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DECAY OF RESONANT STATES AND DETERMINATION OF THEIR QUANTUM NUMBERS

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1. INTRODUCTION

THE main event in the physics of elementary particles in the last decade was the discovery of a large number of resonances in hadron systems.

If we disregard the resonances in the πN system (isobars), an intensive study of resonances started in 1960, when the ρ meson was discovered, followed by the ω meson. At the present time, there are approximately 25 meson and about 40 baryon[70] resonances (if each isomultiplet is regarded as one resonance). These discoveries were unexpected by the theoreticians, but with increasing number of discovered resonances, the number of theoretical models and schemes for resonance classifications increased much faster. First to be mentioned among these are schemes based on the SU(3) symmetry and the Regge-Gribov theory. A detailed theoretical analysis of resonances from the point of view of various models is not within the scope of the present article, and we refer the interested reader to the reviews[61,71,72], where he can find also other references to earlier work. It must be emphasized that both the SU(3) symmetry and the Regge-Gribov theory were not invented specially to explain resonances, but it turned out that many characteristics of resonances are very well explained by them. However, for a final judgement concerning the success of any particular model there is still not enough experimental data on the quantum numbers of the resonances, particularly their spins and parities. We present only two examples. According to the SU(3) symmetry model, the Σ-baryon resonances with spin and parity $1/2^+$ form an octet, a member of which, in particular, should be the resonance $\Sigma(1610)$. However, the spin and parity of this resonance have not yet been established. The same pertains also to the proposed $\Sigma(5/2^+)$ resonance $\Sigma(1915)$ of the octet.

For the Regge-trajectory model, it would be very important to confirm the fact that the meson resonance $A_{2H}(1315)$ actually has spin and parity 2^+, thus permitting this resonance to lie on the so-called R-trajectory[61], which determines, for example, the asymptotic behavior of the cross section of the process $\pi^-p \to \eta^0 n$,[61] etc.

There have been a large number of recent papers dealing with the Veneziano duality model (see the review[73]). This model uses essentially the fact that the number of resonances can become infinite with increasing mass, and the Regge trajectories can become linear. If the reader looks at Rosenfeld's tables[70], he will see that these assumptions do not contradict the existing data; the mass of the last heaviest baryon resonance is 3230 MeV, but its spin and parity are known.

Thus, it becomes urgently necessary to obtain methods of determining the quantum numbers of the resonances at arbitrarily large resonant masses. Such methods were developed and used many times in recent years, and are the subject of the present review.

2. GENERAL FORMULATION OF THE PROBLEM

This review is devoted to methods of determining the spin and parity of resonances. Methods of determining other quantum numbers (mass, width, isotopic spin) can be found, for example, in the book[52] and in the reviews[24-26].

Assume that the resonance x of interest to us is produced in the reaction

$$b_1+b_2\to x+c_1+c_2+\ldots+c_n, \tag{2.1}$$

and then decays via one of the channels

$$x\to a_1+a_2, \tag{2.2}$$

$$x\to a_1+a_2+a_3, \tag{2.3}$$

or via several channels of the type (2.2) and (2.3) (4-particle decay of x will be considered only under the condition that it has a cascade character, i.e., one of the particles in the reaction (2.2) and (2.3) decays in turn into two or three particles).

We introduce some definitions concerning the reaction (2.1), which will be used throughout the review:

1) The production plane x is the plane containing the momenta of the incoming particle and the resonance x in the l.s. of the reaction (2.1).

2) The Z system (Y system) is the coordinate system in which the Z (Y) axis is chosen to be the direction normal to the x production plane.

3) The Capps condition is defined as the situation wherein parity is conserved in the reaction (2.1), the particles b_1 and b_2 are not polarized, and the momenta and spin states of the particles $c_1, c_2, \ldots, c_n$ are not measured.

We are interested in the determination of the spin and parity of the resonance x. To this end we can use the following information obtained in the experimental study of the reaction (2.1) and the decay (2.2) or (2.3) (we note, however, that in practice, in concrete experiments, it is possible to obtain only part of the information listed below):

1) The angular distribution of x in the c.m.s. of the reaction (2.1).

2) The angular distribution of the x decay products and the angular distribution of the polarization of the x decay products in the rest system of x.

3) The energy distribution (the distribution on the

Dalitz plot) of the products of the decay (2.3) in the rest system of x.

In the study of many resonances, successful use was made of the methods of determining j and η_x, based on the assumption that the momenta of the decay products of x are small. These methods include first of all the analysis of the distribution of the decay products (2.3) on the Dalitz plot. These methods can be used when the energy release in the decay of x is given by

$$\Delta E \leqslant \mu, \tag{2.4}$$

where μ is the pion mass. For a detailed description of these methods see[50-54].

Another method used successfully in practice is that of Adair, for the use of which it is necessary to select cases in which x is emitted forward or backward relative to the incident beam within a range of small angles

$$\vartheta \leqslant \mu/k_x, \tag{2.5}$$

where k_x is the momentum of x in the c.m.s. of the production reaction. Adair's method is described in[24,25,50,52].

It is clear that when the energy in reaction (2.1) increases and the mass x increases, the conditions (2.4) and (2.5) for the applicability of the aforementioned methods are violated. Therefore to determine the quantum numbers of the resonances at high energies one can use only methods based on the general conservation laws of quantum mechanics—the law of conservation of the angular momentum and parity in reaction (2.1) and in the decays (2.2) and (2.3).

The development of such methods is the subject of a large number of theoretical and experimental investigations, where use is made of the relations derived for these quantities from the conservation laws. Although a large number of such relations have been obtained, it is difficult for the experimenter to cope with them in applications, owing to the lack of a unified theoretical approach to the problem. The point is that many of these relations contain identical information on the quantum numbers of the resonance and are essentially equivalent, but different methods of experimental-data reduction are needed for their application. Furthermore, in some relations the experimental information is not completely used.

In the prevent review we develop a universal method, based on the conservation laws, for investigating the quantum numbers of resonances, namely the method of polarization moments. Besides providing a unified theoretical approach to the problem, this method has also the advantage that it makes it possible to use the entire experimental information, and also to unify and simplify to the largest extent the computer reduction of the experimental data (this is discussed in detail in the review and in Appendix III). This method can be used for any decay of the type (2.2) or (2.3).

The gist of the method of polarization moments consists in the following. The distribution of the products of the decay (2.2) or (2.3) depends on the spin state of x, which we specify with the aid of polarization moments (PM). We note that resonances with large masses decay, as a rule, in cascade fashions, i.e., the particles a_k, produced in the reactions (2.2) and (2.3) are in turn unstable. The distribution of the decay products of a_k (which is a function of the direction of the emission of a_k in the rest system of x, and also of the direction of the emission of the decay products of a_k in the rest system of a_k) is conveniently characterized by quantities which we shall call cascade polarization moments (CPM), which are natural generalizations of PM. We note that the set of CPM contains complete information on the distribution in question (just as the Fourier coefficients define completely a periodic function).

The CPM are determined in simple manner in experiment, and they are connected by relations that follow from the conservation laws. The coefficients in these relations depend on j and η_x—the spin and parity of x. Substituting the experimentally obtained CPM in the indicated relations, we can obtain the value of j and η_x at which the relations are satisfied—this indeed is the procedure of determining j and η_x.

The method of polarization moments as applied to the physics of resonances is discussed in a number of papers[1,14,16,28,33,35,38-40,56], in the review[24-26], and in the Byers review[57] (where decays of fermions with parity nonconservation are considered). Recognizing that the two reviews were published only in the form of preprints and are not readily available at present, we have undertaken to describe consistently and compactly the method of polarization moments. At the same time, considerable space has been allotted in our review to concrete decays, and also to experimental investigations with which the application of the method is illustrated.

The present review does not consider one of the most important methods of determining the quantum numbers of baryon resonances, namely the phase-shift analysis. This method has long been used by experimenters and is well known to them. The latest accomplishments in the field of reduction of experimental data on πN scattering by the phase-shift analysis method is the subject of a review by Shchegel'skiĭ[74].

The review was organized in the following manner.

In Ch. 3 we consider the spin state of one particle and present two methods of its description—with the aid of the density matrix and with the aid of the PM. It also deals with very important properties of PM, which will be needed later on, and with the method of determining the PM from the experimental data. At the end of this chapter, the spin state of two particles is discussed.

In Ch. 4 we consider the general theory of two-particle decay of the resonant state. It is first necessary to write down the amplitude of the decay in terms of the most convenient parameters. Such parameters are the helicity amplitudes of the decay, A_λ, the properties of which are discussed in detail. Knowing A_λ and the PM of the resonance x, we can determine the spin state of the decay product of x, namely the PM of the particle a_1. If a_1 itself decays, then the angular distribution of the products of this decay depends on the PM of the particle a_1. It is convenient to introduce here the CPM, $t^{LM}_{l\mu}$. We then establish a relation between the CPM, which can be used for the determination of the spin and parity of the resonance x, and consider the question of the experimental determination of the CPM.

Ch. 5 is devoted to concrete decays: $j \to 0 + 0$, $j \to 1 + 0$, $j \to 1/2 + 0$, $j \to 3/2 + 0$.

In Ch. 6 we investigate three-particle decay of resonances. Just as for two-particle decay, we first introduce convenient parameters, in terms of which the scattering amplitude $f_{\tilde{m}\lambda}(\omega_1, \omega_2)$ is expressed. We then again introduce CPM, and from the general properties of the invariance against rotations and reflection we obtain relations between CPM. These relations make it possible to determine the spin and parity of the resonance x. For an experimental determination of the CPM it is necessary to have the angular distribution of the decay (2.3), and also the angular distribution of the decay of a_1 (if the spin of a_1 is not equal to zero). The general formulas are used for concrete decays: a) decay into three spinless particles, b) decay into a fermion and two spinless particles.

In Ch. 7 we show how to use the laws of conservation in the reaction of production of the resonance for the determination of its quantum numbers. In particular, we investigate the problem of employing a polarized proton target for the determination of the quantum numbers of an isobar decaying into a non-strange baryon and spinless particles.

3. SPIN DENSITY MATRIX AND POLARIZATION MOMENTS

a) The Density Matrix and Its Properties

Assume that the system containing the investigated resonance x is described by a wave function $\psi(m, \alpha)$, where m is equal to the projection of the spin of x on the Z axis in a specified coordinate system XYZ, and α is the aggregate of all the remaining variables describing the system.

We assume for concreteness that x decays into two spinless particles. The wave function of the decay products of x depends only on m (it does not depend on α!) and is equal, as is well known, to $Y_{jm}(\vartheta, \varphi)$, where Y_{jm} is a spherical function; the angles ϑ and φ specify the direction of the momentum of one of the products of the decay of x with respect to the axes XYZ. The probability density of the decay of x in the direction defined by the angles ϑ and φ is obviously

$$\frac{d\omega}{d\Omega} = \sum_{m, m'=-j}^{j} \sum_{\alpha} \psi^*(m', \alpha)\psi(m, \alpha) Y^*_{jm'}(\vartheta, \varphi) Y_{jm}(\vartheta, \varphi) = \sum_{m, m'=-j}^{j} \rho_{mm'} Y^*_{jm'}(\vartheta, \varphi) Y_{jm}(\vartheta, \varphi), \quad (3.1)$$

where

$$\rho_{mm'} = \sum_{\alpha} \psi(m, \alpha)\psi^*(m', \alpha), \qquad -j \leqslant m', \ m \leqslant j. \quad (3.2)$$

We see that to find the probability density of the decay of x in a given direction it is necessary and sufficient to know the matrix $\rho_{mm'}$ (3.2), which is called the spin density matrix or simply the ρ matrix.

The properties of the ρ matrix are discussed in detail in articles and books (see, for example,[5,62]). These properties can be easily obtained from the definition (3.2) and we shall only list them:

$$\rho_{mm'} = \rho^*_{m'm} \quad \text{(hermiticity)} \quad (3.3)$$

$$\sum_{m=-j}^{j} \rho_{mm} = \mathrm{Sp}\,\hat{\rho} = 1 \quad \text{(normalization)} \quad (3.4)$$

$$0 \leqslant \rho_{mm} \leqslant 1, \quad (3.5)$$

$$\sum_{m, m'=-j}^{j} \rho_{mm'}\rho_{m'm} = \mathrm{Sp}\,\hat{\rho}^2 \leqslant 1. \quad (3.6)$$

The equal sign in (3.6) holds only for the pure state x, which is characterized by the fact that the spin state of x does not depend on the remaining variables*. This means that the resonance and the remaining part of the system are described separately by the functions $\varphi(m)$ and $\chi(\alpha)$ (i.e., $\psi(m, \alpha) = \varphi(m)\chi(\alpha)$). In this case we have, in accordance with (3.2),

$$\rho_{mm'} = \varphi(m)\varphi^*(m'). \quad (3.7)$$

In the general case of a partly polarized state, the number of real parameters defining the ρ matrix is

$$r = (2j+1)^2 - 1; \quad (3.8)$$

(3.8) can be easily obtained from (3.3) and (3.4).

In most practical cases, the number of independent parameters is smaller than in (3.8). For example, if the Capps condition is satisfied in the reaction (2.1), then in the Z system

$$\rho_{mm'} = 0 \qquad \text{for odd} \qquad m - m'. \quad (3.9)$$

This theorem was proved by Capps[1]. It reflects the fact that when the Capps condition is satisfied the ρ matrix is a function of vectors lying in the production plane of x. This means that the elements of the matrix $\rho_{mm'}$ remain unchanged upon reflection in the production plane of x. On the other hand, it is known (see, for example,[37]) that $\rho_{mm'} \to (-1)^{m-m'}\rho_{mm'}$ following the reflection in the XY plane. If we choose the production plane of x to be the XY plane, then we get (3.9) from the last two statements.

b) Polarization Moments and Their Properties

In the analysis of the angular distribution of the decay products of the resonance, it is more convenient to specify its polarization state not by means of the ρ matrix, but by means of the polarization moments (PM) T_{LM}, which are connected with $\rho_{mm'}$ by the formula

$$T_{LM} = \sum_{m, m'=-j}^{j} \rho_{mm'} C^{jm}_{jm', LM}, \quad (3.10)$$

where $C^{jm}_{jm', LM}$ are Clebsch-Gordan coefficients (CGC).

The CGC properties used in this review are listed in Appendix I; more details on CGC can be found in the books[5-9].

The PM were used many times to describe the polarization state of nuclei and elementary particles (see, for example,[64-67]).

Let us demonstrate the advantage of parametrization of the polarized state of x with the aid of PM, using the decay of x into two spinless particles as an example. If we represent the products of the spherical functions in (3.1) in the form (II, 7′) of Appendix II, then, with allowance for (3.10), we obtain

$$\frac{d\omega}{d\Omega} = \sum_{L=0}^{2j} \sum_{M=-L}^{L} \sqrt{\frac{2L+1}{4\pi}} C^{j0}_{j0, L0} T_{LM} Y_{LM}(\vartheta, \varphi). \quad (3.11)$$

Let us multiply (3.11) by $Y^*_{LM}(\vartheta, \varphi)$ and integrate with respect to the solid angle, taking into account the

*The pure state is called also the completely polarized state. In the opposite case one says that the resonance (particle) is partly polarized.

orthonormality property of the spherical functions. We obtain

$$T_{LM} = \sqrt{\frac{4\pi}{2L+1}} (C^{j0}_{j0,\,L0})^{-1} \langle Y^*_{LM}(\vartheta, \varphi)\rangle, \tag{3.12}$$

where

$$\langle Y^*_{LM}(\vartheta, \varphi)\rangle = \int Y^*_{LM}(\vartheta, \varphi)\, d\omega(\vartheta, \varphi). \tag{3.13}$$

$\langle Y^*_{LM}(\vartheta, \varphi)\rangle$ is determined from the experimental data by means of the formula

$$\langle Y^*_{LM}(\vartheta, \varphi)\rangle = \frac{1}{n}\sum_{i=1}^{n} Y^*_{LM}(\vartheta_i, \varphi_i), \tag{3.14}$$

where the summation is carried out over all cases of the decay of x into two spinless particles, and n is the total number of cases. It is obvious that (3.14) is the limit of (3.13) as n tends to infinity.

Formulas (3.14) and (3.12) make it possible to obtain quite easily the PM* from the experimental data, where it is much more difficult to determine the ρ matrix from (3.1) (especially in the case of large spin). At any rate, there is no simple algorithm such as (3.12) and (3.14) for the reconstruction of the ρ matrix in the case of arbitrary spin.

We shall verify in what follows that for other types of decay of x it is more convenient to use the PM than the ρ matrix.

The PM can be interpreted in the following manner: the ρ matrix of a particle with spin j (just as the product of wave functions of two particles with spin j) is a superposition of spin functions with values of the spin from zero to 2j. According to the rule for the addition of angular momenta, the condition (3.10) selects from this superposition a function with spin L and projection M. Such an interpretation of the PM illustrates formula (3.12) quite well.

We emphasize that the definitions of the polarized state of x with the aid of the ρ matrix and with the aid of the PM are perfectly equivalent: knowing the ρ matrix we can find the PM from formula (3.10) and, conversely, knowing the PM we can establish the ρ matrix by means of the formula†

$$\rho_{mm'} = \sum_{L=0}^{2j}\sum_{M=-L}^{L} \frac{2L+1}{2j+1} C^{jm}_{jm',\,LM} T_{LM}. \tag{3.15}$$

The properties (1.3)–(1.6) of the ρ matrix correspond to the following properties of the PM:

$$T_{LM} = (-1)^M T^*_{L-M}, \tag{3.16}$$

$$T_{00} = 1, \tag{3.17}$$

$$\min C^{jm}_{jm,\,L0} \leqslant T_{L0} \leqslant \max C^{jm}_{jm,\,L0}\ \ddagger, \tag{3.18}$$

$$\sum_{L=0}^{2j}\sum_{M=-L}^{L} (2L+1)\,|T_{LM}|^2 \leqslant 2j+1. \tag{3.19}$$

*We note that by virtue of (I.6) only T_{LM} with Y_{LM} having even L contribute to the angular distribution (3.11). Consequently it is impossible to determine T_{LM} with odd L from the decay into two spinless particles, i.e., the distribution (3.11) does not contain complete information on the polarization of x.

†Formula (3.15) is obtained by multiplying (3.10) by $[2L + 1)/(2j + 1)] \times C^{jm_1}_{jm_1',\,L_1M_1}$, summing over L_1 and M_1, and using (I.3), (I.4b), and (I.4f).

‡Limitations of the type (3.18) exist also for T_{LM} when $M \neq 0$. For more details concerning limitations of this type see [57].

In (3.18), the maximum and minimum values of $C^{jm}_{jm,\,L0}$ are taken at fixed j and L.

Formulas (3.16)–(3.19) are obtained from (3.3)–(3.6) with allowance for (3.10), (I.4e)–(I.4f), and (I.5).

From the definition (3.10) and (I.1) it follows that

$$T_{LM} = 0, \quad \text{if} \quad |M| > L, \quad \text{or} \quad L > 2j. \tag{3.20}$$

Therefore, with account taken of (3.16) and (3.17), we conclude that the number of independent real parameters specifying all the PM is equal to

$$\sum_{L=0}^{2j} (2L+1) - 1 = (2j+1)^2 - 1,$$

which naturally coincides with the number of ρ-matrix parameters in (3.8).

We present one more useful formula

$$\langle m^2\rangle \equiv \sum_{m=-j}^{j} m^2 \rho_{mm} = \frac{1}{3}[j(j+1)(2j-1)(2j+3)]^{1/2} T_{20} + \frac{1}{3} j(j+1), \tag{3.21}$$

which can be readily obtained with the aid of (I.8) and (3.4).

Just like the ρ matrix, the PM are specified relative to a definite coordinate system. When the coordinate system is rotated through the Euler angles α, β, and γ, the PM are transformed as follows:

$$T'_{LM'} = \sum_{M=-L}^{L} D^L_{MM'}(\alpha, \beta, \gamma) T_{LM}, \tag{3.22}$$

where $T'_{LM'}$ is the PM in the new system, and $D^L_{MM'}(\alpha, \beta, \gamma)$ are the Wigner D functions*.

The Capps theorem (3.9) in terms of the PM reads as follows: if the Capps condition is satisfied, then we have in the Z system

$$T_{LM} = 0 \quad \text{for odd M} \tag{3.23}$$

and in the Y system

$$T_{LM} = (-1)^L T^*_{LM}. \tag{3.24}$$

Formula (3.23) follows from (3.9) and (3.10), while (3.24) is obtained from (3.23) by using (3.22), (II.1), and (II.4′), and recognizing that the Y system is obtained by rotating the Z system through the Euler angles $(0, \pi/2, \pi/2)$.

Assume that the particle a_1 produced in the decay (2.2) or (2.3) has a spin s. Its polarization state can be described both by the density matrix $\rho_{\lambda\lambda'}$ and by the PM $t_{l\mu}$ (this symbol of the PM for a_1 is used to distinguish it from the PM of the resonance T_{LM}). It is clear that $\rho_{\lambda\lambda'}$ and $t_{l\mu}$ have the same properties as $\rho_{mm'}$ and T_{LM}. For example,

$$\rho_{\lambda\lambda'} = \sum_{l=0}^{2s}\sum_{\mu=-l}^{l} [(2l+1)/(2s+1)]\, C^{s\lambda}_{s\lambda',\,l\mu} t_{l\mu}, \tag{3.15′}$$

$$t_{l\mu} = (-1)^\mu t^*_{l-\mu}, \tag{3.16′}$$

$$t_{00} = 1, \tag{3.17′}$$

$$t_{l\mu} = 0, \quad \text{if} \quad |\mu| > l, \quad \text{or} \quad l > 2s. \tag{3.20′}$$

Finally, let us consider by way of an example, the PM

*For details of D functions see the books [6,12,13]. The D functions are defined somewhat differently in these books. We follow Rose's definition [12]. Many useful formulas for the D functions are contained in [10,11]. In Appendix II are given only the D-function formulas used in the present review.

of a particle with spin 1/2. Its ρ matrix is usually written in the form

$$\rho = (1 + p\sigma)/2,$$

where p is the polarization vector. From (3.10) we obtain

$$T_{00} = 1, \quad T_{10} = p_Z/\sqrt{3}, \quad T_{1\pm1} = -(p_X \mp ip_Y)/\sqrt{6}. \qquad (3.25)$$

The remaining T_{LM} vanish in accord with (3.20).

c) The ρ Matrix and PM of Two Particles

The polarization state of two particles with spins j_1 and j_2 is characterized by their common polarization matrix $\rho_{m_1m_1', m_2m_2'}$, which generally speaking does not reduce to a product of the ρ matrices of the two particles (there is a correlation between the polarizations of the two particles). The spin projections m_1 and m_1' of the first particle, and m_2 and m_2' of the second, are specified in terms of the rest systems of the corresponding particles relative to definite coordinate systems.

Just as in the case of one particle, it is possible to characterize the polarization state of two particles by using in place of the ρ matrix the PM $T_{L_1M_1, L_2M_2}$, which are connected with $\rho_{m_1m_1', m_2m_2'}$ by a formula similar to (3.10):

$$T_{L_1M_1,\, L_2M_2} = \sum_{m_1,\, m_1'} \sum_{m_2,\, m_2'} C^{j_1m_1}_{j_1m_1',\, L_1M_1} C^{j_2m_2}_{j_2m_2',\, L_2M_2} \rho_{m_1m_1',\, m_2m_2'},$$

$$-j_1 \leqslant m_1, \quad m_1' \leqslant j_1, \quad -j_2 \leqslant m_2, \quad m_2' \leqslant j_2, \quad 0 \leqslant L_1 \leqslant 2j_1, \quad 0 \leqslant L_2 \leqslant 2j_2. \qquad (3.26)$$

The inversion formula is obtained in analogy with (3.15) and takes the form

$$\rho_{m_1m_1',\, m_2m_2'} = \sum_{L_1=0}^{2j_1} \sum_{L_2=0}^{2j_2} \frac{(2L_1+1)(2L_2+1)}{(2j_1+1)(2j_2+1)} C^{j_1m_1}_{j_1m_1',\, L_1M_1} C^{j_2m_2}_{j_2m_2',\, L_2M_2} T_{L_1M_1,\, L_2M_2}. \qquad (3.27)$$

4. TWO-PARTICLE DECAY OF RESONANT STATE. GENERAL THEORY

a) Decay Amplitude

We consider the decay of the investigated resonance

$$x \to a_1 + a_2, \qquad (4.1)$$

where a_1 and a_2 are particles or resonances with definite quantum numbers. We denote by k_x, k_1, and k_2 the momenta of the corresponding particles, by η_x, η_1, and η_2 their parities, and by j and s the spins of x and a_1. The spin of the particle a_2 will be assumed equal to zero, this being the case of greatest practical interest. At the end of the section we shall generalize the results to the case when a_2 also possesses a spin.

We specify a coordinate system XYZ in the rest system of x. The state of the decay products of x is specified by the following quantities: ϑ and φ—the polar and azimuthal angles of $\mathbf{k}_1$ (relative to the axes XYZ), and λ—the projection of the spin of a_2 on the direction of $\mathbf{k}_2$ (λ is called the helicity of the particle a_1). If the projection of the spin x on the Z axis is equal to m, then the wave function of the decay products of x is equal to

$$\Psi^j_{m\lambda}(\vartheta, \varphi) = [(2j+1)/4\pi]^{1/2} A_\lambda D^{j*}_{m\lambda}(\varphi, \vartheta, 0). \qquad (4.2)$$

The variables of this function are the direction k_1 (the angles ϑ and φ) and the helicity λ of the particle a_1.

It is clear that the decay of x should be characterized by invariant quantities determined by the dynamics of the decay and are independent of the choice of the axes XYZ. Such quantities are the A_λ. They are called the helicity amplitudes of the decay*.

Let us show how (4.2) is derived and let us explain the physical meaning of A_λ.

Let the projection of the spin of x on a certain direction Z′ equal λ. By virtue of the conservation of the momentum in the decay (4.1), the emission of the particle a_1 in the direction Z′ is possible only if the helicity of a_1 is equal to λ. We denote the amplitude of such a decay by $[(2j+1)/4\pi]^{1/2}A_\lambda$ where $[2j+1)/4\pi]^{1/2}$ is a normalization factor. It is obvious that A_λ does not depend on the choice of the coordinate system XYZ.

Let us consider a spin state of x such that the projection of the spin on the Z axis in the XYZ system is equal to m. In this case the spin function x, which has a projection λ on the direction k_1, is equal to $D^{j*}_{m\lambda}(\varphi, \vartheta, 0)$. (As is well known[10-13], in rotation the spin-function transformation matrix is made up of D functions. Consequently, $\langle jm|\hat{R}(\varphi, \vartheta, 0)|j\lambda\rangle = D^j_{m\lambda}(\varphi, \vartheta, 0)$, where R is the rotation operator).

Thus, if x has a spin projection m on the Z axis, then the product of $[(2j+1)/4\pi]^{1/2}A_\lambda$ by $D^{j*}_{m\lambda}(\varphi, \vartheta, 0)$ is none other than the wave function of the decay products of x, i.e., we arrive at (4.2).

In Appendix IV we give a rigorous derivation of (4.2), and also prove the property (4.4), described in the next item, of the helicity amplitudes of the decay.

b) Properties of A_λ

If $j \geq s$, then the number of decay helicity amplitudes A_λ is equal in the general case to the number of the spin states of the particle a_1, i.e., to $(2s+1)$.

On the other hand, if $j < s$, then by virtue of (II.8) the number of A_λ does not exceed $(2j+1)$. Thus,

$$-s \leqslant \lambda \leqslant s, \quad \text{if} \quad j \geqslant s, \qquad (4.3)$$
$$-j \leqslant \lambda \leqslant j, \quad \text{if} \quad j < s.$$

However, not all the amplitudes are independent. If parity is conserved in the decay (4.1), then

$$A_\lambda = \sigma A_{-\lambda}, \qquad (4.4)$$

where

$$\sigma = (\eta_1\eta_2/\eta_x)(-1)^{j-s}. \qquad (4.5)$$

Relations (4.4) are the starting point for the determination of j and η_x. They mean that the A_λ with nonnegative λ are independent. This is a consequence of the fact that in parity-conserving decays (4.1), out of all the possible orbital momenta (which take on values from $|j-s|$ to $j+s$ and whose number is equal to $2s+1$ or $2j+1$), the only admissible ones are those having the same parity, such that $(-1)^l = (\eta_x/\eta_1\eta_2)$.

The A_λ are expressed in terms of the orbital decay

*The helicity amplitudes in different processes were first considered by Zastavenko [63].

amplitudes $A^{(l)}$ by means of the formula (see, for example,[10])

$$A_\lambda = \sum_{l=|j-s|}^{j+s} [(2l+1)/(2j+1)]^{1/2} C^{j\lambda}_{l0,\, s\lambda} A^{(l)}. \tag{4.6}$$

In order for the function (4.2) to be normalized, i.e.,

$$\sum_\lambda \int |\Psi^j_{m\lambda}|^2 \, d\cos\vartheta \, d\varphi = 1, \tag{4.7}$$

it is necessary that A_λ satisfy the normalization condition

$$\sum_\lambda |A_\lambda|^2 = 1. \tag{4.8}$$

In (4.7) and (4.8), the summation is over the values of λ given by (4.3).

c) Connection Between Amplitudes of the Decay x and the Polarization State of the Products of Its Decay. Cascade Polarization Moments

In the preceding subsection we have noted that the relations (4.4) between the decay helicity amplitudes A_λ are the starting point for the determination of j and η_x. It is clear that the amplitudes A_λ influence above all the polarization state of the particle x_1, which can be determined experimentally.

The polarization state of the particle a_1 will be characterized either by the density matrix $\rho^{(a_1)}_{\lambda\lambda'}$ or by the PM $t_{l\mu}$. Both $\rho^{(a_1)}_{\lambda\lambda'}$ and $t_{l\mu}$ are determined in the rest system of a_1 relative to the axes X′Y′Z′ shown in Fig. 1.

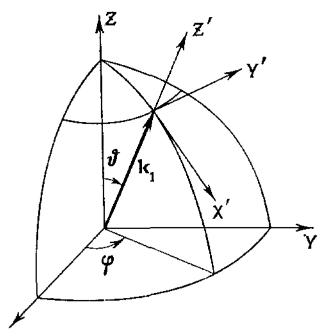

FIG. 1

If the system containing the resonance x is described by the function $\psi(m, \alpha)$ (m is the projection of the spin of x on the Z axis and α are the remaining variables of the system), then the angular distribution of the decay products of x are determined from the formula

$$I(\vartheta, \varphi) = \frac{d\omega}{d\Omega} = \sum_{m, m'=-j}^{j} \sum_\lambda \sum_\alpha \psi^*(m', \alpha)\, \psi(m, \alpha)\, \Psi^{j*}_{m'\lambda}(\vartheta, \varphi)\, \Psi^j_{m\lambda}(\vartheta, \varphi). \tag{4.9}$$

Taking into account the definition (3.2) of the resonance ρ matrix, we rewrite (4.9) in the form

$$I(\vartheta, \varphi) = \sum_{m, m'=-j}^{j} \sum_\lambda \rho_{mm'} \Psi^{j*}_{m'\lambda}(\vartheta, \varphi)\, \Psi^j_{m\lambda}(\vartheta, \varphi). \tag{4.10}$$

The density matrix of the particle a_1 is obtained in analogy with (4.10):

$$\rho^{(a_1)}_{\lambda\lambda'} = \frac{1}{I(\vartheta, \varphi)} \sum_{m, m'=-j}^{j} \rho_{mm'} \Psi^j_{m\lambda}(\vartheta, \varphi)\, \Psi^{j*}_{m'\lambda'}(\vartheta, \varphi). \tag{4.11}$$

The ρ matrix (4.11) is normalized in accordance with (3.4) and, of course, is a function of ϑ and φ.

We make the following transformations in (4.11): we express the ρ matrix of x in terms of the PM T_{LM} in accordance with formula (3.5), and replace $\Psi^j_{m\lambda}$ and $\Psi^{j*}_{m'\lambda'}$ by their values (4.2). After simple algebraic calculations in which formulas (II.4), (II.7), (I.2), and (I.4) are used, we obtain

$$I(\vartheta, \varphi)\, \rho^{(a_1)}_{\lambda\lambda'} = (A_\lambda A^*_{\lambda'}/4\pi) \sum_{L=0}^{2j} \sum_{M=-L}^{L} (2L+1)\, C^{j\lambda}_{j\lambda',\, L\lambda-\lambda'} D^{L*}_{M,\, \lambda-\lambda'}(\varphi, \vartheta, 0)\, T_{LM}. \tag{4.12}$$

Relation (4.12) connects the polarization state of the particle a_1 with the decay amplitude A_λ and with T_{LM}, which are the PM of x. From (4.12) we can obtain a relation more convenient for practical applications. To this end, we multiply (4.12) by $D^L_{M,\lambda-\lambda'}(\varphi, \vartheta, 0)$ and integrate over the solid angle with allowance for the orthogonality property of the D functions (II.6). We obtain

$$\int \rho^{(a_1)}_{\lambda\lambda'} D^L_{M,\, \lambda-\lambda'}(\varphi, \vartheta, 0) \times I(\vartheta, \varphi)\, d\cos\vartheta\, d\varphi = C^{j\lambda}_{j\lambda',\, L\lambda-\lambda'} A_\lambda A^*_{\lambda'} T_{LM}. \tag{4.13}$$

We have already noted that the polarization state of the particle is best described with the aid of the PM, since the PM are easier to determine from experiment than the ρ matrix. We therefore express the ρ matrix of the particle a_1 in (4.13) in terms of its PM $t_{l\mu}$, using formula (3.15′). We arrive at the result

$$\sum_{l=0}^{2s} [(2l+1)/(2s+1)]\, C^{s\lambda}_{s\lambda',\, l\mu} t^{LM}_{l\mu} = C^{j\lambda}_{j\lambda',\, L\mu} A_\lambda A^*_{\lambda'} T_{LM}, \tag{4.14}$$

where

$$t^{LM}_{l\mu} = \int t_{l\mu} D^L_{M\mu}(\varphi, \vartheta, 0)\, I(\vartheta, \varphi)\, d\cos\vartheta\, d\varphi. \tag{4.15}$$

The quantities $t^{LM}_{l\mu}$ will be called the cascade polarization moments (CPM) of the particle a_1. This name is most appropriate for the following reason.

As seen from (4.15), to find the CPM it is necessary to have the angular distribution of $t_{l\mu}$ in the reaction wherein the particle a_1 is produced* (in this case, in the decay of the resonance x). In Ch . 3 we have noted that $t_{l\mu}$ can be determined from the angular distribution of the decay products of a_1 (if a_1 is stable, then to determine $t_{l\mu}$ it is necessary to scatter a_1 by an analyzer target). Thus, to find the CPM it is necessary to have the distribution function in the cascade of the production and decay (or the scattering) of the particle a_1, this function being dependent both on the direction of the momentum of a_1 and on the direction of the momenta of the decay products of a_1.

*The definition (4.15) of the CPM does not require that a_1 must be produced in the decay (4.1). If, for example, a_1 is produced in a four-particle reaction, then it is still possible to introduce a CPM in accordance with formula (4.15), with ϑ and φ specifying the direction of the momentum of a_1 in the c.m.s. of this reaction.

d) Experimental Determination of the CPM

The easiest to determine experimentally is T_{00}^{LM}. Indeed, from (3.17′) and (4.15) we obtain

$$t_{00}^{LM} = \langle D_{M0}^{L}(\varphi, \vartheta, 0)\rangle. \tag{4.16}$$

In (4.16) and throughout this review we use the following notation:

$$\langle t_\alpha \rangle = \frac{1}{n}\sum_{i=1}^{n} t_{\alpha_i}, \tag{4.17}$$

where the summation is over all cases of the investigated decay (which can be also of the cascade type), and t_{α_i} is the value of t_α in the i-th case. In (4.16) we have $t_\alpha = D_{M_0}^{L}(\varphi, \vartheta, 0)$.

Equation (4.16) is the limit of (4.15) at $l = \mu = 0$), when n tends to infinity.

The rms error of the quantity $a\langle t_\alpha\rangle$ is calculated from the formula

$$\delta(a\langle t_\alpha\rangle) = \frac{a}{n}\Big[\sum_{i=1}^{n} t_{\alpha_i}^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} t_{\alpha_i}\Big)^2\Big]^{1/2}. \tag{4.18}$$

Unlike t_{00}^{LM}, the other CPM of the particle a_1 can be determined experimentally only if the momenta of the decay products of a_1 are measured.

Let a_1 decay into two particles. The direction of the *momentum of one of the decay products will be specified by the angles* ϑ' and φ' in the rest system of a_1 relative to the axes X′Y′Z′, which must be chosen in the manner shown in Fig. 1.

We shall show later on that the PM of a_1 are determined from the experimental data by means of the formula

$$t_{l\mu} = \gamma_l \langle D_{\mu 0}^{l}(\varphi', \vartheta', 0)\rangle, \tag{4.19}$$

where the coefficient γ_l depends on the concrete decay. Thus, for example, if a_1 decays into two spinless particles, *then it follows from* (3.12), (II.2), *and* (4.19) *that*

$$\gamma_l(s \to 20) = (C_{s0,\,l0}^{s0})^{-1}. \tag{4.20}$$

From (4.19) and (4.15) we obtain the following formula for the experimental determination of the CPM*

$$t_{l\mu}^{LM} = \gamma_l \langle D_{M\mu}^{L}(\varphi, \vartheta, 0)\, D_{\mu 0}^{l}(\varphi', \vartheta', 0)\rangle. \tag{4.21}$$

We recall once more that we use the notation (4.17) in (4.21).

The errors of the CPM are calculated by means of formula (4.18).

e) Properties of the CPM

Let us note certain properties of the CPM.

It follows from (4.15), (II.8), and (3.20′) that

$$t_{l\mu}^{LM} \equiv 0, \quad \text{if} \quad |\mu| > L, \quad \text{or} \quad |M| > L, \quad \text{or} \quad |\mu| > l, \quad \text{or} \quad l > 2s. \tag{4.22}$$

From (4.14) and (3.20) we conclude that

$$t_{l\mu}^{LM} \equiv 0, \quad \text{if} \quad L > 2j. \tag{4.23}$$

From (4.15), (3.16′), and (II.4) we obtain the relation

$$t_{l-\mu}^{L-M} = (-1)^{M} t_{l\mu}^{LM*}, \tag{4.24}$$

which is always satisfied, regardless of the process in which the particle a_1 was produced. Therefore the CPM with non-negative M (or μ) are independent.

From (4.14) we can express the CPM with the aid of (I.2) by means of the formula

$$t_{l\mu}^{LM} = T_{LM}\Big(\sum_{\lambda,\lambda'=-s}^{s} C_{s\lambda',\,l\mu}^{s\lambda} C_{j\lambda',\,L\mu}^{j\lambda} A_\lambda A_{\lambda'}^*\Big). \tag{4.25}$$

If parity is conserved in the decay (4.1), then we obtain from (4.25), (4.4), and (I.4f)

$$t_{l-\mu}^{LM} = (-1)^{L+l} t_{l\mu}^{LM}. \tag{4.26}$$

It follows from (4.24) and (4.26) that if the particle a_1 was produced in a parity-conserving decay (4.1), then the $t_{l\mu}^{LM}$ with non-negative M and μ are independent.

It is clear from (4.15) that CPM are specified with respect to definite axes XYZ (in the rest system of x) and X′Y′Z′ (in the rest system of a_1). Whereas the directions of the axes X′Y′Z′ must be chosen as shown in Fig. 1, the directions of XYZ are in general arbitrary. However, if the Capps condition is satisfied in the reaction (2.1) of x production, then the XZ axes are best chosen in the plane of this reaction (i.e., the CPM are best specified in the Y system). When XYZ is so chosen, we get the relation

$$t_{l\mu}^{LM} = (-1)^{l}\, t_{l\mu}^{LM}. \tag{4.27}$$

This relation can be easily derived from (3.24), (4.25), (4.24), and (4.26).

f) CPM and Determination of the Quantum Numbers of the Resonance

This is the main problem in Ch. 4. Assume that parity is conserved in the decay (4.1). We then get from (4.4)

$$A_\lambda A_{\lambda'}^* - \sigma A_\lambda A_{-\lambda'}^* = 0. \tag{4.28}$$

Substituting (4.14) in this formula, we obtain relations for the determination of j and η_X:

$$C_{j-\lambda',\,L\mu'}^{j\lambda} \sum_{l=0}^{2s}{}' (2l+1)\, C_{s\lambda',\,l\mu}^{s\lambda} t_{l\mu}^{LM} - \sigma C_{j\lambda',\,L\mu}^{j\lambda} \sum_{l'=0}^{2s}{}' (2l'+1)\, C_{s-\lambda',\,l'\mu'}^{s\lambda} t_{l'\mu'}^{LM} = 0. \tag{4.29}$$

The prime at the summation sign means that the summation must be carried out only over even l and l' or only over odd ones*.

The use of (4.29) for the determination of j and η_X is based on the fact that the coefficients of these relations depend on j via $C_{j\lambda,\,L\mu}^{j\lambda}$) and σ.

The procedure for determining j and η_X is as follows: in (4.29) it is necessary to substitute the experimentally obtained CPM and to choose j and σ such as to satisfy (4.29). After j and σ are determined, the parity is de-

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*It is seen from (4.20) that in the decay $\alpha_1 \to 20$ at odd l we have $\gamma_l \to \infty$. A similar situation occurs also in the parity-conserving decay $a_1 \to 1/2 + 0$. It is clear that in these cases it is impossible to use formula (4.21) to determine $t_{l\mu}^{LM}$ with odd l. This question will be discussed in detail later.

*The fact that (4.29) is satisfied separately for even and odd l and l' can be verified with the aid of (4.26) and (I.4f). It is, however, very easy to obtain this relation if the Capps conditions are satisfied in the reaction (2.1). In this case relation (4.27) holds, from which it follows that at even l and l' the CPM are real, and at odd ones they are imaginary.

termined by means of formula (4.5).

Such a simple procedure is realized if the errors and the background are quite small. If the errors are not small, then only the most probable values of j and η_x can be determined. To this it is necessary to calculate by the χ^2 method the probability of satisfaction of (4.29) for different values of j and σ (details on the χ^2 method are given in Appendix III). We note that as a rule (4.29) is more sensitive to σ than to j. Therefore in the experiments the parity of the resonance is determined more definitely than the spin.

Relations (4.29) (as well as (4.14), from which they are derived) are valid when $t_{l\mu}^{LM}$ is averaged over a region of the angles of the production of x. It is therefore possible to employ the entire statistics obtained in the experiment.

At different values of L and M, the relations in (4.29) are independent. In addition, at fixed L and M, the relations for which

$$0 \leqslant \lambda' \leqslant \lambda \leqslant s \tag{4.30}$$

are independent. This follows from (4.24) and (4.26).

g) Determination of the Polarized State of a Resonance and Its Decay Amplitudes

After determining j and η_x, the experimentally obtained CPM can be used to calculate the PM of x, T_{LM}:

$$T_{LM} = \sum_{l=0}^{2s}{}' \sum_{\lambda=-s}^{s} \frac{2l+1}{2s+1} (C_{s\lambda,\, l0}^{s\lambda} / C_{j\lambda,\, L0}^{j\lambda})\, t_{l0}^{LM}. \tag{4.31}$$

This formula is obtained from (4.14) when account is taken of (4.8).

The prime in (4.31) denotes that it is necessary to sum $t_{l_0}^{LM}$ for which L and l have the same parity (by virtue of (I.4f), the $t_{l_0}^{LM}$ with different parities of L and l cancel out in the sum (4.31). It is clear that formula (4.31) is valid only if $C_{j\lambda,\, L_0}^{j\lambda} \neq 0$ for all λ.

To calculate the decay amplitudes it is possible to use formulas (4.14). We note that with the aid of (4.14) it is possible to find A_λ only accurate to within a common phase.

We present one more useful formula

$$T_{LM} = \gamma_L \langle D_{M0}^L (\varphi, \vartheta, 0) \rangle, \tag{4.32}$$

where

$$\gamma_L = \Big(\sum_{\lambda=-s}^{s} C_{j\lambda,\, L0}^{j\lambda} |A_\lambda|^2 \Big)^{-1}. \tag{4.33}$$

These formulas are obtained from (4.25) and (4.16).

If the amplitudes of the decay of a resonance are determined during the course of its investigation, then formula (4.33) makes it possible to determine γ_L. It is then possible to use (4.32) for an experimental determination of the PM of this resonance, if the latter is produced by decay of another resonance (i.e., if it itself plays the role of the particle a_1 in the decay (4.1)). In this case (4.32) is written in the form (4.19) with the obtained values of γ_l.

The general theory of two-particle decays is developed in [14].

In the next chapter we shall apply the general theory to concrete decays.

h) Decay of a Resonance Into Two Particles with Nonzero Spin

Let us generalize the principal results of the present section to include the case with both particles a_1 and a_2 produced in the decay (4.1) have nonzero spins s_1 and s_2.

The polarized state of a_1 and a_2 will be characterized by the PM $t_{l_1\mu_1,\, l_2\mu_2}$, analogous to (3.26). The indices l_1, μ_1, and l_2, μ_2 are determined in the rest systems of the corresponding particles; the axes $X_1'Y_1'Z_1'$ and $X_2'Y_2'Z_2'$ have opposite directions, with $X_1'Y_1'Z_1'$ directed in the same manner as X′Y′Z′ in Fig. 1.

The CPM are determined in analogy with (4.15):

$$t_{l_1\mu_1,\, l_2\mu_2}^{LM} = \int t_{l_1\mu_1,\, l_2\mu_2} D_{M,\, \mu_1-\mu_2}^{L} (\varphi, \vartheta, 0)\, I(\vartheta, \varphi)\, d\cos\vartheta\, d\varphi \tag{4.34}$$

The properties of the CPM are analogous to (4.22)–(4.24):

$$t_{l_1-\mu_1,\, l_2-\mu_2}^{LM} = (-1)^M t_{l_1\mu_1,\, l_2\mu_2}^{LM*}. \tag{4.35}$$

If parity is conserved in the decay (4.1), then we have in analogy with (4.26)

$$t_{l_1-\mu_1,\, l_2-\mu_2}^{LM} = (-1)^{L+l_1+l_2} t_{l_1\mu_1,\, l_2\mu_2}^{LM}. \tag{4.36}$$

In the generalization of (4.27) it is necessary to replace $(-1)^l$ by $(-1)^{l_1+l_2}$.

For a generalization of (4.29) for the determination of j and η_x, and for their application to the decays $j \to 1/2 + 1/2$ and $j \to 1 + 1/2$ see [14].

5. TWO-PARTICLE DECAY OF THE RESONANT STATE. CONCRETE DECAYS

a) Decay of a Boson into Two Spinless Particles

We have shown earlier (see Ch. 3, b) that in the decay of a boson resonance into two spinless particles there is no asymmetry with respect to the reflection of the produced particles in the angular distribution of the decay products, and no T_{LM} with odd L appear in the decay of x. For even L, the PM of x are determined in the experiment by means of the formula

$$T_{LM} = (C_{j0,\, L0}^{j0})^{-1} \langle D_{M0}^L (\varphi, \vartheta, 0) \rangle, \tag{5.1}$$

which can be readily obtained from (3.12) and (II.2).

What conclusion can be drawn with respect to j and η_x on the basis of an analysis of the angular distribution of the products of the decay of x? If parity is conserved in the decay, then

$$\eta_x = \eta_1\eta_2 (-1)^j. \tag{5.2}$$

As to j, it is possible to establish for it only a lower limit. By virtue of (4.23)

$$j \geqslant L_{\max}/2, \tag{5.3}$$

where $L_{\max}$ is the largest value of L for which $\langle D_{M_0}^L(\varphi, \vartheta, 0)\rangle$ exceeds the level of the errors. More information concerning the value of j than is contained in (5.3) can be obtained only in individual cases. Let us stop to discuss this in greater detail.

Assume that in the analysis of the decay of x it turns out that $L_{\max} = 2$. We then obtain from (3.15), (I.7), (I.8), (5.1), (II.1), and (II.3)

$$\frac{1}{2}(\rho_{mm}+\rho_{-m-m})=\frac{1}{(2j+1)}\left[1-5\frac{3m^2-j(j+1)}{j(j+1)}\langle P_2(\cos\vartheta)\rangle\right]. \quad (5.4)$$

From (3.5) and (5.4) we obtain the inequality

$$-1/5\leqslant\langle P_2(\cos\vartheta)\rangle\leqslant(j+1)/5(2j-1), \quad (5.5)$$

which can help establish the spin of x*. A similar analysis can be carried out in the case when $L_{max} = 4$.

Let us consider by way of an example the experiment of Carmony and Van der Walle[15]. They investigated the process $\pi^+p \rightarrow \pi^+p\pi^0$ at an incoming-pion momentum 1.25 GeV/c. 1684 events were selected for the dipion mass in the ρ-meson region $27\ m_\pi^2 \leq M^2 \leq 33\ m_\pi^2$ and for the recoil proton momentum 100–400 MeV/c. The authors obtained the cross section

$$d\sigma/d\cos\vartheta=[(26.4\pm 2.4)\cos^2\vartheta-(1.0\pm 1.4)\cos\vartheta+(6.9\pm 0.7)]\ \text{mb}, \quad (5.6)$$

where ϑ is the angle between the incoming π^+ and outgoing π^+ in the dipion rest system.

From (5.6) we get

$$\langle P_2(\cos\vartheta)\rangle=\left(\int\frac{d\sigma}{d\cos\vartheta}P_2(\cos\vartheta)\,d\cos\vartheta\right)\Big/\left(\int\frac{d\sigma}{d\cos\vartheta}\,d\cos\vartheta\right)=0.224\pm 0.019. \quad (5.7)$$

This value of $\langle P_2(\cos\vartheta)\rangle$ satisfies the inequality (5.5) only if j = 1.

We note that $\langle P_2(\cos\vartheta)\rangle$ and the errors must be calculated directly from the experimental data, using formulas such as (4.17) and (4.18), whereas we have calculated $\langle P_2(\cos\vartheta)\rangle$ by using the result (5.6) given by the authors of[15].

b) Decay of a Boson into a γ Quantum and a Spinless Particle

A photon cannot have zero helicity. Therefore in the decay $x \rightarrow \gamma + 0$ only the decay amplitudes A_1 and A_{-1} differ from zero. Taking this circumstance into account, we obtain from (4.32), (4.33), (4.8), and (I.4f) for even L

$$\langle D_{M0}^L(\varphi,\vartheta,0)\rangle=C_{j1,\,L0}^{j1}T_{LM}. \quad (5.8)$$

Let x have two decay channels, $x \rightarrow \gamma + 0$ and $x \rightarrow 0 + 0$. By investigating the angular distribution of the decay of x through both channels, we can establish j (the parity of x is determined from formula (5.2)). Indeed, comparing (5.1) and (5.8), and taking (I.16) into account, we obtain

$$\langle D_{M0}^L(\varphi,\vartheta,0)\rangle_{(x\rightarrow\gamma+0)}-\left[1-\frac{L(L+1)}{2j(j+1)}\right]\langle D_{M0}^L(\varphi,\vartheta,0)\rangle_{(x\rightarrow 0+0)}=0. \quad (5.9)$$

In the derivation of this relation we took it into account that the PM of x (T_{LM}) do not depend on the channel through which x decays.

In determining j with the aid of (5.9), the experimental data must be reduced by the χ^2 method.

c) Decay of a Boson Into Particles with Spin 0 and 1

Let us consider the case when the particle a_1 produced in the decay of the resonance (4.1) has spin 1 (the spin of particle a_2 is 0). According to (4.3), there are three decay amplitudes A_1, A_0, and A_{-1} (if $j \geq 1$). By virtue of parity conservation in the decay of x, we have from (4.4) and (4.5)

$$A_1=\sigma A_{-1}, \quad (5.10)$$

$$A_0=\sigma A_0, \quad (5.11)$$

$$\sigma=(\eta_1\eta_2/\eta_x)(-1)^{j-1}. \quad (5.12)$$

In order to determine the spin and the parity of x with the aid of relations (4.29), it is necessary to determine experimentally the CPM of the particle a_1. According to (4.22), at s = 1 there are only t_{00}^{LM}, $t_{1\mu}^{LM}$, $t_{2\mu}^{LM}$.

Let a_1 decay in turn into two spinless particles. In Sec. a of this chapter we have noted that no PM with odd l appear in such a decay. Therefore $t_{1\mu}^{LM}$ cannot be determined from experiment. On the other hand, it is possible to determine from experiment t_{00}^{LM} and $t_{2\mu}^{LM}$. The former is determined from (4.16), and a formula for the latter is obtained from (4.21) and (4.20) by substituting s = 1*:

$$t_{2\mu}^{LM}=-\sqrt{5/2}\,\langle D_{M\mu}^L(\varphi,\vartheta,0)\,D_{\mu 0}^2(\varphi',\vartheta',0)\rangle. \quad (5.13)$$

Thus, in order to determine j and η_x in the decay $x \rightarrow 1 + 0$ (with subsequent decay $1 \rightarrow 0 + 0$), we can use relation (4.29) only for even l and l'.

Let us write down (4.29) for even l and l' at s = 1.

If $\sigma = 1$, then substitution in (4.29) of $\lambda = \lambda' = 0$ and also $\lambda = 1$ and $\lambda' = 0$ results in identities. On the other hand, if $\sigma = -1$, then, at the indicated values of λ and λ' we obtain respectively the relations

$$t_{00}^{LM}-\sqrt{10}\,t_{20}^{LM}=0,\quad L\ \text{even}, \quad (5.14)$$

$$t_{21}^{LM}=0. \quad (5.15)$$

Substituting in (4.29) $\lambda = \lambda' = 1$ and summing over even l and l' with the aid of (I.16) and (I.17), we obtain

$$\left[\frac{L(L+1)}{(L-1)(L+2)}\right]^{1/2}\left(t_{00}^{LM}+\sqrt{\frac{5}{2}}\,t_{20}^{LM}\right)-\sigma\sqrt{15}\left[\frac{L(L+1)}{2j(j+1)}-1\right]t_{22}^{LM}=0, \quad (5.16)$$

where L are even*.

When j and η_x are determined by the χ^2 method, it is first necessary to find the probability of satisfying the hypothesis (5.14) and (5.15) (see Appendix III). If this probability is small, then it can be stated that $\sigma = 1$. In this case it is necessary to find the probability of satis-

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*We note that if x is emitted forward in the production reaction (Adair's situation), then the angular distribution of the decay products of x contains additional information on the spin of x. This question, however, is outside the scope of the present review (Adair's analysis for a boson resonance decaying into two spinless particles can be found, for example, in [24,52]).

*We recall that the angles ϑ and φ determine the direction of the momentum of the particle a_1 in the rest system of x relative to arbitrary axes XYZ, and ϑ' and φ' determine the direction of the momentum of one of the decay products of a_1 relative to the axes X'Y'Z' (which must be chosen in the manner shown in Fig. 1) in the rest system of a_1. If a_1 is a vector meson that decays into three pseudoscalar mesons, then formulas (4.16) and (5.13) remain valid as before, except that the angles ϑ' and φ' must define now the direction of the normal to the plane of the decay of the vector meson in its rest system.

*We note that in the case of odd L all the CPM entering in (5.16) and (5.14) are equal to zero. This can be readily verified with the aid of (1.4), (5.11), and (4.25).

fying the hypothesis (5.16) for different values of j only $\sigma = 1$, and then to choose the most probable j. On the other hand, if the probability of satisfying the hypothesis (5.14) and (5.15) turns out to be large, then no conclusions can be drawn with respect to σ, since (5.14) and (5.15) can be satisfied also when $\sigma = 1$ (relations (5.14) and (5.15) are a consequence of the fact that when $\sigma = -1$ we always have $A_0 = 0$, but for an "accidental" reason it may turn out that $A_0 = 0$ also at $\sigma = 1$). In this case it is necessary to find the probability of satisfying (5.16) for different j at both values $\sigma = \pm 1$ and to choose the most probable values of j and σ.

Formulas (5.14)–(5.16) were obtained by Chang[16].

The described procedure of determining j and η_x will be illustrated with concrete experiments somewhat later, after we consider a few more general questions.

It may turn out that in the decay (4.1) the spinless particle is identical with one of the products of the decay of particle a_1. Then, owing to the interference of the identical particles, the formulas obtained in this subsection must be revised. It is clear that the revision is meaningful when the width of the resonance a_1 greatly exceeds the experimental resolution (for example in the case of the ρ meson).

Let us consider, for example, the decay

$$x \rightarrow \rho^0 \pi^- \rightarrow \pi_1^- \pi^+ \pi_2^-. \tag{5.17}$$

The interference of π^- mesons takes place only when the invariant $\pi_1^- \pi^+$ and $\pi_2^- \pi^+$ masses lie in the ρ band (i.e., in the case of intersection of the ρ band on the Dalitz plot).

Since the distribution of the ρ-meson decay products in the absence of interference is symmetrical with respect to interchange of the momenta (i.e., with respect to reflection of the momenta of the ρ mesons in the rest system of ρ), we can use relations (5.14)–(5.16) to determine j and η_x. However, when calculating $t_{l\mu}^{LM}$ by formulas (4.16) and (5.13) it is necessary to discard the cases when the ρ bands intersect, and the cases in which the invariant mass of the π^- mesons lies in the ρ band must be taken with weight 2. If in the decay (5.17) there are cases of intersections of ρ bands, when the invariant mass of the π^- mesons also lies in the ρ band (i.e., cases when the invariant mass of each pair of π mesons lies in the ρ band, as is the case for the decay of the A_2 meson), then their contribution to $t_{l\mu}^{LM}$ is taken into account in a more complicated manner (for more details see[17]).

The correction of $t_{l\mu}^{LM}$ for the decay A_2 meson can be found in[18], and turned out to be not larger than 5%.

Let us stop to discuss the determination of the PM of x (T_{LM}). From (4.31), (4.16), (5.13), and (I.16) we obtain

$$T_{LM} = (3C_{j1,\,L0}^{j1})^{-1} \Big\langle D_{M0}^{L}(\varphi, \vartheta, 0)\Big[3 - \frac{L(L+1)}{2j(j+1)} - \frac{5L(L+1)}{2j(2j+1)} D_{00}^{2}(\varphi', \vartheta', 0)\Big]\Big\rangle, \tag{5.18}$$

where the L are even.

If $\eta_x = \eta_1 \eta_2 (-1)^j$, then $A_0 = 0$ in accord with (5.11) and (5.12), and we obtain from (4.32), (4.33), (4.8), and (I.4f)

$$T_{LM} = (C_{j1,\,L0}^{j1})^{-1} \langle D_{M0}^{L}(\varphi, \vartheta, 0)\rangle, \tag{5.19}$$

where the L are even. The errors δT_{LM} are calculated from a formula similar to (4.18).

If x decays into two pseudoscalar mesons, or into a pseudoscalar and a vector meson, then according to (5.2) we have $\eta_x = (-1)^j$. From a comparison of (5.1) with (5.19) we obtain, with the aid of (1.16), additional relations for the determination of j:

$$\langle D_{M0}^{L}(\varphi, \vartheta, 0)\rangle_{x\rightarrow 1^- + 0^-} - \Big[1 - \frac{L(L+1)}{2j(j+1)}\Big] \langle D_{M0}^{L}(\varphi, \vartheta, 0)\rangle_{x\rightarrow 0^- + 0^-} = 0 \quad (L - \text{even}). \tag{5.20}$$

It is clear from (5.13) and (4.16) that the CPM depend on the choice of the axes XYZ, relative to which the angles ϑ and φ are defined. Equations (5.14)–(5.16) are valid for any choice of XYZ. It is most convenient, however, to direct the Y axis along the normal to the plane of x production. In this case, in accordance with (4.27), all the t_{00}^{LM} and $t_{2\mu}^{LM}$ should be pure real in the absence of a background. If the x-production reaction has a peripheral character, then the Z axis is best directed along the momentum of the incoming particle in the rest system of x, and the Y axis along the normal to the reaction plane. Such a system of coordinates will be called the Treiman-Yang system[59].

By way of an example let us consider the experimental work of Ascoli et al.[60], who investigated the quantum numbers of the B meson in the reaction

$$\pi^- + p \rightarrow B^- + p \tag{5.21}$$

at an incoming π^--meson momentum 5 GeV/c. The produced B meson decays in accordance with the scheme

$$B^- \rightarrow \pi^- + \omega, \quad \omega \rightarrow \pi^+ \pi^- \pi^0. \tag{5.22}$$

The method described above can be used for the decay (5.22), with the angles ϑ' and φ' in (5.13) defining the direction of the normal to the plane of decay of the vector meson ω in its rest system relative to the axis X'Y'Z' (see Fig. 1).

In[60], the CPM was calculated in the Treiman-Yang system, and the authors subtracted the background (we shall not discuss the procedure used in[60] to subtract the background, since this question is beyond the scope of the present survey). In Table I are given those CPM values of[60] which are outside the limits of errors.

Table I

CPM	t_{20}^{00}	t_{00}^{20}	$\tilde{t}_{20}^{20} = \Big\langle\Big(\frac{5}{2}\sin^2\vartheta' - 1\Big) \times D_{00}^{2}(\varphi, \vartheta, \varphi')\Big\rangle$	$\mathrm{Re}\,\tilde{t}_{22}^{20} = \mathrm{Re}\,\langle D_{02}^{2}(\varphi, \vartheta, \varphi')\rangle$
Numerical value and error	0.222 ± 0.079	-0.136 ± 0.056	-0.166 ± 0.056	0.076 ± 0.038

It is easy to show that the CPM $\tilde{t}_{20}^{20}$ and $\tilde{t}_{22}^{20}$ given in Table I are connected with the CPM t_{20}^{20} and t_{22}^{20} (see formula (5.13)) by means of the formulas*

$$\frac{3}{2}\tilde{t}_{20}^{20} = t_{00}^{20} + \sqrt{\frac{5}{2}}\, t_{20}^{20}, \tag{5.23}$$

*We note that (5.24) is valid if there is no background in the decay of the vector meson. If the background of the vector meson is not very small, then it is more convenient to use t_{22}^{20} than $\tilde{t}_{22}^{20}$, for the influence of the background is smaller in the calculation of t_{22}^{20}.

$$\tilde{t}_{22}^{20} = -\sqrt{\frac{5}{3}}\, t_{22}^{20}. \qquad (5.24)$$

To determine the spin and parity of the B meson it is first necessary to calculate the probability of the hypothesis (5.14) and (5.15). As reported by the authors of[60], all the t_{21}^{LM} are smaller than the errors; the hypothesis (5.15) are therefore likely. With the aid of Table I and (5.23) it is easy to verify that at L = 2 and M = 0 we get $\chi^2 < 1$ for the hypothesis (5.14) (i.e., the hypothesis is likely). At L = 0, Eq. (5.14) takes the form

$$1 - \sqrt{10}\, t_{20}^{00} = 0. \qquad (5.14')$$

From the value of t_{20}^{00} (and of the error) given in Table I it is easy to find that $\chi^2 = 1.5$ for the hypothesis (5.14′); the corresponding probability is 22%, i.e., the hypothesis (5.14′) is likely.

We have already explained that when the hypotheses (5.14) and (5.15) are satisfied it is impossible to draw any definite conclusion concerning the value of σ. Therefore the hypothesis (5.16) must be investigated by the χ^2 test at different values of j and at both values $\sigma = \pm 1$. In terms of $\tilde{t}_{20}^{20}$ and $\tilde{t}_{22}^{20}$, Eq. (5.16) takes the form

$$\left[\frac{L(L+1)}{(L-1)(L+2)}\right]^{1/2} \tilde{t}_{20}^{20} + \sigma\left[\frac{L(L+1)}{j(j+1)} - 2\right]\tilde{t}_{22}^{20} = 0. \qquad (5.16')$$

Table II lists the results of an investigation of the hypothesis (5.16′) by the χ^2 test for different values of the spin and parity of the B meson. The same table gives the values of the PM (T_{20}) for the B meson, calculated for those values of j and η_B which are compatible with the experimental data. T_{20} was calculated for 1^+ by formula (5.18), and for 2^+ and 3^- by formula (5.19). The last column of Table II gives the mean-square values of the B-meson spin projection on the Z axis, as calculated by formula (3.21). (The Z axis was chosen to be the direction of the momentum of the incident pion in the rest system of B.) In addition, the density matrix element ρ_{00} is given for 1^+.

Table II

j^{η_B}	χ^2	Probability, %	Conclusion	T_{20}	$\langle m^2\rangle^{1/2}$
1^+	2.7	10	The hypothesis (5.16) is compatible with the experimental data.	-0.45 ± 0.18	$\rho_{00} = 0.81\pm0.19$
2^-	11.7	0.07	The hypothesis (5.16) is incompatible with the experimental data.	—	—
3^+	11.5	0.07	Ditto	—	—
1^-	11.7	0.07	Ditto	—	—
2^+	2.7	10	The hypothesis (5.16) is compatible with the experimental data.	0.51 ± 0.21	1.96 ± 0.2
3^-	1.1	29	Ditto	0.35 ± 0.145	2.6 ± 0.22

We see from Table II that the method of polarization moments has made it possible in[60] to refute the quantum numbers 1^-, 2^-, 3^+, etc., but did not permit a strict choice between 1^+, 2^+, 3^-, etc.

However, it is seen from the same Table II that in the case of 1^+ the element ρ_{00} predominates in the ρ matrix of the B meson, and in the case of 2^+ and 3^- the predominant elements are ρ_{mm} with $m \neq 0$. Therefore, if the quantum numbers of the B meson are 2^+, 3^-, etc., then a drop should occur in the distribution with respect to t at small t ($t = -(k_B - k_\pi)^2$ is the square of the momentum transferred to the B meson in the reaction (5.21). On the other hand, if there is no drop, then only 1^+ is possible*.

In[60], the distribution with respect to t turned out to be proportional to exp(At), where A = 4 GeV^{-2}. Therefore the quantum numbers of the B meson are most likely 1^+.

The reaction (5.21) was investigated at the Institute of Theoretical and Experimental Physics[77] at an incoming pion momentum 3.25 GeV/c. The conclusions concerning the quantum numbers and amplitudes of the B-meson decay in[60] and[77] are in good agreement. But the polarization given for the B meson in the two references is quite different, pointing to a different mechanism of the reaction (5.21) at incoming-pion momenta 5 and 3.25 MeV/c.

d) Decay of an Isobar into a Baryon and a Spinless Particle

Let us consider the decay (2.1) in the case when x is an isobar, a_1 a baryon (with spin 1/2), and a_2 is a spinless particle. According to (4.3), there are two helicity amplitudes of the decay, $A_{1/2}$ and $A_{-1/2}$. By virtue of parity conservation, they are connected by a relation analogous to (4.4) and (4.5):

$$A_{1/2} = \sigma A_{-1/2}, \qquad (5.25)$$

$$\sigma = (\eta_x/\eta_1\eta_2)(-1)^{j-1/2}. \qquad (5.26)$$

We shall discuss two problems: 1) the determination of j and η_x and 2) the determination of the polarization state of the isobar x.

Let us consider the first problem. The relations for the determination of j and η_x are obtained from (4.29) substituting (I.12) and the explicit values of $C^{1/2\,\lambda}_{1/2\,\lambda,1\mu}$ (and also by taking into account the fact that the particles with spin 1/2 have only t_{00}^{LM} and $t_{1\mu}^{LM}$ as CPM):

$$(2j+1)\, t_{10}^{LM} - \sigma\sqrt{2L(L+1)}\, t_{11}^{LM} = 0. \qquad (5.27)$$

From (4.25), (5.25), (I.4f), and (I.12) it follows that

$$t_{1\mu}^{LM} = 0, \quad \text{if L is even}, \qquad (5.28)$$

Therefore (5.27) can be used for the determination of j and η_x only for odd L. To this end it is necessary to determine experimentally t_{10}^{LM} and t_{11}^{LM} for the baryon.

If the baryon a_1 decays in turn without conserving parity, in accordance with the scheme

$$a_1 \to N + \Pi \qquad (5.29)$$

(N—baryon, Π—pseudoscalar meson), then its polariza-

*It can be thought that at an incoming pion energy 5 GeV the reaction (5.21), albeit in a crude approximation, is described by the Regge-pole model (see the review [61]). Then the matrix element of the reaction is proportional to $(-t)^{m/2}$. But in any other peripheral model the cross section of the reaction (5.21) should decrease at small values of t if the B meson is produced with a spin projection $m \neq 0$.

tion state (and CPM) can be determined from this decay.

As is well known, when parity is not conserved in the reaction (5.29) the angular distribution of N (in the rest system of a_1) is given by

$$d\omega/d\Omega = (1 + \alpha \mathbf{p}\boldsymbol{\nu})/4\pi, \tag{5.30}$$

where Ω is the solid angle in the a_1 rest system, **p** is the a_1 polarization vector, and ν is a unit vector in the direction of the momentum of N (in the rest system of a_1).

The asymmetry coefficient α is different for concrete decays (5.29). Table III lists the values of α for a number of decays. The data were borrowed from the review of Rosenfeld et al.[70] From (5.30) we easily obtain a formula for the experimental determination of p:

$$\mathbf{p} = (3/\alpha)\langle \boldsymbol{\nu} \rangle. \tag{5.31}$$

From (5.31) and (3.25) we obtain a formula for the experimental determination of the PM of a_1

$$t_{10} = \frac{\sqrt{3}}{\alpha}\langle \cos\vartheta' \rangle \equiv \frac{\sqrt{3}}{\alpha}\langle D^1_{00}(\varphi', \vartheta', 0)\rangle,$$

$$t_{1\pm 1} = \mp\sqrt{\frac{3}{2}}\frac{1}{\alpha}\langle \sin\vartheta' e^{\mp i\varphi'}\rangle \equiv \frac{\sqrt{3}}{\alpha}\langle D^1_{\pm 10}(\varphi', \vartheta', 0)\rangle. \tag{5.32}$$

We took into account in (5.32) the fact that the PM of a_1 are designated $t_{l\mu}$ (and not T_{LM}) and that they are defined relative to the axes X′Y′Z′ (see Fig. 1).

From (5.32) and (4.15) we obtain a formula for the experimental determination of the CPM of a_1

$$t^{LM}_{1\mu} = \frac{\sqrt{3}}{\alpha}\langle D^L_{M\mu}(\varphi, \vartheta, 0)\, D^1_{\mu 0}(\varphi', \vartheta', 0)\rangle. \tag{5.33}$$

The errors of the CPM are determined by formula (4.18). The notation (4.17) is used in (5.31)–(5.33).

Thus, substituting the experimentally obtained CPM (5.33) in (5.27) we can determine j and η_x. The experimental data must be reduced by the χ^2 method (see Appendix III).

The described method was used to determine j and η_x of the isobars $\Sigma^*(1385)$ (see, for example,[29,75]), $\Xi^*(1530)$ (see, for example,[30,76]). The results of[30] were used by us in Appendix III to illustrate the χ^2 method.

Relations (5.27) were derived by Byers and Fenster[28].

We emphasize that the method of polarization moments makes use of all the information concerning the spin and parity of the isobar, and could therefore be used successfully in investigations employing other methods (for example the work of Shafer and coworkers[31]) who investigated the angular distribution of the longitudinal and "magic" polarization of the Λ hyperon produced in the $\Sigma^*(1385) \to \Lambda + \pi$) decay.

Table III. Asymmetry coefficient in strange-baryon decays

Decay	Asymmetry coefficient α
$\Lambda \to p + \pi^-$	0.6455 ± 0.0159
$\Sigma^+ \to p + \pi^0$	-0.9547 ± 0.0696
$\Sigma^+ \to n + \pi^+$	-0.0175 ± 0.0390
$\Sigma^- \to n + \pi^-$	-0.0604 ± 0.0469
$\Xi^- \to \Lambda + \pi^-$	-0.4070 ± 0.0370

In conclusion we present formulas for the experimental determination of the PM of x (these formulas can be easily obtained from (4.25), (5.25), and (5.33)):

$$T_{LM} = (C^{j1/2}_{j1/2,\,L0})^{-1}\, t^{LM}_{00} = (C^{j1/2}_{j1/2,\,L0})^{-1}\langle D^L_{M0}(\varphi, \vartheta, 0)\rangle, \tag{5.34}$$

where L are even, and

$$T_{LM} = \sqrt{3}\,(C^{j1/2}_{j1/2,\,L0})^{-1}\, t^{LM}_{10} = 3\,(\alpha C^{j1/2}_{j1/2,\,L0})^{-1}\langle \cos\vartheta' D^L_{M0}(\varphi, \vartheta, 0)\rangle, \tag{5.35}$$

where L are odd.

We recall that in (5.32)–(5.35) the angles ϑ and φ, (ϑ' and φ') define the direction of the momentum of a_1(N) in the rest system of x(a_1) relative to the axes XYZ (X′Y′Z′). If the direction of XYZ is arbitrary (but is frequently convenient to choose XZ to be the plane of x production), then X′Y′Z′ are directed as shown in Fig. 1.

We see from (5.34) and (5.35) that to determine T_{LM} with even L it is necessary to have only the angular distribution of the decay of x, and to determine T_{LM} with odd L it is necessary to have the angular distribution of the polarization of a_1 (i.e., besides the x decay it is also necessary to investigate the a_1 decay (5.29), where α is the asymmetry coefficient in the latter decay). Thus, if a_1 is a stable baryon, then only the PM with even L can be determined.

Incidentally, if the baryon a_1 is stable, then it is impossible to determine j and η_x with the aid of (5.27), and a polarized proton target must be used for this purpose (see Ch. 7).

In this review we do not consider fermion decay with parity nonconservation, since the hyperons Λ, Σ, and Ξ have already been well investigated, and the statistics are still patently inadequate for the study of the Ω hyperon. The decay of fermions with parity nonconservation is discussed in the reviews[57,58,24] and in the original papers[23,27,45-47].

e) The Decay j → 3/2 + 0

Assume that the decay (4.1) of the isobar x produces an isobar a_1 with spin 3/2 (and known parity) and a spinless particle a_2. According to (4.3) there are four helicity amplitudes of the decay (if $j \geq 3/2$), $A_{\pm 1/2}$ and $A_{\pm 3/2}$. By virtue of parity conservation, they are connected by relations similar to (4.4) and (4.5):

$$A_{-3/2} = \sigma A_{3/2}, \quad A_{-1/2} = \sigma A_{1/2}, \quad \sigma = (\eta_x/\eta_1\eta_2)(-1)^{j-3/2}. \tag{5.36}$$

The relations for the determination of j and η_x are obtained by substituting in (4.29) the equations (I.11)–(I.15) and the explicit values of $C^{3/2\,\lambda}_{3/2\,\lambda',\,l\mu}$:

$$\sqrt{(L-1)(L+2)}\, t^{LM}_{22} - \sigma(j + 1/2)\, t^{LM}_{21} = 0, \tag{5.37}$$

$$\sqrt{70(L-1)(L+2)}\, t^{LM}_{32} - \sigma(j+1/2)(3\sqrt{2}\, t^{LM}_{11} + 2\sqrt{7}\, t^{LM}_{31}) = 0; \tag{5.38}$$

in (5.37) and (5.38) the L are even;

$$\begin{gathered}(j+1/2)\sqrt{(L-1)(L+2)}\, t^{LM}_{21} - \sigma[L(L+1) - 2(j+1/2)^2]\, t^{LM}_{22} = 0,\\ (j+1/2)\sqrt{(L-1)(L+2)}\,(3\sqrt{2}\, t^{LM}_{11} + 2\sqrt{7}\, t^{LM}_{31}) \\ - \sigma[L(L+1) - 2(j+1/2)^2]\sqrt{70}\, t^{LM}_{32} = 0,\end{gathered} \tag{5.39}$$

$$\left[\frac{(2j+1)}{(L-2)L(L+1)(L+3)}\right]^{1/2}(3\sqrt{3}\, t^{LM}_{10} + \sqrt{7}\, t^{LM}_{30}) \tag{5.40}$$

$$-\sigma\left[\frac{\sqrt{(L-1)(L+2)}}{(j-1/2)(j+3/2)}-\frac{3}{\sqrt{(L-1)(L+2)}}\right]2\sqrt{35}\,t_{33}^{LM}=0, \tag{5.41}$$

$$(2j+1)(t_{10}^{LM}-\sqrt{21}\,t_{30}^{LM})+\sigma\sqrt{L(L+1)}(2\sqrt{7}\,t_{31}^{LM}-2\sqrt{2}\,t_{11}^{LM})=0; \tag{5.42}$$

in (5.39)–(5.42) the L are odd.

Formulas (5.37)–(5.42) were derived by Shafer[33]; in his paper, relation (5.41) contains an error and is numbered (37). He also gives relations obtained under the assumption that the amplitude with the smallest orbital angular momentum is dominant in the $j \to 3/2 + 0$ decay.

To determine j and η_x with the aid of relations (5.37)–(5.42) it is necessary to determine $t_{l\mu}^{LM}$, the CPM of a_1.

Let the isobar a_1 decay with parity conservation into a baryon and a spinless particle in accordance with the scheme

$$a_1 \to N' + \Pi'. \tag{5.43}$$

Then $t_{2\mu}$, the PM of a_1, are determined from this decay by formula (5.34) (except that T_{LM} in (5.34) must be replaced by $t_{l\mu}$, and ϑ and φ by ϑ' and φ', since in the decay (5.43) a_1 plays the role of x and N′ plays the role of a_1). We thus obtain from (5.34) and (4.15) a formula for the experimental determination of $t_{2\mu}^{LM}$:

$$t_{2\mu}^{LM}=-\sqrt{5}\langle D_{M\mu}^{L}(\varphi,\vartheta,0)\,D_{\mu 0}^{2}(\varphi',\vartheta',0)\rangle. \tag{5.44}$$

Thus, by studying the angular distribution in the decays (2.1) and (5.43) it is possible to determine from formulas (4.16) and (5.44) the values t_{00}^{LM} and $t_{2\mu}^{LM}$ of the CPM of a_1. In this case it is possible to use for the determination of j and η_x only the relations (5.37) and (5.39).

To use the remaining relations (5.38) and (5.40)–(5.42) it is necessary to determine experimentally $t_{1\mu}^{LM}$ and $t_{3\mu}^{LM}$. This is possible only if the baryon N′ produced in the decay (5.43) is in turn unstable and decays with parity nonconservation in accordance with the scheme

$$N' \to N'' + \Pi''. \tag{5.45}$$

Then $t_{1\mu}$ and $t_{3\mu}$ are determined by formula similar to (5.35) (with T_{LM} replaced by $t_{l\mu}$ and ϑ and φ by ϑ' and φ', as explained above):

$$t_{l\mu}=3\,(\alpha' C_{3/2\;1/2,\,l0}^{3/2\;1/2})^{-1}\langle\cos\vartheta'' D_{\mu 0}^{l}(\varphi',\vartheta',0)\rangle, \tag{5.46}$$

where α' is the asymmetry coefficient in the decay (5.45), ϑ'' is the angle between the direction of the momentum of N′ (in the rest system of a_1) and the direction of the momentum of N″ (in the rest system of N′).

From (5.46) and (4.15) we obtain formulas for the experimental determination of $t_{1\mu}^{LM}$ and $t_{3\mu}^{LM}$:

$$t_{1\mu}^{LM}=(3\sqrt{15}/\alpha')\langle D_{M\mu}^{L}(\varphi,\vartheta,0)\,D_{\mu 0}^{1}(\varphi',\vartheta',0)\cos\vartheta''\rangle, \tag{5.47}$$

$$t_{3\mu}^{LM}=-(\sqrt{35}/\alpha')\langle D_{M\mu}^{L}(\varphi,\vartheta,0)\,D_{\mu 0}^{3}(\varphi',\vartheta',0)\cos\vartheta''\rangle. \tag{5.48}$$

We recall that in (5.44) and (5.46)–(5.48) the angles ϑ and φ (ϑ' and φ') specify the direction of the momentum of a_1 (N′) in rest system of x (a_1) relative to the axis XYZ (X′Y′Z′). The direction of XYZ is arbitrary (but it is frequently convenient to choose XZ in the x production plane) and X′Y′Z′ must be chosen in a manner shown in Fig. 1.

6. THREE PARTICLE DECAY OF A RESONANT STATE

a) Three-particle Amplitudes Decay and Their Properties

Assume that in the decay

$$x \to a_1 + a_2 + a_3 \tag{6.1}$$

the particles a_2 and a_3 are spinless, the spin of x is j, and the spin of a_1 is s. This is the case of greatest practical interest. The momenta and the parities of the particles are respectively k_x, k_1, k_2, k_3; η_x, η_1, η_2, η_3.

We choose the coordinate axes XYZ in the rest system of x. The state of the decay products of x are specified by the following quantities: the angles ϑ and φ, which determine the direction of the momentum of one of the particles (for example, $\mathbf{k}_2$) relative to the axes XYZ, the angle Φ, which specifies the position of the x decay plane (Fig. 2), the energies ω_1 and ω_2 (ω_3 is determined by the energy conservation law), and finally the helicity λ of the particle a_1.

If the projection of the spin of x on the Z axis is m, then the wave function of the decay products of x is given by (see Appendix IV)

$$\Psi_{m\lambda}(\varphi,\vartheta,\Phi;\omega_1,\omega_2)=\frac{\sqrt{j+(1/2)}}{2\pi}\times\sum_{\tilde m=-j}^{j} f_{\tilde m\lambda}(\omega_1,\omega_2)\,D_{\tilde m,m}^{j*}(\varphi,\vartheta,\Phi). \tag{6.2}$$

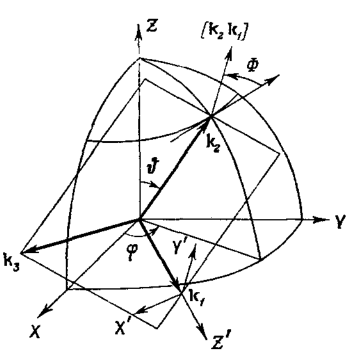
The decay amplitudes $f_{\tilde m\lambda}(\omega_1,\omega_2)$ do not depend on the choice of the axes XYZ; they can differ from zero only if

$$-j\leqslant\tilde m\leqslant j,\qquad -s\leqslant\lambda\leqslant s. \tag{6.3}$$

$f_{\tilde m\lambda}(\omega_1,\omega_2)$ has a physical meaning analogous to A_λ for the two-particle decay (see Ch. IV), namely, $(\sqrt{j+1/2}/2\pi)f_{\tilde m\lambda}$ is the amplitude of the decay of x with a spin projection $\tilde m$ on the Z axis, if the momentum $\mathbf{k}_2$ is directed along Z, the plane of the decay (6.1) coincides with XZ, and the helicity of the particle a_1 is λ.

If the function (6.2) is normalized, then we obtain, taking (II.5) into account

$$\sum_{\tilde m=-j}^{j}\sum_{\lambda=-s}^{s}\int|f_{\tilde m\lambda}(\omega_1,\omega_2)|^2\,d\omega_1\,d\omega_2=1. \tag{6.4}$$



The Y′ axis is parallel to $\mathbf{k}_2 \times \mathbf{k}_1$.

FIG. 2

The condition (6.4) is analogous to the normalization condition (4.8) for the amplitudes of the two-particle decay.

By virtue of parity conservation in the decay (6.1), the following relation holds between the decay amplitudes

$$f_{\widetilde{m}\lambda}(\omega_1, \omega_2) = I(-1)^{\lambda-\widetilde{m}} f_{-\widetilde{m}-\lambda}(\omega_1, \omega_2), \tag{6.5}$$

$$I = (\eta_x/\eta_1\eta_2\eta_3)(-1)^{j-s}. \tag{6.6}$$

For a proof see Appendix IV.

We note that other parametrizations of the wave function of three particles are possible (see, for example, the article of Berman and Jacob[11] and the review[28]), but we shall not use them in this review.

b) Cascade Polarization Moments for Three-particle Decay

The polarization states of x and a_1 will be characterized by the polarization moments T_{LM} and $t_{l\mu}$ in their rest systems relative to the axes XYZ and X′Y′Z′, respectively. Whereas the choice of the axes XYZ is arbitrary (except that for convenience XZ should be chosen in the x-production plane), the axes X′Y′Z′ must be chosen as shown in Fig. 2.

It will be convenient in what follows to use the cascade polarization moments (CPM) $t_{l\mu}^{LM\widetilde{M}}$ of the particle a_1 in the three-particle decay (6.1). These are given by

$$t_{l\mu}^{LM\widetilde{M}} = \int t_{l\mu} D^L_{M\widetilde{M}}(\varphi, \vartheta, \Phi)\, I(\varphi, \vartheta, \Phi; \omega_1, \omega_2)\, d\cos\vartheta\, d\varphi\, d\Phi\, d\omega_1\, d\omega_2, \tag{6.7}$$

where $I(\varphi, \vartheta, \Phi; \omega_1, \omega_2)$ is the distribution of the products of the decay (6.1) with respect to the angles and the energies. We note that (6.7) is analogous to the definition (4.15) of the CPM for the two-particle decay.

The CPM of a_1 can be determined experimentally. Thus, for example, by virtue of (3.17′) we obtain from (6.7) a formula for the experimental determination of $t_{00}^{LM\widetilde{M}}$:

$$t_{00}^{LM\widetilde{M}} = \langle D^L_{M\widetilde{M}}(\varphi, \vartheta, \Phi)\rangle. \tag{6.8}$$

Other $t_{l\mu}^{LM\widetilde{M}}$ are determined experimentally, depending on the method of determining $t_{l\mu}$. For example, if a_1 is a baryon decaying with parity nonconservation in accordance with the scheme (5.29), then we obtain from (5.32) and (6.7) a formula for the experimental determination of $t_{1\mu}^{LM\widetilde{M}}$:

$$t_{1\mu}^{LM\widetilde{M}} = \frac{\sqrt{3}}{\alpha}\langle D^1_{\mu 0}(\varphi', \vartheta', 0)\, D^L_{M\widetilde{M}}(\varphi, \vartheta, \Phi)\rangle, \tag{6.9}$$

where α is the asymmetry coefficient in the decay (5.29), and ϑ' and φ' are the angles that determine the direction of the momentum of N, which is produced in the decay (5.29), in the rest system of a_1 relative to the axes X′Y′Z′ (see Fig. 2).

In (6.8) and (6.9) we use the notation (4.17). The errors are calculated in accordance with (4.18). From (6.2) it follows that

$$t_{l\mu}^{LM\widetilde{M}} = T_{LM}\Big(\sum_{\lambda, \lambda'=-s}^{s}\ \sum_{\widetilde{m}, \widetilde{m}'=-j}^{j} C^{s\lambda}_{s\lambda', l\mu} C^{j\widetilde{m}}_{j\widetilde{m}', L\widetilde{M}} \widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'}\Big), \tag{6.10}$$

where

$$\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'} = \int f_{\widetilde{m}\lambda}(\omega_1, \omega_2)\, f^*_{\widetilde{m}'\lambda'}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2. \tag{6.11}$$

The derivation of (6.10) and (6.11) is analogous to the derivation of (4.25) from (4.2).

It follows from (6.11), (6.4), and (6.5) that

$$\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'} = \widetilde{f}^*_{\widetilde{m}'\lambda', \widetilde{m}\lambda}, \tag{6.12}$$

$$\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'} = (-1)^{\lambda-\lambda'+\widetilde{m}'-\widetilde{m}} \widetilde{f}_{-\widetilde{m}-\lambda, -\widetilde{m}'-\lambda'}, \tag{6.13}$$

$$\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'} = I(-1)^{\lambda'-\widetilde{m}'} \widetilde{f}_{\widetilde{m}\lambda, -\widetilde{m}'-\lambda'}, \tag{6.14}$$

$$0 \leqslant \widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}\lambda} \leqslant 1, \quad \sum_{\widetilde{m}, \lambda} \widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}\lambda} = 1, \quad |\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'}|^2 \leqslant \widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}\lambda} \widetilde{f}_{\widetilde{m}'\lambda', \widetilde{m}'\lambda'}. \tag{6.15}$$

Let us note certain properties of the CPM.

From (6.7), (3.16), (3.16′), and (II.4) it follows that

$$t_{l\mu}^{LM\widetilde{M}*} = (-1)^{M+\widetilde{M}+\mu} t_{l-\mu}^{L-M-\widetilde{M}}. \tag{6.16}$$

From (6.10), (6.13), and (I.4f) it follows that

$$t_{l\mu}^{LM\widetilde{M}} = (-1)^{L+l+\widetilde{M}+\mu} t_{l-\mu}^{LM-\widetilde{M}}. \tag{6.17}$$

If the Capps condition is satisfied in the x production reaction and if XZ are chosen in the plane of this reaction, then it follows from (6.10), (6.12), (6.13), (3.24), and (I.4e) that

$$t_{l\mu}^{LM\widetilde{M}*} = (-1)^l t_{l\mu}^{LM\widetilde{M}}. \tag{6.18}$$

The properties (6.16)–(6.18) correspond to the properties (4.24), (4.26), and (4.27) of the CPM for the two-particle decay.

c) Determination of the Spin and Parity of a Resonance Decaying into Three Particles

From (6.10) we express $\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'}$ with the aid of (I.2) and (I.4):

$$\widetilde{f}_{\widetilde{m}\lambda, \widetilde{m}'\lambda'} = \sum_{L=0}^{2j}\sum_{l=0}^{2s} \frac{(2L+1)(2l+1)}{(2j+1)(2s+1)} C^{j\widetilde{m}}_{j\widetilde{m}', L\widetilde{M}} C^{s\lambda}_{s\lambda', l\mu}\left(t_{l\mu}^{LM\widetilde{M}}/T_{LM}\right). \tag{6.19}$$

From (6.14) and (6.19) we obtain a relation for the determination of j and η_x:

$$\begin{aligned}\sum_{L=0}^{2j}{\sum_{l=0}^{2s}}' (2L+1)(2l+1)\, C^{j\widetilde{m}}_{j\widetilde{m}', L\widetilde{M}} C^{s\lambda}_{s\lambda', l\mu}\left(t_{l\mu}^{LM\widetilde{M}}/T_{LM}\right) \\ - I(-1)^{\lambda'-\widetilde{m}'} \sum_{L'=0}^{2j}{\sum_{l'=0}^{2s}}' (2L'+1)(2l'+1)\, C^{j\widetilde{m}}_{j-\widetilde{m}', L'\widetilde{M}'} C^{s\lambda}_{s-\lambda', l'\mu'} \\ \times\left(t_{l'\mu'}^{L'M'\widetilde{M}'}/T_{L'M'}\right) = 0.\end{aligned} \tag{6.20}$$

The prime at the summation sign denotes that the summation must be carried out only over even L + l and L′ + l', or only over odd ones*.

Since the CPM $t_{l\mu}^{LM\widetilde{M}}$ are determined experimentally, (6.20) is a system of equations linear in $(1/T_{LM})$, and the coefficients of these equations depend on j (via $C^{j\widetilde{m}}_{j\widetilde{m}', L\widetilde{M}}$) and I. As a rule, the number of equations exceeds the number of unknowns (this will be verified with concrete examples). The most probable values of j and η_x can therefore be determined by the χ^2 method,

*The fact that (6.20) is satisfied separately for even and odd l + L and l' + L′ can be verified with the aid of (6.17) and (I.4f).

by choosing the values of j and η_x in best agreement with the system (6.20).

It follows from (6.10) that the relations (6.20) remain valid when $t_{l\mu}^{LM\widetilde{M}}$ and T_{LM} are averaged over any region of the angles of production of x, and therefore in the determination of j and η_x we can use the entire statistics obtained in the experiment. By virtue of (6.17), the independent relations are (6.20) with non-negative $\widetilde{M}$ and $\widetilde{M}'$, i.e.,

$$0 \leqslant \widetilde{m}' \leqslant \widetilde{m} \leqslant j. \tag{6.21}$$

In (6.20) it is possible for each L any value $M \geq 0$, and an independent equation is obtained for each M.

If the x production reaction satisfies the Capps conditions, then $t_{l\mu}^{LM\widetilde{M}}$ and T_{LM} are pure real or pure imaginary in accordance with (6.18) and (3.16), and therefore (6.20) is a system of equatinos with real coefficients relative to real unknowns.

The angles ϑ and φ in (6.2) and (6.7) can be used to define the direction of any vector lying in the plane of the decay (6.1). The values of the CPM obtained in each case are kinematically independent; consequently, in each case independent systems (6.20) are obtained (with the exception of the case j = 1, I = −1). Thus, by specifying by means of the angles ϑ and φ the directions of the different vectors, it is possible to increase the number of independent equations (6.20), while the number of unknown T_{LM} remains the same as before.

It may turn out that for certain L all the $T_{LM} = 0$. Then the unknowns in (6.20) should be taken to be the quantities $\tau_{l\mu}^{L\widetilde{M}} = (t_{l\mu}^{LM\widetilde{M}}/T_{LM})$. The number of unknowns in (6.20) increases, but if it is still smaller than the number of equations, then j and η_x can be determined.

The T_{LM} are determined incidentally when j and η_x are determined from (6.20). It should be verified whether the T_{LM} satisfy the condition that the ρ matrix of x is non-negative[2]. In addition, formula (6.19) can be used to calculate $f_{\widetilde{m}\lambda, \widetilde{m}'\lambda'}$ and to verify whether conditions (6.15) are satisfied.

The general theory of three-particles decays was developed in[11,34,35]. Let us apply the general formulas to concrete decays.

d) Decay of a Boson Into Three Spinless Particles

In the decay j → 30, the only CPM of a_1 are $t_{00}^{LM\widetilde{M}}$, and these are determined from experiment by means of formula (6.8). Taking this circumstance into account, we write down the system (6.20) for the determination of the spin and parity of x:

$$\sum_{L=0}^{2j}{}'(2L+1)\,C_{j\widetilde{m}',\,L\widetilde{M}}^{j\widetilde{m}}\left(\frac{\langle D_{M\widetilde{M}}^{L}\rangle}{T_{LM}}\right) - I\sum_{L'=0}^{2j}{}'(2L'+1)\,C_{j-\widetilde{m}',\,L'\widetilde{M}'}^{j\widetilde{m}}\left(\frac{\langle D_{M'\widetilde{M}'}^{L'}\rangle}{T_{L'M'}}\right) = 0,$$

$$0 \leqslant \widetilde{m}' \leqslant \widetilde{m} \leqslant j, \qquad I = \eta_x\eta_1\eta_2\eta_3(-1)^j. \tag{6.22}$$

The prime at the summation sign denotes that the summation must be carried over even L and L′ or over odd ones. In (6.22) and some of the following formulas, the arguments φ, ϑ, and Φ of the D functions are omitted, but they are of course implied. According to the general properties (6.17) of the CPM, the following relations should be satisfied:

$$\langle D_{M-\widetilde{M}}^{L}(\varphi, \vartheta, \Phi)\rangle = (-1)^{L+\widetilde{M}}\langle D_{M\widetilde{M}}^{L}(\varphi, \vartheta, \Phi)\rangle. \tag{6.23}$$

If the Capps conditions are satisfied in the x production reaction and if XZ is chosen to be the plane of this reaction, then according to (6.18) the values of $\langle D_{M\widetilde{M}}^{L}(\varphi, \vartheta, \Phi)\rangle$ should be real.

By way of illustration let us consider the system (6.22) for the decay of 2 → 30, I = +1. At $\widetilde{m} = \widetilde{m}' = 0$ and I = 1, Eq. (6.22) becomes an identity. At the remaining $\widetilde{m}$ and $\widetilde{m}'$ satisfying (6.21), we obtain the following equations from (6.22):

a) for even L:

$$5\left(\langle D_{M0}^{2}\rangle - \sqrt{6}\,\langle D_{M2}^{2}\rangle\right)/T_{2M} + 12\left(\langle D_{M'0}^{4}\rangle + \sqrt{\frac{5}{2}}\,\langle D_{M'2}^{4}\rangle\right)\Big/ T_{4M'} = 0, \tag{6.24}$$

$$10\,\langle D_{M0}^{2}\rangle/T_{2M} + 3\left(\langle D_{M'0}^{4}\rangle - \sqrt{70}\,\langle D_{M'4}^{4}\rangle\right)/T_{4M'} = 0, \tag{6.25}$$

$$\sqrt{10}\,\langle D_{M1}^{2}\rangle/T_{2M} + \sqrt{3}\left(\langle D_{M'1}^{4}\rangle + \sqrt{7}\,\langle D_{M'3}^{4}\rangle\right)/T_{4M'} = \tag{6.26}$$

b) for odd L

$$\langle D_{M1}^{1}\rangle/T_{1M} + \sqrt{\frac{7}{2}}\left(\langle D_{M'1}^{3}\rangle + \sqrt{\frac{5}{3}}\,\langle D_{M'3}^{3}\rangle\right)\Big/ T_{3M'} = 0. \tag{6.27}$$

In order to confirm or reject the hypothesis j = 2, I = 1 it is necessary to find χ^2 for the system (6.24)–(6.26), by choosing the optimal ratio $T_{2M}/T_{4M'}$. The number of independent equations can be increased by using the angles ϑ and φ to specify the directions of different vectors lying in the x decay plane. It is possible to obtain in the same manner several independent equations (6.27) (with the same unknowns T_{1M} and $T_{3M'}$) and to investigate them likewise by the χ^2 method.

In[35] is given the system (6.22) for the cases j = 1, 2 and I = ±1.

An equivalent method of determining j and η_x in the j → 30 decay was developed in[36].

e) Decay of an Isobar Into a Baryon and Two Spinless Particles

In this decay, the polarization of the produced baryon is best characterized by the polarization vector **p**. We shall accordingly consider the following CPM of the baryon: $p_{X'}^{LM\widetilde{M}}$, $p_{Y'}^{LM\widetilde{M}}$, $p_{Z'}^{LM\widetilde{M}}$, which are determined by means of the formula

$$p_{X',Y',Z'}^{LM\widetilde{M}} = \int p_{X',Y',Z'}D_{M\widetilde{M}}^{L}(\varphi, \vartheta, \Phi) \times I(\varphi, \vartheta, \Phi;\ \omega_1, \omega_2)\,d\cos\vartheta\,d\varphi\,d\Phi\,d\omega_1\,d\omega_2. \tag{6.28}$$

The $p_{X'Y'Z'}^{LM\widetilde{M}}$ are connected with $t_{1\mu}^{LM\widetilde{M}}$ by formulas analogous to (3.25):

$$t_{10}^{LM\widetilde{M}} = \frac{1}{\sqrt{3}}\,p_{Z'}^{LM\widetilde{M}}, \qquad t_{1\pm1}^{LM\widetilde{M}} = \mp\frac{1}{\sqrt{6}}\left(p_{X'}^{LM\widetilde{M}} \mp i\,p_{Y'}^{LM\widetilde{M}}\right), \tag{6.29}$$

with X′Y′Z′ directed as shown in Fig. 2.

In terms of $p_{X'Y'Z'}^{LM\widetilde{M}}$, the properties (6.16) and (6.17) of the CPM have respectively the forms

$$p_{X',Y',Z'}^{LM\widetilde{M}*} = (-1)^{M+\widetilde{M}}p_{X',Y',Z'}^{L-M-\widetilde{M}}, \quad t_{00}^{LM\widetilde{M}*} = (-1)^{M+\widetilde{M}}\,t_{00}^{L-M-\widetilde{M}}, \tag{6.30}$$

$$p_{X',Z'}^{LM\widetilde{M}} = (-1)^{L+\widetilde{M}+1}p_{X',Z'}^{LM-\widetilde{M}}, \quad p_{Y'}^{LM\widetilde{M}} = (-1)^{L+\widetilde{M}}\,p_{Y'}^{LM-\widetilde{M}},$$
$$t_{00}^{LM\widetilde{M}} = (-1)^{L+\widetilde{M}}t_{00}^{LM-\widetilde{M}}. \tag{6.31}$$

These formulas were obtained with allowance for (6.29).

If the Capps condition is satisfied in the x production reaction and if X and Z lie in the plane of this reaction, then we conclude from (6.18) and (6.29) that $t_{00}^{LM\widetilde{M}}$ and $p_{Y'}^{LM\widetilde{M}}$ are real, while $p_{X'Z'}^{LM\widetilde{M}}$ are imaginary. If the baryon decays with parity nonconservation in accordance with the scheme (5.29), then we obtain from (5.31) and (6.28) a formula for the experimental determination of $p_{X'Y'Z'}^{LM\widetilde{M}}$:

$$p_{X', Y', Z'}^{LM\widetilde{M}} = (3/\alpha)\langle \nu_{X', Y', Z'} D_{M\widetilde{M}}^{L}(\varphi, \vartheta, \Phi)\rangle, \tag{6.32}$$

where α is the asymmetry coefficient in the decay (5.29) and ν is the unit vector along the momentum of N in the a_1 rest system. t_{00}^{LMM} is determined in experiment by formula (6.8), where the notation (4.17) is employed.

Substituting in (6.20) the values of $C_{1/2\lambda' l, \mu}^{1/2\lambda}$ and taking (6.29) into account, we obtain relations for the determination of j and η_X

$$\sum_{\substack{L=0 \\ \text{even}}}^{2j-1} (2L+1) C_{j\widetilde{m}', L\widetilde{M}}^{j\widetilde{m}} (t_{00}^{LM\widetilde{M}}/T_{LM}) - iI(-1)^{\widetilde{m}'+1/2} \times \sum_{\substack{L'=1 \\ \text{odd}}}^{2j} (2L'+1) C_{j-\widetilde{m}', L'\widetilde{M}'}^{j\widetilde{m}} (p_{Y'}^{L'M'\widetilde{M}'}/T_{L'M'}) = 0, \tag{6.33}$$

$$\sum_{\substack{L=0 \\ \text{even}}}^{2j-1} (2L+1) C_{j\widetilde{m}', L\widetilde{M}}^{j\widetilde{m}} (p_{Z'}^{LM\widetilde{M}}/T_{LM}) - I(-1)^{\widetilde{m}'-1/2} \times \sum_{\substack{L'=0 \\ \text{even}}}^{2j-1} (2L'+1) C_{j-\widetilde{m}', L'\widetilde{M}'}^{j\widetilde{m}} (p_{X'}^{L'M'\widetilde{M}'}/T_{L'M'}) = 0.$$

These equations are valid if the parities of L and L′ are reversed simultaneously.

By way of illustration let us consider the decay 1/2 → 1/2 + 0 + 0. From (6.33) we obtain the following relations:

$$\left.\begin{array}{ll} p_{Z'}^{1M0} + I\sqrt{2}\, p_{X'}^{1M1} = 0, & M = 0, 1; \\ p_{X'}^{1M0} - I\sqrt{2}\, p_{Z'}^{1M1} = 0, & M = 0, 1; \\ ip_{Y'}^{000} + I\sqrt{6}\,(t_{00}^{1M1}/T_{1M}) = 0, & M = 0, 1; \\ 1 - iI\sqrt{6}\,(p_{Y'}^{1M1}/T_{1M}) = 0, & M = 0, 1. \end{array}\right\} \tag{6.34}$$

According to (6.6)

$$I = (\eta_x/\eta_1\eta_2\eta_3).$$

A system of Eqs. (4.33) for the decay 3/2 → 1/2 + 0 + 0 is given in[35].

7. PRODUCTION OF A RESONANCE IN A FOUR-PARTICLE REACTION AND THE DETERMINATION OF ITS QUANTUM NUMBERS

a) Reaction Amplitudes and Their Properties

Let the resonance x be produced in the parity conserving reaction

$$b_1 + b_2 \to c + x. \tag{7.1}$$

The spins, parities, and momenta of the particles participating in the reaction (7.1) will be denoted respectively s_{b_1}, s_{b_2}, s_c, j; $\eta_{b_1}, \eta_{b_2}, \eta_c, \eta_x$; $k_{b_1}, k_{b_2}, k_c, k_x$.

The spin states of the particles will be specified by the projections m on axes lying in the plane of the reaction (7.1) in the c.m.s. (the Y axis is directed along the normal to the reaction plane).

The reaction (7.1) is described by the amplitudes

$$F_{m_{b_1}m_{b_2};\, m_c m_x}(k_{b_1}, k_{b_2}; k_c, k_x) = \langle k_c m_c, k_x m_x | \hat{T} | k_{b_1} m_{b_1}, k_{b_2} m_{b_2}\rangle. \tag{7.2}$$

By virtue of parity conservation, the transition operator $\hat{T}$ commutes with the operator of reflection in the plane of the reaction (7.1). From this we obtain relations between the amplitudes (7.2):

$$F_{m_{b_1}m_{b_2};\, m_c m_x} = \xi(-1)^{m_{b_1}+m_{b_2}-m_c-m_x} F_{-m_{b_1}-m_{b_2};\, -m_c-m_x}, \tag{7.3}$$

$$\xi = (\eta_x\eta_c/\eta_{b_1}\eta_{b_2})(-1)^{j+s_c-s_{b_1}-s_{b_2}}. \tag{7.4}$$

The proof of (7.3) is perfectly analogous to the proof of (6.5), which is given in Appendix IV.

We have omitted in (7.3) the momenta of the particles in the reaction amplitudes, since the momenta are the same in the left and right sides (the momenta are not changed by reflection in the reaction plane!)

Relations of the type (7.3), which follow from parity conservation in the reaction (7.1), were first obtained by A. Bohr[37] (except that in[37] the spin-quantization axis was chosen to be normal to the reaction plane).

With the aid of (7.3) (or the Bohr formulas) we obtain relations between the polarization effects in parity-conserving reactions[19,38], which can be used to determine the spin and parity of x.

Let us consider particular cases.

b) Use of a Spinless Target to Determine the Spin and Parity of a Boson Resonance

In the reaction of the type

$$0 + 0 \to x + 0 \tag{7.5}$$

the transition amplitudes satisfy, in accord with (7.3) and (7.4), the relation

$$F_m = \xi(-1)^m F_{-m}, \tag{7.6}$$

where ξ is equal to the product of the parities of the particles taking part in the reaction (7.5), multiplied by $(-1)^j$; m is the projection of the spin of x on an axis lying in the plane of the reaction (7.5).

The density matrix of x is obviously

$$\rho_{mm'} = F_m F_{m'}^{*}/(\sum_{m=-j}^{j} |F_m|^2). \tag{7.7}$$

From (7.6) and (7.7) we obtain

$$\rho_{mm'} = \xi(-1)^{m'} \rho_{m-m'}. \tag{7.8}$$

Substituting (3.15) in (7.8) we obtain a relation that can be used to determine j and η_X[39,40]:

$$\sum_{L=0}^{2j}{}' (2L+1) C_{jm', LM}^{jm} T_{LM} = \xi(-1)^{m'} \sum_{L'=0}^{2j}{}' (2L'+1) C_{j-m', L'M'}^{jm} T_{L'M'}. \tag{7.9}$$

The prime at the summation sign denotes that the summation can be carried out only over even L and L′ or else only over odd ones (this follows from (3.24))*.

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*We note that relations (7.9) remain in force also if the boson resonance is produced in the reaction Π + p → x + p (Π is a spinless particle at high energies and if the main contribution to the amplitude of this reaction is made by poles in the j plane of only the vacuum group (P, P′, ρ, R, ω, φ–poles [61]), with $\xi = \eta_\Pi \eta_X (-1)^j$.

Equations (7.9) are obviously valid when the T_{LM} are averaged over any region of the x-production angles in the reaction (7.5), so that to determine j and η_x we can use the entire statistics in the reaction (7.5).

By virtue of (3.16), the T_{LM} with non-negative M are independent. Therefore relations (7.9) are independent when

$$0 \leqslant m' \leqslant m \leqslant j. \tag{7.10}$$

If x decays in accordance with the scheme $x \to 20$ or $x \to \gamma + 0$ or else $x \to 1 + 0$, then, substituting in (7.9) the formulas (5.1) or (5.8) or (5.18) and (5.19), respectively, for the experimental determination of T_{LM}, we obtain the relations between the observed quantities. Specifying different values of j and η_x we can obtain the most probable j and η_x from these relations by the χ^2 method.

If x decays into three spinless particles, then the addition of relations (7.9) to the system (6.22) increases the number of equations without changing the number of the unknown T_{LM}. Therefore the reliability with which j and η_x are determined increases.

If we eliminate T_{LM} from (7.9) and (6.22), then we obtain, generally speaking, nonlinear relations between the observed quantities $\langle D^L_{M\tilde{M}}(\varphi, \vartheta, \Phi)\rangle$. However, linear relations will also hold (at least up to j = 4) (for linear relations at j = 1, 2 see[40]). By way of illustration, we present one of the linear relations for j = 2, $I = -1$, $\xi = -1$*

$$\langle D^2_{00}(\varphi, \vartheta, \Phi)\rangle + \sqrt{6}\,(\langle D^2_{02}(\varphi, \vartheta, \Phi)\rangle + \langle D^2_{20}(\varphi, \vartheta, \Phi)\rangle) + 6\,\langle D^2_{22}(\varphi, \vartheta, \Phi)\rangle = \frac{2}{7}. \tag{7.11}$$

If x is produced in the reaction (7.5) and decays into two spinless particles, then, as shown by Peshkin[41]

$$\langle P_{2j}(\cos\Theta)\rangle \neq 0, \tag{7.12}$$

where P_{2j} is a Legendre polynomial and θ is the angle between the normal to the plane of the reaction (5.5) and the direction of the momentum of one of the decay products of x (in the x rest system).

Using Peshkin's idea, Ryndin has shown[42] that (7.12) is valid also in the case when x, produced in the reaction (7.5) decays into three spinless particles. In this case Θ is the angle between the normals to the plane of the reaction (7.5) and to the decay plane of x (in the rest system of x). Table IV gives the limits obtained in[42] for $\langle P_{2j}(\cos\Theta)\rangle$ at j = 0, 1, 2, 3 (I_R and I_D denote the products of the parities of the particles in the reaction (7.5) and in the decay (6.1) respectively).

Table IV

j \ I_R, I_D	$I_R = I_D = 1$ ($I_R I_D = 1$)	$I_R = I_D = -1$ ($I_R I_D$) $= 1$	$I_R I_D = 1$
0	$\langle P_0\rangle = 1$	Forbidden	Forbidden
1	$\langle P_2\rangle = 2/5$	$\langle P_2\rangle = 1/10$	$\langle P_2\rangle = -1/5$
2	$1/126 \leqslant \langle P_4\rangle \leqslant 36/126$	$\langle P_4\rangle = 8/63$	$-12/63 \leqslant \langle P_4\rangle \leqslant -2/63$
3	$9/429 \leqslant \langle P_6\rangle \leqslant 100/429$	$1/1716 \leqslant \langle P_6\rangle \leqslant 225/1716$	$-50/286 \leqslant \langle P_6\rangle \leqslant -1/286$

The procedure for determining j consists simply of determining the highest $\langle P_L(\cos\Theta)\rangle$; then $j = L_{max}/2$. Moreover, if $\langle P_{2j}(\cos\Theta)\rangle$ lies between the upper or between the lower limits for the cases $I_R = I_D = 1$ and $I_R = I_D = -1$, then it is possible to determine also the parity of x. It is necessary to bear in mind, however, that the lower limit of $|\langle P_{2j}(\cos\Theta)\rangle|$ is quite small and can be the result of the background. Therefore the most reliable method of determining j and η_x is to solve the systems (7.9) and (6.22).

c) Use of a Polarized Proton Target to Determine the Spin and Parity of an Isobar

If the isobar decays into a stable baryon (nucleon) and a spinless particle or into a stable baryon and two spinless particles, then j and η_x cannot be determined with the aid of relations (5.27) and (6.33), since the CPM of a stable baryon, $t^{LM}_{1\mu}$ and $p^{LM\tilde{M}}_{X'Y'Z''}$, cannot be determined from experiment.

We shall show that the spin and parity of the isobar can be determined also without the aforementioned CPM, provided the isobar is produced in a reaction with a polarized proton target[43,44]

$$0 + p \to 0 + x. \tag{7.13}$$

This question deserves attention, since the method of polarizing proton targets has been extensively used recently.

The polarization of the proton and of the isobar will be specified in their rest systems relative to the axes $X_1Y_1Z_1$ and XYZ respectively, with the coordinate axes directed as follows:

$$Z \parallel \mathbf{k}_x, \quad Y \parallel Y_1 \parallel [\mathbf{k}_0, \mathbf{k}_x],\ Z_1 \parallel \mathbf{k}_0, \tag{7.14}$$

where $\mathbf{k}_x$ and $\mathbf{k}_0$ are the l.s. momenta of the isobar and of the incoming particle in the reaction (7.13). With such a choice of axes, the amplitudes of the reaction (7.13) satisfy, in accord with (7.3) and (7.4), the relation

$$F_{\nu, m} = \xi(-1)^{\nu - m} F_{-\nu, -m}, \tag{7.15}$$

$$\xi = (\eta_x/\eta_p)\, I'\, (-1)^{j-1/2}, \tag{7.16}$$

where I′ is the product of the parities of the spinless particles taking part in the reaction (7.13); $m(\nu)$ is the projection of the isobar (proton) spin on the axis Z (Z_1).

The isobar density matrix is expressed in terms of the amplitudes of the reaction (7.13) in accordance with the formula

$$\mathcal{F}\rho_{mm'} = \sum_{\nu, \nu' = -1/2}^{1/2} F_{\nu, m} F^*_{\nu', m'} \cdot \frac{1}{2}(1 + \mathbf{p}\sigma)_{\nu\nu'}, \tag{7.17}$$

where $\sigma_{\nu\nu'}$ are Pauli matrices, $\mathbf{p}$ is the polarization vector of the proton target, so that $(1 + \mathbf{p}\cdot\boldsymbol{\sigma})_{\nu\nu'}/2$ is its density matrix, and $\mathcal{F}$ is the square of the matrix element summed over the spin states. Accurate to the phase volume, $\mathcal{F}$ is equal to the differential cross section of the reaction (7.13).

Assume that we investigate the production of an isobar in reaction (7.13) in the same plane and at equal

*In (7.11), just as in (6.22) and (7.9), the angles φ, ϑ, Φ should be chosen as shown in Fig. 2; the Y axis is directed along the normal to the plane of the reaction (7.5).

angles on opposite sides of the incident beam. It follows from (7.14) that $p_{X_1} > 0$ and $p_{X_1} < 0$ when the isobar is scattered to one side and to the other, respectively. Accordingly, we shall call positive the side for which $p_{X_1} > 0$ and negative the one for which $p_{X_1} < 0$.

We denote by $\mathcal{F}^{\pm}$ and $T^{\pm}_{LM}$ the probability of the transition and the PM of the isobar for scattering in the positive or negative direction respectively (at the same angle and in the same plane).

Substituting (3.15) in (7.17) we obtain, taking (7.15) and (I.4f) into account

$$\frac{1}{|p_{X_1}|}\sum_{\text{чет } L=0}^{2j-1}(2L+1)\,C^{jm}_{jm',\,LM}(\mathcal{F}^{+}\operatorname{Im}T^{+}_{LM}-\mathcal{F}^{-}\operatorname{Im}T^{-}_{LM})$$
$$=\xi(-1)^{m'+1/2}(1/p_{Z_1})\sum_{\text{чет } L'=0}^{2j-1}(2L'+1)\,C^{jm}_{j-m',\,L'M'}(\mathcal{F}^{+}\operatorname{Im}T^{+}_{L'M'}+\mathcal{F}^{-}\operatorname{Im}T^{-}_{L'M'})$$
$$=(2j+1)\operatorname{Im}(F_{1/2,\,m}F^{*}_{-1/2,\,m'}+F_{-1/2,\,m}F^{*}_{1/2,\,m'}). \tag{7.18}$$

If the target is polarized then, according to (3.24), $\operatorname{Im} T_{LM} = 0$ for even L.

By virtue of (3.16), the T_{LM} with non-negative M are independent, and therefore (7.18) are independent if

$$0 < m' < m \leqslant j. \tag{7.19}$$

From this condition we can easily calculate that the number of independent relations (7.18) is $(j^2 - 1/4)$.

The T_{LM} with even L can be determined experimentally by means of formula (5.34) without measuring the polarization of the baryon produced as a result of the isobar decay. Therefore relations (7.18) can be used to determine the spin and parity of an isobar decaying into a stable baryon and a spinless particle.

Substituting (5.34) in (7.18) we obtain

$$p_{Z_1}\sum_{\text{чет } L=0}^{2j-1}(2L+1)\,(C^{jm}_{jm',\,LM}/C^{j1/2}_{j1/2,\,L0})\operatorname{Im}\{\sum_{i=1}^{n^+}D^{L}_{M0}(\varphi_i,\vartheta_i,0)$$
$$-\sum_{j=1}^{n^-}D^{L}_{M0}(\varphi_j,\vartheta_j,0)\}=\xi(-1)^{m'+1/2}\sum_{\text{чет } L'=0}^{2j-1}(2L'+1)\,(C^{jm}_{j-m',\,L'M'}/C^{j1/2}_{j1/2,\,L0})$$
$$\times\operatorname{Im}\{\sum_{i=1}^{n^+}|p_{X_1}|_i D^{L'}_{M'0}(\varphi_i,\vartheta_i,0)+\sum_{j=1}^{n^-}|p_{X_1}|_j D^{L'}_{M'0}(\varphi_j,\vartheta_j,0)\}, \tag{7.20}$$

where the index i is used to renumber the cases of isobar production in the positive side, and j for the negative side.

Since relations (7.18) remain in force if their average over any region of isobar production angles in the reaction (7.13), the summation in (7.20) can be carried out likewise over any region of isobar production angles. We recall that the angles ϑ and φ specify the direction of the baryon momentum in the rest system of x relative to the axis XYZ (see Fig. 1). The axes XYZ and $X_1Y_1Z_1$ are chosen in accordance with (7.14).

In (7.20), account was taken of the fact that p_{Z_1} is the same for all cases of isobar production.

By specifying different values of j and η_x we can determine, from the $(j^2 - 1/4)$ relations (7.20), the most probable values of j and η_x by the χ^2 method.

We present an example. If j = 3/2, then two relations (7.20) are obtained

$$p_{Z_1}\operatorname{Im}\{\sum_{i=1}^{n^+}D^{2}_{10}(\varphi_i,\vartheta_i,0)-\sum_{j=1}^{n^-}D^{2}_{10}(\varphi_j,\vartheta_j,0)\}$$
$$=\xi\operatorname{Im}\{\sum_{i=1}^{n^+}|p_{X_1}|_i D^{2}_{20}(\varphi_i,\vartheta_i,0)+\sum_{j=1}^{n^-}|p_{X_1}|_j D^{2}_{20}(\varphi_j,\vartheta_j,0)\},$$

$$p_{Z_1}\operatorname{Im}\{\sum_{i=1}^{n^+}D^{2}_{20}(\varphi_i,\vartheta_i,0)-\sum_{j=1}^{n^-}D^{2}_{20}(\varphi_j,\vartheta_j,0)\}$$
$$=-\xi\operatorname{Im}\{\sum_{i=1}^{n^+}|p_{X_1}|_i D^{2}_{10}(\varphi_i,\vartheta_i,0)+\sum_{j=1}^{n^-}|p_{X_1}|_j D^{2}_{10}(\varphi_j,\vartheta_j,0)\}. \tag{7.21}$$

If x decays into a baryon and two spinless particles, then according to (6.10) and (6.8) we have

$$T_{LM}=A_L\langle D^{L}_{M0}(\varphi,\vartheta,\Phi)\rangle, \tag{7.22}$$

where

$$A_L=(\sum_{\nu=-1/2}^{1/2}\sum_{\tilde{m}=-j}^{j}C^{j\tilde{m}}_{j\tilde{m},\,L0}\tilde{f}_{\tilde{m}\nu,\,\tilde{m}\nu})^{-1} \tag{7.23}$$

are determined by the dynamics of the decay.

Substituting (7.22) in (7.18), we obtain $(j^2 - 1/4)$ linear homogeneous equations with respect to $(j - 1/2)$ unknown A_L (L even, L ≠ 0). These equations differ from (7.20) in that $(C^{j\,1/2}_{j\,1/2,L0})^{-1}$ and $D^{L}_{M0}(\varphi, \vartheta, 0)$ are replaced by A_L and $D^{L}_{M0}(\varphi, \vartheta, \Phi)$ (we recall that the angles φ, ϑ, and Φ are defined in the rest system of x, as shown in Fig. 2). Choosing A_L and satisfying in the best manner the obtained equations, it is possible to determine the most probable values of j and η_x by the χ^2 method. For j = 3/2 the relations (7.21) hold, with the substitution $D^{2}_{M0}(\varphi, \vartheta, 0) \to D^{2}_{M0}(\varphi, \vartheta, \Phi)$ (the A_2 cancel out).

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APPENDICES

I. CLEBSCH-GORDAN COEFFICIENTS

The Clebsch-Gordan coefficients (CGC) $C^{jm}_{j_1m_1,j_2m_2}$ are produced upon quantum-mechanical addition of the angular momenta. They differ from zero if

$$m_1+m_2=m,\qquad |j_1-j_2|\leqslant j\leqslant j_1+j_2, \tag{I.1}$$

which agrees with the well known rule for the addition of angular momenta.

The CGC satisfy the orthogonality relations

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}C^{jm}_{j_1m_1,\,j_2m_2}C^{j'm'}_{j_1m_1,\,j_2m_2}=\delta_{jj'}\delta_{mm'}, \tag{I.2}$$
$$\sum_{j=|j_1-j_2|}^{j_1+j_2}\sum_{m=-j}^{j}C^{jm}_{j_1m_1,\,j_2m_2}C^{jm}_{j_1m_1',\,j_2m_2'}=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{I.3}$$

and the symmetry relations

$$C^{jm}_{j_1m_1,\,j_2m_2}=(-1)^{j-j_1-m_2}\left[\frac{2j+1}{2j_1+1}\right]^{1/2}C^{j_1-m_1}_{j_2m_2,\,j-m} \tag{I.4a}$$
$$=(-1)^{j-j_2+m_1}\left[\frac{2j+1}{2j_2+1}\right]^{1/2}C^{j_2-m_2}_{j-m,\,j_1m_1} \tag{I.4b}$$

$$=(-1)^{j_1+j_2-j}\, C^{jm}_{j_2m_2,\, j_1m_1} \tag{I.4c}$$

$$=(-1)^{j_1-m_1}\left[\frac{2j+1}{2j_2+1}\right]^{1/2} C^{j_2-m_2}_{j_1m_1,\, j-m} \tag{I.4d}$$

$$=(-1)^{j_2+m_2}\left[\frac{2j+1}{2j_1+1}\right]^{1/2} C^{j_1-m_1}_{j-m,\, j_2m_2} \tag{I.4e}$$

$$=(-1)^{j_1+j_2-j}\, C^{j-m}_{j_1-m_1,\, j_2-m_2}. \tag{I.4f}$$

We present some particular values of the CGC:

$$C^{jm}_{jm,\, 00}=1, \tag{I.5}$$

$$C^{j0}_{j0,\, L0}=\frac{1+(-1)^L}{2}(-1)^{L/2}\left[\frac{(2j-L)!\,(2j+1)}{(2j+L+1)!}\right]^{1/2}\frac{L!\,(j+L/2)!}{(j-L/2)!\left(\frac{L}{2}!\right)^2}, \tag{I.6}$$

$$C^{jm}_{jm,\, 10}=-C^{j-m}_{j-m,\, 10}=\frac{m}{\sqrt{j(j+1)}}, \tag{I.7}$$

$$C^{jm}_{jm,\, 20}=C^{j-m}_{j-m,\, 20}=\frac{3m^2-j(j+1)}{\sqrt{j(j+1)(2j-1)(2j+3)}}, \tag{I.8}$$

$$C^{j1/2}_{j1/2,\, L0}=2(-1)^{\frac{L-1}{2}}\left[\frac{(2j-L)!}{(2j+1)(2j+L+1)!}\right]^{1/2}\frac{L!\left(j+\frac{L}{2}\right)!}{\left(j-\frac{L}{2}\right)!\left[\frac{(L-1)}{2}!\right]^2} \tag{I.9}$$

(in (I.9), L are odd),

$$C^{j1/2}_{j1/2,\, L0}=2(-1)^{\frac{L}{2}}\left[\frac{(2j-L)!}{(2j+1)(2j+L+1)!}\right]^{1/2}\frac{L!\left(j+\frac{L+1}{2}\right)!}{\left(j-\frac{L+1}{2}\right)!\left(\frac{L}{2}!\right)^2} \tag{I.10}$$

(in I.10, L are even).

In this review we use the following relations between the CGC:

$$C^{j3/2}_{j3/2,\, L0}=\frac{1}{(j+3/2)(j-1/2)}\{(j+3/2)(j-1/2)-L(L+1)$$
$$+(j+1/2)^2[1+(-1)^{L+1}]\}\, C^{j1/2}_{j1/2,\, L0}, \tag{I.11}$$

$$C^{j1/2}_{j-1/2,\, L1}=-\frac{[1+(-1)^{L+1}]}{2\sqrt{L(L+1)}}(2j+1)\, C^{j1/2}_{j1/2,\, L0}, \tag{I.12}$$

$$G^{j3/2}_{j1/2,\, L1}=\frac{L(L+1)-(j+1/2)^2[1+(-1)^{L+1}]}{\sqrt{(j+3/2)(j-1/2)L(L+1)}}\, C^{j1/2}_{j1/2,\, L0}, \tag{I.13}$$

$$C^{j3/2}_{j-1/2,\, L2}=\frac{(2j+1)[1+(-1)^{L+1}-L(L+1)]}{2\sqrt{(j+3/2)(j-1/2)(L-1)L(L+1)(L+2)}}\, C^{j1/2}_{j1/2,\, L0}, \tag{I.14}$$

$$C^{j3/2}_{j-3/2,\, L3}=\frac{[1+(-1)^{L+1}](j+1/2)(L-1)(L+2)}{\sqrt{(L-2)(L-1)L(L+1)(L+2)(L+3)}}\, C^{j1/2}_{j1/2,\, L0}, \tag{I.15}$$

$$C^{j1}_{j1,\, L0}=\left[1-\frac{L(L+1)}{2j(j+1)}\right] C^{j0}_{j0,\, L0}, \tag{I.16}$$

$$G^{j1}_{j-1,\, L2}=-\left[\frac{L(L+1)}{(L-1)(L+2)}\right]^{1/2} C^{j0}_{j0,\, L0}, \tag{I.17}$$

$$C^{j1}_{j0,\, L1}=\frac{1}{2}\left[\frac{L(L+1)}{j(j+1)}\right]^{1/2} C^{j0}_{j0,\, L0}. \tag{I.18}$$

(in (I.16)–(I.18), L are even.)

A description of the existing tables of CGC is given in the book[7], pp. 26–28. In the book[8] are given tables of the CGC for $j_1 + j_2 + j_3 \leq 16$, $j_{max} \leq 6$, and also algebraic formulas for specified values of j_2 and m_2.

II. WIGNER FUNCTIONS

The D functions $D^j_{mn}(\varphi, \vartheta, \Phi)$ are produced when spherical functions (spherical spinors) are rotated through the Euler angles

$$D^j_{mn}(\varphi, \vartheta, \Phi)=e^{-im\varphi}d^j_{mn}(\vartheta)\, e^{-in\Phi}. \tag{II.1}$$

D functions with integer j are connected with the spherical functions by means of the formula

$$Y_{jm}(\vartheta, \varphi)=\sqrt{\frac{2j+1}{4\pi}}\, D^{j*}_{m0}(\varphi, \vartheta, 0), \tag{II.2}$$

so that, for example,

$$P_j(\cos\vartheta)=d^j_{00}(\vartheta). \tag{II.3}$$

The D functions satisfy the symmetry relations

$$D^j_{mn}(\varphi, \vartheta, \Phi)=(-1)^{m-n}D^{j*}_{-m-n}(\varphi, \vartheta, \Phi)=D^{j*}_{-n-m}(\Phi, \vartheta, \varphi), \tag{II.4}$$

$$d^j_{mn}(\vartheta)=(-1)^{j+m}d^j_{m-n}(\pi-\vartheta), \tag{II.4'}$$

$$d^j_{mn}(0)=(-1)^{j+m}d^j_{m-n}(\pi)=\delta_{mn}, \tag{II.4''}$$

and also the orthogonality relations

$$\int D^j_{mn}(\varphi, \vartheta, \Phi)\, D^{j'*}_{m'n'}(\varphi, \vartheta, \Phi)\, d\cos\vartheta\, d\varphi\, d\Phi=\frac{8\pi^2}{2j+1}\delta_{jj'}\delta_{mm'}\delta_{nn'}, \tag{II.5}$$

$$\int D^j_{mn}(\varphi, \vartheta, 0)\, D^{j'*}_{m'n}(\varphi, \vartheta, 0)\, d\cos\vartheta\, d\varphi=\frac{4\pi}{2j+1}\delta_{jj'}\delta_{mm'}. \tag{II.6}$$

There is a formula for the expansion of a product of two D functions,

$$D^j_{mn}(\varphi, \vartheta, \Phi)\, D^{j'}_{m'n'}(\varphi, \vartheta, \Phi)=\sum_{L=|j-j'|}^{j+j'} C^{Lm+m'}_{jm,\, j'm'}C^{Ln+n'}_{jn,\, j'n'}D^L_{m+m',\, n+n'}(\varphi, \vartheta, \Phi). \tag{II.7}$$

From (II.2), (II.4), and (II.7) we get

$$Y_{jm}(\vartheta, \varphi)\, Y^*_{jm'}(\vartheta, \varphi)=\sum_{L=0}^{2j}\sqrt{\frac{2L+1}{4\pi}}\, C^{j0}_{j0,\, L0}C^{jm}_{jm',\, LM}Y_{LM}(\vartheta,\varphi), \tag{II.7'}$$

$$D^j_{mn}(\varphi, \vartheta, \Phi)\equiv 0, \quad \text{если} \quad |m|>j, \quad \text{либо} \quad |n|>j. \tag{II.8}$$

III. REDUCTION OF EXPERIMENTAL DATA BY THE χ^2 METHOD

In determining the spin and parity of a resonant state by the χ^2 method, it is necessary first to find the lower limit of j. From the condition (4.23) we conclude that for Fermion resonance

$$j \geqslant L^{odd}_{max}/2, \qquad j \geqslant (L^{even}_{max}+1)/2, \tag{III.1}$$

and for boson resonance

$$j \geqslant L^{even}_{max}/2, \qquad j \geqslant (L^{odd}_{max}+1)/2, \tag{III.1'}$$

where L^{odd}_{max} and L^{even}_{max} are the maximum values of L for which there are nonzero CPM.

In order to find L_{max}, we number for each fixed value of L the independent CPM (since the real and imaginary parts of the CPM are independent, we assign to them different numbers) and denote them by t_α. We find the error matrix by means of the formula

$$U_{\alpha\alpha'}=\frac{1}{n^2}\sum_{i=1}^{n}\{t_{\alpha_i}-\langle t_\alpha\rangle\}\{t_{\alpha'_i}-\langle t_{\alpha'}\rangle\}, \tag{III.2}$$

where t_{α_i} is the value of t_α in the i-th case of the decay, and n is the number of decay cases:

$$\langle t_\alpha\rangle=\frac{1}{n}\sum_{i=1}^{n}t_{\alpha_i}. \tag{III.2'}$$

Calculating the inverse matrix $U^{-1}_{\alpha\alpha'}$, we find χ^2 by means of the formula

$$\chi^2=\sum_{\alpha,\,\alpha'=1}^{m}\langle t_\alpha\rangle\, u^{-1}_{\alpha\alpha'}\langle t_{\alpha'}\rangle, \tag{III.3}$$

where m is the rank of the error matrix, equal to the number of independent CPM determined in the experiment.

For the obtained value of χ^2 and the number of degrees of freedom we determine, using the tables of the probability integral (see, for example,[48]), the probability that the deviation of $\langle t_\alpha \rangle$ from zero (at a given L) is equal to or larger than the value found in our experiment. The maximum value of L for which this probability turns out to be small must be substituted in (III.1) and (III.1′).

It is now necessary to establish the value of j and η_x for which the experimental data satisfy best the relations (4.29) or (6.20), the only admissible values of j being those satisfying (III.1) and (III.1′).

For fixed value of j and η_x, one calculates the probability of satisfying the hypothesis (4.29) or (6.20). The procedure for calculating this probability coincides fully with the one just described.

More details on the χ^2 method can be found in[68,69].

By way of an example, Table V presents the results obtained by the Shlein group[30], who investigated the decay $\Xi^*(1530) \to \Xi + \pi$.

Table V

Hypothesis	χ^2	Number of degrees of freedom	Probability	Conclusion
$t_0^{2M}=0$	16.5	3	0.0003	
$t_{1\mu}^{3M}=0$	12.0	6	0.036	
$t_0^{2M}=0,\ t_{1\mu}^{3M}=0$	29.2	9	0.0002	
$t_0^{4M}=0$	7.6	5	0.11	$j_{\min}=3/2$
$t_{1\mu}^{5M}=0$	7.6	10	0.58	
$t_0^{4M}=0,\ t_{1\mu}^{5M}=0$	18.2	15	0.20	
$j=\frac{3}{2}$: $(2j+1)\,t_{10}^{LM}-\sqrt{2L(L+1)}\,t_{11}^{LM}$	10.3	4	0.016	$D_{3/2}$ unlikely
$j=\frac{3}{2}$: $(2j+1)\,t_{10}^{LM}+\sqrt{2L(L+1)}\,t_{11}^{LM}$	1.5	4	0.65	$P_{3/2}$ likely
$j=\frac{5}{2}$: $(2j+1)\,t_{10}^{LM}-\sqrt{2L(L+1)}\,t_{11}^{LM}$	9.5	4	0.023	$F_{5/2}$ unlikely
$j=\frac{5}{2}$: $(2j+1)\,t_{10}^{LM}+\sqrt{2L(L+1)}\,t_{11}^{LM}$	0.9	4	0.84	$D_{5/2}$ likely

It is seen from this table that the parity is determined more definitely than the spin.

IV. AMPLITUDES OF TWO-PARTICLE AND THREE-PARTICLE DECAY AND THEIR PROPERTIES

Assume that in the parity-conserving decay (6.1) the spins of the particles a_1, a_2, and a_3 are respectively s_1, s_2, and s_3.

The state of the produced particles can be characterized in the rest system of x by the following quantities: the angles ϑ and φ, which determine the direction of the momentum of one of the particles (for example, a_2), the angle Φ, which specifies the position of the plane of the decay (6.1) (see Fig. 2), the energies ω_1 and ω_2 (ω_3 is determined by the energy-conservation law), and also the helicities λ_1, λ_2 and λ_3, which characterize the spin state of the particles a_1, a_2, and a_3.

If x is in a state with a spin projection on the Z axis equal to m, then the wave function of the products of the decay (6.1) is obviously

$$\Psi_{m\lambda_1\lambda_2\lambda_3}(\varphi, \vartheta, \Phi; \omega_1, \omega_2)=\langle \varphi, \vartheta, \Phi; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3 | \hat{T} | jm\rangle, \tag{IV.1}$$

where $\hat{T}$ is the operator of the transition (6.1).

We denote by $\hat{R}(\alpha, \beta, \gamma)$ the operator of rotation through the Euler angles α, β, and γ and carry out certain transformations in the right-hand side of (IV.1):

$$\begin{aligned}\langle \varphi, \vartheta, \Phi; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3 | \hat{T} | jm\rangle &= (\langle jm | \hat{T}^+ | \varphi, \vartheta, \Phi; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle)^* \\ &= (\langle jm | \hat{T}^+\hat{R}(\varphi, \vartheta, \Phi) | 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle)^* \\ &= (\langle jm | \hat{R}(\varphi, \vartheta, \Phi)\hat{T}^+ | 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle)^*. \end{aligned} \tag{IV.2}$$

It can be readily seen from Fig. 2 that $|0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle$ is such a state of the particles a_1, a_2, and a_3, wherein the momentum $\mathbf{k}_2$ is directed along the Z axis, and the plane of the decay (6.1) coincides with XZ. It is obvious that

$$\hat{R}(\varphi, \vartheta, \Phi) | 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle = | \varphi, \vartheta, \Phi; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle.$$

Since $|j\tilde{m}\rangle$ $(-j \le \tilde{m} \le j)$ is the total set of the states of the particle x, it follows that (IV.2) can be further transformed:

$$\begin{aligned}(\langle jm | \hat{R}(\varphi, \vartheta, \Phi)\hat{T}^+ | 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle)^* &= \sum_{\tilde{m}=-j}^{j} (\langle jm | \hat{R}(\varphi, \vartheta, \Phi) | j\tilde{m}\rangle)^* (\langle j\tilde{m} | \hat{T}^+ | 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle)^* \\ &= \sum_{\tilde{m}=-j}^{j} \langle 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3 | \hat{T} | j\tilde{m}\rangle (\langle jm | \hat{R}(\varphi, \vartheta, \Phi) | j\tilde{m}\rangle)^*. \end{aligned} \tag{IV.3}$$

As is well known, the transformation matrix of the states $|jm\rangle$ upon rotation is made up of D functions, so that

$$\langle jm | \hat{R}(\varphi, \vartheta, \Phi) | j\tilde{m}\rangle = D^j_{m\tilde{m}}(\varphi, \vartheta, \Phi). \tag{IV.4}$$

From (IV.1)–(IV.4) it follows that

$$\Psi_{m\lambda_1\lambda_2\lambda_3}(\varphi, \vartheta, \Phi; \omega_1, \omega_2) = (\sqrt{j+1/2}/2\pi) \sum_{\tilde{m}=-j}^{j} f_{\tilde{m}\lambda_1\lambda_2\lambda_3}(\omega_1, \omega_2)\, D^{j*}_{m\tilde{m}}(\varphi, \vartheta, \Phi), \tag{IV.5}$$

where

$$f_{\tilde{m}\lambda_1\lambda_2\lambda_3}(\omega_1, \omega_2) = \frac{2\pi}{\sqrt{j+1/2}} \langle 0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3 | \hat{T} | j\tilde{m}\rangle. \tag{IV.6}$$

From parity conservation in the decay (6.1) we obtain relations between the decay amplitudes

$$\begin{gathered} f_{\tilde{m}\lambda_1\lambda_2\lambda_3}(\omega_1, \omega_2) = I(-1)^{\lambda_1+\lambda_2+\lambda_3-\tilde{m}} f_{-\tilde{m}-\lambda_1-\lambda_2-\lambda_3}(\omega_1, \omega_2), \\ I = (\eta_x/\eta_1\eta_2\eta_3)(-1)^{j-s_1-s_2-s_3}. \end{gathered} \tag{IV.7}$$

To prove (IV.7), let us consider the operator

$$\hat{Y} = \hat{P}\hat{R}(0, \pi, 0), \tag{IV.8}$$

where $\hat{P}$ is the inversion operator.

$\hat{R}(0, \pi, 0)$ is obviously the operator of rotation through an angle π about the Y axis. It is therefore clear from (IV.8) that $\hat{Y}$ is the operator of reflection in the XZ plane.

In the parity-conserving decay, $\hat{T}$ and $\hat{Y}$ commute, so that

$$\hat{T} = \hat{Y}\hat{T}\hat{Y}^{-1}. \tag{IV.9}$$

We substitute (IV.9) in (IV.6) and recognize that $\hat{Y}$, acting on the states $|j\tilde{m}\rangle$, $|0, 0, 0; \omega_1, \omega_2; \lambda_1, \lambda_2, \lambda_3\rangle$, does not change the momenta of the particles and that the spin states of the particles in accordance with (IV.4) are altered in the following manner:

$$\begin{aligned}\hat{Y} | s, \lambda\rangle = \hat{P}\hat{R}(0, \pi, 0) | s, \lambda\rangle &= \eta \sum_{\lambda'=-s}^{s} D^s_{\lambda', \lambda}(0, \pi, 0) | s, \lambda'\rangle \\ &= \eta \sum_{\lambda'=-s}^{s} (-1)^{s-\lambda}\delta_{\lambda', -\lambda} | s, \lambda'\rangle = \eta(-1)^{s-\lambda} | s, -\lambda\rangle. \end{aligned} \tag{IV.10}$$

After the indicated substitution we obtained (IV.7), q.e.d.

We note that in the decay (6.1) the angles φ, ϑ, and Φ can be defined in a manner different from than that in the present review. The wave function of the products of the decay will have, as before, the form (IV.5). The new decay amplitudes will be linear combinations of the old ones. The relations that follow from the parity conservation will, of course, differ from (IV.7) (for more details see[11,26]).

In the two-particle decay (4.1), the state of the particles a_1 and a_2 (the spins of which are equal to s_1 and s_2) can be characterized by the angles ϑ and φ, which specify the direction of the momentum of one of the particles (in the rest system of x), and the helicities λ_1 and λ_2.

If x is in a state with a spin projection on the Z axis equal to m, then the wave function of the decay products (4.1) is equal to

$$\Psi_{m\lambda_1\lambda_2}(\vartheta,\varphi)=[(2j+1)/4\pi]^{1/2}A_{\lambda_1\lambda_2}D^{j*}_{m,\,\lambda_1-\lambda_2}(\varphi,\vartheta,0), \qquad \text{(IV.11)}$$

where

$$A_{\lambda_1\lambda_2}=[4\pi/(2j+1)]^{1/2}\langle 0,0;\lambda_1,\lambda_2|\hat{T}|j,\lambda_1-\lambda_2\rangle. \qquad \text{(IV.12)}$$

(IV.11) and (IV.12) can be obtained in exactly the same manner as (IV.5) and (IV.6), if it is recognized that

$$|\varphi,\vartheta;\lambda_1,\lambda_2\rangle=\hat{R}(\varphi,\vartheta,0)|0,0;\lambda_1,\lambda_2\rangle, \qquad \text{(IV.13)}$$

and that the decay of x along the Z axis is possible only if $m = \lambda_1 - \lambda_2$.

If parity is conserved in the decay (4.1), then the amplitudes of the decay $A_{\lambda_1\lambda_2}$ satisfy the relations

$$A_{\lambda_1\lambda_2}=\sigma A_{-\lambda_1-\lambda_2},\quad \sigma=(\eta_1\eta_2/\eta_x)(-1)^{j-s_1-s_2}. \qquad \text{(IV.14)}$$

(IV.14) can be proved in exactly the same manner as (IV.7).

SYMBOLS AND DEFINITIONS USED IN THE REVIEW

1. Symbols

x—resonance whose quantum numbers are to be determined;

j, η_x—spin and parity of the resonance x;

a_k—(k = 1, 2 or k = 1, 2, 3)—products of the decay of the resonance x;

s—spin of particle a_1;

$\rho_{mm'}$—spin density matrix of x;

$\rho_{\lambda\lambda'}$—spin density matrix of a_1;

T_{LM}—polarization moments (PM) of x;

$t_{l\mu}$—polarization moments (PM) of a_1;

$D^L_{MM'}(\alpha,\beta,\gamma)$—Wigner functions;

A_λ—helicity amplitudes of two-particle decay of x (4.1) if particle a_2 is spinless;

$\sigma=(\eta_x/\eta_1\eta_2(-1)^{j-s}$, where η_1 and η_2 are the parities of the particles in the decay (4.1);

$t^{LM}_{l\mu}$—cascade polarization moments (CPM) of the particle a_1 in the decay (4.1);

$\langle t_\alpha\rangle=\frac{1}{n}\sum_{i=1}^{n}t_{\alpha_i}$ with summation carried out over all cases of the investigated decay, t_{α_i}—value of t_α in the i-th case, n—total number of decay cases;

N—baryon;

Π—pseudoscalar meson;

α—asymmetry coefficient in the decay (5.29);

P—polarization vector of a particle with spin 1/2;

$\boldsymbol{\nu}$—unit vector in the direction of the momentum of N in the decay (5.29);

$f_{\tilde{m}\lambda}(\omega_1,\omega_2)$—amplitude of three-particle decay (6.1);

$I=(\eta_x/\eta_1\eta_2\eta_3)(-1)^{j-s}$; η_1, η_2, η_3—parity of particles in the decay (6.1);

$t^{LM\tilde{M}}_{l\mu}$—cascade polarization moments of the particle a_1 in the decay (6.1).

2. Definitions

Plane of production of x—plane containing the momenta of the incoming particle and of the resonance x in the l.s. of the reaction (2.1).

Z system (Y system)—coordinate system in which the Z(Y) axis is chosen to be a normal to the plane of production of x.

Treiman-Yang system—Y system in which the Z axis is directed along the momentum of the incoming particle in reaction (2.1) in the rest system of x.

Capps condition—a situation wherein the particles b_1 and b_2 in reaction (2.1) are not polarized, the momenta and the spin states of the particles c_1, c_2, ..., c_n are not measured, and is conserved.

Plane of decay of x—plane in the rest system of x, containing the momenta of the products of its decay in the case of three-particle decay (6.1) of x.

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