory.

To avoid misunderstanding, we emphasize that not only do we not intend to furnish a proof of the theorems of ergodic theory, but not even to formulate them ex-

actly. Our fundamental aim is to explain as graphically as possible, if we can so say it, the physical meaning of ergodic theory.

We recommend to readers who are interested in the properly mathematical side of this theory, first of all, the book by Halmos,[23] which is apparently the most accessible of the specialized texts. The current state of the theory has been presented rather fully in the reviews by Rokhlin and by Anosov and Sinaĭ.[16]

3.2. Methods of Describing Motion in Ergodic Theory

The starting point is the phase trajectory of the motion, i.e., the trajectory in the phase space x(t) of the system, where x denotes the phase vector (the set of coordinates in phase space). For the elementary model of (3.1), $x = (\varphi, \psi)$, while the time is discrete: $t = n$ (integer). We shall call such a motion a cascade.[16] Motion in continuous time, as given, e.g., by differential equations, is called a flow.

Another way of describing motion involves use of some function f(x) of the point in phase space, which is generally complex. The latter can be any quantity, e.g., the energy of the system. However, we shall assume below for the sake of graphic presentation that f(x) is the distribution in phase space that is usual in statistical mechanics, or the (phase) density.[22]

A singular density ($f = \overline{\delta(x - x_0(t))}$) is equivalent to description in terms of a single trajectory. In ergodic theory, one usually uses a non-singular density f(x), excluding special trajectories in quantity of measure zero. However, we note that a non-singular density is unobservable in principle (unrealizable)*, and therefore it can be used only as an auxiliary, intermediate concept.

The time variation of the phase density f(x, t) is determined by the motion of the system along phase trajectories:

$$f(x, t + T) = \hat{S}_T f(x, t) \equiv f(\hat{T}x, t); \tag{3.2}$$

Here $\hat{T}$ is the operator for displacement of the phase point along the trajectory in the time T, and $\hat{S}'_T$ is the operator for the variation in the function during the same time (the classical analog of the S-matrix; see, e.g.[24]). For a cascade, the operator $\hat{T}$ is given by a transformation, e.g., (3.1). The meaning of introducing the new operator $\hat{S}_T$ is that it is linear, while it is also unitary for a transformation that is reversible and that preserves measure.

An important characteristic of the operator $\hat{S}_T$ is its spectrum. First let us define the scalar product of two vectors in Hilbert space as

$$\langle g, f \rangle = \int gf \, d\Gamma. \tag{3.3}$$

It will be convenient for us henceforth to introduce the correlation coefficient of the functions g, f†: $\rho(g, f) = \langle g, f \rangle - \langle g \rangle \langle f \rangle$.

In the special case in which $g = \hat{S}_T f$, one speaks of autocorrelation. According to the theorem of Khinchin,[25,26] this determines the spectrum $S(\omega)$:

$$\langle \hat{S}_T f, f \rangle = \int_{-\infty}^{\infty} e^{i\omega T} S(\omega) \, d\omega; \tag{3.4}$$

Here T is a variable parameter, so that the autocorrelation is a function of T and a functional of f. Of course, the correlation spectrum $S(\omega)$ depends on f, but the type of spectrum does not depend on f within certain limits.[16] Below we shall deal with two fundamental types of spectra, discrete and continuous. In the former case, the spectral density $S(\omega) = \sum_k C_k \delta(\omega - \omega_k)$. In the latter case, the function $S(\omega)$ is continuous and finite.

3.3. Ergodicity of Motion, or How Probability Appears in Dynamic Theory

Ergodicity of motion is perhaps the best-known property of a statistical system. This concept was introduced into physics by Boltzmann, and it is usually associated with the idea of trajectories of motion that uniformly fill (according to an invariant measure) all of the accessible phase space. Moreover, ergodicity implies that in the limit (as $t \to \infty$) the fraction of the time (t_A/t) that the system spends in any element (A) of phase space is proportional to the phase volume Γ_A of this element:

$$w_A = \lim_{t \to \infty} \frac{t_A}{t} = \frac{\Gamma_A}{\Gamma}. \tag{3.5}$$

The existence of the limit in (3.5) is the central (Birkhoff-Khinchin) theorem of the classical ergodic theory, and it permits one to introduce the concept of probability for a single dynamic trajectory.[25] In (3.5), w_A denotes the probability that the system will occupy the region A.* The agreement between probability as thus introduced with our empirical concept of it will be discussed in Sec. 3.6.

The Birkhoff-Khinchin theorem[25] implies the existence of the limit in (3.5) only from the existence of a finite invariant measure. Hence, all Hamiltonian systems are ergodic in a certain sense. In order to explain this assertion, which seems strange at first glance, we must make more precise the concept of "accessible phase volume" that figures in the definition of ergodicity given above. In some cases, e.g., the elementary model of (3.1), this can be all of phase space. However, usually only a certain subspace is "accessible," as defined by single-valued integrals of motion.[28] In an autonomous (closed) system, there is always at least one such integral, the energy W, which defines the energy surface $W = \text{const}$ in phase space. It has the number of dimensions $2N - 1$, where N is the number of degrees of freedom of the system.

*This is implied even by the fact that a non-singular density does not obey the inverse Poincaré theorem, which is valid for any Hamiltonian system (see Sec. 3.4).

†Sometimes one uses also the normalized correlation coefficient: $(g, f) = [\langle g, f \rangle - \langle g \rangle \langle f \rangle] / [(\langle g^2 \rangle - \langle g \rangle^2)(\langle f^2 \rangle - \langle f \rangle^2)]^{1/2}$. Here and below: $\langle f \rangle = \int f d\Gamma$; sometimes one uses another notation: $\langle f, 1 \rangle$(3.3).

*For a system that consists of a very large number $N \to \infty$ of identical subsystems, or "particles", one can use another definition of the probability in phase space of a single "particle" (the so-called μ-space): $w_a = \lim_{N \to \infty} (N_a/N)$. This is the usual way of introducing the distribution function in μ-space. However, in this case one must adopt additional assumptions on the permissible initial conditions that w_a depends on for any $N \to \infty$. [27] The phase space of the entire system in which the probability (3.5) is defined is called the Γ-space.

We shall call a system maximally stable if it possesses N single-valued integrals. Then the "accessible phase volume" is reduced to a surface of N dimensions. In terms of the separable variables of action and angle, this is an N-dimensional torus:

$$I = \text{const}, \quad \theta_h = t\omega_h \quad (I_1, \ldots, I_N) + \theta_{h0}.$$

A larger number of single-valued integrals, up to $2N - 1$, is possible only on resonance tori $\sum_{k=1}^{N} n_k\omega_k = 0$ (where the n_k are integers),[28] the measures of which in the phase space of the system are zero.

On non-resonance tori that have measures of unity in phase space, the phase trajectory is not closed, and it covers the entire torus. The latter property is called transitivity (on the torus). According to the Birkhoff theorem,[25] transitivity is equivalent to ergodicity. In a somewhat simplified fashion, one can formulate this theorem as follows: if a trajectory fills any region of phase space, then it fills it uniformly (in terms of the measure imposed on it, see the following remark). However, in physics, one understands as ergodicity a case in which the motion is transitive on the energy surface.* If this condition is not satisfied, then the system can always be divided into ergodic components. This means that one can always represent all of the phase space as a sum (union) of a finite or countable number, or even a continuum of subspaces. In each of the subspaces, motion is transitive, and hence ergodic. An example of this is the above-mentioned multitude (continuum) of tori of a maximally stable system.

Thus, the problem of ergodicity is reduced to seeking the maximum region of transitive motion.

In spectroscopic language, ergodicity is equivalent to the condition that the number unity should be a non-degenerate eigenvalue of the operator $\hat{S}_T$. This condition ensures transitivity of motion. Otherwise, subspaces arise that are invariant during motion, and the system breaks down into ergodic components in number equal to the number of different eigenfunctions corresponding to the eigenvalue unity.

Uniform filling of phase space by the trajectory leads to relaxation of any initial distribution function $f(x, t)$ to a constant;[23] the latter is always an eigenfunction of the operator $\hat{S}_T$ having the eigenvalue unity, as is implied by the expression (3.2). This property of ergodic motion can be expressed by the equality

$$\left.\begin{aligned} \overline{f(x,t)} \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{S}_{kT} f(x,t) = \langle f(x,t)\rangle, \\ \hat{S}_{kT} f(x,t) = f(x, t + kT), \end{aligned}\right\} \qquad (3.6)$$

Here, as we see from the definition, the overline denotes averaging over the time in terms of the step T. The equality $\bar{f} = \langle f \rangle$ holds almost everywhere in phase space, i.e., perhaps except for a set of points of measure zero. The latter qualification is typical of the entire ergodic theory.

*Since the total phase volume is invariant, the imposed measure is determined in this case by the volume of the layer between two nearby energy surfaces having $\Delta W = \text{const} \to 0$, or $\Gamma(S) \propto S/\text{grad } W(x)$, where S is the area of the energy surface.

The relation (3.6) expresses the well-known equality in statistical mechanics of time ($\bar{f}$) and phase ($\langle f \rangle$) averages. However, it is essential to emphasize that relaxation in an ergodic system occurs only in the time average, whereas the function $f(x, t)$ generally varies quasiperiodically, since the spectrum can be discrete (see above).

The condition of ergodicity (in a given set) can also be expressed by using the correlation coefficient of the functions $f(x, t)$ and $g(x)$ ($\rho_T(f, g) \equiv \rho(S_T f, g)$) (Sec. 3.2) in the form:

$$\overline{\rho_T(f,g)} \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho_{kT}(f,g) = 0. \qquad (3.7)$$

In the special case $g(x) = f(x, 0)$, we get an analogous relation for the autocorrelation.

Finally, we shall introduce another important quantity: the transition probability $w_{ik}(T)$ between two regions of phase space A_i and A_k. The basis for introducing such a quantity is the fact that trajectories that link the two regions always exist for ergodic motion. Let $\hat{T}A_i$ be the region into which the region A_i transforms in the time T by motion along the trajectories. Its measure of overlap with the region A_k is*: $\Gamma_{ik}(T) = \Gamma(\hat{T}A_i \cap A_k)$, and we shall call the transition probability:

$$w_{ik}(T) \equiv \Gamma_{ik}(T).$$

It turns out[23] that ergodicity implies that

$$\overline{w_{ik}(T)} \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \Gamma_{ik}(lT) = \Gamma_i \Gamma_k = w_i w_k, \qquad (3.8)$$

where w_l is the probability that the trajectory will enter the region A_l, which is equal to the measure Γ_l of the region ($\Gamma = 1$) (see above).

We can derive the relation (3.8) directly from (3.7) by taking as the functions $f(x, 0)$ and $g(x)$ the so-called characteristic functions of the regions A_i and A_k, i.e., functions that are equal to unity within the corresponding regions, and zero outside.

3.4. Motion Involving Mixing, or Turbulence of Phase Flow

According to the preceding section (Sec. 3.3), the spectrum of an ergodic system can be discrete, and this means that its motion is quasiperiodic. A characteristic feature of such motion is its regularity. In particular, this implies that the trajectory returns to the starting point within a given accuracy after a definite interval of time having an upper bound. An example of this is motion on the surface of a two-dimensional torus with constant frequencies ω_1 and ω_2 whose ratio is irrational. The frequencies of the spectrum in this case are: $\omega_{kl} = k\omega_1 + l\omega_2$, where k and l are integers. When $\omega_1 \ll \omega_2$ or $\omega_2 \ll \omega_1$, the regularity of the circuit of the torus is especially graphic. The pattern of motion then resembles the scanning of the ray on a television screen, and it has nothing in common with our intuitive notion of a random process. Hence, ergodicity is too weak a property from the standpoint of creating a model of a random process. To use a hydrodynamic analogy, we can say that the phase flow, i.e., the set of all trajector-

*I.e., the common part of the regions $\hat{T}A_i$ and A_k.

ies in the phase space of the system is laminar in this case. A turbulent phase flow is called mixing[23] (see Sec. 2.4). This is one of the fundamental concepts of modern ergodic theory, as contrasted with the classical theory, which stops with ergodicity. In the language, respectively, of relaxation of density, correlation, and transition probability, an exact definition of mixing has the form:

$$\left.\begin{aligned} &\lim_{t\to\infty} f(x,t) = \langle f(x,t)\rangle, \\ &\lim_{T\to\infty} \rho_T(f,g) = 0, \\ &\lim_{T\to\infty} w_{ik}(T) = w_i w_k. \end{aligned}\right\} \tag{3.9}$$

A characteristic feature of these relations is the irreversibility of the relaxation process in mixing, in distinction from the quasiperiodic variation in time of the quantities f(x, t), $\rho_T(f, g)$, and $w_{ik}(T)$ when only the ergodicity of (3.6)—(3.8) exists. In other words, in ergodicity, the limiting values are attained only in the time average, while in mixing, they arise asymptotically after a long enough time. This is already considerably closer to our intuitive notion of a random process. In particular, mixing gives rise not only to the concept of probability, but also to that of statistically independent events (for a single trajectory!). For example, the latter might be the passage of the trajectory into different regions of phase space (the last relation in (3.9)). We note that the second relation in (3.9) is the well-known law in statistical mechanics of uncoupling of time correlations.

We shall introduce another auxiliary concept that will help in explaining more deeply the mechanics of mixing. This is the so-called weak mixing, as defined by the equivalent conditions

$$\left.\begin{aligned} &\overline{|f(x,t) - \langle f(x,t)\rangle|} = 0, \\ &\overline{|\rho_T(f,g)|} = 0, \\ &\overline{|w_{ik}(T) - w_i w_k|} = 0. \end{aligned}\right\} \tag{3.10}$$

At first glance, this is very similar to ergodicity (cf. (3.6)—(3.8)). However, the essential difference consists in the fact that here we take the absolute values of the quantities as approaching zero. Hence, in fact, the property of weak mixing is closer to mixing in (3.9) than to ergodicity.

With weak mixing, recurrence of the density f(x, t) is possible. That is, it can deviate strongly from the equilibrium (limiting) value for any $t \to \infty$. However, the frequency of these deviations, or their duration, must approach zero as $t \to \infty$.

According to ergodic theory,[23] weak mixing is equivalent to a continuous spectrum of the operator $\hat{S}_T$. This implies that strong mixing also has a continuous spectrum. This is also implied directly by the irreversible nature of the relaxation process in (3.9).

According to the relations (3.9), irreversibility of the relaxation process does not contradict the inverse Poincaré theorem,[23] since the latter deals with the phase trajectory of the motion, rather than with the distribution function (cf. Sec. 3.2).* We shall take up this problem in somewhat greater detail, because a false understanding of the inverse theorem has become widespread in the physics literature as being an assertion of quasiperiodicity of motion (see, e.g.[19,29,30]). If we take the latter term literally, it is equivalent to a discrete spectrum of motion*[36] and of autocorrelations. Nevertheless, the spectrum is continuous in mixing, although the inverse Poincaré theorem, which rests only on the existence of an invariant measure, remains valid, of course. Perhaps this error is terminological to a considerable extent, and arises from stretching the concept of quasiperiodic motion. It is important to emphasize that recurrence of trajectories is substantially different in nature when the spectrum of motion is discrete, or continuous. In the former case, recurrence is regular in the sense that there is a lower non-zero bound for the frequency of recurrence for a given accuracy of recurrence. In other words, recurrence necessarily occurs in the course of a finite time interval. In the latter case, the lower bound of the frequency of recurrence is zero, in spite of the fact that recurrence will occur an infinite number of times, according to the Poincaré theorem.[23] The motion along the trajectory is irregular ("aperiodic") in the latter case, and to no extent contradicts a "real" random process, and in particular, a state of statistical equilibrium (with fluctuations!). Hence, the statements that one sometimes encounters that statistical notions are inapplicable to large enough time intervals that exceed the Poincaré cycle[30] are erroneous in their very essence.† The relations (3.9) show that the situation is just the opposite: the longer the time interval, the more exact statistical ideas are in mechanics.

3.5. K-entropy and the Kinetic Equation

Mixing brings about relaxation to a state of statistical equilibrium. However, this property generally does not suffice for deriving the kinetic equation that describes the relaxation process itself. The point is that a dynamic system can be described statistically only after limiting relations like (3.9) have been satisfied to a given degree of accuracy. Let this occur over the time $\sim\tau_n$. Then the characteristic diffusion time that is defined by the kinetic equation must be much larger: $\tau_D \gg \tau_n$. This inequality can be satisfied, inasmuch as the diffusion and mixing can refer to different coordinates of phase space. For example, such a situation occurs in our basic model of (2.12). Here diffusion occurs in I, and it is characterized by a time scale: $\tau_D \sim \epsilon^{-2}$ (in steps). However, mixing affects fundamentally the phase θ, and it occurs in a time $\tau_n \sim 1$ (for a more detailed discussion of this problem, see Chap. 7).

The law of relaxation (uncoupling of correlations) upon mixing in (3.9) is also of no little importance. The point is that, under usual conditions (in particular, in a limited volume), diffusion leads to exponential relaxa-

*We note that this theorem is sometimes formulated imprecisely (see, e.g. [51]).

*More exactly, the term quasiperiodic even denotes a certain special type of discrete spectrum having a finite number of fundamental frequencies: $\omega_{n_1 \ldots n_N} = n_1\omega_1 + \ldots + n_N\omega_N$. In the general case of a discrete spectrum, one refers to an almost-periodic function.

†However, we note that the kinetic equation actually becomes inapplicable for large times, since it does not describe the fluctuations, which are substantial for these times. Of course, however, this does not mean that a statistical description is generally inapplicable.

tion. Hence, clearly, if we want to have a statistical description with sufficient accuracy, then uncoupling of correlations at least must also follow an exponential law (but, of course, with a shorter characteristic time). It turns out that real mechanical systems usually actually obey such a mixing law. This fact was first established by Krylov.[13] Later, independently of Krylov's work, Kolmogorov[31] introduced a special quantity, the entropy of a dynamic system (h), which is equal in order of magnitude to the reciprocal of the time for uncoupling of correlations. Below we shall call this quantity the Krylov-Kolmogorov entropy, or the K-entropy.*

Strict definition of K-entropy is conveniently associated with the property of local instability of motion (Sec. 2.4), which characterizes the divergence of neighboring trajectories. One can study local instability by means of linearized equations of motion, or variational equations, which are also called tangential transformation.*

First we shall examine the case of a cascade defined by some transformation. Let l and l' be the length of the tangential vector (the distance between the two trajectories) before and after the transformation. Then the K-entropy of the cascade can be defined as[15]

$$h=\overline{\lim_{l\to 0}\ln\left(\frac{l'}{l}\right)}, \tag{3.11}$$

Here the overline denotes time-averaging (Sec. 3.3), i.e., averaging along the trajectory of motion. This expression is convenient for finding the entropy experimentally, e.g., by numerical solution of the equations of motion (Chap. 4). In the given case of a cascade, according to (3.11), the K-entropy characterizes the mean divergence per step of neighboring trajectories.

Now we shall proceed to a flow, which we shall envision as the limit of a cascade whose step duration $T \to 0$. Then we have from (3.11):

$$h=\overline{\lim_{T\to 0}\left(\frac{1}{T}\lim_{l\to 0}\ln\left(\frac{l'}{l}\right)\right)}. \tag{3.11a}$$

This expression immediately gives rise to an exponential law of growth of local instability:

$$l(t)=l(0)e^{ht}.$$

According to (3.11a), the K-entropy equals the mean increment of this instability.

The theory[15] permits one to derive an expression that is more convenient for analytically calculating the K-entropy directly in terms of the parameters of the system. For a cascade, this can be done as follows. Like every linear transformation, a tangential transformation is characterized by a matrix whose product of eigenvalues $\prod_i \lambda_i = 1$, because of conservation of phase volume. If the transformation coefficients, and this implies also the λ_i, were constant, then instability would correspond simply to real λ_i, while complex conjugate values of λ_i would correspond to local stability. The elementary model of (3.1) with $f(\psi) = \psi$ gives an example of such a very simple case. Uncomplicated calculations give the eigenvalues and eigenvectors:

$$\left.\begin{aligned}\lambda&=1+\frac{K}{2}\pm\sqrt{K\left(1+\frac{K}{4}\right)}\to\begin{cases}K,\\K^{-1},\end{cases}\\ \operatorname{tg}\theta&=\frac{K}{\lambda-1}\to\begin{cases}1-1/K,\\-K,\end{cases}\end{aligned}\right| \tag{3.12}$$

where $K(\psi) = kf'(\psi) = k$ in the given case; θ is the angle between the ψ axis and the eigenvector having $\lambda > 1$; the latter expressions give the limiting values as $K \to \infty$. Local stability corresponds to the interval

$$-4<K<0. \tag{3.13}$$

The motion is unstable for the remaining values of K.

Figure 2 depicts schematically the transformation of a small region of phase space in the latter case. The region expands exponentially ($\sim\lambda^n$) along the eigenvector having $\lambda^+ > 1$, and it contracts along the vector having $\lambda^- < 1$. The mixing process begins when the length of the region attains the maximum dimensions of the phase region of the system (Fig. 2, cf. Sec. 2.4).

We can easily see that, as $t \to \infty$, the direction of almost any tangential vector will approach the direction of the expansion eigenvector. Hence, (3.11) goes over into

$$h=\ln\lambda^+. \tag{3.14}$$

In the typical case in which the coefficients of the tangential transformation are variable (in phase space), the problem of local instability is considerably complicated. On the one hand, instability can occur even for complex-conjugate values of λ, owing to parametric resonance. On the other hand, real $\lambda(x)$ do not yet guarantee local instability, since expansion of a phase region can turn into compression, owing to rotation of the eigenvectors. In turn, this can lead to limited oscillations of neighboring trajectories, i.e., local stability.* In this case, we must find the solution of the linearized equations over the entire time axis. We note that an analogous procedure has already been applied for a long time in hydrodynamics[32] to obtain the criterion for appearance of turbulence.

On the other hand, local instability is possible even when h = 0, although it does not grow exponentially in this case. For example, a local instability $l \propto t$ formally exists for any non-linear oscillations, owing to the frequency shift between trajectories. However, by all reasonable criteria, such motion should be considered stable. On the contrary, there are examples of mixing (i.e., unquestionably unstable motion) having h = 0. This shows that the property of local instability must be used with some caution. A rigorous mathematical study of this problem for an extensive class of dynamic systems has been made only relatively recently by Anosov and Sinaĭ.[14-16] They showed that exponential local instability for all initial conditions, both in the fundamental and tangential spaces, entails statistical properties: positive K-entropy,* mixing, and ergodicity.

The Anosov-Sinaĭ theory permits us to amplify somewhat the result given above for the elementary model. Namely, Oseledets and Sinaĭ have shown that the motion of this model shows statistical properties if

*See below with regard to its relation to thermodynamic entropy.

*One says that this transformation acts in tangential space.

*In particular, the latter effect can be observed in the example [86] of a special non-linear transformation constructed by McMillan.

*We shall henceforth say simply K-entropy, always assuming that $h > 0$.

$$|f'(\psi)| \geqslant C, \tag{3.15}$$

where $C > 0$ is a certain constant. The meaning of the latter condition consists in the idea that there are no stable regions in (3.13) if $k > 4/C$ is large enough.

This result is the most that modern ergodic theory can give for an oscillatory system. Owing to taking the fractional part in (3.1), the perturbation proves not to be smooth under the condition (3.15). In the real case, stable regions can exist near the points $f' = 0$ for any $k \to \infty$. This rules out the possibility of rigorous application of the modern theory.* On the other hand, the dimensions of these regions approach zero with increasing k:

$$\Delta\psi \approx \frac{4}{kf''}. \tag{3.16}$$

Hence, we can assume that one can actually neglect the effect of the stable regions, in spite of the inapplicability of the rigorous theory, provided only that $k \gg 1$. Numerical experiments with the elementary model ultimately confirm this conclusion, although they show that it is far from trivial (Chap. 4). Upon adopting this conclusion, we can determine that the stochasticity boundaries for the elementary model lie in order of magnitude at $K \sim 4$. By using (2.13), we get for the basic model of (2.12): $s \sim 2$. This agrees with the initial estimate of (2.11). There is no point in refining the latter estimate, since the very concept of the stochasticity boundary is approximate. Indeed, there is an entire transitional zone ($s \sim 1$) having a very complex structure in the phase plane.

In the case being treated where $\lambda(x)$ is variable, the K-entropy is expressed in terms of some value of $\lambda^+(x)$ averaged over phase space.[15] The overall situation is somewhat simplified for the elementary model when $K \gg 1$, because the direction of the eigenvectors remains almost constant here (3.12), except for a narrow phase interval ($\Delta\psi \sim 1/k$) near the stability region of (3.13). The general theory[15,8] leads to the expression

$$h \approx \langle \ln \lambda^+(x) \rangle \approx \langle \ln |K(\psi)| \rangle. \tag{3.17}$$

Analogously, for the basic model we get

$$h \approx \ln |\varepsilon\omega' I_0 T| + \langle \ln |f'(\theta)| \rangle. \tag{3.18}$$

Figure 2 demonstrates graphically that local instability of motion combined with limited phase space of a system leads to mixing of trajectories. Hence we can say that local instability is the original cause of the mixing process, and this implies also that it serves thus for all the statistical properties of the dynamic system. This is why the K-entropy is the most essential statistical characteristic of a dynamic system, as Krylov[13] also has emphasized.

As Anosov showed,[14] exponential local instability of motion gives rise to another highly important property of a stochastic system: its coarseness, or structural stability. As we know (see, e.g.,[50]), the latter implies that the topological structure of the phase trajectories is independent of small variations in the equations of motion. This makes it possible to restrict the treatment in studying stochasticity to the first approximation of the perturbation theory, and also to apply widely various approximate models.

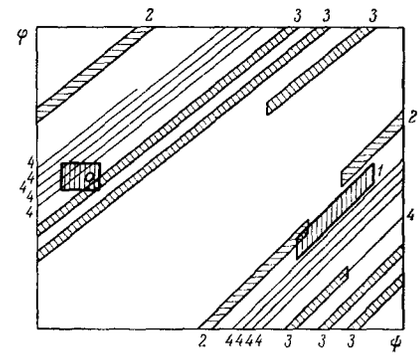

FIG. 2. A schematic representation of the mixing process for the elementary model of (3.1) with $f(\psi) = \psi$ ($k \gg 1$). The initial region is the square; the numerals give the step number. The direction of extension approximately coincides with the diagonal of the phase square, while the direction of compression coincides with the φ axis.

The example in Fig. 2 shows that coarse mixing of large (~ 1) regions of phase space occurs in a time $\tau_n \sim h^{-1}$. In addition, the overall pattern of development of local instability lets us conclude that fine mixing, i.e., mixing of small regions, occurs with a delay: $t(\Delta_0) \sim (|\ln \Delta_0|)/h$, where Δ_0 is the original size of the region. This estimate also characterizes the dependence of the spatial scale of mixing on the time ($\Delta_0(t)$).

In conclusion, let us take up the relation of K-entropy to ordinary thermodynamic entropy.[8] As we know, the latter characterizes the statistical state of a system, and it depends only on the distribution function:[22]

$$H = -\int d\Gamma f \ln (f\Gamma_0); \tag{3.19}$$

Here Γ_0 is a certain constant that has the dimensions of volume of phase space. In statistical physics, this constant is defined by quantizing phase space: $\Gamma_0 = (2\pi h)^N$. The physical meaning of the quantity Γ_0 is that it characterizes the minimum cell of subdivision of phase space. In the classical case, one can also introduce a minimum cell by the following considerations. In its physical meaning, the entropy characterizes the statistical properties of the system that arise from the mixing process. However, the latter is characterized by the scale $\sim\Delta_0$ (see above). Hence, we can assume that $\Gamma_0 \sim \Delta_0(t)$.

The new entropy $H(f, \Delta_0)$ thus defined now characterizes not only the statistical state of the system (as a function of f), but also the dynamics of mixing (as a function of $\Delta_0(t)$). For an arbitrary fixed Δ_0, we return to the ordinary thermodynamic entropy of (3.19), which is defined in classical physics apart from an arbitrary constant ($-\ln \Gamma_0$). This means that we neglect here the residual dynamic correlations in regions of the scale Δ_0. When f is fixed (i.e., the statistical state of the system is fixed), we get another entropy, which characterizes the mixing process. In the latter case, it is convenient to choose $f = 1$, i.e., the state of statistical equilibrium. The entropy $H(1, \Delta_0)$ thus defined increases monotonically for any system that shows mixing. However, in the case of exponential local instability of mo-

*To fill out the picture, we note that there are no stable regions in another mechanical system: colliding spheres, which are a model in a certain sense for molecular systems. This permitted Sinaĭ [34] to prove rigorously its statistical properties.

tion, the quantity $H(1, \Delta_0)$ proves to be asymptotically proportional to the time. Hence, it is natural to introduce its mean rate of variation as a characteristic of the mixing:

$$-\lim_{t\to\infty}\frac{1}{t}\int d\Gamma \ln \Delta_0(t) = h. \qquad (3.20)$$

And this is the dynamic entropy that Kolmogorov introduced to describe the process of mixing.[31]

3.6. Is Motion Involving Mixing a True Random Process?

This question lies outside the framework of purely mathematical theory, but it is essential for physics. A formal definition of a random process includes two fundamental requirements (see, e.g.[37]). The first is the possibility of introducing probability; this ensures the existence of the limit in (3.5) for an ergodic system. The second requirement, the so-called irregularity of the process, ultimately implies that this process lacks an algorithm. Evidently, the second requirement is violated by definition in dynamic systems. However, this requirement is not convincing from the standpoint of physics. Moreover, we can suppose that the imagined property of irregularity belongs to those properties that are unobservable in principle. In fact, any (temporally) finite realization of even an "absolutely" random process can always be approximated by some regular function, e.g., a Fourier or Taylor series. Of course, we can convince ourselves that some process is regular by predicting its future course. However, any apparent irregularity may prove actually to be the realization of a rather complex algorithm. Hence, we assume that the requirement of irregularity can be eliminated from the definition of a random process.*[8] Instead of this, we can base the definition of a random process on the motion of an ergodic dynamic system that shows mixing and K-entropy. In ergodic theory, such systems are called K-systems after Kolmogorov, who discovered them.[31]

In fact, the motion of a K-system is regular in a certain sense, and this is manifested, in particular, in residual autocorrelations. However, these correlations decay exponentially with time, so that we can neglect them under the condition $ht \gg 1$. One might object that such a definition of a random process would be imperfect. However, this is just we observe experimentally, since a process of statistical relaxation always preserves a contribution of the initial state, although it is exponentially small.

A "true" random process, as fixed, say, by the probability of a transition (a Markov process) can be treated here as the limiting case of the motion of a K-system, in which we can neglect the regular phase of growth of exponential instability (see Sec. 3.5), and this implies also neglecting the residual correlations. Correspondingly, the entropy of a "true" random process $h = \infty$.

Thus, as we see it, the idea is not ruled out that any random process in nature is realized by the motion of some K-system. However, we are far from thinking that this is the only possibility at present. Hence, we use the term stochasticity for a K-system and its motion, and keep for the concept randomness its former, somewhat mystical content.

Stochasticity is possible only in a non-linear system, since a linear system is always maximally stable (Sec. 3.3). Hence, stochasticity makes it possible to construct a non-linear model of statistical laws. To fill out the picture, let us recall that there is also a quasilinear model of statistical laws, which was introduced by Bogolyubov[38], and which has now been made the basis of statistical mechanics. It is based on using an infinitely large thermostat ($N \to \infty$) with random initial conditions.[40,27] The two models have been compared in detail in[8].

4. NUMERICAL EXPERIMENTS

As we noted above (Sec. 3.5), the problem of the motion of even the elementary model of (3.1) with a smooth perturbation proves to be insolvable by modern ergodic theory because of the regions of stability near the points $f' = 0$. The difficulties that arise here prove to be very deep, and they involve the very complicated structure of the phase space of the system.[39] On the other hand, if $f(\psi)$ is smooth, the dimensions in terms of phase of the stable regions approach zero ($\Delta\psi \sim 1/k$) (3.15) as $k \to \infty$. Hence it is reasonable to assume, as has been done above (Sec. 3.5), that we can neglect the stable regions when $k \gg 1$. One naturally resorts to an experiment, "real" or numerical, to test this assumption. The fundamental problem here is to select the best model. We think that one need not work with very complex models that are necessarily close to real mechanical systems, although some control experiments with such models are highly desirable (see Chap. 5).* We think that the elementary model exhibits fully enough the phenomenon of stochastic instability in all its complexity and variety. This will become especially evident somewhat later on, when we have become acquainted with the stochastic layer near the separatrix (Sec. 6.1).

The detailed numerical experiments with the elementary model described in[39] confirm that the motion of the elementary model becomes stochastic when $k \gg 1$ (actually already when $k > 6$).†

5. STOCHASTIC FERMI ACCELERATION

Before we proceed to a more detailed description of the mechanism of stochastic instability (Chap. 6), we shall examine a simple example of an actual mechanical system in which this instability arises.

In 1949, in order to explain the source of cosmic rays, Fermi proposed a mechanism of stochastic acceleration upon collision of charged particles with moving magnetic clouds in interstellar space.[49] In explaining this mechanism, Fermi used an analogy with molecular collisions, and treated the magnetic cloud as a gigantic particle. However, because such a cloud is a macroscopic object whose motion would seem to be determined by dynamic laws, the question arises of whether statistical laws are applicable here, and if so, to what

*Postnikov [37] had come to an analogous conclusion earlier.

*Stochastic instability has recently been studied in a large series of experiments with an electron beam in an accumulator. [84]

†We recall that the stochasticity boundary lies at $k \sim 4$ (Sec. 3.5).

extent. Only in 1961 did Ulam try to elucidate this question by numerical experiments on a very simple model.

5.1. The One-dimensional Model of Ulam[44]

The model is a particle moving between two plane-parallel, infinitely heavy, and absolutely elastic walls. One of them is fixed, while the other oscillates according to a definite given law. Thus the latter serves as a model for a moving magnetic cloud. A numerical calculation of the motion of this system[44] gave a negative result: acceleration was hardly observed. The velocity of the particle sometimes became as much as 3—4 times the velocity of the wall, and in most cases was of the order of magnitude of the velocity of the wall. However, according to the Fermi mechanism, the mean velocity of the particle should have increased indefinitely in proportion to the time.[49]

We give below a brief analysis of the motion of this model, following[45].

Let the wall oscillate according to a "sawtooth" law in such a way that its velocity varies linearly with the time within each half-period. Further, let the minimum distance between the walls be l, and the amplitude of the oscillations of one of them be a. Then the motion of the particle between them can be described by the following approximate transformation,* which is valid under the condition $a \ll l$, the case which proves most interesting (see below):

$$\left.\begin{aligned} \bar{v} &= v + V(\psi - 1/2), \\ \bar{\psi} &\approx \left\{\psi + \frac{lV}{16a\bar{v}}\right\}; \end{aligned}\right\} \qquad (5.1)$$

Here v and $\bar{v}$ are the absolute values of the velocity of the particle before and after collision with the wall; V/4 is the amplitude of the velocity of the wall; and ψ is the phase of the oscillation of the wall at the moment of impact, as defined on the interval (0, 1). The second equation of (5.1) is approximate; it takes account of the phase shift only for the minimum path length of the particle between collisions of $2l$, the additional small segment ~a being neglected.

The transformation (5.1) is of the same type as the basic model of (2.12). We can find the condition for stochasticity most simply by using the relation (2.14):

$$s^2 \sim |K| = \left|\frac{d\bar{\psi}}{d\psi} - 1\right| \approx \frac{l}{16a}\left(\frac{V}{v}\right)^2 \gtrsim 4. \qquad (5.2)$$

We find therefrom the stochasticity region in terms of velocity:

$$\frac{4v}{V} \lesssim \frac{1}{2}\sqrt{\frac{l}{a}}. \qquad (5.3)$$

A rather unexpected condition ($a \ll l$) must be satisfied in order to get an appreciable acceleration ($v \gg V$).

Figure 3 shows the velocity distribution function of the particles as expressed in terms of the parameter K for different values of the ratio a/l. These data were obtained in[45] by numerical integration of the exact equations of motion of Ulam's model.

The distribution function was constructed from a single trajectory, and was defined as the ratio of the number of collisions of the particle with the wall that put the particle into the given velocity range to the total number of collisions, which was 10^5. We see that the distribution function breaks off rather exactly at the value $|K| = 4$.

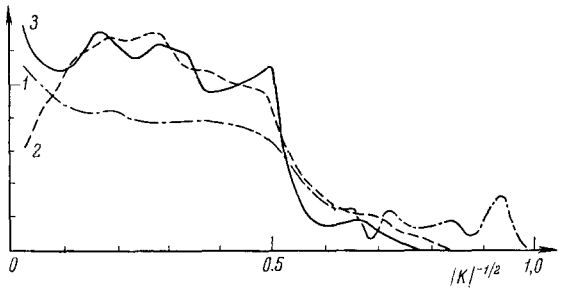

FIG. 3. The velocity distribution function of the particles in the Ulam model $v \propto |K|^{-1/2}$(5.2). 1—$a/l = 2.5 \times 10^{-3}$; 2—$a/l = 10^{-4}$; 3—$a/l = 2.5 \times 10^{-5}$.

5.2. The Multidimensional Case

The situation changes substantially for the case of two (or more) dimensions. In particular, Sinaĭ has rigorously proved[34] that stochasticity always exists in the elastic collision of any objects with a convex surface. We can rather graphically imagine this result to arise from exponential divergence of close trajectories due to scattering by the convex surface. On the contrary, existence of concave regions on the surface can produce regions of stability. A very simple example might be a modification of Ulam's model in which one of the walls is slightly concave and the multidimensional problem is treated. Evidently, the transverse motion will be stable here, and hence a stochasticity limit will exist, as in the one-dimensional case.

5.3. The Stochatron

As early as 1948, Burshteĭn, Veksler, and Kolomenskiĭ proposed using ordinary accelerators like the proton synchrotron or synchrotron in a stochastic mode.[46] To do this, they proposed applying an accelerating high-frequency voltage having a random phase. By using stochastic instability, one can achieve such a mode of acceleration with an ordinary (regular) high-frequency voltage, which is apparently more convenient in practice.

Let us derive the conditions for stochastic acceleration, while restricting the treatment for the sake of simplicity to the case of a homogeneous magnetic field B.* We can write the equations of motion in the form of a transformation of the type of the basic model:

$$\left.\begin{aligned} \bar{W} &= W + eU\cos\psi, \\ \bar{\psi} &= \psi + \frac{2\pi\omega}{\omega_B(\bar{W})}, \end{aligned}\right\} \qquad (5.4)$$

Here W is the total energy of the particle; U and ω are the amplitude and frequency of the accelerating voltage; and $\omega_B = eBc/W$ is the Larmor frequency of the relativistic particle. We derive the condition for stochasticity in a way analogous to (5.2): $|K| = 2\pi q(eU/W)|\sin\psi| \gtrsim 4$, or

*For the exact equations of motion for this model, see [45].

*More realistic estimates are given in [8].

$$q \gg \frac{W}{eU}, \quad U \gg \frac{Bc}{\omega}, \tag{5.5}$$

where $q = \omega/\omega_B$ is the high-frequency ratio.

We can easily see that the stochasticity boundary in (5.5) corresponds to the condition for the so-called microtron acceleration[47] in which a monotonic variation of the energy occurs, as is highly unusual for a non-linear system.

The inequality in (5.5) shows that a microtron is converted into a stochatron by increasing the accelerating voltage of frequency, or by decreasing the magnetic field.

Finally, we mention another distinctive variant of the stochatron, which was applied in[48] for preliminary heating of a plasma in a stellarator.

6. THE STOCHASTIC LAYER AND THE CONDITION OF OVERLAPPING RESONANCES

We shall now proceed to a more detailed description of the mechanism of stochastic instability. As we shall see below, the central point in elucidating this mechanism is to study the behavior of the system in the region in the phase plane that lies near the separatrix of a non-linear resonance. We shall conduct this analysis on the example of the motion of a charged particle in the field of two plane waves.[21]

6.1. Motion of a Particle in the Field of Two Plane Waves

On the one hand, the system being treated is described by the general equations of non-linear resonance (2.1). On the other hand, it is of independent interest in plasma physics. Let one of the waves be a small perturbation. We shall write the equations of motion in the form

$$m\ddot{x} = -eE_0 \sin k_0 x - eE_1 \sin(k_1 x - \nu t), \quad \varepsilon = E_1/E_0 \ll 1$$

or in dimensionless form,

$$\left.\begin{aligned} \tau_0 \frac{dv}{dt} &= -\sin\xi - \varepsilon \sin\left(\frac{k_1}{k_0}\xi - \nu t\right), \\ \xi &= k_0 x, \quad v = \tau_0 k_0 \dot{x}, \quad \tau_0 = \sqrt{m eE_0 k_0}, \end{aligned}\right\} \tag{6.1}$$

Here $2\pi\tau_0$ is the period of the small unperturbed oscillations. The trajectories of unperturbed motion in the phase plane are analogous to those depicted in Fig. 1, provided that ξ and v are plotted as the abscissa and ordinate, respectively. Using the usual operations, we can introduce the new variables of action (I) and angle (θ), and rewrite Eq. (6.1) in the form

$$\left.\begin{aligned} \dot{I} &= -\frac{\varepsilon}{\omega\tau_0^2} v \sin\left(\frac{k_1}{k_0}\xi - \nu t\right), \\ \dot{\theta} &= \omega, \end{aligned}\right\} \tag{6.2}$$

Here $\omega = \omega(I)$ is the non-linear frequency of the unperturbed oscillations of the particle in the potential well. Near the bottom of the well, $\omega \to 2\pi/\tau_0$, while $\omega \to 0$ as we approach the separatrix.

We shall describe qualitatively the perturbed behavior of the particle near the separatrix (here $\omega\tau_0 \ll 1$). The particle spends relatively little time ($\sim\tau_0$) near the center of the well, and its velocity v here is close to its maximum value. Conversely, most of the time ($\sim 2\pi/\omega \gg \tau_0$), the particle is found near the turning points, where its velocity is near zero. This fact can also easily be established formally by exact solution of the unperturbed problem. If now we substitute on the right-hand side of (6.2) the unperturbed frequency v, then the force acting on the particle can be represented in the form of a succession of very narrow pulses that follow one another at a very large interval $\sim\omega^{-1}$. The change in the action resulting from a pulse can be written in the form

$$\left.\begin{aligned} \Delta I &= \frac{\varepsilon}{\tau_0\omega(I)} \int dt v(t) \sin\vartheta, \\ \vartheta &= \frac{k_1}{k_0}\xi - \nu t, \end{aligned}\right\} \tag{6.3}$$

Here the integration is performed over an interval of time containing one pulse. Actually, the fundamental contribution to the integral comes from a narrow region of t near the maximum of v(t).[20] Hence, the phase n in (6.3) is taken at the point of the extremum $\xi = \xi_0$. The motion in (6.2) can be written approximately in the form of the following canonical transformation (per step):

$$\left.\begin{aligned} \bar{I} &= I - \frac{\partial V(\bar{I}, \vartheta)}{\partial\vartheta}, \\ \bar{\vartheta} &= \vartheta - \pi\nu/\omega(\bar{I}) + \frac{\partial V(\bar{I}, \vartheta)}{\partial\bar{I}}, \end{aligned}\right\} \tag{6.4}$$

Here the generating function V is equal to

$$V = -\frac{\varepsilon}{\tau_0\omega(\bar{I})} \int dt v(t) \cos\vartheta.$$

Since the perturbation $V \sim \epsilon$, we can replace $V(\bar{I}, \vartheta)$ by $V(I, \vartheta)$ in the equations (6.4) to an accuracy of terms $\sim\epsilon^2$.

The system of (6.4) is analogous to the basic model of (2.12). We can find the condition for stochasticity from (6.4) by using the expression (2.14):

$$K(\vartheta) = \left|\frac{d\bar{\vartheta}}{d\vartheta} - 1\right| \approx \pi\nu \frac{\omega'}{\omega^2} \Delta I \gtrsim 4. \tag{6.5}$$

Under rather general conditions near the separatrix, we have the asymptotic expression

$$\omega \approx \pi/\tau_0 \ln\left(\frac{32W_0}{|W - W_0|}\right), \tag{6.6}$$

where W is the energy of the particle, and W_0 is the energy at the separatrix. If we take account of the fact that $\Delta I \sim \epsilon/\tau_0\omega$, under the condition $\nu\tau_0 \lesssim 1$, we get from (6.5) the stochasticity boundary in the form[8,21]

$$\left.\begin{aligned} \omega &\leqslant \pi/\tau_0 \ln\frac{16\pi}{\varepsilon}, \\ |I_0 - I| &\leqslant \varepsilon I_0, \quad |W - W_0| \leqslant \varepsilon W_0, \end{aligned}\right\} \tag{6.7}$$

Here I_0 is the value of the action of the particle at the separatrix. The stochasticity boundary is defined by (6.7) symmetrically with respect to the separatrix for captured and escaping particles. Thus we can state generally that a layer of finite width is formed about the separatrix in which the integrals of motion break down. We shall call it the stochastic layer (see Fig. 1). When $\nu\tau_0 \gg 1$, the relative size of the stochastic layer is exponentially small, and is of the order of:[8,21]

$$|W - W_0| \sim W_0 e^{-\nu\tau_0}. \tag{6.7a}$$

The width of the stochastic layer according to the estimates of (6.7) and (6.7a) agrees in order of magnitude with the splitting of the separatrix as derived by Mel'nikov.[62] We note that Mel'nikov's very thorough

studies nevertheless did not permit him to estimate the actual width of the unstable layer near the separatrix; the size of the splitting of the latter gives only a lower limit of this width. Moreover, serious risks existed that instability could cover practically the entire resonance region. The estimates of (6.7) and (6.7a) show, however, that these risks were trivial, and that the width of the stochastic layer is generally small.

The problem under discussion is also of interest from the standpoint of the general theory of non-linear resonance, since it can be interpreted as being a case of interaction of two resonances. In this interpretation, the frequency spacing between resonances equals ν, while according to (2.8), the width of each of the resonances is equal, respectively, to $4/\tau_0$ and $4/\tau_1 = \sqrt{eE_1k_1/m}$. The resonances overlap completely when $\epsilon \sim 1$ and $\gamma\tau_0 \lesssim 1$. Simultaneously, according to (6.7a), they break down completely.* This result confirms the criterion of stochasticity in (2.11), and makes it possible to extend it to the general case of interaction of resonances. We note that, when $\epsilon \ll 1$, the relative width of the stochastic layer is always small, so that the inner region of the resonances remains stable, regardless of the spacing between resonances. Hence, the criterion of stochasticity (2.11) implies overlap of resonances of the same order of magnitude. In the general case, stochastic instability can arise from intersection of narrow stochastic layers, but only under special initial conditions. The criterion for such a weak stochasticity has been studied in[64,65,8].

6.2. Brownian Movement of a Particle in the Field of a Wave Packet

The study conducted above is a fundamental element that makes it possible to proceed to construct the general condition for appearance of stochastic instability. We can assume that, when a particle moves in complicated fields, the phase trajectories of the particle have the following topology in the phase plane: the entire plane is divided into a large number of cells associated with different separatrices and into a region between the separatrices. Each separatrix is associated with a stochastic layer that is delimited by the "good" trajectories that have conserved integrals of motion. The individual separatrices can intersect, and then a broader region of stochastic motion is formed. Thus the criterion of stochastic instability is the condition of overlapping separatrices, or in other words, the condition of overlapping resonances of (2.11).

We shall illustrate the aforesaid with the example of motion of a particle in the field of the wave packet

$$\ddot{x} = \frac{e}{m}\sum_k E_k \cos(kx - \omega_k t) \tag{6.8}$$

having a characteristic spacing Δk between the wave numbers of adjacent harmonics entering into the packet. Each plane wave gives rise in the phase plane to a separatrix encompassing the trajectories of particles captured by the wave. The dimensions of the separatrix also determine the width of the resonance between the particle and the wave. In order to determine the width of the resonance, we shall rewrite Eq. (6.8) in the form

$$\left.\begin{aligned} \dot{v} &= \sum_k V_k \cos\vartheta_k, \qquad V_k = \frac{e}{m}E_k, \\ \dot{\vartheta}_k &= kv - \omega_k \equiv \omega_k(v), \quad \vartheta_k = kx - \omega_k t, \end{aligned}\right\} \tag{6.9}$$

Hence we see that the velocity is analogous to the variable I in the equations (6.2). Evidently the width of the separatrix is

$$\delta v_k \sim \sqrt{V_k \Big/ \frac{d\omega_k(v)}{dv}} = \sqrt{V_k/k} = \frac{1}{k\tau_k},$$
$$\frac{1}{\tau_k} = \sqrt{keE_k/m}.$$

The spacing between adjacent resonances is

$$\Omega_k = \omega_{k+\Delta k}(v) - \omega_k(v) = \Delta k\,(v - d\omega_k/dk),$$

and the condition for appearance of stochastic instability acquires the form[21,52]

$$K = \left(\frac{k\delta v_k}{\Omega_k}\right)^2 = \frac{1}{\tau_k\Omega_k} \gg 1. \tag{6.10}$$

Let the criterion (6.10) be satisfied by all the waves in the packet having phase velocities from some v_{min} to v_{max}. Then, evidently, the phases ϑ_k are stochastic in this velocity range, while the motion of the particle is analogous to Brownian movement, and its velocity increases with time on the average.

We note that stochasticity can vanish for very large $K \gtrsim N$ (where N is the number of waves).[8] This is easy to understand in the limiting case where $K \gg N$, in which all the waves form for the particle a single potential well with slowly varying parameters. In this regard, we should supplement the criterion (6.10) with the inequality

$$\frac{1}{\tau_k\Omega_k} \ll N, \tag{6.10a}$$

which was previously derived in[53] from somewhat different considerations.

The inequalities (6.10) and (6.10a) are the conditions for applicability of the so-called quasilinear approximation in non-linear plasma theory.[53]

6.3. Discussion of the Properties of the Basic Model

The examples given in this chapter allow us to advance the following notions:

1) Breakdown of the separatrix is described by the equations (6.4), which are equivalent to the basic model of (2.12).

2) Overlap of separatrices (resonances) in (2.10) is the condition for appearance of stochastic instability of motion of a dynamic system.

3) Whenever the motion of the system can be reduced to the form (2.12), one can estimate the phase correlation function:[54]

$$R_n = R(t) = \frac{1}{2\pi}\int_0^{2\pi} d\psi_m e^{-i\psi_m} e^{i\psi_{m+n}} \sim e^{-n\ln K} = e^{-t\Omega\ln K} \tag{6.11}$$

and determine the mixing time

$$\tau_n \sim (\Omega \ln K)^{-1}, \tag{6.12}$$

and hence also the K-entropy (see Chap. 3). This information quite suffices, not only for transforming to a statistical description of the system using the kinetic

*This conclusion was recently confirmed in [87] by a numerical experiment.

equation (Chap. 7), but also for analyzing some fine details of this transformation.

4) We shall define the region of applicability of the basic model. Let the perturbation of the system be such that the change in the state of the system within the time intervals T_1, T_2, ..., is adiabatic (I = const.), while a considerable (not exponentially small) change in the action I occurs at the boundaries of the intervals T_i. Let us denote by Δt_i the time region within which I varies considerably. Then, under the condition

$$\Delta t \ll T \tag{6.13}$$

we can transform from the differential equations to finite-difference equations like (2.12), i.e., to the basic model. This is achieved as follows:[54] the solution is written in the WKB approximation within the intervals T_i, and the solutions in adjacent intervals T_i are linked by taking account of the change in the action ΔI and the phase $\Delta\vartheta$. If the condition (6.13) is not satisfied, then one cannot calculate the mixing time and the K-entropy in general form. Nevertheless, the condition of overlapping resonances always permits an estimate, however crude, and the existence of mixing can be established.

5) When there is no overlap of resonances ($s \ll 1$), the stochastic region degenerates into a system of exponentially narrow (see (6.7a)) stochastic layers. In the one-dimensional case of (2.1), these layers do not intersect in the phase plane, and the motion becomes stable for any $t \to \infty$. This result has been proved rigorously in the Kolmogorov-Arnol'd-Mozer theory for sufficiently small, but finite s. Numerical experiments (Chap. 4) show that the boundary of such a permanent stability lies near the stochasticity boundary at $s \sim 1$.

In the multidimensional case, the stochastic layers of different resonances always intersect, however small s is. This can lead to diffusional motion of the system along these layers within a rather large volume of phase space. The first example of this stability was devised by Arnol'd.[35] This instability, which has become called Arnol'd diffusion, may prove to be essential for various physical applications.[8]

7. THE RANDOM-PHASE APPROXIMATION AND THE FUNDAMENTAL KINETIC EQUATION (MASTER EQUATION)

7.1. Analysis of Loss of Memory of the Initial Conditions

The results given above permit a new approach to derivation of the fundamental kinetic equation in statistical mechanics. The central point in deriving such equations is the hypothesis that the phases are random at the initial instant of time (see, e.g.,[27]). This circumstance is usually called the random-phase approximation (RPA), and its formal content appears as follows for the special case discussed below. Let $f(I, \vartheta, t)$ be the distribution function of the set of actions $(I_1, I_2, \ldots)$ and phases $(\vartheta_1, \vartheta_2, \ldots)$. Then the RPA consists in the idea that f does not depend on the phases at t = 0, i.e.,

$$f(I, \vartheta, 0) = f(I). \tag{7.1}$$

Further, the examples treated above show that one can establish directly from the equations of motion the criterion for stochastic phases, and as we shall see below, one can find a criterion under which the condition (7.1) holds to a required accuracy. A feature of this approach is a more thorough analysis of the motion of the system than has been performed thus far.

We shall illustrate with a simple example how an analysis of the stochastic properties of a system permits one to derive the kinetic equation under arbitrary initial conditions[69] (see also[64,20]).

We shall treat again a non-linear oscillator perturbed by periodic δ-function pulses:

$$\left.\begin{aligned} \dot I &= \varepsilon I \sum_{k=-\infty}^{\infty} \delta(t - kT)\sin\vartheta \equiv \varepsilon V(I, \vartheta, t), \\ \dot\vartheta &= \omega(I), \end{aligned}\right\} \tag{7.2}$$

Here ϵ is a small parameter, and we shall write the continuity equation in phase space

$$\frac{\partial f}{\partial t} + \omega\frac{\partial f}{\partial\vartheta} + \varepsilon\frac{\partial}{\partial I}(Vf) = 0. \tag{7.3}$$

Let us expand f and V in Fourier series:

$$\left.\begin{aligned} f(I, \vartheta, t) &= \sum_{n=-\infty}^{\infty} f_n(I, t)\, e^{in\vartheta}, \\ V(I, \vartheta, t) &= \sum_{n,\,k=-\infty}^{\infty} V_{n,k}(I)\, e^{i(n\vartheta + k\Omega t)} \qquad (\Omega = 2\pi/T) \end{aligned}\right\} \tag{7.4}$$

(the case being treated has only terms with n = +1). Upon substituting (7.4) into (7.3), we get the equation for the f_n:

$$\frac{\partial f_n}{\partial t} + in\omega f_n = -\varepsilon\frac{\partial}{\partial I}\sum_{n',\,k} V_{n',k} f_{n-n'}. \tag{7.5}$$

We note that the passage in Eq. (7.5) to the interaction representation is performed by the transformation

$$f_n \to f_n \exp\left\{-in\int_0^t \omega\, dt\right\},$$

This is because the frequency depends on the time owing to the non-linearity, according to the equations of motion (7.2). This leads to added difficulties in deriving the kinetic equation. However, henceforth we shall use the inequality

$$\left.\begin{aligned} &|\Delta\vartheta - \omega t| \ll \omega t, \\ &\Delta\vartheta = \vartheta(t) - \vartheta(0) = \int_0^t dt\omega. \end{aligned}\right\} \tag{7.6}$$

The condition (7.6) is actually equivalent to the VKB approximation.

Now it is convenient to go over to the Laplace representation for the f_n:

$$g_n(p) = \int_0^\infty e^{-pt} f_n(I, t)\, dt,$$

and convert Eq. (7.5) to the following form, with account taken of the inequality (7.6):

$$g_n(p) = \frac{f_n(0)}{p} - \frac{\varepsilon}{p}\frac{\partial}{\partial I}\sum_{m,\,k}[V_{mk} g_{n-m}(p + i(n-m)\omega - ik\Omega)], \tag{7.7}$$

Here we have denoted in abbreviated form:

$$g_n(I, p) \equiv g_n(p), \quad f_n(I, 0) \equiv f_n(0).$$

From now on, we shall be interested in the asymptotic behavior as $t \to \infty$, which is equivalent to $p \to 0$. Taking this into account, we shall carry out the iteration of Eq.

(7.7), and keep only the main (resonance) terms. This gives for g_0:

$$g_0(p)=\frac{1}{p}f_0(p)-\frac{\varepsilon}{p}\frac{\partial}{\partial I}\sum_{n,\,k>0}\left[\frac{V_{n,k}f_n(0)}{p+i(n\omega-k\Omega)}+\frac{V_{-n,-k}f_{-n}(0)}{p-i(n\omega-k\Omega)}\right]$$
$$+2\frac{\varepsilon^2}{p}\frac{\partial}{\partial I}\sum_{n,\,k>0}\frac{|V_{n,k}|}{p^2+(n\omega-k\Omega)^2}\frac{\partial}{\partial I}|V_{n,k}|f_0(0)+O(\varepsilon^3),$$

or, if we return to the t-representation as $p \to 0$:

$$\frac{\partial f_0}{\partial t}=-\varepsilon\frac{\partial}{\partial I}\sum_{n,\,k>0}[V_{n,k}f_n(0)e^{i(n\Delta\vartheta-k\Omega t)}+V_{-n,-k}f_{-n}(0)e^{-i(n\Delta\vartheta-k\Omega t)}]$$
$$+2\pi\varepsilon^2\frac{\partial}{\partial I}\sum_{n,\,k>0}|V_{n,k}|\delta(n\omega-k\Omega)\frac{\partial}{\partial I}|V_{n,k}|f_0+O(\varepsilon^3). \tag{7.8}$$

We see from (7.8) that the terms of the order of ϵ (and also all the rest of the terms that are odd in ϵ) contain oscillating coefficients that enter simultaneously with the initial conditions $f_n(0)$ (with $n \neq 0$, since V depends on ϑ). Hence, in the linear case in which ω = const and $\Delta\vartheta = \omega t$, it follows directly that the difference $(n\omega - k\Omega)$ can be as small as is desired (resonance), and one cannot eliminate the oscillating terms in (7.8) by any suitable procedure of averaging over a given finite interval of time. This constitutes the fundamental difficulty in deriving the fundamental kinetic equation to be solved by using the RPA. In fact, the condition (7.1) for f_n takes on the form $f_n(0) = f_0(0)\delta_{n,0}$.

The situation becomes quite different in the non-linear case in which $\omega = \omega(I)$. In order to understand this, let us introduce a discrete time scale with the interval T for large $t \approx NT$ ($N \gg 1$), and express $\vartheta(t) = \vartheta_N$ in terms of the initial phase $\vartheta(0) = \vartheta_0$. We can do this by starting with Eqs. (7.2). To do this, we must integrate a system analogous to (2.12):

$$\left.\begin{aligned}\vartheta_{m+1}&\approx\vartheta_m+\omega T+K_m\sin\vartheta_m,\\ I_{m+1}&=I_m+\varepsilon I_m\sin\vartheta_m,\end{aligned}\right\}$$
$$K_m=\varepsilon\frac{I_m}{\Omega}\frac{d\omega(I_m)}{dI_m}. \tag{7.9}$$

We note that the transformation (7.9) is obtained approximately from the basic model of (2.12) by taking account in the latter of the smallness of the variation $|\bar{I} - I| \ll I$. A transformation in the form of (7.9) proves to be convenient in estimating the correlation function

$$R(t)\equiv R_N=\frac{1}{2\pi}\int_0^{2\pi}d\vartheta_0 e^{i(\vartheta_N-\vartheta_0)}.$$

One can show[54] that when $K \gg 1$,

$$R(t)\sim e^{-\Omega+\ln K+i\omega t}. \tag{7.10}$$

If now we introduce the distribution function

$$F=\frac{1}{2\pi}\int_0^{2\pi}f_0\,d\vartheta_0,$$

then for $K \gg 1$, (7.8) directly gives the kinetic equation for F:

$$\frac{\partial F}{\partial t}=2\pi\varepsilon^2\frac{\partial}{\partial I}\sum_{n,\,k>0}|V_{n,k}|\delta(n\omega-k\Omega)\frac{\partial}{\partial I}|V_{n,k}|F. \tag{7.11}$$

It is useful to note the following two facts: 1) when $K \ll 1$, averaging over ϑ_0 has no effect, since then $R(t) \sim e^{i\omega t}[1 + O(K)]$; 2) since the motion is ergodic when $K \gg 1$, averaging over ϑ_0 is equivalent to averaging over a time much greater than T. The latter implies that we can average Eq. (7.8) over ϑ_0 if the diffusion time in (7.1) is much greater than T (Sec. 3.5). In other words, the kinetic equation (7.11) is valid for times $t \gg T$. In this way, it differs from the kinetic equation derived by the RPA method, which is valid for $t \gg t_{in}$, where the interaction time t_{in} in this case is the duration of an impulse ($t_{in} \ll T$). Thus, the additional assumption that the phases are random at the initial instant of time makes it possible to extend the region of applicability of the kinetic equation with respect to time.

7.2. Time Scales

As we have noted, the principal feature of the existing methods of deriving the fundamental kinetic equation is the use of the RPA in some form or another (see the review[70]). Without discussing the problem of the expedience of the random-phase hypothesis in constructing the fundamental kinetic equation, we note only that in the ordinary theories it is not only convenient, but perhaps even necessary. The latter involves the fact that the analysis to be conducted of the dynamic equations of motion of the system is essentially linear, and it does not permit one to take account of the instability that leads to mixing. Formally, this involves the fact that the frequencies are considered constant in solving the equations of motion. Hence, we can call this approach the quasilinear model of statistical laws.

Modern ergodic theory permits us not only to fill this gap, but also to perform a more detailed analysis of those distribution functions for which we can derive the fundamental kinetic equation. Since mixing in ergodic theory essentially involves non-linearity of the system, it is natural to speak in this case of the non-linear model of statistical laws.

We have already seen in Sec. 7.1 that a kinetic equation of the Markov type is derived for the "coarse" distribution function F(I, t), which is obtained from f(I, f(I, ϑ, t) by averaging over the phase ϑ in the interval from 0 to 2π. Here we distinguish the process of dynamic mixing from statistical relaxation (Sec. 3.5). However, the mixing process actually involves not only the phase ϑ, but also I to a lesser degree. Hence, in the general case, the kinetic equation is valid for a distribution function that has also been averaged over a finite interval Δ_I. The equation of motion (7.9) implies that when ϑ is extended by $\sim 2\pi$, the action I varies by

$$\Delta_I^0\sim 2\pi\varepsilon I/K. \tag{7.12}$$

Evidently this quantity determines the cell for averaging the distribution function. Here, according to (7.10), mixing occurs after a time

$$\tau_n\sim T/\ln K, \tag{7.13}$$

i.e., practically after one step. If we decrease the dimensions Δ_I of the cell in comparison with (7.12), then mixing shows an additional retardation (see Sec. 3.5) of the time (number of steps)

$$n_0\approx\left(\ln\frac{2\pi\varepsilon I}{K\Delta_I}\right)\Big/\ln K. \tag{7.14}$$

The maximum permissible retardation is determined by the condition that the kinetic equation should be applicable:

$$n_0T\leqslant\tau_D\sim T/\varepsilon^2, \tag{7.15}$$

This directly gives the minimum dimensions of the averaging cell

$$\min \Delta_I \sim \frac{2\pi\varepsilon I}{K} \exp\left(-\frac{\ln K}{\varepsilon^2}\right). \quad (7.16)$$

Now we shall note the characteristic time scales that are used in deriving the kinetic equation, and arrange them in order of increasing times.

1) t_{in} is the interaction time, which in this example is the duration of the impulse, and is the smallest time.

2) $\tau_n = T/\ln K$ is the mixing time, or the time for uncoupling of phase correlations for the maximum averaging cell of (7.12).

3) $\tau_\Delta = n_0 T + T/\ln K$ is the mixing time for the cells $\Delta_I = 2\pi\epsilon I/K^{n_0}$ with account taken of retardation. When $n_0 = 0$, we have $\tau_\Delta = \tau_n$. However, if n_0 is defined by (7.14), then $\tau_\Delta = \tau_D$. In general, $\tau_n \lesssim \tau_\Delta \lesssim \tau_D$.

4) $\tau_D = T/\epsilon^2$ is the diffusion time.

We have discussed above only some very simple problems in which stochastic instability plays a role. We shall also mention the destruction of magnetic surfaces in the stellarator under the action of various perturbations;[64,67,20,68] interaction of non-linear waves and the Fermi-Pasta-Ulam problem;[71-77,81-83] weak turbulence,[69] etc. We hope that this brief review will attract attention to the discussed problem and will foster its further development.

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[1] N. M. Krylov and N. N. Bogolyubov, Vvedenie v nelineĭnuyu mekhaniku (Introduction to Non-linear Mechanics), Izd-vo AN UkrSSR, Kiev, 1937 (Engl. Transl., Princeton Univ. Press (excerpted)).

[2] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii (Asymptotic Methods in the Theory of Non-linear Oscillations), Fizmatgiz, M., 1958 (Engl. Transl., Hindustan Publ. Corp., Delhi, 1961).

[3] I. I. Putilin, Malye planety (The Small Planets), Gostekhizdat, M., 1953.

[4] V. I. Veksler, Dokl. Akad. Nauk. SSSR 43, 346; 44, 393 (1944).

[5] E. M. McMillan, Phys. Rev. 68, 143 (1945).

[6] A. A. Kolomenskiĭ and A. N. Lebedev, Teoriya tsiklicheskikh uskoriteleĭ (Theory of Cyclic Accelerators), Fizmatgiz, M., 1962 (Engl. Transl., Wiley, New York, 1966).

[7] B. V. Chirikov, Dokl. Akad. SSSR 125, 1015 (1959) [Sov. Phys.-Doklady 4, 390 (1959)].

[8] B. V. Chirikov, Doctoral Dissertation, Novosibirsk, 1969.

[9] F. K. Goward, Proceedings of the CERN Conference on the 25-GeV Proton Synchrotron with Strong Focusing (Geneva, October, 1953); M. G. Hine, ibid.

[10] B. V. Chirikov, Atomnaya énergiya 6, 630 (1959); Candidate's Dissertation, Novosibirsk, 1959.

[11] G. M. Zaslavskiĭ, Statisticheskaya neobratimost' v nelineinykh sistemakh (Statistical Irreversibility in Nonlinear Systems), Nauka, M., 1970.

[12] G. Hedlund, Bull. Am. Math. Soc. 45, 241 (1939); E. Hopf, Math. Ann. 117, 590 (1940); Usp. Mat. Nauk 4, 129 (1949).

[13] N. S. Krylov, Raboty po obosnovaniyu statisticheskoĭ fiziki (Studies on the Foundation of Statistical Physics), Izd-vo AN SSSR, M.-L., 1950.

[14] D. V. Anosov, Geodezicheskie potoki na zamknutykh rimanovykh mnogoobraziyakh otritsatel'noi krivizny (Geodesic Flows on Closed Riemann Manifolds of Negative Curvature), Trudy MIAN im. V. A. Steklova 90, 3 (1967).

[15] Ya. G. Sinaĭ, Izv. AN SSSR, ser. matem. 30, 15 (1966).

[16] D. V. Anosov and Ya. G. Sinaĭ, Usp. Mat. Nauk 22, 107 (1967); V. A. Rokhlin, ibid., 22, 3 (1967).

[17] G. I. Budker and S. T. Belyaev, in Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 2, Izd-vo AN SSSR, M., 1958, p. 330.

[18] R. Brout and I. Prigogine, Physica 22, 621 (1956).

[19] S. Chandrasekhar, Problems in Physics and Astronomy, Revs. Modern Phys. 15, 1 (1943).

[20] N. N. Filonenko, R. Z. Sagdeev and G. M. Zaslavsky, Nucl. Fusion 7, 253 (1967).

[21] G. M. Zaslavskiĭ and N. N. Filonenko, Zh. Eksp. Teor. Fiz. 54, 1590 (1968) [Sov. Phys.-JETP 27, 851 (1968)].

[22] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), M., Gostekhizdat, 1951 (Engl. Transl., Addison-Wesley, Reading, Mass., 1969).

[23] P. R. Halmos, Lectures on Ergodic Theory, Mathematical Society of Japan, Tokyo, 1956 (Russ. Transl., IL, M., 1959).

[24] K. P. Gurov, Osnovaniya kineticheskoĭ teorii (metod N. N. Bogolyubova) (Foundations of the Kinetic Theory (Method of N. N. Bogolyubov)), Nauka, M., 1966.

[25] V. V. Nemytskiĭ and V. V. Stepanov, Kachestvennaya teoriya differentsial'nykh uravneniĭ (Qualitative Theory of Differential Equations), Gostekhizdat, M., 1947 (Engl. Transl., Princeton Univ. Press, 1960).

[26] A. A. Kharkevich, Spektry i analiz (Spectra and Analysis), Fizmatgiz, M., 1962 (Engl. Transl., Consultants Bureau, New York).

[27] I. Prigogine, Non-Equilibrium Statistical Mechanics, Interscience, New York, 1962 (Russ. Transl., Mir, M., 1964).

[28] L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Fizmatgiz, M., 1958 (Engl. Transl., Pergamon Press, New York, 1960).

[29] G. E. Uhlenbeck and G. W. Ford, Lectures in Statistical Mechanics, Am. Math. Soc., Providence, R. I., 1963 (Russ. Transl., Mir, M., 1965).

[30] R. Balescu, Physica 36, 433 (1967).

[31] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 119, 861 (1958).

[32] A. N. Kolmogorov, ibid. 30, 299 (1941); L. D. Landau, ibid. 44, 339 (1944); C. C. Lin, Theory of Hydrodynamic Instability (Russ. transl.), IL, M., 1958.

[33] B. B. Kadomtsev, in Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 4, Atomizdat, M., 1964, p. 188.

[34] Ya. G. Sinaĭ, Dokl. Akad. Nauk SSSR 153, 1261 (1963); [Sov. Math.-Doklady 4, 1818 (1963)]; Usp. Mat. Nauk 25, 141 (1970).

[35] V. I. Arnol'd, Dokl. Akad. Nauk SSSR 156, 9 (1964) [Sov. Math.-Doklady 5, 581 (1964)]; paper given at the

International Congress of Mathematics, Moscow, 1966.
[36] H. A. Bohr, Almost-Periodic Functions, Chelsea Publ. Co., New York, 1947; B. M. Levitan, Pochti-periodicheskie funktsii (Almost-Periodic Functions), Gostekhizdat, M., 1953).
[37] A. G. Postnikov, Arithmetic Simulation of Random Processes, Trudy MIAN im. V. A. Steklova 57, 3 (1960).
[38] N. N. Bogolyubov, O nekotorykh statisticheskikh metodakh v matematicheskoĭ fizike (On Some Statistical Methods in Mathematical Physics), Izd-vo AN UkrSSR, 1945.
[39] F. M. Izraĭlev and B. V. Chirikov, Stokhastichnost' prosteĭsheĭ dinamicheskoĭ modeli s razdelennym fazovym prostranstvom (Stochasticity of a Very Simple Model Having a Divided Phase Space), Preprint of the Institute of Nuclear Physics of the Siberian Department of the Academy of Sciences of the USSR, Novosibirsk, 1968.
[40] L. Van Hove, Physica 21, 517 (1955).
[41] B. Chirikov, E. Keil, and A. Sessler, Stochasticity in Many-Dimensional Non-Linear Oscillating Systems, Cern Report, 1969.
[42] A. V. Komin, L. S. Krasitskaya, and V. P. Minaev, Magnitnye poverkhnosti toroidal'nogo magnitnogo polya vblizi separatrisy (Magnetic Surfaces of a Toroidal Magnetic Field Near the Separatrix), Preprint No. 225 of the Institute of Nuclear Physics of the Siberian Department of the Academy of Sciences of the USSR, Novosibirsk, 1968.
[43] L. J. Laslett, A Computational Investigation of a Nonlinear Algebraic Transformation, 1967 (unpublished).
[44] S. M. Ulam, Proc. 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol. 3, 1961, p. 315 (Russ. Transl., Matematika 7, 137 (1963)).
[45] M. Zaslavskiĭ and B. V. Chirikov, Dokl. Akad. Nauk SSSR 159, 306 (1964).
[46] É. L. Burshteĭn, V. I. Veksler, and A. A. Kolomenskiĭ, A Stochastic Method of Accelerating Particles, in Nekotorye voprosy teorii tsiklicheskikh uskoriteleĭ (Some Problems of the Theory of Cyclic Accelerators), Izd-vo AN SSSR, M., 1955.
[47] S. P. Kapitsa and V. N. Melekhin, Mikrotron (The Microtron), Nauka, M., 1969.
[48] V. N. Bocharov, V. I. Volosov, A. V. Komin, and V. M. Panasyuk, Uderzhanie plazmy v stellaratore pri razlichnykh znacheniyakh dliny svobodnogo probega (Containment of a Plasma in the Stellarator at Different Values of the Mean Free Path), Fusion-III, D-7, Novosibirsk, 1968.
[49] E. Fermi, Phys. Rev. 75, 1169 (1949).
[50] A. A. Andronov, A. A. Vitt, and S. É. Khaikin, Teoriya kolebaniĭ (Theory of Oscillations), Fizmatgiz, M., 1959; A. Blaquiere, Nonlinear System Analysis, Academic, 1966.
[51] I. Prigogine, Introduction to Nonequilibrium Statistical Physics, Proc. Internat. School of Nonlinear Mathematics and Physics, Physics Session, Munich, 1966.
[52] R. Z. Sagdeev and A. A. Galeev, Nonlinear Plasma Theory, Benjamin, New York, 1969.
[53] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion 1, 82 (1961).
[54] G. M. Zaslavskiĭ, PMTF 2, 16 (1967).
[55] L. A. Artsimovich, Upravlyaemye termoyadernye reaktsii (Controlled Thermonuclear Reactions), Fizmatgiz, M., 1961 (Engl. Transl., Oliver and Boyd, Edinburgh, 1964).
[56] G. I. Budker, Usp. Fiz. Nauk 89, 533 (1966) [Sov. Phys.-Uspekhi 9, 534 (1967)].
[57] S. N. Rodionov, Atomnaya énergiya 6, 623 (1959).
[58] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 98, 527 (1954).
[59] V. I. Arnol'd, Usp. Mat. Nauk 18, 91 (1963).
[60] J. Moser, A Rapidly Convergent Iteration Method and Nonlinear Differential Equations. II, Annali della Scuola Normale Superiore di Pisa. Serie III, Vol. 20, No. III, 1966.
[61] H. Poincaré, Les méthodes nouvelles de la mécanique celeste, Vols. 1–3, 1892, 1893, 1899 (Engl. Transl., NASA, Washington, 1967).
[62] V. K. Mel'nikov, Dokl. Akad. Nauk SSSR 148, 1257 (1963) [Sov. Math.-Doklady 4, 266 (1963)].
[63] V. M. Alekseev, ibid. 177, 495 (1967) [Sov. Math.-Doklady 8, 1421 (1967)]; Matem. Sbornik 76 (118), 72 (1968); 77 (119), 545 (1968); 78 (120), 3 (1969).
[64] M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavskiĭ, Nuclear Fusion 6, 297 (1966).
[65] G. Contopoulos, Bull. Astronomique, Ser. 3, 2, 223 (1967).
[66] A. Schoch, CERN Report 57–21 (1958).
[67] A. Gibson, Physics of Fluids 10, 1553 (1967).
[68] B. V. Chirikov, Dokl. Akad. Nauk SSSR 174, 1313 (1967) [Sov. Phys.-Doklady 12, 612 (1967)].
[69] G. M. Zaslavskiĭ and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. 52, 1081 (1967) [Sov. Phys.-JETP 25, 718 (1967)].
[70] J. Chester, (Theory of Irreversible Processes), (Russ. transl.), Nauka, M., 1966.
[71] E. Fermi, J. Pasta, and S. Ulam, Studies of Nonlinear Problems. I, Los Alamos Scientific Report LA-1940 (1955).
[72] J. Ford, J. Math. Phys. 2, 387 (1961); J. Ford and J. Waters, ibid. 4, 1293 (1963).
[73] E. A. Jackson, J. Math. Phys. 4, 551, 686 (1963).
[74] F. M. Izraĭlev and B. V. Chirikov, Dokl. Akad. Nauk SSSR 166, 57 (1966) [Sov. Phys.-Doklady 11, 30 (1966)].
[75] F. M. Izraĭlev, A. I. Khisamutdinov, and B. V. Chirikov, Chislennye éksperimenty s nelineĭnoĭ tsepochkoĭ (Numerical Experiments with a Non-Linear Chain), Preprint No. 252, Institute of Nuclear Physics, Siberian Department of the Academy of Sciences of the USSR, Novosibirsk, 1968.
[76] F. M. Izraĭlev, Issledovanie stokhastichnosti kolebaniĭ struny s kvadratichnoĭ nelineinost'yu (A Study of Stochasticity of Oscillations of a String with Quadratic Non-Linearity), Preprint No. 77, Institute of Nuclear Physics, Siberian Department of the Academy of Sciences of the USSR, Novosibirsk, 1966.
[77] H. Hirooka and N. Saito, J. Phys. Soc. Japan 26, 624 (1969).
[78] W. E. Drummond and D. Pines, Nucl. Fusion, Suppl., Pt. 3, 1049 (1962).
[79] A. A. Galeev and V. I. Karpman, Zh. Eksp. Teor. Fiz. 44, 592 (1963) [Sov. Phys.-JETP 17, 403 (1963)].
[80] N. J. Zabusky, J. Math. Phys. 3, 1028 (1962).
[81] G. M. Zaslavskiĭ and N. N. Filonenko, Zh. Eksp. Teor. Fiz. 56, 1064 (1969) [Sov. Phys.-JETP 29, 572 (1969)].
[82] N. N. Filonenko, Preprint No. 282, Institute of Nuclear Physics, Siberian Department of the Academy

of Sciences of the USSR, Novosibirsk, 1969.
[83] G. P. Berman and G. M. Zaslavskiĭ, Zh. Eksp. Teor. Fiz. 58, 1453 (1970) [Sov. Phys.-JETP 31, 778 (1970)].
[84] G. N. Kulipanov, S. I. Mishnev, and A. N. Skrinskiĭ, Izuchenie stokhasticheskoĭ neustoĭchivosti betatronnykh kolebaniĭ élektronnogo puchka v nakopitele (A Study of Stochastic Instability of Betatron Oscillations of an Electron Beam in an Accumulator), Report at the International Conference on Accelerators, Erevan, 1969.
[85] J. Ford and G. H. Lunsford, Phys. Rev. A1, 59 (1970).
[86] E. M. McMillan, Some Thoughts on Stability in Nonlinear Periodic Focusing Systems, UCRL-17795, Berkeley (1967); L. J. Laslett, E. M. McMillan, and J. Moser, Long-term Stability for Particle Orbits, NYO-1480-101, Courant Institute of Mathematical Sciences, New York, 1968.
[87] G. H. Walker and J. Ford, Phys. Rev. 188, 416 (1969).

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