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Integrability structures in string theory

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Abstract. **This review is a collection of various methods and observations relevant to structures in three-dimensional systems similar to those responsible for the integrability of two-dimensional systems. Particular focus is on Nambu structures and loop variables naturally appearing in membrane dynamics. While reviewing each topic in more detail, we emphasize connections among them and speculate on possible relations to membrane integrability.**

Keywords: integrability, Nambu structure, loop algebra, membranes, M-theory

1. Introduction

Most generally speaking, a dynamical system is said to be integrable when the number of degrees of freedom required to describe its dynamics is twice smaller than the dimension of the phase space. In other words, the system has a set of conserved charges allowing dynamical equations to be integrated along the corresponding directions. While this understanding of integrability applies most directly to mechanical systems with finite phase space, for field theories, the above becomes too vague and stricter criteria have to be introduced. In particular, we are talking about classical integrability in the Liouville sense, when an infinite set of conserved charges can be generated by making use of the Lax pair approach. The Green–Schwarz superstring on the $AdS_5 \times \mathbb{S}^5$ background, that is, a two-dimensional supersymmetric sigma model, is definitely not the simplest one, however relevant it is to the present discussion on examples of a classically integrable field theory system. The notion of integrability naturally extends to quantum systems, where it means the existence of an infinite set of commuting operators, one of which can be set as the Hamiltonian. This set can be generated by quantum Lax operators constructed using the algebraic Bethe ansatz and Yang–Baxter relations, or the thermodynamic Bethe ansatz, or by constructing a quantum spectral curve. There are plenty of introductory lectures and reviews explaining the notion of quantum integrability and these approaches (see, e.g., [1–5]); here, we will be mainly focused on the structures responsible for classical integrability.

The narrative of the present review develops from the integrability of the Green–Schwarz superstring on $AdS_5 \times \mathbb{S}^5$, its deformations preserving integrability and the algebraic structures that emerge in this procedure. The first integrability of this system was observed in [6] by explicit construction of a flat Lax connection that generates an infinite set of conserved currents. In particular, interest in the integrability of the string on specific backgrounds stated in this work was an extension of the AdS/CFT duality beyond the correspondence between weakly coupled strings and strongly coupled gauge theories. A strong indication of the integrability of the string is the conjectured integrability of its holographic partner, the $\mathcal{N}=4$ $d=4$ $SU(N)$ super Yang–Mills theory. This, in turn, was initially based on the observation that the large N dilatation operator of the gauge

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theory, when restricted to the sector of operators built out of scalars, can be regarded as the Hamiltonian of an integrable quantum spin chain [7]. The string on $AdS_5\times\mathbb{S}^5$ is known to belong to a family of integrable sigma models that can be obtained via its deformation. In particular, one finds the so-called Lunin–Maldacena (LM) β-deformed background [8], dual to a certain Leigh–Strassler deformation of the $\mathcal{N}=4$ $d=4$ supersymmetric Yang–Mills theory [9]. The integrability of the string on the LM background was shown in [10, 11] (nonintegrability for complex deformations was shown in [12]). For a review of the interplay between AdS/CFT correspondence and integrability, see [13]. When speaking about the open Green–Schwarz superstring, we are able to find boundary conditions that preserve integrability [14], allowing us to speak about integrable configurations of D-branes. Notably, one finds a D3–D5-brane system dual to a defect CFT known to be integrable [15] and a D2–D4-brane system dual to ABJM theory in the presence of a half-BPS domain wall also known to be integrable [16]. When uplifted to M-theory, the latter gives an M2–M5-system, strongly arguing in favor of the possibility of defining membrane integrability. Note, however, [17, 18], where nonintegrability of string motion on certain D-brane backgrounds were shown. These results could either restrict the possibility of defining integrable structures for D-branes or indicate that a better choice of variables must be made.

A more systematic approach to generating integrable two-dimensional sigma models is based on Yang–Baxter deformations of sigma models developed in [19] for principal chiral models and in [20] for sigma models on symmetric spaces. In [21], an integrable Yang–Baxter deformation of the $AdS_5\times\mathbb{S}^5$ superstring was presented, which significantly spurred development of the methods described in the present review. The deformation is parametrized by a matrix r that is a solution to the classical Yang–Baxter equation (CYBE) [22, 23]

$$r^{b_1[a_1}r^{a_2|b_2|}f_{b_1b_2}{}^{a_3]}=0\,, \tag{1.1}$$

where $f_{ab}{}^{c}$ denote structure constants of the (super)algebra.[1] The natural question of whether the deformed sigma model can be interpreted as a superstring on a supergravity background has found its answer in [25], where the corresponding background has been presented and shown to violate equations of $D=10$ supergravity. The proper set of equations satisfied by the background, currently referred to as ABF (Arutyunov, Borsato, Frolov), was found in [26] and is now referred to as generalized supergravity [27]. We will not cover such generalizations of supergravity equations here; the interested reader can find more details in review [28] and references therein. Important for us here is the result of [29, 30], where a rule for performing Yang–Baxter deformations for a general background beyond coset spaces was formulated. The rule is based on the open-closed string map and more conveniently can be formulated as a local $O(10,10)$ transformation generated by a bi-vector $\beta=r^{a_1a_2}k_{a_1}\wedge k_{a_2}$, where $k_a=k_a{}^{\mu}\partial_{\mu}$ is a set of Killing vectors of the initial background. The bi-vector $\beta^{\mu\nu}$ plays the role of noncommutative parameter of the corresponding open-closed string map. In [31, 32], it was shown that, for the deformed background to satisfy equations of supergravity, it is sufficient to impose a classical Yang–Baxter equation on $r^{a_1a_2}$ and the so-called unimodularity condition $r^{a_1a_2}f_{a_1a_2}{}^{b}=0$, discovered in [33]. Breaking the latter gives solutions to equations of generalized supergravity. An important side comment here is that precisely the classical Yang–Baxter matrix r can be used to generate Poisson brackets of an integrable system, given a pair of Lax operators (see Section 1.2).

Written in the form of linear $O(10,10)$ transformations, Yang–Baxter deformations of 10D backgrounds (2D σ-model) can be naturally generalized to deformations of 11D backgrounds (3D σ-model), which was done in [34–36]. The corresponding generalization of the classical Yang–Baxter equation (gCYBE) was presented in [37, 38] following an algebraic approach based on the so-called exceptional Drinfeld algebra, the generalizing classical Drinfeld double.[2] Verification that gCYBE (together with the unimodularity constraint) is enough for a deformation to generate a solution was performed in [36] for general backgrounds. In the case of 11D backgrounds, a deformation is parametrized by a tri-vector $\Omega=\rho^{a_1a_2a_3}k_{a_1}\wedge k_{a_2}\wedge k_{a_3}$, which now requires an object $\rho^{a_1a_2a_3}$ with three indices rather than a matrix $r^{a_1a_2}$. At the moment, no interpretation of gCYBE as a classical limit of an equation similar to the quantum Yang–Baxter equation is known, although certain attempts were made in [39] to construct the quantum equation by hands and in [35] to link it to the Zamolodchikov tetrahedron equation. Despite that, a side comment similar to the one above can be made: a Nambu bracket of a dynamical system can be generated using $\rho^{a_1a_2a_3}$ and a triple of Lax operators.

Naturally, a set of questions arises here. To what extent is this system integrable? Can Liouville integrability be formulated for three-dimensional field-theoretical systems? Does tri-vector deformation preserve the integrability of the 2D σ-model in the usual sense? This review aims at collecting and describing in a uniform language several attempts to make sense of three-dimensional integrability and of the integrability of Nambu–Poisson systems, as these seem to have a close relation to the algebraic structures arising in tri-vector deformations of M-theory backgrounds. We will try to emphasize these relations in each case and speculate on possible further developments. The text is structured as follows. In this section, the standard approach to Liouville integrability using Lax pairs and the Lax connection is briefly reviewed mainly to introduce notations and to make the text self-contained. In Section 2, Nambu systems and approaches to their integrability are discussed. As a particular example relevant for 3D integrability, we focus on the Kadomtsev–Petviashvili (KP) hierarchy. In Section 3, we briefly review the integrability of the superstring, mainly focusing on integrable deformations and their interpretation as a Poisson–Lie T-duality. Section 4 describes tri-vector deformations and related algebraic structures and contains a description of several approaches to the integrability of the 3D membrane. We describe approaches to membrane dynamics based on loop algebra variables which seem natural and emphasize that the deformation tensor $\Omega^{a_1a_2a_3}$ can be interpreted as a loop noncommutativity parameter for membranes, while the

[1] Note that the deformation of [21] does not uniquely define the deformed theory, and freedom in defining the r-matrix remains. In [24], a unimodular η-deformation of $AdS_5\times\mathbb{S}^5$ satisfying the inhomogeneous (modified) Yang–Baxter equation was constructed and shown to generate a solution to supergravity equations.

[2] Earlier, this condition was found in [34] in the form of a condition sufficient for R-flux to vanish. However, since the condition derived was sufficient rather than necessary, certain terms could have been added, and the final form of the gCYBE equation still had to be determined.

deformation map has the same form as the open-closed membrane map. At the end of Section 4, we discuss possible relations to the tetrahedron equation of Zamolodchikov and a particular way to define a Wilson surface in terms of loops. Finally, Section 5 presents the results and observations discussed in the main text in the form of lists to present the picture in a clearer way, as far as that is possible.

1.1 Liouville integrability

Let us start with a brief reminder of the standard Lax pair approach to classical Liouville integrability of mechanical systems. The presentation below mainly follows [40]; however, the same can be found in any review or textbook on integrable systems. We start with a dynamical system defined by a set of equations of motion $\dot{x}^i = f^i(x)$. Given the Hamiltonian H of the system, the equations of motion can be written as

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{1.2}$$

Equivalently, $\dot{p}_i = \{H, p_i\}$, $\dot{q}^i = \{H, q^i\}$, where the Poisson bracket for a pair of arbitrary functions $f(p,q)$ and $g(p,q)$ of dynamical variables is defined as usual as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}. \tag{1.3}$$

Suppose the system has nontrivial integrals of motion defined as $\dot{I}_i = 0$, or equivalently

$$\{H, I_i\} = 0. \tag{1.4}$$

Each integral of motion I allows turning to the so-called action–angle variables and completely integrating the dynamical equations for a pair of coordinates (p,q). Hence, if the number of integrals of motion is equal to the total number of degrees of freedom, the system is completely integrable. To summarize, a system with $2n$-dimensional phase space is said to be Liouville integrable if it possesses n integrals of motion I_i, all of which are in involution, i.e., $\{I_i, I_j\} = 0$.

Let us consider action–angle variables in more detail, for which we take a single integral of motion given by a function $F(p,q) = f = \text{const}$. We solve this equation for p and define a 1-form

$$\alpha = p\,\mathrm{d}q, \tag{1.5}$$

where p is understood as a function $p = p(q,f)$ of the coordinate q and the integral of motion f. Then, the 2-form $\omega = \mathrm{d}\alpha = \mathrm{d}p \wedge \mathrm{d}q$ defines a symplectic form in the phase space. Let us now define the action S as

$$S[q,f] = \int_{q_0}^{q} \alpha = \int_{q_0}^{q} p(q,f)\,\mathrm{d}q. \tag{1.6}$$

By construction, the action is a function of two variables q and f, and one may define new dynamical variables by writing

$$p = \frac{\partial S}{\partial q}, \qquad \psi = \frac{\partial S}{\partial f}, \tag{1.7}$$

where the first equality is straightforward and the second simply defines the angle variable ψ. Such a defined transformation from the variables (p,q) to the action–angle variables (f,ψ) is canonical, i.e., preserves the symplectic form. Indeed, considering

$$0 \equiv \mathrm{d}^2 S = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}f \wedge \mathrm{d}\psi, \tag{1.8}$$

one finds that ω does not change.

The dynamics of the system becomes particularly simple in terms of such defined action–angle variables, allowing the equations of motion for the integral $I = F(p,q)$ to be explicitly solved. Indeed, we write

$$\begin{aligned} \dot{f} &= \{H, f\} = 0, \\ \dot{\psi} &= \{H, \psi\} = \frac{\partial H}{\partial f} = \omega(f) = \text{const}. \end{aligned} \tag{1.9}$$

These can be easily solved to give

$$\begin{aligned} f &= \text{const}, \\ \psi &= \omega(f)\,t + \psi_0, \end{aligned} \tag{1.10}$$

i.e., the angle variable ψ corresponding to the action given by the integral of motion f evolves linearly with time. Given n integrals of motion, the above procedure can be repeated for all n pairs of variables (p_i, q^i), allowing the equations of motion to be solved completely in terms of the corresponding action–angle variables (f_i, ψ^i). This is basically the explicit manifestation of integrability of the system: linear evolution and no chaos.

As the simplest example illustrating the above general principle, consider one-dimensional harmonic oscillator $H = (1/2)(p^2 + \omega^2 q^2)$. Hamiltonian equations of motion read

$$\dot{p} = -\omega^2 q, \qquad \dot{q} = p. \tag{1.11}$$

This system has only one integral of motion, which is the energy $H(p,q) = E$. Solving this equation, we get for p the following:

$$p = \sqrt{2E - \omega^2 q^2}. \tag{1.12}$$

The action is then $S = \int_0^q \sqrt{2E - \omega^2 z^2}\,\mathrm{d}z$ and the angle variable then reads

$$\psi = \frac{\partial S}{\partial E} = \int_0^q \frac{\mathrm{d}z}{2\sqrt{2E - \omega^2 z^2}} = \frac{1}{2\omega}\arctan\left(\frac{\omega q}{p}\right). \tag{1.13}$$

Hence, we have

$$\begin{aligned} &\frac{1}{2\omega}\arctan\left(\frac{\omega q}{p}\right) = \frac{1}{2}\,t, \\ &p^2 + \omega^2 q^2 = 2E, \end{aligned} \tag{1.14}$$

which gives the standard solution $q = \sin(\omega t)$, $p = \omega\cos(\omega t)$.

1.2 Lax pair

Evidently, the above procedure is not algorithmic, as it deals with solving differential equations at various steps, for which reason it would be desirable to formulate an approach that allows generating integrals of motion from a single expression and thus guarantee integrability in the considered sense. Such an approach is known as the Lax–Zakharov–Shabat formalism and starts with an assumption that we have managed to write equations of motion for a dynamical system in the form

$$\dot{L} = [L, M], \tag{1.15}$$

where L, M are some matrices which might additionally depend on some (spectral) parameter(s) u. Then, integrals of motion can be generated by simply taking a trace of various matrix powers of L, i.e.,

$$F_k := \operatorname{Tr} L^k \implies \dot{F}_k = 0\,. \tag{1.16}$$

The pair of matrices L, M is referred to as the Lax pair. Altogether, the above equations imply that dependence on time for these matrices is given by

$$\begin{aligned} &L(t) = g(t)L(0)g(t)^{-1}\,,\\ &M(t) = \dot{g}(t)g(t)\,. \end{aligned} \tag{1.17}$$

Roughly speaking, if we have managed to find a Lax pair for a dynamical system, i.e., to rewrite its equations of motion as above, the system is integrable. Certainly, this step is no more algorithmic as the standard action–variables method. However, once the Lax pair is found, integrals of motion are generated automatically. For example, for a one-dimensional harmonic oscillator, the Lax pair can be chosen as follows:

$$\begin{aligned} &L = p\sigma_3 + \omega q\sigma_1 = \begin{bmatrix} p & \omega q \\ \omega q & -p \end{bmatrix},\\ &M = -\frac{\mathrm{i}}{2}\,\omega\sigma_2 = \begin{bmatrix} 0 & -\dfrac{1}{2}\,\omega \\ \dfrac{1}{2}\,\omega & 0 \end{bmatrix}, \end{aligned} \tag{1.18}$$

where σ_1, σ_2, σ_3 are the standard Pauli matrices. The only integral of motion is then $H = (1/4)\operatorname{Tr} L^2$.

To check whether the integrals $\{F_i\}$ are in involution, one has to ensure $\{F_i, F_j\} = 0$. For further discussion, it is convenient to consider the Lax pair as the starting point and to generate a Poisson bracket for the system using a classical Yang–Baxter r-matrix. To do so, we start with a matrix $L \in \mathfrak{gl}(d)$ and define the Poisson bracket as

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]\,, \tag{1.19}$$

where $[\,,]$ is the usual commutator in the algebra $\mathfrak{gl}(d)$, and the matrices $L_{1,2}$ are defined as

$$\begin{aligned} &L_1 = L \otimes \mathbb{1}\,,\\ &L_2 = \mathbb{1} \otimes L\,. \end{aligned} \tag{1.20}$$

In other words, both L_1 and L_2 belong to $\mathfrak{gl}(d) \otimes \mathfrak{gl}(d)$. Such a defined bracket is antisymmetric by construction and must additionally satisfy the Jacobi equation that is ensured by the (modified) classical Yang–Baxter equation for the matrix r. To see this, we have to consider three copies of the algebra $\mathfrak{gl}(d)$ for which we define

$$\begin{aligned} &L_1 = L \otimes \mathbb{1} \otimes \mathbb{1}\,,\\ &L_2 = \mathbb{1} \otimes L \otimes \mathbb{1}\,,\\ &L_3 = \mathbb{1} \otimes \mathbb{1} \otimes L\,, \end{aligned} \tag{1.21}$$

and similarly for the r-matrix, i.e., $r_{12} = r \otimes \mathbb{1}$ and so on. The Jacobi identity for the Poisson bracket requires

$$\big\{L_1, \{L_2, L_3\}\big\} + \big\{L_3, \{L_1, L_2\}\big\} + \big\{L_2, \{L_3, L_1\}\big\} = 0\,. \tag{1.22}$$

Using definition (1.19) from the above Jacobi identity, we obtain the following equation [41]:

$$\begin{aligned} &\big[L_1, \{L_2, r_{13}\} - \{L_3, r_{12}\} + [r_{12}, r_{13} + r_{23}] + [r_{32}, r_{13}]\big]\\ &+ \big[L_2, \{L_3, r_{21}\} - \{L_1, r_{23}\} + [r_{23}, r_{21} + r_{31}] + [r_{13}, r_{21}]\big]\\ &+ \big[L_3, \{L_1, r_{32}\} - \{L_2, r_{31}\} + [r_{31}, r_{32} + r_{12}] + [r_{21}, r_{32}]\big] = 0\,. \end{aligned} \tag{1.23}$$

For the case of a constant and antisymmetric matrix $r_{ij} = -r_{ji}$, the above is satisfied in the classical Yang–Baxter equation (cYBE)

$$[r_{23}, r_{12}] + [r_{23}, r_{13}] + [r_{13}, r_{12}] = 0\,. \tag{1.24}$$

To see that, let us start with the first term, which gives

$$\begin{aligned} \big\{L_1, \{L_2, L_3\}\big\} &= \big[r_{23}, \{L_1, L_2\}\big] + \big[r_{23}, \{L_1, L_3\}\big]\\ &= \big[r_{23}, [r_{12}, L_1]\big] + \big[r_{23}, [r_{12}, L_2]\big] + \big[r_{23}, [r_{13}, L_1]\big]\\ &+ \big[r_{23}, [r_{13}, L_3]\big]\,. \end{aligned} \tag{1.25}$$

Writing the other two cyclic permutations contributing the identity, we find three types of terms: those acting on L_1, L_2, and L_3, all of which have the same form. Taking for concreteness the first one, we have

$$\big[r_{23}, [r_{12}, L_1]\big] + \big[r_{23}, [r_{13}, L_1]\big] + \big[[r_{13}, r_{12}], L_1\big] = 0\,, \tag{1.26}$$

where we used the Jacobi identity for the commutator. Now, we notice that the following holds true:

$$\big[r_{23}, [r_{12}, L_1]\big] = \big[[r_{23}, r_{12}], L_1\big]\,, \tag{1.27}$$

since r_{23} acts only on the second and third copies of $\mathfrak{gl}(d)$, r_{12} acts only on the first and second copies, while L_1 belongs to the first copy. Hence, loosely speaking, the commutators are independent. This allows us to finally write

$$\begin{aligned} &\big[[r_{23}, r_{12}], L_1\big] + \big[[r_{23}, r_{13}], L_1\big] + \big[[r_{13}, r_{12}], L_1\big]\\ &\quad = -c^2\big[[L_2, L_3], L_1\big]\,, \end{aligned} \tag{1.28}$$

which is the (modified) classical Yang–Baxter equation ((m)CYBE). The right-hand side together with the remaining two cyclic permutations is rendered as zero due to the Jacobi identity for the commutator. When $c = 0$, the modified CYBE becomes the usual CYBE and can be written in the nice form

$$[r_{23}, r_{12}] + [r_{23}, r_{13}] + [r_{13}, r_{12}] = 0\,, \tag{1.29}$$

which will be useful later.

It is straightforward to check that the integrals F_i are indeed in involution with respect to such a defined Poisson bracket. Certainly, this bracket defines the same evolution as (1.15):

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \{F_k, L\} = [M_k, L]\,, \qquad M_k = -k\operatorname{Tr}_1\,[{L_1}^{k-1} r_{21}]\,, \tag{1.30}$$

where the subscript indicates that the trace is taken with regard to the first factor. A theorem states that eigenvalues of the Lax matrix L (the conserved quantities F_k) are in involution if and only if there exists a function r_{21} that satisfies CYBE and defines the Poisson bracket as above.

For details of the theorem, see lectures [40]; here, we only mention that the r-matrix is a natural attribute of a classical integrable system.

As a final note in this subsection, let us provide some expressions in an explicitly chosen basis $\{T^i{}_j\} = \mathrm{bas}\,\mathfrak{gl}(d)$. For the r-matrix $r \in \mathfrak{gl}(d) \wedge \mathfrak{gl}(d)$, we then have the following component form:

$$r = r^{i_1}{}_{j_1}{}^{i_2}{}_{j_2} T^{j_1}{}_{i_1} \wedge T^{j_2}{}_{i_2} \,. \tag{1.31}$$

Equation (1.19) then becomes

$$\begin{aligned}\{L^{i_1}{}_{j_1}, L^{i_2}{}_{j_2}\} &= r^{i_1}{}_{k_1}{}^{i_2}{}_{j_2} L^{k_1}{}_{j_1} - L^{i_1}{}_{k_1} r^{k_1}{}_{j_1}{}^{i_2}{}_{j_2} \\ &\quad - r^{i_1}{}_{j_1}{}^{i_2}{}_{k_2} L^{k_2}{}_{j_2} + L^{i_2}{}_{k_2} r^{i_1}{}_{j_1}{}^{k_2}{}_{j_2} \,. \end{aligned} \tag{1.32}$$

We observe that the classical r-matrix naturally has two pairs of indices, each acting on some linear space. Let us illustrate the above by the usual example of a harmonic oscillator whose matrices L and M have been presented previously. The classical r-matrix can be written in the following form:

$$\begin{aligned} r &= \frac{1}{q}\big(F\otimes E - E\otimes F\big) \\ &= \frac{1}{q}\begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes\begin{bmatrix}0&1\\0&0\end{bmatrix} - \frac{1}{q}\begin{bmatrix}0&1\\0&0\end{bmatrix}\otimes\begin{bmatrix}0&0\\1&0\end{bmatrix}, \end{aligned} \tag{1.33}$$

where F and E are generators of $\mathfrak{sl}(2)$. To derive the corresponding Poisson bracket, we calculate on the one hand

$$[r, L\otimes \mathbb{1} + \mathbb{1}\otimes L] = -\omega(\sigma_3\otimes\sigma_1 - \sigma_1\otimes\sigma_3)\,, \tag{1.34}$$

and on the other hand,

$$\{L\otimes\mathbb{1}, \mathbb{1}\otimes L\} = \omega\{p,q\}\sigma_3\otimes\sigma_1 + \omega\{q,p\}\sigma_1\otimes\sigma_3\,. \tag{1.35}$$

Comparing the two, we have $\{q,p\} = 1$. Note that, since the r-matrix (1.33) does not solve the CYBE (1.24), the above example illustrates the most general picture, when fulfillment of Jacobi identity (1.22) is guaranteed by (1.23).

1.3 Quantum Yang–Baxter equation

For a quantum system, integrability basically means the same as above: the existence of an infinite set of conserved charges Q_s commuting with each other. When speaking about a quantum system, one is mainly interested in deriving its full spectrum, which is usually simple for free theories and becomes an incredibly complicated problem for interacting systems. For integrable quantum models, powerful methods have been developed to compute the spectrum: the algebraic and coordinate Bethe ansatz, the thermodynamic Bethe ansatz, and the spectral curve. Of particular relevance to the present discussion is the approach of thermodynamic Bethe ansatz, which allows computing the spectrum of an integrable quantum system using scattering data and the Yang–Baxter equation. Let us focus on this one, referring the reader to reviews [1–5] for more detailed descriptions of other methods in application to AdS/CFT integrability and spin-chain models.

For scattering processes, the integrability of a quantum system means that there is no particle production. For theories in dimension $d > 2$, the Coleman–Mandula theorem states that the S-matrix is trivial $S = 1$ if there is even a single charge that is a second or higher order tensor. In contrast, in dimension $d = 1+1$, the S-matrix remains nontrivial although pretty much restricted:

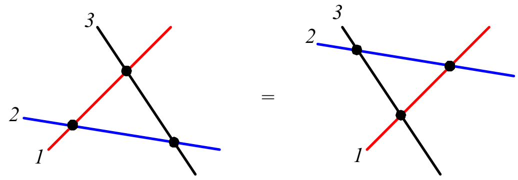

Figure 1. Graphical representation of quantum Yang–Baxter equation governing scattering of three particles. Equation states that the S-matrix does not depend on the mutual position of the lines, in particular, on the position of the black line (*3*) in the picture.

- no particle production;
- the initial set of momenta $\{p_i\}_{\text{in}} = \{p_i\}_{\text{out}}$ is the same as the final set;
- the scattering factorizes.

Factorized S-matrices as exact solutions to $1+1$-dimensional quantum field theories were first considered in [42] and then used to develop the method of the thermodynamic Bethe ansatz in [43]. The factorization property of the S-matrix means that the S-matrix of n particles decomposes into a product of S-matrices for all pairs of particles. In general, such decomposition can be performed in multiple ways, all of which must be equivalent, leading to consistency constraints. For 3-to-3 particle scatter, this can be illustrated by Fig. 1, where the scattering of red, blue, and black particles labeled *1*, *2*, and *3*, respectively, can be factorized into pair interactions in two ways. Setting the time direction to run from left to right in Fig. 1 and denoting the S-matrix $R_{ij}(u)$ for particles with labels i and j and mutual rapidity u, we obtain the following equation:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)\,. \tag{1.36}$$

This is the quantum Yang–Baxter equation (qYBE), introduced in [44] to solve an eigenvalue problem for an N-particle system using the algebraic Bethe ansatz and independently in [45] to compute the partition function for a certain lattice matrix model. For more details, see, e.g., [46, 47].

Suppose each particle, in addition to rapidities, is described by a linear space V of its states, then R-matrices $R_{ij} \in \mathrm{End}\,(V_i\otimes V_j)$ act on the vector space of states of two particles, encoding their interaction. The most well-known example would be to set V to a two-dimensional set of states of a particle with spin $\hbar/2$. Then, R-matrix would take values in the product of groups $\mathrm{SL}(2,\mathbb{C})$. Such R-matrices and their generalization to $\mathrm{SL}(n,\mathbb{C})$ and to quantum groups have found a wide variety of applications in knot theory and braid groups [48–50]. For us, the quantum Yang–Baxter equation will be relevant in two aspects:

- it describes scattering of the superstring states on $\mathrm{AdS}_5\times\mathbb{S}^5$;
- its quasiclassical limit gives the classical Yang–Baxter equation discussed above and relevant to integrable deformations.

Keeping the former outside of the scope of the review, let us consider the latter in more detail.

First, one should be careful with the interpretation of qYBE in terms of matrices, since each R-matrix acts only in the product of two vector spaces and it is convenient to define an operator $\hat{R} \in \mathrm{End}\,(V\otimes V)$. Hence, $\hat{R}_{12}(u) \in \mathrm{End}\,(V^{(1)}\otimes V^{(2)})$ provides interaction between particles *1* and *2* with quantum states encoded by the spaces $V^{(1)}$ and $V^{(2)}$ scattered at mutual rapidity u. Note that the linear spaces are identical, $V^{(1)} = V^{(2)}$, and the numbers are there just to explicitly distinguish the particles. The same can be done just by keeping track of the place at which the operator stands. In what follows, we adopt the former notation. Hence, we write

$$R_{12}(u) = \hat{R}_{12}(u) \otimes \mathbb{1}\,, \tag{1.37}$$

where $\mathbb{1}$ is the identity operator on V. For matrix notations, we choose a basis in V as $\{e_\alpha\} = \operatorname{bas} V$ and write

$$\begin{aligned} &R_{12}(u)(e_\alpha \otimes e_\beta \otimes e_\gamma) = R_{12}(u)_{\alpha\beta\gamma}^{\delta\epsilon\phi} e_\delta \otimes e_\epsilon \otimes e_\phi\,, \\ &R_{12}(u)_{\alpha\beta\gamma}^{\delta\epsilon\phi} = R(u)_{\alpha\beta}^{\delta\epsilon} \delta_\gamma^\phi\,. \end{aligned} \tag{1.38}$$

In matrix notations, (1.36) is an equation for the matrix with 4 indices $R(u)_{\alpha\beta}^{\delta\epsilon}$ depending on the spectral parameter u.

Usually, the full quantum Yang–Baxter equation is very complicated to analyze and to solve, and one proceeds by taking a quasi-classical limit. For this, what is first noticed is that $R(u) = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ trivially solves the qYBE and hence it is natural to expand around this point in the space of solutions

$$R_{12}(u) = \mathbb{1} + \hbar\, r_{12}(u)\,. \tag{1.39}$$

Here, $\hbar$ is the expansion parameter and $r_{12}(u)$ is constructed from the algebra $\mathfrak{g}$ of the group $G = \operatorname{End}(V)$. Substituting this expansion into the initial equation, it is found that, in the orders $\hbar^0$ and $\hbar^1$, all terms cancel and the equation is satisfied trivially. Hence, the first nontrivial equation one encounters at level $\hbar^2$ reads

$$\big[r_{12}(u-v), r_{13}(u)\big] + \big[r_{13}(u), r_{23}(v)\big] + \big[r_{12}(u-v), r_{23}(v)\big] = 0\,. \tag{1.40}$$

Here, $[x, y]$ is understood as $[x, y] = x \cdot y - y \cdot x$; hence, one should be careful with the definition of $r_{12}(u - v)$. Indeed, since $R_{12}(u) \in \operatorname{End}(V \otimes V \otimes V) \equiv G \times G \times \mathbb{1}$, it is natural to define the matrix $r_{12}(u)$ as $r_{12}(u) \in \phi(\mathfrak{g}) \otimes \phi(\mathfrak{g}) \otimes \mathbf{1}$. Here,

$$\phi : \mathfrak{g} \to A \tag{1.41}$$

is a map to an associative algebra A with unit $\mathbf{1}$ defined such that

$$\phi(a)\phi(b) - \phi(b)\phi(a) = \phi\big([a, b]\big)\,, \tag{1.42}$$

where $[a, b]$ is the Lie bracket in the algebra $\mathfrak{g}$. Hence, equation (1.40) is understood as an equation on $A \otimes A \otimes A$, which, however, can be consistently restricted to $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and hence does not depend on the choice of A [51]. In what follows, for definiteness we will choose the algebra A to be the universal enveloping algebra $A = U(\mathfrak{g})$ and drop ϕ for clarity of notations.

Solutions of the classical Yang–Baxter equation for the r-matrix with a nontrivial spectral parameter can be propagated by a simple shift of a given solution $r_{12}(u)$ as $r'_{12}(u) = r_{12}(u) + r_{12}$ (and similarly for other spaces), where r_{12} must satisfy the constant classical Yang–Baxter equation [51]

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0\,. \tag{1.43}$$

In this paper, we always refer to this equation when mentioning the CYBE. Each element r_{12}, r_{23}, r_{13} above can be decomposed regarding the basis of the algebra $\{t_a\} = \operatorname{bas} \mathfrak{g}$,

$$\begin{aligned} r_{12} &= r^{ab} t_a \otimes t_b \otimes \mathbf{1}\,, \\ r_{13} &= r^{ab} t_a \otimes \mathbf{1} \otimes t_b\,, \\ r_{23} &= r^{ab} \mathbf{1} \otimes t_a \otimes t_b\,. \end{aligned} \tag{1.44}$$

In what follows, we assume $r^{ab} = -r^{ba}$. Substituting these decompositions back into the CYBE (1.43), one obtains

$$r^{ab} r^{cd} \Big([t_a, t_c] \otimes t_b \otimes t_d + t_a \otimes t_c \otimes [t_b, t_d] + t_a \otimes [t_b, t_c] \otimes t_d \Big) = 0\,. \tag{1.45}$$

Replacing $[t_a, t_b] = f_{ab}{}^c t_c$ and properly relabelling the indices, one obtains

$$e_{[a} \otimes e_b \otimes e_{c]} r^{ae} r^{bf} f_{ef}{}^c = 0\,, \tag{1.46}$$

which boils down to the CYBE recovered in deformations of the Type IIA/B supergravity

$$r^{e[a} r^{b|f|} f_{ef}{}^{c]} = 0\,. \tag{1.47}$$

As mentioned above, this equation indeed belongs only to the $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ inside $A \otimes A \otimes A$ and does not depend on the chosen algebra A or the precise form of the map. Out of curiosity, the same is not true for the tetrahedron equation, which is a direct analogue of the qYBE for scattering straight strings. Moreover, a well-defined quasi-classical limit does not exist in this case. We will discuss this more in Section 4.6.2.

1.4 Volume preserving flows and the action principle

An important property of a Hamiltonian system is preservation of its phase volume under evolution. For a given function G, a Hamiltonian H defines a flow, which in the infinitesimal form can be written as

$$G \to G + \{H, G\}\,. \tag{1.48}$$

Consider the phase space distribution of the system $\mathrm{d}w = \rho(p, q)\, \mathrm{d}^n p\, \mathrm{d}^n q$, where the distribution function $\rho(p, q)$ is the probability of having the system in the phase volume $\mathrm{d}^n p\, \mathrm{d}^n q$ at the point (p, q). The Liouville equation states that the distribution function is constant:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q^i}\, \dot{q}^i + \frac{\partial f}{\partial p_i}\, \dot{p}_i = 0\,. \tag{1.49}$$

Equivalently, we can write this as the evolution equation for the density function: $\dot{\rho} = \{H, \rho\}$. Let us now show that, for Hamiltonian systems, the phase volume is preserved. To do so, we define a phase vector at arbitrary time t:

$$x_t = \big(p_1(t), \ldots, p_n(t), q^1(t), \ldots, q^n(t)\big)\,. \tag{1.50}$$

This vector is related to the vector x_0 at $t = 0$ by a coordinate transformation in the phase space, whose Jacobian is given by

$$J = \det \frac{\partial x_t^I}{\partial x_0^J} = \det M^I{}_J\,, \tag{1.51}$$

where $I = 1, \ldots, 2n$. Hence, preservation of the phase volume under evolution is equivalent to this defined Jacobian being independent of time. For that, we calculate

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \mathrm{Tr}\left(M^{-1}\,\frac{\mathrm{d}M}{\mathrm{d}t}\right)J = J\,\frac{\partial x_0^I}{\partial x_t^J}\,\frac{\partial \dot{x}_t^J}{\partial x_0^I} = J\,\frac{\partial \dot{x}_t^I}{\partial x_t^I}\,. \tag{1.52}$$

For Hamiltonian systems,

$$\frac{\partial \dot{x}_t^I}{\partial x_t^I} = -\frac{\partial}{\partial p_i}\,\frac{\partial H}{\partial q^i} + \frac{\partial}{\partial q^i}\,\frac{\partial H}{\partial p_i} = 0\,; \tag{1.53}$$

hence, the flow preserves the phase space volume. In what follows, we show that the same is true for Nambu mechanical systems, i.e., defined in terms of tri-brackets.

To define the action of the system, we start with a vector field $\tilde{L}$ that corresponds to the Hamiltonian evolution flow:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{H, f\} = \tilde{L}(f)\,. \tag{1.54}$$

The vector field can be represented as $\tilde{L} = \partial_t + L$, and components of L read

$$L^p = -\frac{\partial H}{\partial q}\,, \quad L^q = \frac{\partial H}{\partial p}\,. \tag{1.55}$$

This vector field is the line field $\mathrm{d}\omega$ that is a derivative of the so-called Poincaré–Cartan 1-form ω on the phase space that defines the action. For a $1+1$-dimensional system, the 1-form ω can be written as

$$\omega = p\,\mathrm{d}q - H\,\mathrm{d}t\,. \tag{1.56}$$

To show that $\tilde{L}$ is indeed the line field, i.e., $\imath_{\tilde{L}}(\mathrm{d}\omega) = 0$, we simply write

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}p\wedge\mathrm{d}q - \frac{\partial H}{\partial p}\,\mathrm{d}p\wedge\mathrm{d}t - \frac{\partial H}{\partial q}\,\mathrm{d}q\wedge\mathrm{d}t\\ &= \mathrm{d}p\wedge\mathrm{d}q - L^q\,\mathrm{d}p\wedge\mathrm{d}t + L^p\,\mathrm{d}q\wedge\mathrm{d}t\,.\end{aligned} \tag{1.57}$$

The 1-form ω defines an integral invariant $\int_\gamma \omega$ that is usually referred to as the action of the system. This expression is invariant under different choices of 1-chains along the evolution flow. Consider a 1-chain c in the phase space; its image at t under Hamiltonian evolution is given by $g^t(c)$. A tube of phase trajectories is given by a 2-chain,

$$J^t c = \left\{g^\tau(c), 0 \leqslant \tau \leqslant t\right\}. \tag{1.58}$$

Simply speaking, one takes a closed curve c in the phase space and drags it along the Hamiltonian flow from 0 to t. Given the Stokes theorem and the fact that $\imath_{\tilde{L}}\,\mathrm{d}\omega = 0$, we have

$$\int_c \omega - \int_{g^t(c)} \omega = \int_{J^t(c)} \mathrm{d}\omega = 0\,, \tag{1.59}$$

where we used the fact that $\partial(J^t(c)) = c - g^t(c)$. Then, the extrema of the integral

$$A(\gamma) = \int_\gamma \omega = \int_\gamma p\wedge\mathrm{d}q - H\,\mathrm{d}t \tag{1.60}$$

give trajectories of the system (for more details see [52]). Interestingly enough, a generalization exists of the above for the Nambu mechanics attributed to Takhtajan [53], which can be uplifted to an action of a membrane. We will return to this later.

1.5 Integrability in field theory

For field theories, i.e., when canonical variables depend on a continuous variable, the concept of integrability is more tricky. The phase space of such models is infinitely dimensional and one would have to require a continuous set of integrals of motion to be able to speak about Liouville integrability. For a mechanical system, integrability means the possibility of turning to action–angle variables, which allows explicitly integrating equations of motion. For field theory, we will similarly be talking about the exact solvability of equations and methods, allowing us to construct such solutions. For quantum systems, we are usually talking about the exact spectrum of operators in the theory, the exact energy spectrum of the system, and the exact S-matrix, which for integrable systems are usually trivial. Let us list a few well-known integrable field theories:

• Korteweg–de Vries (KdV) equation, a mathematical model of waves on shallow water [54]:

$$\dot{h} = 6hh' - h'''\,. \tag{1.61}$$

• Nonlinear Schrödinger equation, used to describe propagation of light in a nonlinear medium [55]:

$$\mathrm{i}\dot{\psi} = -\psi'' + 2\kappa|\psi|^2\psi\,. \tag{1.62}$$

• Sine-Gordon equation, used in the theory of crystal dislocation, Bloch-wall motion, and magnetic flux in the Josephson effect [56]:

$$\ddot{\phi} - \phi'' + m^2\sin\phi = 0\,. \tag{1.63}$$

• Principal chiral model on a compact group manifold, which for us is the simplest model of a string. The action is given by

$$S = \int \mathrm{Tr}\left[(g^{-1}\,\mathrm{d}g)\wedge *(g^{-1}\,\mathrm{d}g)\right]. \tag{1.64}$$

Integrable field equations have in common the property that the scattering of their solitonic solutions is factorizable, i.e., after scattering two or more solitons, their shape is restored. This property is described by the Yang–Baxter equation, for which reason the Lax–Zakharov–Shabat formalism can be repeated for field theories with appropriate changes.

A 2-dimensional field theory is called integrable if its equations can be written in the form of the flatness condition on the so-called Lax connection $A = A_\alpha\,\mathrm{d}\sigma^\alpha$:

$$\mathrm{d}A + A\wedge A = 0\,. \tag{1.65}$$

In general, the connection may depend on an additional (spectral) parameter $u\in\mathbb{C}$. For example, components of the Lax connection for the KdV equation take the form

$$\begin{aligned}A_t &= \begin{bmatrix} -h' & -4u^{-1} - 2h\\ 4u^{-2} - 2u^{-1}h + h'' - 2h^2 & h'\end{bmatrix},\\ A_x &= \begin{bmatrix} 0 & -1\\ u^{-1} - h & 0\end{bmatrix}.\end{aligned} \tag{1.66}$$

To construct the Lax pair and to derive an infinite set of conserved currents, one has to construct variables that depend only on time t, which is possible due to the flatness condition. Indeed, it guarantees that a parallel transport operator

$$U(u;\sigma_1;\sigma_0) = \mathrm{Pexp}\left[\int_{t_0,x_0}^{t_1,x_1} A(u)\right], \qquad (1.67)$$

defined as a Wilson line, does not depend on continuous variations in the integration path. Endpoints in the integral may be identified with boundaries of the system, say, endpoints of a spin chain. A Lax pair constructed of such a defined parallel transport operator in general depends on the boundary conditions. The simplest case is the periodic boundary conditions $\sigma^1 \sim \sigma^1 + L$, when the Lax pair is defined as

$$\begin{aligned} T(u) &= \mathrm{Pexp}\left[\oint A(u)\right], \\ M(u) &= A_t(u)\Big|_{\sigma^1=0}. \end{aligned} \qquad (1.68)$$

It is easy to see that these satisfy precisely the desired equations

$$\dot{T}(u) = \left[T(u), M(u)\right]. \qquad (1.69)$$

Hence, the continuous family of conserved charges is defined as before as

$$F_k(u) = \mathrm{Tr}\; T(u)^k. \qquad (1.70)$$

Note that, as before, the classical r-matrix can be used to define canonical brackets.

In what follows, we will be interested in generalizing these integrability structures to 3-dimensional theories, motivated by certain similarities between the classical Yang–Baxter equation and what we call the generalized Yang–Baxter equation. It is straightforward to assume that, in this case, the 1-form Lax connection A must be replaced by either a 2-form (gerbe) connection or a connection in a loop space, which is actually equivalent, given the transgression map. Both these possibilities can be justified to some extent from a string/M-theory point of view; however, in any case, it is pretty clear that the mechanics behind these structures must be defined by Nambu brackets. Before proceeding with a discussion of Nambu mechanics and the extent to which integrable structures can be generalized, let us discuss another representation of integrable field theories, that is, integrable Lax hierarchies.

Note, however, that to have a higher dimensional integrable system does not necessarily require turning to higher Nambu mechanics. An example of a 1+2-dimensional integrable system is the so-called Kadomtsev–Petviashvili (KP) equation

$$\left(-4\dot{u} + u_{xxx} + 3uu_x\right)_x + 3u_{yy} = 0, \qquad (1.71)$$

where the subscripts denote derivative. This is a two dimensional generalization of the KdV model of waves on shallow water. It appears that this theory belongs to a larger (actually, infinite) family of integrable equations, each defined by its own Hamiltonian flow. Such a system of commuting integrable Hamiltonian flows is referred to as an integrable hierarchy (see [57, 58] for a more detailed review). In practice, integrable hierarchies are highly symmetric infinite sets of nonlinear evolution equations of the Lax type for infinitely many functions u_i of infinitely many variables t_n, $n = 1, 2, \ldots$. The Lax equations have the following form:

$$\frac{\partial L}{\partial t_m} = [B_m, L], \quad m = 1, 2, \ldots, \qquad (1.72)$$

where L and B_m are some pseudo-differential operators depending on the variables u_i. The Lax equations (1.72) can be written in the form of the zero-curvature condition

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \qquad (1.73)$$

which is usually referred to as the Zakharov–Shabat equation.

Let us illustrate the formalism using the example of the KP hierarchy. In this case,

$$\begin{aligned} L &= \partial + \sum_{i=1}^{\infty} u_i \partial^{-i} = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \ldots, \\ B_n &= (L^n)_{\geqslant 0}, \end{aligned} \qquad (1.74)$$

where $\partial = \partial/\partial x$ is a differential operator, ∂^{-1} is formal integration, and the subscript $\geqslant 0$ in the definition of B_n means that only non-negative powers of ∂ must be kept. Let us go through the first few levels of the hierarchy. For $n = 1$, we have $B_1 = \partial$ and

$$[B_1, L] = \sum_{i=1}^{\infty} \partial_x u_i\, \partial^{-i}. \qquad (1.75)$$

The equation is then simply $\partial u_i/\partial t_1 = \partial_x u_i$, which means $t_1 = x$. The Zakharov–Shabat equation when $m = 1$ then becomes

$$\partial_x B_n = [\partial, B_n], \qquad (1.76)$$

which is simply the action of the momentum operator.

The actual KP equation can be derived from the Zakharov–Shabat equation when $m = 2, n = 3$. For that, we calculate

$$\begin{aligned} B_2 &= \partial^2 + 2u_1, \\ B_3 &= \partial^3 + 3u_1' + 3u_1\partial + 3u_2, \end{aligned} \qquad (1.77)$$

where the prime denotes the derivative with respect to the variable x. The Zakharov–Shabat equation has terms proportional to ∂^0 and ∂^1, leading to two equations, which, denoting $y = t_2$, $t = t_3$, $u = 2u_1$, $v = u_2$, can be written as

$$\begin{aligned} &\dot{u}_x - \frac{3}{2}u_{xxy} - 3v_{yx} + \frac{1}{2}u_{xxxx} + 3v_{xxx} - \frac{3}{2}(uu_x)_x = 0, \\ &-\frac{3}{4}u_y + 3v_x + \frac{3}{4}u_{xx} = 0. \end{aligned} \qquad (1.78)$$

Now, taking derivatives ∂_y and ∂_{xx} of the second equation (1.78), we rewrite the first equation in the following form:

$$\dot{u}_x - \frac{3}{4}u_{yy} - \frac{1}{4}u_{xxxx} - \frac{3}{2}(uu_x)_x = 0. \qquad (1.79)$$

This is precisely the KP equation.

Similarly, an integrable hierarchy can be constructed for the KdV equation, which itself is part of an integrable hierarchy. To do this, we define

$$L = \partial^n + u_{n-2}\partial^{n-2} + \ldots + u_1\partial + u_0\,,$$
$$B_m = (L^{m/n})_{\geqslant 0}\,. \tag{1.80}$$

For $n = 2$ at level $m = 3$, we recover the KdV equation, while for $n = 3$ at level $m = 2$, we recover the so-called Boussinesq equation

$$3\ddot{u} = -u''' - 4(uu')'\,. \tag{1.81}$$

To a certain extent, these structures can also be generalized to the case with more than two Lax operators, which is one of the natural ways to generalize the Lax–Zakharov–Shabat approach to Nambu systems. In particular, the KP equation becomes a part of the hierarchy constructed using Lax triples; however, it seems to be not quite integrable.

2. Nambu mechanics

The Hamiltonian mechanics described in terms of Poisson brackets in the previous sections appear to be a particular case of more general Nambu mechanics. The dynamics of a Nambu system is determined in terms of a flow generated by $n-1$ Hamiltonians; correspondingly, the Poisson bracket is replaced by the Nambu bracket, which takes n entries. For the purposes of the present review, we are interested in algebraic structures relevant to M-theory that appear to be 3-brackets when speaking about M2-branes, and 5-brackets when speaking about M5-branes. The history of employing higher algebraic structures, such as n-algebras, to describe the dynamics of membranes starts with the work of Basu and Harvey [59], where an equation was proposed that describes N M2-branes ending on M5-branes and generalizes Hahm's equation that describes D1-branes ending on D3-branes. In string theory, k D1-branes ending on D3-branes from the point of view of the 4-dimensional world-volume theory manifest themselves as infinite spikes [60, 61]. On the other hand, this k monopole system satisfies the Bogomolnyi equation, which turns out to be the Nahm equation for the moduli space of monopoles in the gauge theory [62, 63]. The generalization proposed by Basu and Harvey involves a Nambu 3-bracket[3] instead of the usual Lie 2-bracket in Nahm's equation. Based on these results in [64, 65], a world-volume theory of multiple M2-branes has been proposed. This is a Chern–Simons-like theory based on a Nambu 3-bracket. Although this theory, known as BLG (Bagger–Lambert–Gustavsson), was later rewritten in the form of a more conventional gauge theory in [66] that does not involve 3-algebras, it is clear that a theory of brane dynamics must be formulated in terms of these kinds of higher algebraic structures. We will return to a more detailed discussion of these structures later in Section 4, while here we proceed with a review of Nambu mechanics and approaches to generalizations of integrability structures for such systems.

2.1 Nambu structure

A generalization of Poisson mechanics to a three-dimensional phase space with the evolution defined by two Hamiltonians was proposed by Nambu in [67]. Later, a detailed investigation of the geometry behind the Nambu mechanical system was performed by Takhtajan in [53]. In particular, it appears that Nambu systems are much more rigid than a Poisson system, which, in terms of M-theoretical degrees of freedom, manifests itself in the fact that BLG theory describes a stack of two M2-branes rather than an arbitrary number.

The dynamics of a Nambu system is defined by the following equation of motion:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H_1, \ldots, H_{n-1}, f\}\,, \tag{2.1}$$

where $H_1, \ldots, H_{n-1}$ denote Hamiltonians of the system, and $\{\ldots\}$ is an n-bracket that satisfies Nambu fundamental identities

$$\begin{aligned}
&\big\{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\big\} \\
&\quad + \big\{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\big\} + \ldots \\
&\quad + \big\{f_n, \ldots, f_{2n-2}, \{f_1, \ldots, f_{2n-1}\}\big\} \\
&\quad = \big\{f_1, \ldots, f_{n-1}, \{f_n \ldots f_{2n-1}\}\big\}\,.
\end{aligned} \tag{2.2}$$

These ensure that, for each of n functions f_i satisfying the Nambu equation, the bracket $\{f_1, \ldots, f_n\}$ also satisfies the same equation. As in the case of Poisson mechanics, the Nambu bracket can be realized in terms of an n-vector $\Omega \in \Gamma(\wedge^n TM)$, where M is the configuration space:

$$\Omega(\mathrm{d}f_1, \ldots, \mathrm{d}f_n) = \{f_1, \ldots, f_n\}\,. \tag{2.3}$$

In coordinate notations we have

$$\Omega^{m_1 \ldots m_n}\partial_{m_1} f_1 \ldots \partial_{m_n} f_n = \{f_1, \ldots, f_n\}\,. \tag{2.4}$$

A manifold M endowed globally with such an n-vector is called a Nambu–Poisson manifold and Ω is referred to as a Nambu–Poisson structure. Equivalently, we say that M is a Nambu–Poisson manifold if an $\mathbb{R}$-multilinear map

$$\{\ldots\} : \left[C^\infty(M)\right]^{\otimes n} \to C^\infty(M) \tag{2.5}$$

is defined on the algebra of (infinitely differentiable) functions $C^\infty(M)$. Given $n-1$ Hamiltonians $H_1, \ldots, H_{n-1}$, the n-bracket defines the evolution of a function f, or the so-called Nambu–Hamiltonian flow g^t.

Here is revealed the crucial difference between the Nambu and Poisson structure on a manifold, which is a much stronger set of constraints imposed on the n-vector by the fundamental identity. Acting by the n-vector Ω twice and imposing the fundamental identity, we have constraints following from terms with second and first derivatives of functions separately. For the former, we have an algebraic constraint

$$N_{MN} + P(N_{MN}) = 0\,, \tag{2.6}$$

where $M = \{m_1, \ldots, m_n\}$ and $N = \{n_1, \ldots n_n\}$ represent multi-indices, the tensor N_{MN} is defined as

$$\begin{aligned}
N_{m_1 \ldots m_n, n_1 \ldots n_n} &= \Omega^{m_1 \ldots m_n}\Omega^{n_1 \ldots n_n} + \Omega^{n_n m_1 m_3 \ldots m_n}\Omega^{n_1 \ldots n_{n-1} m_2} + \ldots \\
&\quad + \Omega^{n_n m_2 \ldots m_{n-1} m_1}\Omega^{n_1 \ldots n_{n-1} m_n} - \Omega^{n_n m_2 \ldots m_n}\Omega^{n_1 \ldots n_{n-1} m_1}\,,
\end{aligned} \tag{2.7}$$

and P interchanges the first and $(n+1)$th indices, i.e., m_1 and n_1. It can immediately be noticed that for $n = 2$ the condition is identically satisfied, while for $n \geqslant 3$ it is nontrivial.

[3] Strictly speaking, a 4-bracket, but one entry is always fixed.

The condition descending from terms linear in derivatives of functions reads

$$\sum_{l=1}^{n}\Big(\Omega^{lm_2\ldots m_n}\partial_l\Omega^{n_1\ldots n_n}+\Omega^{n_n lm_3\ldots m_n}\partial_l\Omega^{n_1\ldots n_{n-1}m_2}+\ldots$$
$$+\,\Omega^{n_n m_2\ldots m_{n-1}l}\,\partial_l\Omega^{n_1\ldots n_{n-1}m_n}\Big)=0\,. \tag{2.8}$$

For $n=2$, this is simply

$$\Omega^{l[m}\partial_l\Omega^{nk]}=0\,. \tag{2.9}$$

Hence, it can be concluded that, in contrast to Poisson manifolds, no totally antisymmetric globally defined tensor $\Omega^{m_1\ldots m_n}$ on a manifold is capable of defining a Nambu structure. On top of the usual differential constraints, one has to satisfy algebraic constraints.

An observable $F\in C^\infty(M)$ is called an integral of motion if

$$\{H_1,\ldots,H_{n-1},F\}=0\,. \tag{2.10}$$

The first $n-1$ integrals of motion are the Hamiltonians; the fundamental identity ensures that the Nambu bracket of integrals of motion is again an integral of motion. Naively, one can extend the notion of Liouville integrability to Nambu systems, defining an integrable Nambu system as having n integrals of motion, each in involution with regard to the Nambu bracket. However, the analogy does not go much further, since it is not evident how the action–angle variables can be introduced to completely integrate equations of motion. The same holds for the naive generalization of the proof that Nambu flow preserves phase space volume, although, for some cases, it can be shown explicitly.

2.2 Examples of Nambu systems

Although the construction of Nambu mechanics might seem rather exotic, it describes mechanical systems, many of which are familiar and even integrable in the usual sense. As the first example, consider an n-dimensional harmonic oscillator with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}(p_i^2+x_i^2)\,. \tag{2.11}$$

According to [68], this can be written as a Nambu system using other integrals of motion as additional Hamiltonians. Consider for example the case $n=2$, that is, a harmonic oscillator in two dimensions. We choose the following set of integrals:

$$\begin{aligned}H_1&=\frac{1}{2}\,(p_1^2+x_1^2)\,,\\ H_2&=\frac{1}{2}\,(p_2^2+x_2^2)\,,\\ H_3&=x_1p_2-x_2p_1\,.\end{aligned} \tag{2.12}$$

The Nambu bracket describing the system can then be chosen as

$$\{H_1,H_2,H_3,f\}=\frac{1}{p_1p_2+x_1x_2}\,\frac{\partial(H_1,H_2,H_3,f)}{\partial(p_1,p_2,x_1,x_2)}\,. \tag{2.13}$$

A simple check shows that the above reproduces equations of motion of the two-dimensional oscillator, and the bracket satisfies all the necessary conditions. This system is integrable in the usual sense.

Consider now the example presented by Nambu in the original paper, which describes the rotational dynamics of a rigid body with principle axes of inertia I_i and angular momenta L_i, where $i=1,2,3$. This system is commonly referred to as the Euler asymmetric top. Equations of motion are

$$\begin{aligned}\frac{\mathrm{d}L_1}{\mathrm{d}t}&=\frac{I_3-I_2}{I_2I_3}\,L_2L_3\,,\\ \frac{\mathrm{d}L_2}{\mathrm{d}t}&=\frac{I_1-I_3}{I_1I_3}\,L_1L_3\,,\\ \frac{\mathrm{d}L_3}{\mathrm{d}t}&=\frac{I_2-I_1}{I_1I_2}\,L_1L_2\,.\end{aligned} \tag{2.14}$$

These can be written in terms of Nambu equations for a system with the following two Hamiltonians:

$$H_1=\frac{L_1^2}{2I_1}+\frac{L_2^2}{2I_2}+\frac{L_3^2}{2I_3}\,,\quad H_2=\frac{1}{2}\left(L_1^2+L_2^2+L_3^2\right). \tag{2.15}$$

These are the full energy and the full momentum of the top, and the equations of motion can be written as

$$\frac{\mathrm{d}L_i}{\mathrm{d}t}=\epsilon^{ijk}\partial_jH_1\,\partial_kH_2\,, \tag{2.16}$$

where $\partial_i=\partial/\partial L_i$. This suggests the following definition for the Nambu bracket:

$$\{H_1,H_2,f\}=\epsilon^{ijk}\partial_iH_1\,\partial_jH_2\,\partial_kf, \tag{2.17}$$

the most natural choice for a three-dimensional system.

Equations of motion for the asymmetric Euler top have SU(2) symmetry and are also known under a different name, the Nahm system, when they arise in the theory of static monopoles. As we discuss in subsequent sections, such equations naturally appear in the description of branes ending on branes as world-volume spike-like monopoles. Their generalization, known as the Basu–Harvey equation, underlies the so-called BLG theory describing the dynamics of two M2-branes [64, 65]. The Nahm system is usually written as the following set of equations of motion:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_2x_3\,,\quad \frac{\mathrm{d}x_2}{\mathrm{d}t}=x_1x_3\,,\quad \frac{\mathrm{d}x_3}{\mathrm{d}t}=x_1x_2\,, \tag{2.18}$$

which can be expressed in the Nambu form, given $H_1=x_1^2-x_2^2$, $H_2=x_1^2-x_3^2$. This system is also integrable in the usual sense.

The least action principle can be extended to the Nambu mechanics, leading to an action presumably describing movement of open membrane boundaries. Following Takhtajan [53], we define

$$\omega_2=x^1\,\mathrm{d}x^2\wedge\mathrm{d}x^3-H_1\,\mathrm{d}H_2\wedge\mathrm{d}t\,, \tag{2.19}$$

the Poincaré–Cartan integral invariant 2-form for Nambu mechanics on the phase space $\tilde{X}$ parametrized by coordinates $\{x^1,x^2,x^3,t\}$. We now follow the same lines as in Section 1.4, where an invariant action for a Poisson system was constructed. Define vector field $\tilde{L}=\partial_t+L$ using

$$L=L^i\partial_i\,,\quad L^i=\frac{1}{2}\,\epsilon^{ijk}\,\frac{\partial(H_1,H_2)}{\partial(x^j,x^k)}\,. \tag{2.20}$$

Nambu equations then simply become $\dot{f} = \tilde{L}(f)$. The vector field $\tilde{L}$ is a line field of the 3-form $\mathrm{d}\omega_2$, i.e., $\iota_{\tilde{L}}\,\mathrm{d}\omega_2 = 0$, which simply follows from the explicit expression for the derivative

$$\mathrm{d}\omega_2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \mathrm{d}H_1 \wedge \mathrm{d}H_2 \wedge \mathrm{d}t\,. \tag{2.21}$$

Now, for a given 2-chain c in $\tilde{X}$, we denote $g^t(c)$ its Nambu–Hamiltonian phase flow, then a tube of phase trajectories will be given by $J^t c = \{g^\tau(c), 0 \leqslant \tau \leqslant t\}$. Finally, since $\partial J^t c = c - g^t(c)$ and given $\iota_{\tilde{L}}\,\mathrm{d}\omega_2 = 0$, we have

$$\int_c \omega_2 - \int_{g^t(c)} \omega_2 = \int_{J^t c} \mathrm{d}\omega_2 = 0\,, \tag{2.22}$$

which basically demonstrates invariance of the integrated two-form. Hence, for the action A of the system, we take the following integral:

$$A[c] = \int_c \omega_2 = \int_c x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - H_1 \wedge \mathrm{d}H_2 \wedge \mathrm{d}t\,. \tag{2.23}$$

The first term of the Takhtajan's action (2.23) has precisely the form of the Wess–Zumino term of the action of an M2-brane ending on an M5-brane. In the Nambu–Goto form and dropping all possible world-volume gauge fields, we can write for the M2-brane

$$S_{\mathrm{M2}} = \int_\Sigma \mathrm{d}^3\xi\, \det(-G) + \int_\Sigma C_3\,, \tag{2.24}$$

where G denotes the metric pull-back and $C_3 = C_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ denotes the 3-form living in the target space. Supposing the Wess-Zumino term dominates, and that the 3-form varies slowly along the boundary $\partial\Sigma$ of the M2-brane, we write for the boundary action

$$S_{\partial\mathrm{M2}} \propto C_{123} \int_{\partial\Sigma} x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,, \tag{2.25}$$

which is precisely expression (2.23). This provides yet more evidence that the Nambu mechanics with all its structures must be relevant to membrane dynamics in M-theory. A more detailed discussion of the above narrative can be found in [69]. We will return to the description of membrane dynamics in later sections.

2.3 Lax pair and generalized Yang–Baxter equation

In Section 2.2, we have seen that many dynamical systems, integrable in the usual sense, possess a Nambu structure. This makes it natural that integrability structures, such as the Lax pair, can be reformulated in terms of Nambu brackets, probably giving a criterion for three-dimensional integrability. To our knowledge, the program of defining integrability for Nambu dynamical systems has not been completed at least to the level of understanding we have for Poisson systems; however, certain progress has been made. The overall aim of this review is to collect observations that give hints at the integrability in the theory of membranes, or more generally, in three dimensional systems. Hence, we start with a Nambu system with Hamiltonians H_1 and H_2 and equations of motion given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H_1, H_2, f\} \tag{2.26}$$

and try to generalize the Lax pair construction. Naturally, the very first attempt would be to introduce a Nambu tri-bracket and to rewrite the above in terms of a Lax triple,

$$\dot{L} = [L, M, N]\,, \tag{2.27}$$

for a somehow defined Nambu bracket $[\,,\,,\,]$, which is indeed a useful construction for defining Nambu hierarchies. We will discuss these in a moment, as it is worthwhile to start with a different generalization that has more transparent links to M-theory.

Consider a Lax pair, that is, a pair of matrices $L, M \in \mathfrak{g}$, where $\mathfrak{g}$ is an algebra, such that

$$\dot{L} = [L, M]\,. \tag{2.28}$$

Given a tensor $\rho_{123} \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$, we define a 3-bracket

$$\{L_1, L_2, L_3\} = [\rho_{123}, L_1] + [\rho_{123}, L_2] + [\rho_{123}, L_3]\,, \tag{2.29}$$

where we denote

$$L_1 = L \otimes \mathbb{1} \otimes \mathbb{1}\,, \quad L_2 = \mathbb{1} \otimes L \otimes \mathbb{1}\,, \quad L_3 = \mathbb{1} \otimes \mathbb{1} \otimes L\,. \tag{2.30}$$

We impose a fundamental identity on this defined 3-bracket, that is, turn it into a Nambu structure. This restricts ρ_{123} to satisfy a condition similar to the classical Yang–Baxter equation. Let $\{T_a\} = \mathrm{bas}\,\mathfrak{g}$ denote the basis of the algebra, $f_{ab}{}^c$ denote its structure constants, and $\rho_{123} = \rho^{abc}\, T_a \wedge T_b \wedge T_c$. Then, the condition that is often referred to as the generalized Yang–Baxter equation can be written in the component form as

$$\rho^{a_1[a_2|a_6|}\rho^{a_3a_4|a_5|} f_{a_5a_6}{}^{a_7]} - \rho^{a_2[a_1|a_6|}\rho^{a_3a_4|a_5|} f_{a_5a_6}{}^{a_7]} = 0\,. \tag{2.31}$$

The 3-bracket then defines a Nambu system, whose integrals of motion can be expressed in the usual form $F_k = \mathrm{Tr}\, L^k$. It is a simple calculation to check that these are in involution with regard to this constructed Nambu bracket,

$$\{F_i, F_j, F_k\} = 0\,. \tag{2.32}$$

If there was a procedure allowing us to introduce action–angle variables and completely solve equations of motion using these integrals of motion, we would say that such a constructed system is integrable. However, the authors are not aware if these kinds of constructions for Nambu systems exist.

Equations (2.31) are fascinating in a different respect: they were first derived when investigating U-dualities of M-theory in [37, 38] (and earlier in [34] in the form of a vanishing R-flux). To be more precise, equations of 11-dimensional supergravity are known to be symmetric under a set of particular transformations, called Nambu–Lie U-dualities, whose underlying algebraic structure is formulated in terms of the so-called exceptional Drinfeld algebras. A particular subset of such generalized U-dualities (so-called deformations) can be parametrized by tensor ρ^{abc}, which has precisely the same meaning as above. The condition for such a deformed supergravity background to satisfy equations of 11d supergravity is precisely equation (2.31). We will return to this in more detail later, while here it is important to mention that these are mainly generalizations of similar structures in string theory conforming to the usual classical Yang–Baxter equation (see the summary section of [38]). The most important observation is that deformations parametrized by the matrix r^{ab} satisfying the classical Yang–Baxter equation preserve the integrability of the string sigma model. Hence, starting with different phenomena involving integrability and generalizing them in the more or less same way, we arrive at the same equation (2.31), which allows speculating further on these matters. We will do so in Section 4.6.

2.4 Lax triples and volume preserving flows

As has already been mentioned in Section 2.3, a more straightforward generalization of the Lax construction to Nambu systems is to introduce a Lax triple (multiple). It is convenient to proceed with a construction of integrable hierarchies based on the Lax triple and a Nambu 3-bracket. As in previous explicit examples, we will find that familiar systems, the hierarchy KP in this case, can be formulated in terms of such generalized structures. The approach we will be following here is the one advocated by Guha in [70] and further applied to various examples in [71, 72]. The idea is to generalize the method of [73, 74] developed to study the area preserving diffeomorphic KP equation. This approach, in turn, originated from studying self dual Einstein equations. Generalized flows of [70] briefly reviewed below are integrable in the same sense as the SDiff(2) flows of [73, 74], i.e., in the sense of the nonlinear graviton construction.

Consider a triple of generalized Lax operators L, M, N that are Laurent series in a spectral parameter λ with coefficients being functions of some variables p, q. By definition, the volume preserving integrable hierarchy is given by

$$\begin{aligned} \frac{\partial L}{\partial t_n} &= [B_{1n}, B_{2n}, L]\,, \\ \frac{\partial M}{\partial t_n} &= [B_{1n}, B_{2n}, M]\,, \\ \frac{\partial N}{\partial t_n} &= [B_{1n}, B_{2n}, N] \end{aligned} \tag{2.33}$$

with the additional involution constraint $[L, M, N] = 0$ ensuring volume preservation. Here, $[\,,\,,]$ is a Nambu tribracket satisfying the fundamental identity. As in the case of ordinary integrable hierarchies, we restrict the operators B_{1n} and B_{2n} to have only positive values of L, M:

$$B_{1n} = (L^n)\Big|_{n \geqslant 0}\,, \quad B_{2n} = (M^n)\Big|_{n \geqslant 0}\,. \tag{2.34}$$

The condition for the flows to commute boils down to an analogue of the Zakharov–Shabat equation:

$$\begin{aligned} &[\partial_m B_{1n}, B_{2n}, \bullet] - [\partial_n B_{1m}, B_{2m}, \bullet] + [B_{1n}, \partial_m B_{2n}, \bullet] \\ &\quad - [B_{1m}, \partial_n B_{2m}, \bullet] = \big[[B_{1n}, B_{2n}, B_{2m}], B_{2m}, \bullet\big] \\ &\quad - \big[[B_{1n}, B_{2n}, B_{1m}], B_{2m}, \bullet\big]\,. \end{aligned} \tag{2.35}$$

An important remark here is that, for SDiff(2) area preserving flows, equation (2.35) is simply the zero-curvature condition. We have already observed the same for Poisson integrable hierarchies, where the Zakharov–Shabat equation (2.35) contained the zero-curvature condition for the Lax connection 1-form. Hence, one would expect that the above contains a three-dimensional analogue of the zero-curvature condition of an analogue of the Lax connection, given by a 2-form.

From the geometric point of view, self-duality simply means a Ricci flat Kähler geometry; hence, SDiff(2) flows are naturally expressed in terms of a Kähler-like 2-form. The analogue here is a 3-form,

$$\begin{aligned} \Omega &= \sum_{n=1}^{\infty} \mathrm{d}B_{1n} \wedge \mathrm{d}B_{2n} \wedge \mathrm{d}t_n = \mathrm{d}\lambda \wedge \mathrm{d}p \wedge \mathrm{d}q \\ &\quad + \sum_{n=2}^{\infty} \mathrm{d}B_{1n} \wedge \mathrm{d}B_{2n} \wedge \mathrm{d}t_n\,, \end{aligned} \tag{2.36}$$

where we used the following notations: $t_1 = \lambda$, $B_{11} = p$, $B_{21} = q$. Given the flow equations (2.33), the 3-form can be expressed simply as

$$\Omega = \mathrm{d}L \wedge \mathrm{d}M \wedge \mathrm{d}N\,. \tag{2.37}$$

The 3-form Ω can be verified to be closed, $\mathrm{d}\Omega = 0$, giving

$$\mathrm{d}\left(M \wedge \mathrm{d}L \wedge \mathrm{d}N + \sum_{n=1}^{\infty} B_{1n} \wedge \mathrm{d}B_{2n} \wedge \mathrm{d}t_n\right) = 0\,. \tag{2.38}$$

Hence, the expression in brackets, at least locally, can be written as an exact form,

$$\mathrm{d}Q = M \wedge\, \mathrm{d}L \wedge\, \mathrm{d}N + \sum_{n=1}^{\infty} B_{1n} \wedge \mathrm{d}B_{2n} \wedge \mathrm{d}t_n\,, \tag{2.39}$$

that is an analogue of the Krichever potential, i.e., contains the action.

Let us now consider an example of hierarchy generated by a volume preserving the Lax triple equations. Here, we follow [72], where the KP hierarchy was first written in terms of Lax triples. The hierarchy is defined as

$$\begin{aligned} &\frac{\mathrm{d}L}{\mathrm{d}t_{mn}} = [B_m, B_n, L]\,, \\ &L = \partial + \sum_{i=0}^{\infty} v_i(t)\partial^{-i-1}\,, \\ &B_n = (L^n)_{\geqslant 0}\,, \quad n \geqslant 0\,, \\ &B_0 = 1\,, \end{aligned} \tag{2.40}$$

where, as before, the subscript $\geqslant 0$ means that only operators with positive powers of ∂ are kept. The standard KP hierarchy is recovered from the above when $m = 0$:

$$\frac{\mathrm{d}L}{\mathrm{d}t_{0n}} = [B_0, B_n, L] \equiv [B_n, L]\,. \tag{2.41}$$

The most interesting question is whether integrable hierarchies can be derived for other cases with $m \neq 0$. According to [72], it has affirmative answer: at least for certain given pairs (B_m, B_n), one obtains integrable equations that are already present in the KP hierarchy. It is tempting to claim that the integrable KP hierarchy can be equivalently written in terms of the usual Lax equation or in terms of the generalized equation for Lax triples. However, a subtlety standing in the way of this interpretation was also observed in [72] when analyzing the hierarchy further for larger values of (m, n): the hierarchy contains equations that do not pass the Painlevé integrability test. Hence, not all equations of the generalized Lax triple hierarchy seem to be integrable; however, those that are not contain soliton solutions.

To conclude, we observe that at least some of the steps in the standard path for constructing integrability structures can be repeated for Nambu systems. In particular, one may introduce infinitely many conserved charges, a Lax operator generating them, an involution condition, volume preserving flows, and hierarchies based on Lax triples. Moreover, a Nambu bracket of a dynamical system can be generated by an analogue ρ of the classical r-matrix, which is no longer a matrix, however naturally it appears in the context of U-duality symmetries of M-theory. The same object is in principle expected to appear in a quasi-classical limit of the tetrahedron equation describing factorization of the scattering process of straight strings. Combining all these observations together, it is tempting to conclude that structures reviewed above must be relevant when describing the integrability of $2+1$-dimensional systems, i.e., membranes. As we will discuss in more detail later in Section 4, one indeed finds similar constructions when approaching from the M-theory and supergravity side. In particular, the dynamics of membranes naturally leads to tri-brackets via the Basu–Harvey equation, the object ρ naturally appears in the open membrane metric, and an analogue of the evolution operator can naturally be constructed using loop algebras. Loop algebras, in turn, appear in the analysis of M5-branes holding boundaries of M2-branes, which scatter precisely as strings in the 6-dimensional world-volume.

3. 10d supergravity and strings

Methods for investigating integrability structures for two-dimensional systems briefly reviewed above can well be applied to the dynamics of a fundamental string propagating on a background defined by a solution to supergravity equations. As we discuss in more detail below, the dynamics of the string on certain backgrounds defined in terms of the two-dimensional nonlinear sigma model (NLSM) is classically integrable, i.e., a Lax connection can be constructed. Among other similar results, the integrability of the string on certain backgrounds is of particular interest, e.g., in the context of holography. Indeed, holographic correspondence basically says that the same system can be described equivalently in terms of very different variables. For example, in the case of AdS/CFT, correspondence equivalence of descriptions in terms of closed and open strings of the near horizon region of a D3-brane results in correspondence between 10d supergravity on $\mathrm{AdS}_5\times\mathbb{S}^5$ and $\mathcal{N}=4$ $d=4$ super Yang–Mills theory. Now, since the string on $\mathrm{AdS}_5\times\mathbb{S}^5$ is known to be integrable [6] (see also [75] for a review), the same can be claimed for the gauge theory, as this is simply a different way of parametrizing the same dynamics. Indeed, integrability structures of $\mathcal{N}=4$ $d=4$ have been addressed from different perspectives, which include, e.g., the thermodynamic Bethe ansatz and the spectral curve. Although intimately related, these approaches stand beyond the scope of our review and have been well covered in various reviews [1–5]. In this section, we focus on integrable nonlinear 2d sigma models on group manifolds and (super-)coset spaces and their continuous Yang–Baxter deformations.[4] Such deformations that preserve the integrability of a 2d NLSM were introduced in [19] for a string on a group manifold and further generalized to coset spaces in [20]. Their extension to general solutions of supergravity suggested first in [29, 30] and further developed in [31, 32] does not seem to have a straightforward relation to integrability; however, it does allow introducing similar structures for membranes, i.e., 3d NLSM, which will be discussed further in Section 4.

3.1 Yang–Baxter deformed 2d sigma models

An approach to finding a Lax pair for every 2d principal sigma model on a simple compact group G based on the inverse scattering method was suggested by Zakharov and Mikhailov in [78]. A particular example of this model is the one with $G=\mathrm{SU}(2)$, which was found to belong to a continuous family of integrable models by Cherednik in [79]. Given $g\in\mathrm{SU}(2)$, the model is defined by the following action:

$$S=-\frac{1}{2}\int d\tau\, d\sigma\ \mathrm{Tr}\left[\mathrm{Ad}(\partial_+ g\,g^{-1})J\,\mathrm{Ad}(\partial_- g\,g^{-1})\right], \tag{3.1}$$

where the diagonal matrix $J=\mathrm{diag}\,[J_1,J_2,J_3]$ stands for a deformation of the Killing form. In [19], it was shown that model (3.1) can be understood in terms of Yang–Baxter sigma models, i.e., sigma models deformed by a classical r-matrix R that satisfies the (modified) classical Yang–Baxter equation

$$[RM,RN]-R\big([RM,N]+[M,RN]\big)=c[M,N], \tag{3.2}$$

where $M,N\in\mathbf{su}(2)$. It is worth mentioning that, for historical reasons, the classical r-matrix defining deformations of 2d sigma models is denoted by a capital letter R, which in the mathematical literature is reserved for the quantum r-matrix, i.e., the one solving the quantum Yang–Baxter equation. To keep notations correlated with the rest of the string theory literature, we follow this historical rule, which, however, should not cause much confusion.

The deformation procedure used in [79] is specific to the SU(2) group manifold and cannot be directly generalized to any group manifold taken as a target space. The approach of [19] suggests that the following action be considered:

$$S=\int\left\langle g^{-1}\partial_+ g,(\mathbb{1}+\epsilon R)^{-1}g^{-1}\partial_- g\right\rangle, \tag{3.3}$$

where the angle brackets denote the Killing form on the Lie algebra $\mathfrak{g}$ of a simple compact Lie group G. Further in [80], this model was shown to be integrable, and the corresponding Lax connection can be written as follows:

$$A_\pm(\lambda)=\left(\epsilon^2\mp\epsilon R-\frac{1+e^2}{1\pm\lambda}\right)\left(\mathbb{1}\pm\epsilon R\right)^{-1}g^{-1}\partial_\pm g, \tag{3.4}$$

where $\lambda\in\mathbb{C}$ is a complex spectral parameter. When $\epsilon=0$, the above reproduces precisely the Lax connection introduced by Zakharov and Mikhailov.

The procedure of deforming principal sigma models preserving integrability was generalized to sigma models on coset spaces in [20]. The action for the deformed sigma model on a coset space G/F now involves the so-called dressed r-matrix $R_g=\mathrm{Ad}g^{-1}\,R\,\mathrm{Ad}g$:

$$S=\int\left\langle (g^{-1}\partial_+ g)^{(1)},\frac{1+\eta^2}{\mathbb{1}-\eta R_g P_1}(g^{-1}\partial_- g)^{(1)}\right\rangle. \tag{3.5}$$

Here, P_1 is the projection onto the subspace $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of G, which corresponds to the value $\sigma=+1$ of an order-2 automorphism $\sigma:\mathfrak{g}\to\mathfrak{g}$. Using this approach, an integrable deformation of the $\mathrm{AdS}_5\times\mathbb{S}^5$ superstring in the

[4] Other examples of integrable strings that we will not focus on here are the λ deformations of [76, 77].

Metsaev–Tseytlin formalism was constructed in [21], which we will discuss in a moment. Two important observations have been made concerning such a deformed superstring: (i) the deformed sigma model can be understood as a string propagating on a metric background (ABF) that does not satisfy supergravity equations [25]; (ii) kappa symmetry of the GS superstring still holds [27]. Remarkably, kappa-symmetry of the GS superstring has been shown to imply a slight generalization of supergravity equations [26], which are precisely the ones solved by the ABF background. Hence, it can be concluded that the space of consistent vacua, at least for the GS superstring, is wider than the space of solutions to supergravity equations, and, moreover, certain points in this space are connected by Yang–Baxter deformations.[5] Later in Section 4, we will see that the same picture holds for 11d supergravity, although several loose ends must be tied up, such as kappa invariance of the membrane on similar deformations.

To date, great progress has been made in understanding Yang–Baxter deformed sigma models on group manifolds and coset spaces and in finding new examples. Let us mention some of the most compelling results. A slight generalization of the q-deformation of [21] has been suggested in [22] that contains twists of the R-operator, allowing us to perform partial deformations affecting only of the sphere part of the $\mathrm{AdS}_5 \times \mathbb{S}^5$ superstring. Recall that the r-matrix describing the standard q-deformations is a so-called Drinfeld–Jimbo type:

$$R_{\mathrm{DJ}} = \alpha \sum_M \frac{1}{\mathrm{Tr}\,[e_M f_M]}\, e_M \wedge f_M\,, \tag{3.6}$$

where the index M labels positive e_M and negative f_M roots of the isometry algebra. Jordanian matrices are constructed by a linear twist of R_{DJ} by an arbitrary (bosonic) root. A general class of integrable deformations of sigma models on coset spaces whose Poisson brackets are related to those of [21] by an analytic continuation was found in [84]. This generalizes earlier studies [76, 85, 86], where integrable deformations were constructed as interpolations between exact WZW CFTs. (For more details see, e.g., reviews [87, 88], PhD thesis [89], and references therein.) Integrable deformations of the string on $\mathrm{AdS}_n \times \mathbb{S}^n$ were intensively investigated in [90–93]. Given the discussion at the beginning of this section it is also worth mentioning [29, 30, 94], where a gauge theory interpretation of integrable deformations was presented using the formalism of Drinfeld twists. This generalizes the known interpretation of Abelian deformations, such as the $\mathrm{U}(1)\times\mathrm{U}(1)$ deformation of Lunin and Maldacena as the twisting of the fields product, to the non-Abelian case (for more details on the Abelian case, see, e.g., [95]). The result is a noncommutative Yang–Mills theory, which is expected, since the generators of the deformations are taken along the AdS space.

3.2 Integrable deformation of the $\mathrm{AdS}_5 \times \mathbb{S}^5$ superstring

Let us illustrate the formalism of integrable Yang–Baxter deformations by the example of η-deformation of the $\mathrm{AdS}_5 \times \mathbb{S}^5$ superstring following [20]. We start by recalling the construction of the Lax connection for the Metsaev–Tseytlin superstring [96]. The Type IIB background $\mathrm{AdS}_5 \times \mathbb{S}^5$ is supported by the nonvanishing self-dual RR five-form flux, and hence the superstring on this background can conveniently by described using the Green–Schwarz formalism. Such a superstring lives in the following symmetric superspace:

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)} \supset \frac{\mathrm{SU}(2,2)\times\mathrm{SU}(4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)} \cong \frac{\mathrm{SO}(4,2)\times\mathrm{SO}(6)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)} = \mathrm{AdS}_5\times\mathbb{S}^5\,. \tag{3.7}$$

The corresponding GS sigma model is formulated in terms of a 1-form $A \in \mathbf{su}(2,2|4)$ built out of a supergroup element $\mathbf{g} \in \mathrm{SU}(2,2|4)$ as

$$A = -\mathbf{g}^{-1}\,\mathrm{d}\mathbf{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}\,. \tag{3.8}$$

Here, the decomposition is due to $\mathbb{Z}_4$-grading of $\mathbf{su}(2,2|4)$ induced by a certain order 4 automorphism. This defined 1-form A is flat:

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0\,. \tag{3.9}$$

The action of the superstring on $\mathrm{AdS}_5\times\mathbb{S}^5$ then takes the form of the so-called Metsaev–Tseytlin superstring [97]:

$$S_{\mathrm{MT}} = -\frac{g}{2}\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\left[\gamma^{\alpha\beta}\,\mathrm{STr}\,(A^{(2)}_\alpha A^{(2)}_\beta) + \kappa\epsilon^{\alpha\beta}\,\mathrm{STr}\,(A^{(1)}_\alpha A^{(3)}_\beta)\right],$$
$$\sigma \in (-r, r)\,, \tag{3.10}$$

with $g = R^2/(2\pi\alpha')$, where R is the $\mathbb{S}^5$ radius and α' is the string slope, $\gamma^{\alpha\beta} = \sqrt{-h}\,h^{\alpha\beta}$, where $h^{\alpha\beta}$ is the inverse worldvolume metric, STr denotes the supertrace, and $\epsilon^{\alpha\beta}$ is the worldvolume totally antisymmetric tensor, $\epsilon^{\tau\sigma} = 1$.

For further discussion, it is convenient to introduce tensors

$$P^{\alpha\beta}_{\pm} = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta})\,, \tag{3.11}$$

which are orthogonal projectors in cases $\kappa = \pm 1$, and four projectors P_k onto the corresponding subspaces of $\mathbf{su}(2,2|4)$ with grade $k = 0, \ldots, 3$, such that $A^{(k)} = P_k A$. Also, we will use the following conventions for projected vectors $V^\alpha_\pm = P^{\alpha\beta}_\pm V_\beta$. In these notations,

$$S_{\mathrm{GS}} = -\frac{g}{2}\int \mathrm{d}\tau\,\mathrm{d}\sigma\,P^{\alpha\beta}_-\,\mathrm{STr}\left(A_\alpha[P_1 + 2P_2 - P_3]A_\beta\right). \tag{3.12}$$

The GS action (3.12) must obey a local fermionic symmetry, called κ-symmetry. It halves the number of world-volume fermionic degrees of freedom, making them consistent with the space–time supersymmetry of the string physical spectrum. Its transformation acts on A as

$$\delta_\epsilon A = -\,\mathrm{d}\epsilon + [A, \epsilon]\,, \quad \epsilon = \epsilon^{(1)} + \epsilon^{(3)}\,, \tag{3.13}$$

with

$$\begin{aligned} \epsilon^{(1)} &= A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}\,,\\ \epsilon^{(3)} &= A^{(2)}_{\alpha,+}\kappa^{(3),\alpha}_- + \kappa^{(3),\alpha}_- A^{(2)}_{\alpha,+}\,. \end{aligned} \tag{3.14}$$

An interesting fact is that the κ-invariance of the action requires $\kappa = \pm 1$, and hence $P^{\alpha\beta}_\pm$ are indeed orthogonal projectors.

[5] See, however, the discussion concerning the consistency of generalized supergravity backgrounds on which the string is only scale invariant or the corresponding FT counterterm seems to be nonlocal [81–83].

Equations of motion for (3.12) can be written in the following compact form:

$$
\begin{aligned}
0 &= \partial_\alpha(\gamma^{\alpha\beta}A^{(2)}_\beta) - \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(2)}_\beta] \\
&\quad + \frac{1}{2}\,\kappa\epsilon^{\alpha\beta}\big([A^{(1)}_\alpha, A^{(1)}_\beta] - [A^{(3)}_\alpha, A^{(3)}_\beta]\big)\,, \\
0 &= P^{\alpha\beta}_-[A^{(2)}_\alpha, A^{(3)}_\beta]\,, \\
0 &= P^{\alpha\beta}_+[A^{(2)}_\alpha, A^{(1)}_\beta]\,.
\end{aligned}
\tag{3.15}
$$

The global $\mathrm{PSU}(2,2|4)$ symmetry of the sigma model corresponds to conservation of the following Noether's current:

$$
J^\alpha = g\mathbf{g}\left[\gamma^{\alpha\beta}A^{(2)}_\beta - \frac{1}{2}\,\kappa\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\right]\mathbf{g}^{-1}, \quad \partial_\alpha J^\alpha = 0\,. \tag{3.16}
$$

This model is classically integrable, meaning that the equations of motion (3.15) together with the flatness condition for A (3.9) are equivalent to the zero curvature condition,

$$
\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0\,, \tag{3.17}
$$

for a Lax connection L_α defined by

$$
L_\alpha = \ell_0 A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2\gamma_{\alpha\beta}\epsilon^{\beta\rho}A^{(2)}_\rho + \ell_3 A^{(1)}_\alpha + \ell_4 A^{(3)}_\alpha\,. \tag{3.18}
$$

The prefactors must be chosen as

$$
\begin{aligned}
&\ell_0 = 1\,, \quad \ell_1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \\
&\ell_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right), \quad \ell_3 = z\,, \quad \ell_4 = \frac{1}{z}\,,
\end{aligned}
\tag{3.19}
$$

where z is the spectral parameter and $\kappa = \pm 1$. This means that the requirement of integrability automatically leads to the same constraints as κ-invariance.

Let us now briefly discuss the results of [21], where the integrability of the superstring on the η-deformed $\mathrm{AdS}_5\times\mathbb{S}^5$ has been demonstrated. Note that for A we use the convention (3.8) of [96], which differs from [21] by a '$-$' sign. The superstring on the η-deformed $\mathrm{AdS}_5\times\mathbb{S}^5$ can be written as the following η-deformation of the Metsaev–Tseytlin superstring (3.12):

$$
S^\eta_{\mathrm{MT}} = -g\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\frac{(1+\eta^2)^2}{2(1-\eta^2)}\,P^{\alpha\beta}_-\,\mathrm{STr}\left(A_\alpha\,P\circ\frac{1}{1-\eta R_{\mathbf{g}}\circ P}(A_\beta)\right), \tag{3.20}
$$

where

$$
P = P_1 + \frac{2}{1-\eta^2}\,P_2 - P_3\,, \quad \tilde{P} = -P_1 + \frac{2}{1-\eta^2}\,P_2 + P_3\,. \tag{3.21}
$$

The crucial ingredient here is a skew-symmetric operator on $\mathfrak{su}(2,2|4)$, which acts as $R_{\mathbf{g}} = \mathrm{Ad}^{-1}_{\mathbf{g}}\circ R\circ\mathrm{Ad}_{\mathbf{g}}$ and solves the modified classical Yang–Baxter equation. Specifically, $\forall M, N\in\mathfrak{su}(2,2|4)$:

$$
[RM, RN] - R\big([RM, N] + [M, RN]\big) = [M, N] \tag{3.22}
$$

and $\mathrm{STr}\,(MRN) = -\mathrm{STr}\,(RMN)$.

For $\eta = 0$, action (3.20) reproduces (3.12). The following vectors,

$$
J_\alpha = \frac{1}{1-\eta R_{\mathbf{g}}\circ P}(A_\alpha)\,, \tag{3.23}
$$

$$
\tilde{J}_\alpha = \frac{1}{1+\eta R_{\mathbf{g}}\circ\tilde{P}}(A_\alpha)\,, \tag{3.24}
$$

allow writing equations of motion for (3.20) in the most convenient way:

$$
0 = P(\partial_\alpha J^\alpha_-) + \tilde{P}(\partial_\alpha\tilde{J}^\alpha_+) - \big[\tilde{J}_{+\alpha}, P(J^\alpha_-)\big] - \big[J_{-\alpha}, \tilde{P}(\tilde{J}^\alpha_+)\big]\,. \tag{3.25}
$$

Finally, we can define

$$
\begin{aligned}
L^\alpha_+ &= \tilde{J}^{\alpha(0)}_+ + \lambda\sqrt{1+\eta^2}\,\tilde{J}^{\alpha(1)}_+ + \lambda^{-2}\,\frac{1+\eta^2}{1-\eta^2}\,\tilde{J}^{\alpha(2)}_+ \\
&\quad + \lambda^{-1}\sqrt{1+\eta^2}\,\tilde{J}^{\alpha(3)}_+\,,
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
M^\alpha_- &= J^{\alpha(0)}_- + \lambda\sqrt{1+\eta^2}\,J^{\alpha(1)}_- + \lambda^2\,\frac{1+\eta^2}{1-\eta^2}\,J^{\alpha(2)}_- \\
&\quad + \lambda^{-1}\sqrt{1+\eta^2}\,J^{\alpha(3)}_-
\end{aligned}
\tag{3.27}
$$

with the spectral parameter λ. Then, the whole set of equations of motion (3.25) and the zero curvature equations (3.9) are equivalent to

$$
\partial_\alpha L^\alpha_+ - \partial_\alpha M^\alpha_- - [M_{-\alpha}, L^\alpha_+] = 0\,. \tag{3.28}
$$

Introducing $\mathcal{L}_\alpha = L_{+\alpha} + M_{-\alpha}$, we obtain the standard Lax equation

$$
\partial_\alpha\mathcal{L}_\beta - \partial_\beta\mathcal{L}_\alpha - [\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0\,. \tag{3.29}
$$

This confirms the integrability of the η-deformed sigma model. Also, it is worth mentioning that action (3.20) is κ-invariant.

3.3 Poisson–Lie T-duality

Yang–Baxter deformations appear to be a particular example of Poisson–Lie T-duality and in particular can be represented as a non-Abelian T-duality with an additional parameter, whose inverse is precisely the deformation parameter [98]. While a detailed review of Poisson–Lie T-dualities is not really necessary to define Yang–Baxter equations, this part of the story is still important for the purposes of the present review. The reason is that there are two known ways to arrive at a 3d generalization of the classical Yang–Baxter equation: use algebraic arguments based on generalizations of the Drinfeld double construction [37, 38] or address deformations from the supergravity side [34, 36]. The former are based on a generalization of the U-duality symmetry of the membrane to the so-called Nambu–Lie symmetry along the same lines that lead from the ordinary T-duality to Poisson–Lie duality. This approach is restricted to only group manifolds, while for a general 11d background, one follows the latter approach, which is based on exceptional field theory and eventually again on U-duality, now understood as a local symmetry of a specially extended space. For this reason, we find it useful to show the relation between Yang–Baxter deformations and Poisson–Lie T-dualities to further exploit the logic in the 11d case.

Poisson–Lie T-duality transformations were suggested in [99] to answer the question of whether an inverse of a non-Abelian T-duality transformation can be constructed. The crucial observation here is that the standard (Abelian) T-duality defined by Buscher rules preserves the U(1) isometries on which it is constructed. To generalize these dualities, one may start with a non-Abelian group of symmetries G of a background rather than the Abelian group of a torus [100]. These non-Abelian T-duality transformations in general break the initial isometries, and it is not very obvious how an inverse transformation can be constructed. To T-dualize backgrounds without isometries in the usual sense, in [99], an algebraic approach was suggested based on the notion of Drinfeld doubles, where the isometry actually exists and is hidden inside the algebraic structure of the double. Let us give more details on the construction focusing primarily on Yang–Baxter deformations inside the Drinfeld double. For a review of non-Abelian T-dualities and their applications as a solution generating technique, see, e.g., [88, 101, 102] (for more details on Poisson–Lie T-dualities, including their realization in double field theory, see, e.g., [103–106]). For a recent discussion related to Poisson–Lie and non-Abelian T-duality symmetry for the quantum superstring, see [107, 108], and for a general approach to solution generating techniques, see [109, 110].

To go beyond T-dualization along isometries defined by conserved charges in [99], a conception of the noncommutative conservation law has been introduced. For a sigma model on a group manifold G, the noncommutative conservation law is defined as

$$\mathrm{d}J_a = \frac{1}{2}\,\tilde{f}_a{}^{bc} J_b \wedge J_c\,, \tag{3.30}$$

where the currents defined by 1-forms J_a correspond to the standard action of the group G on itself. In coordinates, the group action can be written as $\delta x^i = v_a{}^i \epsilon^a$. Under such coordinate shifts, the action of the 2d sigma model transforms as

$$\delta S = \int \mathrm{d}^2\sigma\, \epsilon^a \mathcal{L}_{v_a}(E_{ij})\,\partial x^i \bar{\partial} x^j + \int \mathrm{d}\epsilon^a \wedge J_a\,, \tag{3.31}$$

where $E = G + B$. Integrating the last term in (3.31) by parts and assuming proper boundary conditions, we see that the action stays invariant under the transformation when either the usual conservation law $\mathrm{d}J_a = 0$ or the noncommutative conservation law (3.30) holds together with

$$\mathcal{L}_{v_a}(E_{ij}) = \tilde{f}_a{}^{bc} v_a{}^k v_b{}^l E_{ki} E_{jl}\,. \tag{3.32}$$

The integrability of this constraint implies the following relation between the quantities $\tilde{f}_a{}^{bc}$ and structure constants $f_{ab}{}^c$ of the Lie algebra $\mathfrak{g}$ of the isometry group G:

$$4\tilde{f}_{[a}{}^{a[c} f_{b]e}{}^{d]} - \tilde{f}_e{}^{cd} f_{ab}{}^e = 0\,. \tag{3.33}$$

Together with the integrability condition for (3.30), that is, $\tilde{f}_e{}^{g[a}\tilde{f}_g{}^{bc]} = 0$, and the Jacobi identity for $f_{ab}{}^c$, this has the form of the compatibility condition for the structure of a Lie bi-algebra on $\mathfrak{g}$. In [99], it was shown that, given the algebras $\mathfrak{g}$ defined by $f_{ab}{}^c$ and $\tilde{\mathfrak{g}}$ defined by $\tilde{f}_a{}^{bc}$ for a Drinfeld double $\mathcal{D}$ (to be defined below), sigma models on backgrounds realizing $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are equivalent. Equivalence here is meant in the sense that both sigma models can be obtained translating a d-dimensional linear space $\mathcal{E} = T_e D$ tangent to Drinfeld group D at unity e by either $\exp \mathfrak{g}$ or $\exp \tilde{\mathfrak{g}}$. In more physical terms: the equations of motion are the same.

Let us give more details on the Drinfeld algebra construction, avoiding however the categorical language of commutative diagrams, since, working with explicit backgrounds, one always has to choose a specific basis. Hence, let $\{T_a\} = \mathrm{bas}\,\mathfrak{g}$ and $\{\tilde{T}^a\} = \mathrm{bas}\,\tilde{\mathfrak{g}}$ with $a = 1, \ldots, d$, then the Drinfeld double can be realized as a Manin triple $(T_a, \tilde{T}^a, \eta)$, where η is a nondegenerate quadratic form defined as

$$\eta(T_a, \tilde{T}^b) = \delta_a{}^b\,. \tag{3.34}$$

Commutation relations in this basis read

$$\begin{aligned}
[T_a, T_b] &= f_{ab}{}^c T_c\,,\\
[T_a, \tilde{T}^b] &= \tilde{f}_a{}^{bc}\, T_c - f_{ac}{}^b \tilde{T}^c\,,\\
[\tilde{T}^a, \tilde{T}^b] &= \tilde{f}_c{}^{ab}\, \tilde{T}^c\,.
\end{aligned} \tag{3.35}$$

To define Poisson–Lie T-dualities in these terms, it is convenient to denote the whole basis by $\{T_A\} = \{T_a, \tilde{T}^a\}$ and structure constants by $F_{AB}{}^C$, i.e., $[T_A, T_B] = F_{AB}{}^C T_C$. The quadratic form is then given by the invariant tensor η_{AB} of the $\mathrm{O}(d,d)$ group,

$$\eta_{AB} = \begin{bmatrix} 0 & \delta_a{}^b \\ \delta_c{}^d & 0 \end{bmatrix}. \tag{3.36}$$

Poisson–Lie T-duality transformations are then such $\mathrm{O}(d,d)$ rotations of the basis

$$T'_A = C_A{}^B T_B \tag{3.37}$$

that preserve the Drinfeld double. To construct a geometric realization, one takes the so-called geometric subgroup, which is by definition the one generated by $\mathfrak{g}$, and constructs right-invariant 1-form $r = g^{-1}\,\mathrm{d}g$, where $g \in G$. The dual background is then constructed as the geometric realization of the transformed geometric subgroup generated by $\mathfrak{g}'$. For a more detailed description of this algorithm, see [37, 39, 111]. The search for such a matrix $C_A{}^B \in \mathrm{O}(d,d)$ preserving a Drinfeld double is the most complicated task in constructing Poisson–Lie T-dual backgrounds. Certain classification results for lower dimensional Lie algebras are available in the literature [112–114]. For special matrices $C_A{}^B$ corresponding to inner automorphisms of the $\mathrm{O}(d,d)$ group (factorized T-dualities), one has discrete transformations that are guaranteed to preserve a given Drinfeld double. An example of such a transformation is switching $\mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}$, which is a different way of saying that a given Drinfeld algebra can be decomposed into two Manin triples. Moreover, in [115], examples of Drinfeld doubles were found that can be decomposed into more than three Manin triples, which has been called Poisson–Lie T-plurality.

After this long introduction, we are finally at the point of defining Yang–Baxter deformations in terms of Drinfeld doubles and Poisson–Lie symmetries. Consider a continuous family of deformations of a given Drinfeld algebra $\tilde{f}_a{}^{bc} = r^{d[b} f_{ad}{}^{c]}$, which corresponds to deformation of the Drinfeld algebra with $\tilde{f}_a{}^{bc} = 0$ by the following matrix:

$$C_A{}^B = \begin{bmatrix} \delta_a{}^b & r^{ac} \\ 0 & \delta_d{}^c \end{bmatrix}. \tag{3.38}$$

Such defined dual structure constants $\tilde{f}_a{}^{bc}$ satisfy all compatibility conditions if r^{ab} satisfies classical Yang–Baxter equation

$$r^{e[a}r^{|f|b}f_{ef}{}^{d]} = 0\,. \tag{3.39}$$

In the language of double field theory to be discussed below, this matrix corresponds simply to a special case of generalized diffeomorphisms of extended space [116, 117]. Since such a Yang–Baxter deformation changes the initial Drinfeld algebra, it is strictly speaking not a duality in the usual sense of a relation between two different descriptions of the same physics. However, geometric realization of the deformed Drinfeld double solves supergravity equations, given the initial background is a solution and the so-called unimodularity constraint $r^{ab}f_{ab}{}^c = 0$ holds, which is best seen in the formalism of double field theory, which we now turn to.

3.4 Bi-vector deformations of 10d supergravity backgrounds

As has already been mentioned, the construction above is restricted to group manifolds and coset spaces (in the case of non-Abelian T-duality (NATD) [98]). The reason can be seen in the fact that it is very algebraic in its nature and heavily relies on the usage of right-invariant forms as target-space vielbeins. A more field theoretic approach to Yang–Baxter deformations was suggested in [29, 30] and further developed in [31, 32, 117, 118]. The approach of [29, 30] was based on noticing that Yang–Baxter deformations of a background given by metric G can be represented in the form of an open-closed string map

$$(G^{-1} + \beta)^{-1} = g + b\,, \tag{3.40}$$

where g and b are the deformed metric and deformed 2-form Kalb–Ramon field, and the deformation parameter $\beta = r^{ab}k_a \wedge k_b$ is defined in terms of Killing vectors $k_a = k_a{}^m\partial_m$ of the initial background. Let us note here that the deformation parameter β^{mn} enters equations very similar to the noncommutativity parameter of Seiberg and Witten [119]. Although the deep meaning of this is not clear, exactly the same is observed in 11 dimensions. There, the deformation parameter has 3 indices Ω^{mnk} precisely as the membrane noncommutativity parameter, and generalized Yang–Baxter deformation rules have precisely the form of the open-closed membrane map to be discussed in Section 4.4.

Since we are not able to comment more on this very intriguing relation, we prefer to formulate Yang–Baxter deformations in the $O(10,10)$ covariant language that is ready to generalize to 11 dimensions. This is the language of double field theory, where all supergravity fields depend on a doubled set of coordinates $\{\mathbb{X}^M\} = \{x^m, \tilde{x}_m\}$ subject to the so-called section constraint

$$\eta^{MN}\partial_M \bullet \partial_N \bullet = 0\,, \tag{3.41}$$

where the bullets stand for any of the fields of the theory and their combinations. Basically, the section constraint removes the dependence on half of the coordinates, e.g., on the so-called nongeometric ones $\{\tilde{x}_m\}$. In what follows, we will always assume this choice of the section. The idea of doubling the coordinates follows from the early work by Fradkin and Tseytlin [120], where right and left moving modes of a closed string on a torus are considered independently, hence the alternative reference for $\tilde{x}_m$ as winding coordinates. The notion of the section condition and generalized Lie derivative were introduced in [121, 122]. Full formulation of double field theory was developed in [123] for the NS–NS sector, in [124] for the full bosonic field content of supergravity, and in [125] to include supersymmetry. For the purposes of this review, double field theory simply provides a convenient choice of parametrization of fields for which Yang–Baxter deformations become a linear $O(10,10)$ transformation. Hence, we will provide only the necessary bits of the formalism, and for a more detailed review the reader is referred to [126–128].

In the covariant formalism, the metric and the B-field of supergravity are packed into the so-called generalized metric $\mathcal{H}_{MN} \in O(10,10)/O(1,9)\times O(9,1)$. For our purposes, it is more convenient to introduce a generalized vielbein $\mathcal{H}_{MN} = E_M{}^A E_N{}^B \mathcal{H}_{AB}$, where $\mathcal{H}_{AB}$ is a constant unity matrix. The generalized vielbein in the upper triangular form can be defined by exponentiating the space–time vielbein $e_m{}^a$ and b_{mn} with certain generators of $O(10,10)$. For that purpose, we decompose the generators with regard to the action of the geometric $GL(10)$ subgroup, i.e., parameterize the generators as follows: $\{T_{ab}, T_a{}^b, T^{ab}\} = \text{bas}\,\mathbf{o}(10,10)$. Then, the generalized vielbein is defined as

$$E_M{}^A = \exp\left[e_m{}^a T_a{}^m\right]\exp\left[b_{ab}T^{ab}\right] = \begin{bmatrix} e_m{}^a & b_{mk}e_b{}^k \\ 0 & e_b{}^n \end{bmatrix}. \tag{3.42}$$

Consider now an $O(10,10)$ transformation of the form

$$E'_M{}^A = O_M{}^N E_N{}^A\,, \quad O_M{}^N = \exp\left[\beta^{mn}T_{mn}\right] = \begin{bmatrix} \delta_m{}^n & 0 \\ \beta^{nk} & \delta_l{}^k \end{bmatrix}. \tag{3.43}$$

The generalized metric then transforms linearly by conjugations $\mathcal{H}' = O^{-1}\mathcal{H}O$, and in terms of the space–time fields g, b the transformation has precisely the form of an open-closed string map. Following [29, 30], we assume the bi-Killing ansatz for the bivector

$$\beta^{mn} = r^{ab}k_a{}^m k_b{}^n\,, \tag{3.44}$$

where $r^{ab} = -r^{ba}$ is a constant matrix and $k_a{}^m$ are Killing vectors of the initial background g, b. Now, the advantage of the covariant language is that, to ensure that this transformed background is still a solution to supergravity equations, it is enough to check that the so-called generalized fluxes stay invariant [98]. The fluxes are defined as a generalization of anholonomy coefficients

$$\begin{aligned} &\mathcal{L}_{E_A}E_B = \mathcal{F}_{AB}{}^C E_C\,, \\ &\mathcal{L}_{E_A}d = \mathcal{F}_A\,, \end{aligned} \tag{3.45}$$

where $\mathcal{L}_{E_A}$ denote the generalized Lie derivative along the vielbein $E_A{}^M$ and d is the invariant dilaton. In general, the fluxes $\mathcal{F}_{AB}{}^C$ and $\mathcal{F}_A$ are some combinations of the fields g, b, ϕ, and their derivatives, and become constant in the case of group manifolds. These are then precisely the structure constants $F_{AB}{}^C$ of the corresponding Drinfeld double. Crucial for discussing this feature of double field theory is that, pretty much like general relativity, its action and field equations can be written completely in terms of fluxes and their derivatives [129]. Hence, if the deformation does not change generalized fluxes, it is a solution generating transformation. This boils down to a condition on the matrix r^{ab} that remarkably is the classical Yang–Baxter equation together with the unimodularity constraint:

$$r^{e[a}r^{|f|b}f_{ef}{}^{c]} = 0\,, \quad r^{ab}f_{ab}{}^{c} = 0\,. \tag{3.46}$$

To conclude this section, recall that, from the point of view of the sigma model, Yang–Baxter deformations are the ones that preserve integrability. In terms of T-dualities, these act as a special case of the Poisson–Lie symmetry deforming a given Drinfeld double. Beyond group manifolds, these act as a solution generating transformations preserving generalized fluxes of double field theory. At the moment, the latter two of these approaches to deformations have been generalized to 11 dimensions, apparently without reference to integrability of the membrane.

4. 11d supergravity and membranes

Classical integrability of a two-dimensional system (say, the fundamental string) implies that its equations of motion can be recast in the form of the Lax–Zakharov–Shabat equation requiring a 1-form A, the Lax connection, to be flat. In turn, this implies that an evolution operator can be constructed as a Wilson line that does not depend on the choice of the path. Taking the flatness condition as a starting point, one may use the r-matrix satisfying the classical Yang–Baxter equation to generate Poisson brackets of the system. This equation is a quasi-classical limit of the quantum Yang–Baxter equation, whose solution R defines the S-matrix of the system. This factorization property of the S-matrix means that the theory is integrable. Given an integrable superstring on a supergravity background, classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ can be used to define its integrable deformations. For the general case of a solution to 10d supergravity equations, such (unimodular) Yang–Baxter deformations work as a solution generating transformations. The algebraic structure behind these symmetries is provided by classical Drinfeld double and Poisson–Lie T-dualities.

In Section 2.4, we have seen that, when replacing the Poisson bracket by a Nambu bracket, one naturally arrives at an action structurally similar to that of the membrane. Although the integrability in three dimensions is not a well defined concept, a certain generalization of the structures responsible for integrability in 2d can be made. Consider as before a Lax pair. Then, a Nambu bracket can be generated by making use of the so-called ρ-tensor $\rho \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$, which satisfies a certain generalization of the classical Yang–Baxter equation. Such a ρ-tensor can be used to deform the so-called exceptional Drinfeld algebra, which is a generalization of the classical Drinfeld double standing behind Nambu–Lie U-duality. The condition for the deformation to preserve the algebraic structure is the same generalized Yang–Baxter equation. Beyond group manifolds, one finds generalized Yang–Baxter deformations as transformations generating families of solutions to 11d supergravity equations. Unfortunately, at this moment, we are not able to claim that these deformations preserve the integrability of the membrane due to the lack of its clear description. Formally following the 2d constructions, we face the absence of the proper ordering of points on a 2d surface generalizing Wilson line. The way out might be in turning to loop algebra variables, which seem to be more natural for describing membrane dynamics. In this section, we give a more detailed review of Nambu–Lie U-duality, deformations of exceptional Drinfeld algebras leading to the generalized Yang–Baxter equation, and a generalization of the construction beyond group manifolds; describe in more detail the arguments for Nambu brackets and loop algebraic variables to appear naturally in M-theory; describe the so-called transgression map relating the two; and finally provide some vague considerations following possible definitions of a Wilson surface and quasi-classical limits of tetrahedron equations, which seems to be the proper 3d analogue of the quantum Yang–Baxter equation.

4.1 Nambu–Lie U-duality

M-theory can be understood as the theory whose weak coupling approximation is the perturbative string theory. This statement is supported by the double dimensional reduction in the membrane when it wraps the compact cycle, giving the fundamental string. Given that, one may think of string theory as of a theory of various membranes on 11-dimensional background space–time that can be described in terms of a Polyakov string at certain points of the moduli space of vacua where the coupling g_s is small. Since, in the double dimensional reduction, g_s is determined by the size of the compact cycle, the dimension of the background space at these points is effectively 10. A more detailed discussion of M-theory from this point of view can be found in [130]. Since g_s does not play the role of a coupling constant in M-theory, various branes whose tensions differ by its powers can be mapped into each other by a symmetry that enhances T-duality and includes S-duality. This so-called U-duality (for unity or unified) was first observed in 11d supergravity compactified on a 7-torus in [131], where the resulting 4d equations of motion were shown to be invariant under $E_{7(7)}$ transformations. In general, in compactifying 11d supergravity on a d-torus, one ends up with a theory invariant under $E_{d(d)}$ symmetry, where $E_{3(3)} = SL(2) \times SL(3)$, $E_{4(4)} = SL(5)$, $E_{5(5)} = SO(5,5)$, and for $d \geqslant 9$ the symmetry algebra becomes infinite and special constructions are needed. Taking into account quantum effects breaks this symmetry into $E_{d(d)}(\mathbb{Z})$, as was shown in [132]. A highly detailed discussion of exceptional symmetries of toroidal compactifications of 11d supergravity can be found in [133, 134], and of U-duality symmetries of M-theory, in [135].

Although a construction similar to that of Buscher for the string does not seem to exist for membranes, simply due to the fact that one must simultaneously consider M2 and M5 branes to properly define duality transformations, an extension of the standard Abelian U-duality symmetry to non-Abelian Nambu–Lie U-duality is possible.[6] For that, one has to construct a generalization of the classical Drinfeld double, called the exceptional Drinfeld algebra (EDA), where the $O(d,d)$ symmetry is replaced by one of the U-duality symmetry groups. This has been done in [37–39, 137] and the step-by-step algorithm of constructing a Nambu–Lie dual can be found in [111]. Let us sketch the construction without going into much detail. Generators of the exceptional Drinfeld algebra belong to $\mathbf{10}$, $\mathbf{16}_s$, $\overline{\mathbf{27}}$ for groups $SL(5)$, $SO(5,5)$, $E_{6(6)}$, respectively, so in general one has for the basis

$$T_A = \{T_a, T^{a_1a_2}, T^{a_1\ldots a_5}\}\,. \tag{4.1}$$

[6] Note, however, [136], where invariance of membrane equations of motion are shown for the specific case of a 4-dimensional target space where the M5-brane cannot fit. Dual coordinates then correspond to windings of the M2-brane only.

The generators $\{T_a\}$ form a basis of the so-called geometric subalgebra $\mathfrak{g}$, while the others can be understood as corresponding to windings of the M2- and M5-brane, pretty much as in the classical Drinfeld double case generators $\tilde{T}^a$ defined the dual algebra. The algebraic structure is defined by the following multiplication table:

$$T_A \circ T_B = F_{A,B}{}^C T_C\,, \tag{4.2}$$

where $F_{A,B}{}^C$ are to generalized fluxes of exceptional field theory as structure constants of the classical Drinfeld double are to generalized fluxes of double field theory. More concretely, the multiplication table can be represented in the following form:

$$\begin{aligned}
&T_a \circ T_b = f_{ab}{}^c\, T_c\,,\\
&T_a \circ T^{b_1 b_2} = f_a{}^{b_1 b_2 c}\, T_c + 2 f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,\\
&T_a \circ T^{b_1\ldots b_5} = -f_a{}^{b_1\ldots b_5 c}\, T_c - 10 f_a{}^{[b_1 b_2 b_3}\, T^{b_4 b_5]}\\
&\quad - 5 f_{ac}{}^{[b_1}\, T^{b_2\ldots b_5]c} + 6\, Z_a\, T^{b_1\ldots b_5}\,,\\
&T^{a_1 a_2} \circ T_b = -f_b{}^{a_1 a_2 c}\, T_c + 3 f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta_b^{[c}\, T^{a_1 a_2]}\,,\\
&T^{a_1 a_2} \circ T^{b_1 b_2} = -2 f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c} - f_{c_1 c_2}{}^{[a_1}\, T^{a_2]b_1 b_2 c_1 c_2}\\
&\quad + 3\, Z_c\, T^{a_1 a_2 b_1 b_2 c}\,,\\
&T^{a_1 a_2} \circ T^{b_1\ldots b_5} = 5 f_c{}^{a_1 a_2 [b_1}\, T^{b_2\ldots b_5]c}\,,\\
&T^{a_1\ldots a_5} \circ T_b = f_b{}^{a_1\ldots a_5 c}\, T_c + 10 f_b{}^{[a_1 a_2 a_3}\, T^{a_4 a_5]}\\
&\quad + 20 f_c{}^{[a_1 a_2 a_3}\, \delta_b^{a_4}\, T^{a_5]c} + 5 f_{bc}{}^{[a_1}\, T^{a_2\ldots a_5]c}\\
&\quad + 10 f_{c_1 c_2}{}^{[a_1}\, \delta_b^{a_2}\, T^{a_3 a_4 a_5]c_1 c_2} - 36\, Z_c\, \delta_b^{[c}\, T^{a_1\ldots a_5]}\,,\\
&T^{a_1\ldots a_5} \circ T^{b_1 b_2} = 2 f_c{}^{a_1\ldots a_5 [b_1}\, T^{b_2]c} - 10 f_c{}^{[a_1 a_2 a_3}\, T^{a_4 a_5]b_1 b_2 c}\,,\\
&T^{a_1\ldots a_5} \circ T^{b_1\ldots b_5} = -5 f_c{}^{a_1\ldots a_5 [b_1}\, T^{b_2\ldots b_5]c}\,.
\end{aligned} \tag{4.3}$$

In contrast to classical Drinfeld, the double exceptional Drinfeld algebra is a Leibniz algebra, as structure constants $F_{A,B}{}^C$ are not antisymmetric in lower indices. Consistency requires quadratic relations on the constants $f_{ab}{}^c$, $f_a{}^{bcd}$, Z_a that, written in terms of the covariant object $F_{A,B}{}^C$, repeat quadratic constraints of maximal gauged supergravity [138]. Nambu–Lie U-duality is defined as such an $E_{d(d)}$ transformation $T_A \to C_A{}^B T_B$ that preserves the algebra. One immediately notices the absence of the natural duality, swapping the geometric algebra $\mathfrak{g}$ spanned by T_a in the chosen basis and its dual spanned by the rest, since the dimensions are different and the rest of the generators do not form a Lie algebra. However, an analogue of such swapping was suggested in [139] based on outer automorphisms of $\mathfrak{e}_d$, which allowed generating several examples of non-Abelian U-dual backgrounds in [111].

We are interested here in deformations of exceptional Drinfeld algebras consistent with their structure and defined by analogy with Yang–Baxter deformations of the classical Drinfeld double as follows:

$$f_a{}^{bcd} = \rho^{e[bc} f_{ea}{}^{d]}\,, \quad f_a{}^{a_1\ldots a_6} = \rho^{e[a_1\ldots a_5} f_{ea}{}^{a_6]}\,, \tag{4.4}$$

where $f_a{}^{a_1\ldots a_6} = \epsilon^{a_1\ldots a_6} Z_a$. Such deformations of exceptional Drinfeld algebras were introduced in [37, 38]. In the context of deformations of supergravity backgrounds, these were considered earlier in [34] as a generalization of the open-closed string map to the case of 11d supergravity fields. The condition for the deformation to preserve the exceptional Drinfeld algebra structure is called the generalized Yang–Baxter equations and reads

$$\begin{aligned}
&\rho^{a_1[a_2|a_6|}\rho^{a_3 a_4|a_5|} f_{a_5 a_6}{}^{a_7]} - \rho^{a_2[a_1|a_6|}\rho^{a_3 a_4|a_5|} f_{a_5 a_6}{}^{a_7]}\\
&\quad - 3 f_{a_5 a_6}{}^{[a_1}\rho^{a_2]a_3 a_4 a_5 a_6 a_7} = 0\,,\\
&\rho^{a_1 a_2[a_8}\rho^{a_3 a_4 a_5 a_6 a_7 a_9]} f_{a_8 a_9}{}^{a_{10}} = 0\,.
\end{aligned} \tag{4.5}$$

When restricted to the SL(5) EDA, i.e., four geometric generators, the above condition is precisely the one of [34] obtained from the vanishing R-flux condition (to be discussed in the next section). Note that if $\rho^{a_1\ldots a_6} = 0$ the generalized Yang–Baxter equations in the first line above are exactly conditions (2.31) that ensure that the tri-bracket defined for Lax operators in terms of the ρ-tensor ρ^{abc} is a Nambu bracket, i.e., satisfies the fundamental identity. To our knowledge, in the present context, this was first observed in [39] (Section 4 there), and a candidate for an equation whose quasiclassical limit gives (4.5) was suggested. Although the details are not completely clear, the suggested quantum generalized Yang–Baxter equation looks very similar to the tetrahedron equation in the form of the decorated Yang–Baxter equation (see Section 4.6.2). If proven, this would be a strong hint of the integrability of the membrane.

4.2 Polyvector deformations

As in the cases of Poisson–Lie T-duality and Yang–Baxter deformations, the generalized Yang–Baxter deformations discussed above in the context of exceptional Drinfeld algebras can be extended beyond group manifolds. We will follow here the construction of [35, 36], where generalized Yang–Baxter deformations of a general supergravity background with at least three Killing vectors are given by a certain $E_{d(d)}$ transformation. The transformation acts on fields of exceptional field theory, which is a covariant formulation of supergravity with the field transforming in irreducible representations of $\mathfrak{e}_d$ and in general depend on coordinates X^M parametrizing space–time extended by membrane winding modes. Similarly to double field theory, the consistency of local symmetries requires the section condition, which we will assume to be solved by keeping only geometric coordinates x^m. We will not go into detail about the construction, for which the interested reader is referred to plenty of detailed literature on the subject [140–144].

Since the local symmetry group of exceptional theories changes with dimension d of the so-called internal space entering the split $11 = D + d$, explicit expressions for generalized Yang–Baxter deformations also significantly change. To illustrate the main idea, we take the simplest SL(5) theory, whose deformations might be trivial in the sense discussed below; however, all the main features present. As in double field theory, we focus only on the generalized vielbein $E_M{}^A$ and the corresponding generalized metric $m_{MN} \in \mathrm{SL}(5)/\mathrm{SO}(5) \times \mathbb{R}^+$. Note however, that, in contrast to the full double field theory, exceptional field theory includes gauge fields that transform nontrivially under U-duality. To restrict the discussion to the generalized vielbein and a dilaton-like field ϕ corresponding to a determinant of the external D-dimensional metric, a specific truncation must be performed [35]. Leaving the details aside, we note that generators of SL(5), when decomposed under the action of its GL(4) subgroup, follow the same labeling pattern as generators of the SL(5) EDA,

$$\mathrm{bas}\,\mathfrak{sl}(5) = \{T_A\} = \{T_{abc}, T_a{}^b, T^{abc}\}\,. \tag{4.6}$$

As before, the last 10, i.e., generators of a non-negative level regarding the action of the GL(1) subgroup of GL(4), define the generalized veilbein itself:

$$E_M{}^A = e^{\phi T} \exp\left[e_m{}^a T_a{}^m\right] \exp\left[C_{abc} T^{abc}\right], \tag{4.7}$$

where T is the generator of $\mathbb{R}^+$. The deformation map is then defined by negative level generators and has the following form:

$$\begin{aligned} &E'_M{}^A = O_M{}^N E_N{}^A, \\ &O_M{}^N = \exp\left[\Omega^{mnk} T_{mnk}\right] = \begin{bmatrix} \delta_m{}^n & 0 \\ \epsilon_{npqr}\Omega^{pqr} & 1 \end{bmatrix}. \end{aligned} \tag{4.8}$$

As before, imposing the tri-Killing ansatz

$$\Omega^{mnk} = \rho^{abc} k_a{}^m k_b{}^n k_c{}^k, \tag{4.9}$$

where $k_a{}^m$ denote Killing vectors of the initial background, and requiring that the deformed background be a solution to the supergravity equation in the exceptional field theory form, we arrive at the condition on ρ^{abc} that is precisely the generalized Yang–Baxter equation (4.5) together with the unimodularity constraint

$$\rho^{abc} f_{ab}{}^d = 0. \tag{4.10}$$

To have 6-vector deformations, one has to go to a larger symmetry group. The triviality of tri-vector deformations inside the SL(5) theory mentioned above comes from the fact that, to ensure invariance of generalized fluxes, which equivalently satisfy supergravity equations, the unimodularity condition is enough. This is the same as the O(3, 3) double field theory and is related to the dimension of the internal space, which renders the (generalized) Yang–Baxter equation in the form of the vanishing of R-flux to be equivalent to the unimodularity condition. One can, however, consider Yang–Baxter deformations that are non-unimodular and hence do not solve equations of standard supergravity, instead leading to their generalization [145, 146].

To recap, a generalization of Poisson–Lie T-duality symmetries to an algebra that includes Abelian U-duality transformations naturally leads to the notion of the exceptional Drinfeld algebra that underlies the Nambu–Lie U-duality symmetry. As the geometric realization of the classical Drinfeld double in terms of generalized vielbein leads to a bi-vector defining the Poisson structure, geometric realization of the exceptional Drinfeld algebra leads to a 3-vector and a 6-vector defining a Nambu structure. In the context of generalized Yang–Baxter deformations, these define a deformed background, and the condition for it to satisfy supergravity field equations is precisely equations (2.31), which appear as the condition for a 3-bracket of a system defined by a Lax pair and the tri-vector to satisfy the fundamental identity.

Precisely as in the string case, map (4.8) has the form of an open-closed membrane map of [119]. To see that we should start with a background with no C-field, the deformed background in this language will be precisely the background seen by the open membrane. The tri-vector Ω^{mnk} is then one of the generalized theta-parameters in the notations of [147] and defines open membrane noncommutativity. As we discuss below, the noncommutativity relations are naturally written in terms of loop variables, which suggests that the same be done for generalized Yang–Baxter equations.

4.3 Membranes ending on membranes

In Section 2.2, we have seen that the Nahm system can be equivalently described in terms of Poisson and Nambu structures. In the latter case, two Hamiltonians have to be introduced, one of which is simply one of the conserved charges in the standard Poisson formulation. For our account, the Nahm system is of interest due to its close relation to the dynamics of systems of Dp-branes; speaking more concretely, boundary conditions of the D1–D3-brane system can be described in terms of Nahm equations. When uplifted to M-theory, this becomes the system of M2 and M5 branes, where the former ends on the latter, and the Nahm equation becomes the so-called Basu–Harvey equation. This procedure was first considered in [59], where a generalization of the Nahm equation was proposed, which naturally involves a tri-bracket, and hence possesses a Nambu structure.

Let us first look at the D1–D3-brane system. The starting point here is to notice that the brane intersection locus can be equivalently described by (i) a fuzzy funnel noncommutative geometry interpolating between D1 and D3 brane geometries, (ii) geometric engineering of Yang–Mills monopoles on the D3-brane. It is worth mentioning that this is also true for a more general intersection of the Dp and D(p+2) branes. These two pictures basically correspond to considering the intersection from the point of view of the D1-brane and of the D3-brane, respectively.

D1-brane from the D3-brane point of view. For the second picture, we start with the worldvolume action of an infinitely large D3-brane:

$$\begin{aligned} S^{\mathrm{D3}} &= \int \mathrm{d}^4\xi \exp(-\phi)\sqrt{\det(g+\mathcal{F})} + \int C_4 + C_2\wedge B_2 \\ &\quad + \frac{1}{2}\, C_0\wedge B_2\wedge B_2, \\ \mathcal{F} &= \mathrm{d}A_1 + B_2, \quad \tau = C_0 + \mathrm{i}\exp(-\phi). \end{aligned} \tag{4.11}$$

This action describes a D3-brane that interacts electrically with the fundamental string F1, which can be seen from the A_1 field in the determinant coupled to open string ends. To replace F1 by D1 and hence to describe interaction with the D1 brane, S-duality is performed:

$$\begin{aligned} &\tau \to -\frac{1}{\tau} = -\frac{C_0 - \mathrm{i}\exp(-\phi)}{C_0^2 + \exp(-2\phi)}, \quad g_{\mu\nu} \to |\tau|\, g_{\mu\nu}, \\ &B_2 \to -C_2, \quad A_1 \to -c_1. \end{aligned} \tag{4.12}$$

Assuming the background is generated purely by the D-branes, i.e., $B_2 = 0$, we have

$$\begin{aligned} S'^{\,\mathrm{D3}} &= \int \mathrm{d}^4\xi \exp(-\phi)\sqrt{\det\left(g - |\tau|^{-1}\mathrm{d}c_1 - |\tau|^{-1}C_2\right)} \\ &\quad + \int C_4 - \frac{1}{2}\,|\tau|^{-2} C_0\wedge C_2\wedge C_2. \end{aligned} \tag{4.13}$$

This action describes interaction between the D3-brane and D1-brane in the sense that the D1-brane endpoints are charged with regard to the world-volume field c_1. From the point of view of the world-volume theory, D1-branes are seen as spikes of energy, corresponding to Yang–Mills monopoles carrying magnetic charge. Let us choose $(X^4, \ldots, X^9) = \mathbf{X}_\perp$ to be transverse directions and chose spherical coordinates on the brane: $(X^1, X^2, X^3) = (r, \theta, \phi)$. This theory has a classical

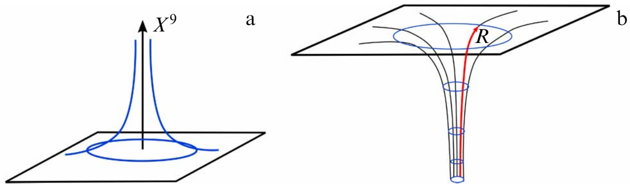

Figure 2. D1-brane ending on a D3-brane from different points of view. (a) As a soliton solution of the D3-brane worldvolume theory. (b) As a throat representing fuzzy sphere geometry around the D1-brane. X^9 denotes coordinate along which the soliton field descends. R denotes physical radius of the fuzzy sphere.

monopole solution,

$$X^9 = \frac{N}{2r}\,, \quad F_{\theta\phi} = -r^{-2}\partial_r X^9\,, \tag{4.14}$$

whose charge is given by

$$Q_m = \frac{1}{2}\int \mathrm{d}\Omega\, F_{\theta\phi} = N\,. \tag{4.15}$$

This is interpreted as N D1-branes ending on the D3-brane at the point $r = 0$ in the chosen coordinates and stretching along X^9. This is schematically depicted in Fig. 2a.

D3-brane from the D1-brane point of view. The opposite picture describing a D3-brane from the point of view of the D1-brane is slightly more subtle and involves non-Abelian Yang–Mills theories. Let us briefly describe the idea here and send the interested reader to [63] for details. We start with a description of N D1-branes stretching along the X^9 direction in terms of $N\times N$ real matrices $(\mathbb{X}^1,\ldots,\mathbb{X}^8)$ (see, e.g., [148]). The worldvolume gauge field also becomes represented by a matrix $\mathbb{A}_i$ and we fix the gauge choice to be $\mathbb{A}_9 = 0$. Now, we are looking for a (supersymmetric) solution to equations of the non-Abelian Yang–Mills theory with $\mathbb{X}^A = (\mathbb{X}^4,\ldots,\mathbb{X}^8) = 0$, which corresponds to the position of the D3-brane (Fig. 2b). Equations of motion together with supersymmetry render [149]

$$\frac{\partial \mathbb{X}^i}{\partial x^9} = \pm\frac{\mathrm{i}}{2}\epsilon^{ijk}\,[\mathbb{X}^j,\mathbb{X}^k]\,, \tag{4.16}$$

where $i,j,k = 1,2,3$, which are Nahm equations. The following solution to this system of equations precisely reproduces the monopole profile obtained before in the opposite approach:

$$\mathbb{X}^i = \pm\frac{1}{2x^9}\,\sigma^i\,, \tag{4.17}$$

where σ^i are the standard Pauli matrices.

The 'coordinates' $\mathbb{X}^i$ are used to measure the physical radius of the sphere around the D1-brane on the surface $\Sigma_\perp$ defined by $\mathbb{X}^A = 0$:

$$R^2 = \frac{2\pi\alpha'}{N}\,\mathrm{Tr}\left[\sum_{i=1}^{3}\mathbb{X}^i\mathbb{X}^i\right] = \frac{\pi\alpha'(N^2-1)}{(x^9)^2}\,. \tag{4.18}$$

We see that the space near the intersection has the geometry of an infinite throat, which at large N indeed matches the previous result (4.14).

The above picture has been uplifted to M-theory in [59] to describe M2–M5 brane junctions. The overall idea is basically the same: from the point of view of the M5-brane theory, the boundary of the M2-brane is described by a string-like BPS (Bogomolnyi–Prasad–Sommerfeld) soliton in the $\mathcal{N} = (2,0)$ supersymmetric gauge theory in $d = 6$. Scalar fields of the theory that correspond to embedding functions of the membrane belong to the supermultiplet that contains a self-dual 3-form, which makes writing a Lagrangian a hard task. Equations of motion for fields of the gauge theory have been obtained in [150, 151] in the so-called superembedding formalism, where the supermanifold describing the M5-brane world-volume is embedded into another supermanifold whose bosonic part is the target space–time. These equations are in the Green–Schwarz form, and the string-like soliton solution was obtained in [60]. Although we cannot provide a detailed review of the formalism without extending the text well beyond its scope, it is wise to give some more details and sketch the main results following [60]. First, we note that all equations below are written in the so-called static gauge, where using superreparametrization of the world-volume bosonic coordinates are identified with six target space–time directions (longitudinal) and 16 out of 32 fermionic fields are set to zero. In searching for classical solutions, we set the remaining fermionic fields to zero as well and write the following bosonic equations:

$$\begin{aligned} &G^{\mu\nu}\nabla_\mu\nabla_\nu X^{a'} = 0\,,\\ &G^{\mu\nu}\nabla_\mu H_{\nu\rho\sigma} = 0\,. \end{aligned} \tag{4.19}$$

Conventions for the indices are the following: $\mu,\nu,\kappa,\ldots = 0,\ldots,5$ and $a,b,c,\ldots = 0,\ldots,5$ are curved and flat world-volume indices; $a',b',c',\ldots = 1',\ldots,5'$ label transverse directions. In what follows, we split $\mu = (0,1,m)$ and $a = (0,1,\alpha)$ with $m = 2,3,4,5$ and $\alpha = 2,3,4,5$, labeling directions transverse to the soliton (curved and flat, respectively). The covariant derivative $\nabla_m = \nabla_m[g]$ is constructed on the metric written in terms of the standard world-volume vielbein $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b\eta_{ab}$. The remaining fields are defined as

$$\begin{aligned} &G_{\mu\nu} = E_\mu{}^a E_\nu{}^b\eta_{ab}\,,\\ &E_\mu{}^a = e_\mu{}^b(m^{-1})_b{}^a\,,\\ &m_a{}^b = \delta_a{}^b - 2h_{acd}h^{bcd}\,,\\ &H_{\mu\nu\rho} = E_\mu{}^a E_\nu{}^b E_\rho{}^c m_b{}^d m_c{}^e h_{ade}\,, \end{aligned} \tag{4.20}$$

where h_{abc} is the world-volume self-dual 3-form. Note that the field $H_{\mu\nu\kappa}$ is not self-dual and, moreover, can be written as $H_{\mu\nu\kappa} = 3\partial_{[\mu}B_{\nu\kappa]}$.

Now, we are looking for a string-like solution that lies in the $(0,1)$-plane, for which we introduce the following ansatz:

$$\begin{aligned} &X^{5'} = \phi\,,\\ &h_{01\alpha} = v_\alpha\,,\\ &h_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta}v^\delta\,. \end{aligned} \tag{4.21}$$

Denoting $H_{01m} = V_m$, we are able to write equations in the following form:

$$\begin{aligned} &\delta^{mn}\partial_m\partial_n\phi = 0\,,\\ &\delta^{mn}\nabla_m V_n = 0\,. \end{aligned} \tag{4.22}$$

The solution describing N string-like BPS solitons then becomes

$$
\begin{aligned}
& H_{01m} = \pm \frac{1}{4}\, \partial_m \phi\,, \\
& H_{mnp} = \pm \frac{1}{4}\, \epsilon_{mnpq} \delta^{qr} \partial_r \phi\,, \\
& \phi = \phi_0 + \sum_{I=0}^{N-1} \frac{2Q_0}{|x - y_I|^2}\,.
\end{aligned} \tag{4.23}
$$

Note that there is no need for a source term; hence, the solution is indeed solitonic, and due to self-duality, it possesses both electric and magnetic charges regarding H_{mnk}, both equal to $\pm Q_0$.

The string soliton solution (4.23) has a nontrivial profile of the $X^{5'}$ field stretching along X^1 and corresponds to the M2-brane stretched along $(015')$ directions, ending on the M5-brane stretching along (012345) directions. To arrive at a generalization of the Nahm's equation for the M2–M5-brane system, we will proceed as before: describe the junction in terms of the fuzzy sphere construction and write a matrix equation whose solution gives the string soliton profile. For this, we need an equation that has SO(4) symmetry rather than the SO(3) symmetry of the Nahm's equation, which is basically the technical reason for the Nambu bracket to appear. The fuzzy 2-sphere describing the D1–D3-brane junction must be generalized to the fuzzy 3-sphere construction presented in [152]. The space is described by four $N \times N$ matrices G^i where

$$
N = \frac{(n+1)(n+3)}{2}\,, \quad n = 2k+1\,, \quad k \in \mathbb{Z}\,. \tag{4.24}
$$

For $n = 1$, these matrices are simply the standard gamma-matrices in four dimensions. The matrices are given explicitly as

$$
\begin{aligned}
& G^i = \mathcal{P}_{\mathcal{R}_+} \sum_{s=1}^{N} \rho_s(\Gamma^i P_-) \mathcal{P}_{\mathcal{R}_-} + \mathcal{P}_{\mathcal{R}_-} \sum_{s=1}^{N} \rho_s(\Gamma^i P_+) \mathcal{P}_{\mathcal{R}_+}\,, \\
& \sum_{s=1}^{n} \rho_s(\Gamma^i) = \left(\Gamma^i \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes \Gamma^i\right)_{\mathrm{sym}}\,,
\end{aligned} \tag{4.25}
$$

where the 'sym' subscript denotes complete symmetrization of the tensor product. The projectors $P_\pm = 1/2(1 \pm \Gamma^5)$, where Γ^5 is the standard gamma-matrix. The projectors $\mathcal{P}_{\mathcal{R}_\pm}$ project on the irreducible representations

$$
\begin{aligned}
& \mathcal{R}_+ = \left(\frac{n+1}{4}, \frac{n-1}{4}\right), \\
& \mathcal{R}_- = \left(\frac{n-1}{4}, \frac{n+1}{4}\right)
\end{aligned} \tag{4.26}
$$

of the $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ group. Finally, we denote $G_5 = \mathcal{P}_{\mathcal{R}_+} + \mathcal{P}_{\mathcal{R}_-}$.

Now, taking $\mathbb{X}^i \in \mathrm{Mat}_N(\mathbb{C})$, the proper generalization of the Nahm's equation can be written as

$$
\frac{\mathrm{d}\mathbb{X}^i}{\mathrm{d}s} + \frac{\lambda M_{11}^3}{8\pi}\, \epsilon_{ijkl} \left[G_5, \mathbb{X}^j, \mathbb{X}^k, \mathbb{X}^l\right] = 0\,, \tag{4.27}
$$

where the Nambu bracket is given by the following sum over permutations:

$$
[A_1, A_2, A_3, A_4] = \sum_\sigma \mathrm{sign}\,(\sigma) A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} A_{\sigma_4}\,. \tag{4.28}
$$

This equation can be interpreted as the Bogomolnyi equation for the membrane theory. As in the Nahm case, its solution can be written in terms of the matrices defining the fuzzy 3-sphere, which, in the large N limit, takes the following form:

$$
\mathbb{X}^i(s) = \frac{\sqrt{2\pi} \mathrm{i}}{\sqrt{\lambda(n+2) s M_{11}^{3/2}}}\, G^i\,. \tag{4.29}
$$

To see the string soliton profile, we first introduce the physical radius of the fuzzy 3-sphere,

$$
R^2 = \frac{1}{N} \left| \mathrm{Tr} \sum_i (\mathbb{X}^i)^2 \right|. \tag{4.30}
$$

Taking $s = X^{5'}$, the above gives the desired result:

$$
X^{5'} = \frac{2\pi N}{\lambda(n+2) M_{11}^{3/2} R^2}\,. \tag{4.31}
$$

Hence, we conclude that, in the attempt to generalize Nahm's equation describing D1–D3-brane junctions to branes of higher dimensions, the Nambu bracket naturally appears and the generalization is commonly referred to as the Basu–Harvey equation. Later, in [64, 65], based on this observation, an action for a stack of (2) M2-branes was suggested that essentially includes the Nambu bracket of world-volume fields. Although later in [66] an alternative formulation of the world-volume theory (ABJM) that requires no Nambu brackets was suggested, there still seem to be traces of this structure inside. We are referring here to the $\mathrm{U}(1)^3$ deformation of the $\mathrm{AdS}_3 \times \mathbb{S}^7$ background first addressed by Lunin and Maldacena in [8], which is holographically dual to the β-deformation of ABJM theory. From the supergravity point of view, this corresponds to an SL(2) transformation of the parameter

$$
\tau = C_{123} + \mathrm{i}\sqrt{G}\,, \tag{4.32}
$$

where (123) are three $\mathbb{S}^1$ directions of the $\mathbb{S}^7$. Alternatively, this is simply a 3-vector deformation described by the generalized Yang–Baxter equation as has been discussed above. We have already seen that this appears naturally when a Lax triple is introduced for a Nambu system.

On the ABJM theory side, β-deformations of Lunin–Maldacena correspond to introducing certain phase factors for fields entering the superpotential [95]

$$
\begin{aligned}
W \to W_\beta = \frac{4\pi}{k}\, \mathrm{Tr} \Bigg[& \exp\left(-\mathrm{i}\, \frac{\pi\beta}{2}\right) A_1 B_1 A_2 B_2 \\
& - \exp\left(\frac{\mathrm{i}\pi\beta}{2}\right) A_1 B_2 A_2 B_1 \Bigg]\,.
\end{aligned} \tag{4.33}
$$

It is appropriate here to consider a similar deformation of the $\mathcal{N} = 4$ SYM theory, the $\mathrm{U}(1)^2$ β-deformation. On the gravity side, this is an Abelian bivector deformation along two of three Abelian Killing vectors of the dual $\mathrm{AdS}_5 \times \mathbb{S}^5$ solution. On the gauge theory side, things get much more interesting, and the deformation corresponds to the deformation of the product of fields

$$
fg \to f * g = \exp\left[\mathrm{i}\pi\beta\left(p_1^f p_2^g - p_2^f p_1^g\right)\right] fg\,, \tag{4.34}
$$

where p_1 and p_2 denote generators of the two U(1) isometries and the superscript denotes whether the generator acts on f

or g. Now, if both isometries are along the AdS space, we end up with a Moyal product and a noncommutative deformation of SYM [95]; if one isometry is along the AdS and one is along the sphere, we get so-called dipole deformations; and when both isometries are along the sphere, we get the β-deformation of Lunin–Maldacena. In this last case, the generators act simply by multiplication by weight of the operator, and the deformed superpotential becomes (see [95] for more details and for the corresponding brane picture)

$$W \to W_\beta = \mathrm{Tr}\left[\exp(\mathrm{i}\pi\beta)\Phi_1\Phi_2\Phi_3 - \exp(-\mathrm{i}\pi\beta)\Phi_1\Phi_3\Phi_2\right]. \tag{4.35}$$

What is most intriguing here is that the bi-vector deformation effectively introduces a nontrivial bracket of the operators

$$[x^1, x^2] = \beta. \tag{4.36}$$

When both isometries are noncompact, the noncommutative parameter on the right-hand side becomes literally the bi-vector deformation parameter naturally appearing from double field theory. Now, for the exceptional case, we have a tri-vector, which presumably must correspond to a non-associativity parameter

$$[x^m, x^n, x^k] = \Omega^{mnk} \tag{4.37}$$

or define a Moyal-like tri-product, whose explicit form has not really been established in the literature (see, however, [153] for a definition of the star-three product in relation to ABJM theory). The precise definition of this tri-product as well as a controllable formulation of a non-associative gauge theory stand among the fascinating directions of research to deepen the understanding of membrane dynamics.

4.4 Open membranes and loop variables

Since the boundary of an M2-brane ending on an M5-brane is string-like, it could be expected that natural variables to define world-volume theory on a membrane are those taking values in a loop algebra. To our knowledge, the first mention of loop algebras in the context of membrane dynamics appears in [154], where an analysis of the canonical Dirac bracket for membrane world-volume fields was performed. The observation was that, in a similar way to how non-commutativity of open string ends appears for D-branes on a constant NS 2-form field background, loop space noncommutativity appears for the case of membranes. The authors define the star product of fields $X^i(s)$ with s parametrizing the boundary loop. The fascinating observation relating this approach to polyvector deformations is that the Ω^{mnk} tensor parametrizing deformations defines the open membrane metric.

Let us start with a brief reminder of expressions related to open-string noncommutativity; for more details, the reader is referred to the original work [119]. We consider the theory of an open string on a background with a nontrivial B-field with the standard second order action

$$S_{F1} = T\int \mathrm{d}^2\sigma\,(\sqrt{h}g_{mn}h^{\alpha\beta} + b_{mn}\epsilon^{\alpha\beta})\,\partial_\alpha X^m\,\partial_\beta X^n, \tag{4.38}$$

where the integration is taken over the string world-sheet Σ with induced metric $h_{\alpha\beta}$ and coordinates $\sigma^\alpha = (\tau, \sigma)$. Given that the metric and the B-field are constant, boundary conditions along a Dp-brane world-volume take the form

$$g_{mn}\partial_{\mathbf{n}}X^n + b_{mn}\partial_t X^n\Big|_{\partial\Sigma} = 0, \tag{4.39}$$

where $\partial_{\mathbf{n}}$ denotes the derivative along a normal vector to $\partial\Sigma$ and ∂_t denotes a tangent derivative. We are interested in the quantum properties of such two-dimensional field theory with a boundary, in particular, in two-point correlation functions $\langle X^m(\tau,\sigma)X^n(\tau',\sigma')\rangle$ that define the propagator of the theory. Restricting the surface Σ to a disk for simplicity and introducing complex coordinates $(z, \bar z)$, as usual, one arrives at the following expression:

$$\begin{aligned}\left\langle X^m(z)X^n(z')\right\rangle &= \gamma^{mn}\log|z-\bar z| - g^{mn}\log|z-z'| \\ &- G^{mn}\log|z-\bar z'|^2 - \Theta^{mn}\log\frac{z-\bar z'}{\bar z - z'} + D^{mn}, \end{aligned}\tag{4.40}$$

where D^{mn} does not depend on world-volume coordinates. The matrices G^{mn} and Θ^{mn} are defined as symmetric and antisymmetric parts of the matrix $(g+b)^{-1}$, respectively.

Now, the fascinating observation is the following. If both of the points z and z' are inside the world-volume and almost coincident, then the correlator behaves like the usual propagator of a two-dimensional CFT of scalar fields X^m. The matrix g_{mn} is then the proper metric for these fields, and we refer to it as the closed string metric. For an open string, however, vertex operators must be inserted on the boundary, i.e., when both z and z' are at the edge of the disk, i.e., $z = \tau$, $z' = \tau'$. We then have

$$\left\langle X^m(\tau)X^n(\tau')\right\rangle = -G^{mn}\log(\tau-\tau')^2 + \frac{\mathrm{i}}{2}\,\Theta^{mn}\epsilon(\tau-\tau'), \tag{4.41}$$

where D^{mn} has been set to a convenient value and $\epsilon(\tau)$ is the function that is $+1$ for the positive argument and -1 for the negative. We see that the matrix G^{mn} can now be interpreted as the metric seen by open string ends, since it provides the correct behavior of the propagator for close points. The object Θ^{mn} is the noncommutativity parameter of the open string ends, meaning that the effective field theory on a Dp-brane on a background with a nonvanishing B-field must be described by the noncommutative Yang–Mills theory. In [119], it was shown that this is indeed the case.

The relation between open and closed string parameters can be recast in the following form:

$$(g+b)^{-1} = G + \Theta, \tag{4.42}$$

which is precisely the bi-vector deformation rule, when Θ is understood as the deformation tensor, and G, as the initial undeformed background. Although it is not completely clear what the underlying reason behind this similarity is, it cannot be merely a coincidence, as one observes precisely the same match between open-closed membrane relations of [147] and tri-vector transformation rules. Before turning to the case of the membrane, it is instructive to mention another origin of the open string metric G^{mn}, the Dp-brane action. Schematically, it has the form

$$S_{\mathrm{Dp}} = T_p\int \mathrm{d}^{p+1}\xi\,\exp(\phi)\sqrt{\det(g+\mathcal{F})} + \int \mathcal{C}_{(p+1)} + \ldots, \tag{4.43}$$

where $\mathcal{F} = \mathrm{d}A + b$, with $A = A_\alpha\,\mathrm{d}\xi^\alpha$ denoting the Born–Infeld world-volume vector field interacting with the open string endpoints, g and b denoting pullbacks of the target space fields, $C_{(p+1)}$ denoting the top RR form interacting with the Dp-brane, and ellipses denoting the remaining terms altogether rendering the action gauge invariant. Varying the action with respect to the scalar fields X^m, we obtain the equation

$$\Box[G]X^m + \ldots = 0\,, \tag{4.44}$$

where the box denotes the world-volume d'Alambertian constructed of the pullback of the open string metric G_{mn}, and the ellipsis denotes various terms containing only linear derivatives $\partial_\alpha X^m$. Hence, we see that the open string metric appears to be the natural metric for the dynamics of scalar fields, which are nothing more than open string excitations transverse to the Dp-brane.

Let us now turn to a three-dimensional sigma model interacting with target space–time metric g_{mn}, a 3-form field C_{mnk}, that will be a model of the M2-brane of M-theory. The action of the model can be written as follows:

$$S = \frac{1}{2l_p^2}\int_\Sigma \mathrm{d}^3\sigma\,\sqrt{-h}g_{mn}h^{\alpha\beta}\partial_\alpha X^m\partial_\beta X^n + \int_\Sigma C_{(3)} + \int_{\partial\Sigma} B_{(2)}\,, \tag{4.45}$$

where Σ denotes the world-volume of the model and $\partial\Sigma$ denotes its space-like boundary. Similarly to open string ends, whose dynamics is effectively described by a Dp-brane, the boundary of the M2-brane is described in terms of the M5-brane world-volume theory. Since this theory is non-Lagrangian, in the sense that its proper Lagrangian description is not known, the task to write an analogue of the DBI action becomes really tough. On the other hand, the standard CFT methods used above to obtain correlations on the boundary of the 2d disk fail here, since the theory is three-dimensional. To circumvent these difficulties, in [155] an elegant approach was suggested: (i) impose a special decoupling limit to freeze out bulk modes keeping the M5-brane world-volume theory nondegenerate, (ii) using the primary constraint of the resulting theory to construct a Dirac bracket that actually encodes the loop noncommutativity of the boundary fields.

To comment on the first step, let us first notice following [155] that in M-theory there is no sense in which the tension of the probe M2-brane is much smaller than that of the background M5-brane. This is in contrast to the open string theory, where the small string coupling $g_\mathrm{s} \ll 1$ results in large Dp-brane tension $T_{\mathrm{Dp}} \sim g_\mathrm{s}^{-1}$, while the fundamental string tension $T_{F1} \sim 1$. To prevent the probe M2-brane from deforming the background, suppose the latter is generated by a large stack of N_5 M5-branes:

$$\mathrm{d}s^2 = H^{-1/3}\,\mathrm{d}x_\parallel^2 + H^{2/3}\,\mathrm{d}x_\perp^2\,, \quad H = 1 + \frac{N_5 l_p^3}{x_\perp^3}\,, \tag{4.46}$$

where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ denote directions parallel and transverse to the M5-branes, respectively. The 3-form field strength F_4 is proportional to the volume form in the transverse space. Now, detach a single M5-brane of the stack and shift it to the position r_0 close to the initial stack. If $N_5 \gg 1$ and r_0 is small enough, interaction between this M5-brane and the remaining stack makes it effectively frozen, such that the M2-brane can probe it without deformation. Introducing $\epsilon \to 0$, we can write the decoupling limit as

$$\begin{aligned} &l_p \sim \epsilon l_p\,, \\ &\frac{N_5}{r_0^3} \sim \epsilon^{-3}\,\frac{N_5}{r_0^3}\,, \end{aligned} \tag{4.47}$$

such that the product hl_p^3 remains finite. Here, h enters the self-dual world-volume field strength $\mathcal{H}_{\mu\nu\rho}$ as

$$\begin{aligned} &\mathcal{H}_{012} = -\frac{h}{\sqrt{1 + l_p^6 h^2}}\,, \\ &\mathcal{H}_{345} = h\,. \end{aligned} \tag{4.48}$$

In this limit, the action to quantize becomes simply

$$S = \frac{1}{3}\int_{\partial\Sigma} \mathrm{d}^2\sigma\,\mathcal{H}_{\mu\nu\rho}X^\mu\dot{X}^\nu X'^\rho\,, \tag{4.49}$$

where the dot and prime denote derivatives concerning the coordinates (τ, σ) on the boundary $\partial\Sigma$. The equal time Dirac brackets between the fields $X^a = \{X^3, X^4, X^5\}$ then become

$$\left[X^a(\tau,\sigma), X^b(\tau,\sigma')\right] = -\frac{1}{h}\,\frac{\epsilon^{abc}X'^c(\sigma)}{\left|X^a(\sigma)\right|^2}\,\delta(\sigma-\sigma')\,. \tag{4.50}$$

We see that the variables $X^a(\sigma)$ parametrized by the world-volume boundary coordinate σ are indeed noncommutative. Equation (4.50) can be understood as the commutation relation for loop algebra variables $X^a(\sigma)$. The same observation will be made in the next section based on the ADHM (Atiyah–Drinfeld–Hitchin–Manin) membrane construction.

Comparing the above commutator to the antisymmetric part of (4.41), we would expect the right-hand side to be interpreted in terms of the metric seen by the open membrane. That is precisely the case, as it can be learned from [147], where parameters $\Theta^{\mu_1\ldots\mu_p}$ were introduced, called there generalized theta parameters. Instead of copying the relevant expressions from this paper, let us illustrate the idea in terms of exceptional field theory parametrization. The open-closed string map $G + \Theta = (g+b)^{-1}$ is nothing but two different ways of writing the same $\mathrm{O}(d,d)/\mathrm{O}(d)\times\mathrm{O}(d)$ coset:

$$\begin{bmatrix} g - bg^{-1}b & bg^{-1} \\ g^{-1}b & g^{-1} \end{bmatrix} = \mathcal{H} = \begin{bmatrix} G & \Theta G \\ G\Theta & G^{-1} - \Theta G\Theta \end{bmatrix}. \tag{4.51}$$

Moreover, decomposing $\mathrm{O}(d,d)$ generators under the action of its $\mathrm{GL}(d)$ subgroup (upper-left and lower-right blocks in the matrix notation) as $\{T_{\mu\nu}, T^\mu{}_\nu, T^{\mu\nu}\}$, the matrix $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{O}^T\begin{bmatrix} G & 0 \\ 0 & G^{-1} \end{bmatrix}\mathcal{O}\,, \quad \mathcal{O} = \exp\left[\Theta^{\mu\nu}T_{\mu\nu}\right]. \tag{4.52}$$

In other words, adding the noncommutative parameter $\Theta^{\mu\nu}$ can be understood as an $\mathrm{O}(d,d)$ transformation. Now, raising this to the exceptional field theory, i.e., replacing the orthogonal group by one of the groups $\mathrm{SL}(5)$, $\mathrm{SO}(5,5)$, or E_d with $d = 6,7,8$, we reproduce precisely the expressions of [147]. Let us illustrate this using the SL(5) example. Let us decompose the generator under the action of its GL(4) subgroup (upper-left block) $\{T_{\mu\nu\rho}, T^\mu{}_\nu, T^{\mu\nu\rho}\}$ and write

$$\begin{aligned} &\mathcal{U} = \exp\left[\Theta^{\mu\nu\rho}T_{\mu\nu\rho}\right], \\ &\mathcal{U}^{-1}\begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix}\mathcal{U} = \begin{bmatrix} G_{\mu\nu} & \epsilon_{\mu\rho_1\rho_2\rho_3}\Theta^{\rho_1\rho_2\rho_3} \\ \epsilon_{\nu\rho_1\rho_2\rho_3}\Theta^{\rho_1\rho_2\rho_3} & 1 - \Theta_{\rho_1\rho_2\rho_3}\Theta^{\rho_1\rho_2\rho_3} \end{bmatrix}. \end{aligned} \tag{4.53}$$

The overall prefactor proportional to powers of $1 - \Theta_{\rho_1\rho_2\rho_3} \times \Theta^{\rho_1\rho_2\rho_3}$ comes from the proper nonlinear realization of the SL(5)/SO(5) coset element in terms of the actual space–time metric and the 3-form C-field.

As we see, three indices of the parameter $\Theta^{\mu\nu\rho}$ naturally descend from three space–time dimensions of the M2-brane, suggesting there must be $\Theta^{\mu_1\cdots\mu_6}$ accompanying it, which appears to be precisely the case. Reducing this to Type IIA, that is, breaking the exceptional group with regard to its $\mathrm{O}(d,d)$ subgroup, we generate the parameters $\Theta^{\mu_1\cdots\mu_p}$ and all the formulas listed in [147]. Hence, the '3-index' feature of the membrane theory can be understood as either the need of the Nambu structure to describe its dynamics in algebraic terms, or loop noncommutativity. It is possible that these two pictures are completely interchangeable, which stands as an interesting area of further research.

4.5 Loop ADHMN construction

The naturalness of the loop algebra description of membrane dynamics can be seen from the membrane ADHM construction. For this, let us return to the Nahm and Basu–Harvey equations and elaborate on the results of studies [156, 157] (see also lectures [158]), where the Basu–Harvey equation has been shown to be a loop-space version of the Nahm equation. The approach of the former starts from noticing that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, which allows understanding the fuzzy 3-sphere describing M2–M5-brane junctions as a couple of fuzzy 2-spheres. The construction of this work, which we will review in more detail below, uses basically the same loop-space variables and is based on the so-called transgression transformation, allowing a finite dimensional gerbe with a 2-form connection to be mapped to an infinite dimensional vector bundle with a 1-form connection taking values in loop-space.

To illustrate the construction of [157], let us start with the case of the ordinary Nahm equation describing the Dirac monopole. In the notations of [157], we write

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{X}^i = \frac{1}{2}\,\epsilon_{ijk}[\mathbb{X}^j, \mathbb{X}^k]\,, \tag{4.54}$$

where $\mathbb{X}^{i\dagger} = -\mathbb{X}^i$ take values in the algebra $\mathfrak{u}(k)$, hence describing k D1-branes. To construct the Dirac monopole solution, we define the Dirac operator

$$\not{\nabla}_s = -\mathbb{1}\,\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes \mathrm{i}\,\mathbb{X}^i\,. \tag{4.55}$$

Defining a Laplace operator $\Delta = \not{\nabla}^\dagger\,\not{\nabla}$, we see that the condition $[\Delta_s, \sigma^i \otimes \mathbb{1}] = 0$ is equivalent to the condition where $\mathbb{X}^i$ solves the Nahm equation. Following the standard ADHMN construction [62, 159], we introduce the following twist of the Dirac operator:

$$\not{\nabla}_{s,x} = -\mathbb{1}\,\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes \left(\mathrm{i}\,\mathbb{X}^i + x^i\,\mathbb{1}\right), \tag{4.56}$$

which preserves the condition $[\Delta_{s,x}, \sigma^i \otimes \mathbb{1}] = 0$ for solutions to the Nahm equation. Now, orthonormalized zero modes of the twisted Dirac operator

$$\begin{aligned} &\not{\nabla}^\dagger_{s,x}\psi_{s,x,\alpha} = 0\,, \quad \alpha = 1, \ldots, N\,, \\ &\delta_{\alpha\beta} = \int \mathrm{d}s\,\psi^\dagger{}_{s,x,\alpha}\,\psi_{s,x,\beta} \end{aligned} \tag{4.57}$$

define the gauge and scalar fields of the monopole. Here, x^i have the meaning of coordinates in the transverse space and N denotes the total number of D3-branes carrying endpoints of k D1-branes. The gauge potential and the Higgs field then read

$$\begin{aligned} &A_i = \int \mathrm{d}s\,\psi^\dagger{}_{s,x}\,\frac{\partial}{\partial x^i}\,\psi_{s,x}\,, \\ &\Phi = -\mathrm{i}\int \mathrm{d}s\,\psi^\dagger{}_{s,x}\,s\,\psi_{s,x}\,, \end{aligned} \tag{4.58}$$

where the indices α labeling D3-branes are hidden. These fields solve the corresponding Bogomolnyi equations $F_{ij} = \epsilon_{ijk}\partial_k\Phi$, which descend from the higher dimensional Yang–Mills self-duality condition. To be precise, we are working in the setup where all fields of the 10d SYM but A_i and $\Phi^6 = \Phi$ vanish.

Let us illustrate this by two simple examples. Start with $N = k = 1$, which corresponds to a single D1-brane ending on a single D3-brane. The D1-brane is stretched along $x^6 = s$. In this case, the solution to the Nahm equation is $\mathbb{X}^i = 0$; zero modes of the twisted Dirac operator read

$$\begin{aligned} &\psi_+ = \exp\left(-sR\right)\frac{\sqrt{R + x^3}}{x^1 - \mathrm{i}x^2}\begin{bmatrix} x^1 - \mathrm{i}x^2 \\ R - x^3 \end{bmatrix}, \\ &\psi_- = \exp\left(-sR\right)\frac{\sqrt{R - x^3}}{x^1 + \mathrm{i}x^2}\begin{bmatrix} R + x^3 \\ x^1 + \mathrm{i}x^2 \end{bmatrix}, \end{aligned} \tag{4.59}$$

where $R^2 = x^i x^i$. For the ψ_+ zero mode, we get the following fields

$$\begin{aligned} &\Phi = -\frac{\mathrm{i}}{2R}\,, \\ &A_i = \frac{\mathrm{i}}{2(x^1 + x^2)^2}\left[x^2\left(1 - \frac{x^3}{R}\right),\; -x^1\left(1 - \frac{x^3}{R}\right),\; 0,\; 0\right], \end{aligned} \tag{4.60}$$

which apparently describe a monopole. For the ψ_- zero mode, we get fields that are related to the above by a gauge transformation everywhere but the points $|x^3| = R$. Similarly, for two D1-branes, i.e., when $k = 2$, $N = 1$, we find $\mathbb{X}^i = (\mathrm{i}/2s)\sigma^i$ and $\Phi = -\mathrm{i}/R$, precisely the solution we started with in the previous section.

The above approach can be applied to the Basu–Harvey equation and the corresponding Bogomolnyi equation for the self-dual 3-form almost without changes up to the point when twisting occurs. The equation reads

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{X}^i = \frac{1}{3!}\,\epsilon^{ijkl}\left[\mathbb{X}^j, \mathbb{X}^k, \mathbb{X}^l\right], \tag{4.61}$$

where $\mathbb{X}^i$ now belongs to a 3-Lie algebra, which is basically a linear space with a Nambu bracket. Motivated by T-dualization and a further uplift of the D1–D3-brane system to an M2–M5-system, we write the following Dirac operator:

$$\not{\nabla}_s = -\gamma_5\,\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\,\gamma^{ij}D(\mathbb{X}^i, \mathbb{X}^j)\,, \tag{4.62}$$

where $D(\mathbb{X}^i, \mathbb{X}^j) = [\mathbb{X}^i, \mathbb{X}^j,\ ]$ is the inner derivative and γ^i are the standard Dirac gamma-matrices. For a more detailed discussion on type IIB and type IIA Dirac operators motivating the uplift, see the original paper [157]. As before, the condition that $\mathbb{X}^i$ satisfy the Basu–Harvey equation can be written in the form $[\Delta_s, \gamma^{ij}] = 0$, where $\Delta_s = \not{\nabla}_s^\dagger\,\not{\nabla}_s$. Next, we need to introduce an appropriate twist of this defined Dirac operator, for which we apparently need something of the form $\gamma^{ij}a_i b_j$, where $a_i \neq \alpha b_i$ in general for some coefficient α.

Here, we use the fact that the vector bundle of a 2-sphere describing a Dirac monopole gets replaced by a gerbe over a 3-sphere, which, by the transgression map, can be understood in terms of loops over $\mathbb{S}^3$. Hence, we introduce fields $x^i(\tau)$ with τ parametrizing the loop that are restricted by $x^i(\tau)x^i(\tau) = R^2$. From this, it follows that $x^i\dot{x}^i = 0$ and, in addition, we impose $\dot{x}^i\dot{x}^i = R^2$. Then, the proper twist of the Dirac operator can be written as

$$\nabla\!\!\!/_{s,x(\tau)} = -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma^{ij}\left[\frac{1}{2} D(\mathbb{X}^i, \mathbb{X}^j) - \mathrm{i}\, x^i(\tau)\dot{x}^j(\tau)\right]. \quad (4.63)$$

We see that loop space variables naturally enter the twisted Dirac operator, while the construction itself pretty much repeats the conventional ADHMN approach. The next step is to construct gauge and scalar fields, now defined in the loop space, using zero modes of the twisted Dirac operator:

$$\begin{aligned}\Phi\big(x(\tau)\big) &= -\mathrm{i}\int \mathrm{d}s\, \psi^\dagger{}_{s,x(\tau)}\, s\, \psi_{s,x(\tau)}\,,\\ A_i\big(x(\tau)\big) &= \int \mathrm{d}s\, \psi^\dagger{}_{s,x(\tau)}\partial_i \psi_{s,x(\tau)}\,,\end{aligned} \quad (4.64)$$

where the derivative is defined as $\partial_i = \int \mathrm{d}\tau\,(\delta/\delta x^i(\tau))$. The field strength of the gauge field $A_i(x(\tau))$ is then defined as usual as $F_{ij} = 2\partial_{[i}A_{j]}$, and if $\mathbb{X}^i$ satisfies the Basu–Harvey equation, it satisfies

$$F_{ij}\big(x(\tau)\big) = \epsilon_{ijkl}\dot{x}^k\partial_l\Phi\big(x(\tau)\big)\,. \quad (4.65)$$

The crucial statement here is that, while this has schematically the form of the Bogomolnyi equation for SYM theory, its loop structure actually makes it the desired Bogomolnyi equation for the $D = 6$ $\mathcal{N} = (2,0)$ theory. The relation between the self-dual 3-form and this defined 2-form field strength has the following form:

$$F(V_1, V_2) = \int \mathrm{d}\tau\, H_{ijk}\big(x(\tau)\big)\dot{x}^k(\tau) V_1{}^i V_2{}^j\,, \quad (4.66)$$

where $V_{1,2}$ are arbitrary vectors. Equation (4.65) is then equivalent to self-duality of the 3-form,

$$H_{05i} = \frac{1}{4}\,\partial_i\Phi\,, \quad H_{ijk} = \frac{1}{4}\,\epsilon_{ijkl}\,\partial_l\Phi\,. \quad (4.67)$$

As an example, let us look at explicit solutions for $N = k = 1$, a single M2-brane attached to a single M5-brane. Then, $\mathbb{X}^i = 0$ and there are eight zero-modes, of which we will need only four, $\gamma_5\psi_{s,x(\tau)} = \psi_{s,x(\tau)}$. This is related to doubling of zero modes when going from Pauli matrices to gamma matrices, which, in turn, is required since the SU(2) symmetry of the Nahm equation gets replaced by the SO(4) symmetry of the Basu–Harvey equation. The remaining zero-modes can be arranged as follows:

$$\psi_{s,x(\tau)} = \exp\left(-R^2 s\right) \times \begin{bmatrix} \mathrm{i}(R^2 + x^2\dot{x}^1 - x^1\dot{x}^2 - x^4\dot{x}^3 + x^3\dot{x}^4) \\ x^3(\dot{x}^1 + \mathrm{i}\dot{x}^2) + x^4(\dot{x}^2 - \mathrm{i}\dot{x}^1) - (x^1 + x^2)(\dot{x}^3 - \mathrm{i}\dot{x}^4) \\ 0 \\ 0 \end{bmatrix}. \quad (4.68)$$

Note the R^2 in the power of the exponent, which renders the correct dependence of the scalar field on the physical distance $\Phi(x) = \mathrm{i}/2R^2$. As before, for the case $N = 1$ $k = 2$, we reproduce the previously discussed solution with $\mathbb{X}^i \propto G^i$.

To summarize, following [157], we have observed that, by turning to fields defined in loop space, one is able to apply the standard ADHMN procedure to the Basu–Harvey equation and to describe self-dual string solitons in a way very similar to that of the SYM monopole. In the process, one replaces gerbes, also appearing naturally in membrane theory, with vector bundles, albeit over a loop space.

4.6 Speculations on integrability in M-theory

In Section 3, we briefly reviewed how string theory as a two-dimensional sigma model becomes (classically) integrable on certain backgrounds. This means it is possible to write equations of motion for the string in terms of a Lax connection or to write a quantum Yang–Baxter equation for its S-matrix. At the classical level, integrability requires the Lax connection to be flat, which translates into the possibility of defining a parallel transport operator, which is basically a Wilson loop calculated on the Lax 1-form. Turning to a theory of two-dimensional membranes and naively generalizing all these structures, one would expect to have a two-dimensional analogue of the Wilson loop, which it is natural to call a Wilson surface.[7] Crucial here is the fact that, on the one hand, there is no naturally defined ordering on a two-dimensional surface, and on the other hand, the 1-form Lax connection should be replaced by a two-form.

Let us first comment on the latter. Overall, it is natural to expect a 2-form in the problem, since the endpoints of a two-dimensional membrane form a string that naturally interacts with the 2-form. In M-theory, the M2-brane ends on an M5-brane; hence, the 6-dimensional world-volume theory of the latter is formulated in terms of a 2-form. Supersymmetry requires it to be self-dual, rendering a Lagrangian formulation really hard to construct (for various approaches, see [163–165]). In moving from a 1-form connection to the 2-form case, one naturally ends up with the notion of a gerbe connection, which appears when gluing co-cycles at the intersection of four or more maps [166, 167]. Hence, one possibility of constructing an analogue of the evolution operator is to use a 2-form gerbe connection.

Although gerbes provide a nice geometric background for the problem, it still remains unclear whether a natural ordering on a 2-dimensional surface exists. One way to parametrize the surface is to swipe it with loops (see, e.g., [168]), and hence the Lax connection naturally becomes a 1-form taking values in the loop space. As we have discussed above, the idea that loop spaces must be relevant to membrane dynamics is long-standing, and, in particular, in [154], it was noticed that, much like endpoints of an open string become noncommutative, string-like boundaries of the M2-brane become loop-space-noncommutative. Moreover, the metric seen by the open M2-brane boundary is precisely the one that appears in the exceptional field theory approach to deformations. As we briefly review below, the loop-space-noncommutativity of open membrane boundaries naturally appears in the analysis of the Basu–Harvey equations describing M2–M5 brane junctions.

Finally, let us consider the quantum Yang–Baxter equation describing factorization in scattering point-like

[7] Another hint comes from higher gauge theories, where Wilson surfaces understood as higher holonomies provide a set of observables [160–162].

particles. Loosely speaking, from the string theory point of view, this is related to the scattering of endpoints of an open string, which is further motivated by the Wilson loop construction discussed above. Speculating further, it can be concluded that, to describe the integrability of a membrane, one must be interested in the factorization of the scattering of strings. Indeed, the corresponding equation has been derived and is known under the names tetrahedron equation, Zamolodchikov equation, or Frenkel–Moore equation. We briefly review progress on relating these structures to 3d integrability and to M-theory below.

4.6.1 Wilson surfaces and loop-space connections. Recall that, to discuss the integrability of a two-dimensional field theory, one introduces Lax connection $A = A_\alpha \, \mathrm{d}\sigma^\alpha$ satisfying flatness condition $F = \mathrm{d}A + A \wedge A = 0$ and constructs an evolution operator that is basically a Wilson line. For periodic boundary conditions, we write

$$T = P\exp\left[\oint_\gamma A\right], \tag{4.69}$$

where integration is performed along a line. The flatness condition, which is a different way of writing equations of motion of such a system, then implies

$$\dot{T} = [T, M], \tag{4.70}$$

where $M = A_0(\sigma^1 = 0)$ and possible dependence on the spectral parameter is undermined. Then traces of various powers of T give conserved charges. Considering this to be a starting point, one is able to generate Poisson brackets of the corresponding integrable system by using r-matrix $r \in \mathrm{End}\,(V \otimes V)$, where V represents the Hilbert space of the system:

$$\{T_1, T_2\} = [r_{12}, T_1] + [r_{12}, T_2]\,. \tag{4.71}$$

Subscripts denote space on which the operator acts, i.e.,

$$T_1 = T \otimes \mathrm{id}\,, \quad T_2 = \mathrm{id} \otimes T\,. \tag{4.72}$$

To generalize these constructions to, say, a three-dimensional theory, one naturally needs a Wilson surface instead of a Wilson line along which a 2-form $B_{\mu\nu}$ is integrated. While the 1-form A represents a connection on a fiber bundle, the 2-form can naturally be thought of as a connection on a gerbe, which in turn can be mapped to 1-form connections in a loop space [169, 170]. The so-called transgression map was used in [171] to rewrite the $\mathcal{N} = (2, 0)$ theory on the M5-brane formulated in terms of Nambu brackets in [172] as a Yang–Mills-like theory in a loop space. The map naturally identifies elements of a 3-Lie algebra with elements of an associated Lie algebra (of inner derivatives). This is similar to the construction we discussed in Section 2.3, where a generalization of the r-matrix $\rho \in \mathrm{End}\,(V \otimes V \otimes V)$ was used to define the Nambu bracket for a system described by Lax pair $\dot{L} = [L, M]$ as

$$\{L_1, L_2, L_3\} = [\rho_{123}, L_1] + [\rho_{123}, L_2] + [\rho_{123}, L_3]\,, \tag{4.73}$$

where the notations are the same as above. The fundamental identity for the 3-bracket is precisely the generalized Yang–Baxter equation (4.5). Now, on the one hand, we have a

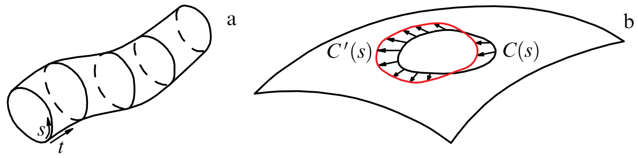

Figure 3. (a) Path in the space of loops parametrized by variable t. Points along each loop are parametrized by $s \in [0, 2\pi]$. (b) Loop $C(s)$ and its deformation $C'(s)$. Vector field tangent to the deformation is depicted by arrows.

formulation of the theory on an M5-brane, that is, a theory of boundaries of M2-branes ending on it. On the other hand, we have a generalization of the classical Yang–Baxter equation that presumably describes the scattering of straight strings that could also be understood as boundaries of the M2-brane. We will return to this point in Section 4.6.2, while we now describe the construction of [171] in more details.

The evolution operator $U = P\exp\int_\gamma A$ does not depend on the path γ if the connection A on a fiber bundle is flat. When considering a surface integral $\int_\Sigma B$ of a 2-form, we face the problem of the absence of a naturally defined ordering of points on a surface. This could be overcome by splitting a cylinder shape surface Σ into a collection of loops $C(t)$ parametrized by $t \in [0, 1]$ varying along the cylinder (Fig. 3). Hence, instead of a curve on a set of points, we consider a curve on a set of loops and instead of the 2-form B we consider a 1-form $\mathcal{A}$ representing a connection in the loop space. To be more precise, consider a space

$$\mathcal{L}M = \{C : \mathbb{S}^1 \to M\} \tag{4.74}$$

of all loops on a manifold M. In a given patch, the curve is defined by coordinate maps $x^\mu = x^\mu(s)$ with $s \in [0, 2\pi]$. At each point of the curve, we can construct a tangent vector $X^m(s)$. A collection of such tangent vectors for a given curve C we will call a vector tangent to a curve (see Fig. 3).[8] Naturally, we have a tangent bundle to the space of loops $\mathcal{L}M$. We denote the basis for vector fields as $\delta/\delta x^\mu(s)$ and the basis of 1-forms as $\delta x^\mu(s)$; then, the usual action of 1-forms reads

$$\delta x^\mu(s)\,\frac{\delta}{\delta x^\nu(s')} = \delta^\mu{}_\nu\,\delta(s - s')\,. \tag{4.75}$$

Following [170, 171], we construct the transgression map $\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)$ relating $k+1$-forms on the manifold M to k-forms in the loop space $\mathcal{L}M$. In a given basis, the map reads

$$(\mathcal{T}\omega)_C\big(v_1(x), \ldots, v_k(x)\big) = \oint_{\mathbb{S}^1} \mathrm{d}s\,\omega\big(v_1(s), \ldots, v_k(s), \dot{x}(s)\big)\,, \tag{4.76}$$

where $x^\mu = x^\mu(s)$ represents coordinates on a loop C. Hence, on the left-hand side we have a k-form evaluated in k vector fields at a point C of $\mathcal{L}M$, while on the right-hand side we have a $k+1$-form evaluated in $k+1$ vector fields at points $x^\mu(s)$ and integrated to keep the information of the whole loop. As a

[8] We intentionally keep the discussion more intuitively clear. For more rigor and a mathematically formal description of these structures, see, e.g., [170] and references therein.

more explicit example, consider the case $k = 2$:

$$\int \mathrm{d}s\,\mathrm{d}t\,(\mathcal{T}\omega)_{\mu,s;\nu,t}v_1^\mu\big(x(s)\big)v_2^\nu\big(x(t)\big)$$
$$= \oint_{\mathbb{S}^1}\mathrm{d}r\,\omega_{\mu\nu\rho}v_1^\mu\big(x(r)\big)v_2^\nu\big(x(r)\big)\,\frac{\mathrm{d}x^\rho}{\mathrm{d}r}\,. \qquad (4.77)$$

Note the integrals over s and t on the left-hand side, which can be understood as an analogue of index contraction when acting by a form on a vector in the loop space, i.e., a form in the loop space has a discrete index $\mu, \nu, \ldots$ and a continuous 'index' $s, t, \ldots$. Inserting Dirac delta-functions on the right-hand side and dropping two integrals, we finally have

$$(\mathcal{T}\omega)_{\mu,s;\nu,t} = \int \mathrm{d}r\,\omega_{\mu\nu\rho}\big(x(r)\big)\,\frac{\mathrm{d}x^\rho}{\mathrm{d}r}\,\delta(sr)\delta(tr)\,. \qquad (4.78)$$

The next step in connecting 3-Lie algebra variables of the $\mathcal{N} = (2,0)$ theory on an M5-brane to a Yang–Mills-like theory is to construct a Lie algebra out of Nambu brackets. For this, we simply consider an associated algebra $\mathfrak{g}_\mathcal{A}$ of inner derivatives, i.e., for any two elements $a, b \in \mathcal{A}$ of the 3-Lie algebra, consider an action

$$D(a,b) \triangleright x \equiv [a,b,x], \quad x \in \mathcal{A}\,. \qquad (4.79)$$

The self-dual 3-form $H_{\mu\nu\rho}$ of the theory takes values in the 3-Lie algebra $\mathcal{A}$ and can be mapped to a 2-form Yang–Mills field strength in the loop space by making use of the transgression map as described above. To do so, we consider a loop C^m taking values in the 3-Lie algebra. Assume that the algebra and loop variables detach, i.e.,

$$C^\mu(s) = C\,x^\mu(s)\,, \quad C \in \mathcal{A}\,. \qquad (4.80)$$

Formally, this can be ensured by imposing $D(C^\mu, C^\nu) = 0$. The transgression-like map for the self-dual 3-form is then defined as

$$(F_{\mu\nu})_C(v_1, v_2) = \oint_{\mathbb{S}^1}\mathrm{d}s\,D\Big(C, H_{\mu\nu\rho}\big(x(s)\big)\dot{x}^\rho(s)\Big)v_1^\mu(s)v_2^\nu(s)\,. \qquad (4.81)$$

In the component form, the action of the 2-form field strength inside the algebra $\mathfrak{g}_\mathcal{A}$ then reads

$$F_{\mu,s;\nu,t} \triangleright \bullet = \oint_{\mathbb{S}^1}\mathrm{d}t\,\big[C, H_{\mu\nu\rho}(r)\dot{x}^\rho(r), \bullet\big]\delta(s-r)\delta(t-r)\,. \qquad (4.82)$$

Given that the 3-form is exact, $H = \mathrm{d}B$, the 2-form $F_{\mu\nu}$ in the loop space can be represented as the field strength for a 1-form defined naturally as

$$A_{\mu,s} \triangleright \bullet = \oint_{\mathbb{S}^1}\mathrm{d}s\,\big[C, B_{\mu\nu}(s)\dot{x}^\nu(s), \bullet\big]\delta(s-t)\,. \qquad (4.83)$$

Here, we are restricted to loops that are covariantly constant regarding this defined 1-form connection:

$$0 = \nabla_\mu C^\nu \equiv \partial_\mu \dot{x}^\nu(s)\,C + \dot{x}^\nu(A_\mu \triangleright C)$$
$$= \oint_{\mathbb{S}^1}\mathrm{d}s\,\frac{\delta}{\delta x^\mu(s)}\,\dot{x}^\nu(s)\,C + \dot{x}^\nu(A_\mu \triangleright C)\,. \qquad (4.84)$$

The integral is apparently zero, and the vanishing of the remaining term effectively implies

$$[C, \bullet, C] = 0\,. \qquad (4.85)$$

The transgression-like map, as above, can also be extended to fermionic and scalar fields of the $\mathcal{N} = (2,0)$ theory, all taking values in the 3-Lie algebra. Hence, the whole formalism, including equations of motion and supersymmetry transformation rules, gets rewritten in loop variables.

Speculating on these results and those from the previous section, one may conclude that loop-space variables are more natural for describing world-volume theory of the M5-brane and hence the dynamics of the open M2-brane boundaries. Although the above construction describes the M5-brane world-volume theory, it gives suggestive hints on how integrability structures for the M2-brane can be formulated. To start with, in previous sections, we have seen a tight relation between the Lax connection for a string on a certain background, the classical Yang–Baxter equation for the r-matrix, and the quantum Yang–Baxter equation. The first is simply the quasi-classical limit of the latter, which, in turn, defines the S-matrix of the string on certain backgrounds. Poisson brackets for evolution operators constructed of the former can be defined in terms of the classical r-matrix as at the beginning of this subsection. Moreover, the classical Yang–Baxter equation appears in relation to bivector deformations given by the open-closed string map, as discussed in Section 3.4. Now, we see the same relation between an open-closed membrane map and the so-called generalized Yang–Baxter equation that appears in relation to tri-vector deformations of 11d backgrounds. The generalized theta parameter $\theta^{\mu\nu\rho}$ that defines the open-closed membrane map also defines the noncommutativity of the membrane on a background with a nonvanishing C-field. Moreover, commutation relations are written for loop space variables; hence, we are talking about loop-space noncommutativity. Similarly to the 2d case, the generalized Yang–Baxter equation guarantees that the bracket defined using the ρ-tensor as in (4.73) is a Nambu bracket, i.e., satisfies the fundamental identity.

To wrap up the above logic, we would like to define an evolution operator using a properly defined Wilson surface or a holonomy in the loop space, which causes the most trouble. A way to define an analogue of the Lax–Zakharov–Shabat construction for higher dimensional theories using loop space variables has been suggested in [173] (for a concise review, see [168]). The main idea is to parametrize the Wilson surface by a collection of loops, each satisfying a parallel transport equation for a loop space connection A_μ. The connection 1-form takes values in a non-Abelian algebra and acts nontrivially on loops. Although this formalism is not completely identical to the one described above, it is similar enough to be of interest for further investigation.

4.6.2 Tetrahedron equation. The whole discussion around open-closed string/membrane parameters, the associated r-matrix or ρ-tensor, and Poisson/Nambu brackets has centered on the classical Yang–Baxter equation and its generalized analogue for the ρ-tensor. As was briefly discussed in Section 3.2, the integrability of the string means not only the possibility of defining a flat Lax connection but also a possibility of writing the string S-matrix in terms of the quantum R-matrix that solves the quantum Yang–Baxter equation (QYBE)

$$R_{12}R_{23}R_{13} = R_{13}R_{23}R_{12}\,. \qquad (4.86)$$

Here, the subscript denotes the Hilbert space on which the R-matrix acts at each intersection. The quantum Yang–Baxter equation describes the scattering of point-like particles, stating that the S-matrix factorizes, and appears to be a particular case of an infinite series of so-called simplex equations (see, e.g., [174]). An n-simplex equation may be understood as describing factorization of the S-matrix corresponding to the scattering of $n-1$-dimensional objects. Hence, we become naturally interested in 2-simplex equations, also known as the Zamolodchikov tetrahedron equation (ZTE), introduced in [175] and further developed in [176, 177] as a description of 3d integrable systems. The 4-simplex equation appeared in [178], and n-simplex equations have been studied, for example, in [174, 179, 180].

Different labeling schemes can be used to write down the 3-simplex equation: label the states of string segments between vertices [175], label the state of vacua between the strings (faces of tetrahedron) [174], or label particles at the edges of the strings [178]. Let us start with the first one and consider the scattering of straight strings. The r-matrix then acts on the space of states V of intersecting strings at each tetrahedron vertex, and hence $R \in \operatorname{End}(V \otimes V \otimes V)$. Then, the equation reads

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123} \,. \tag{4.87}$$

An alternative labeling scheme uses labels for the tetrahedron faces, which are now four in total and hence the scheme is nonlocal [181]. This corresponds to an equation on $V^{\otimes 4}$ rather than $V^{\otimes 6}$ and is referred to as the Frenkel–Moore equation:

$$R_{234} R_{134} R_{124} R_{123} = R_{123} R_{124} R_{134} R_{234} \,. \tag{4.88}$$

Note the important distinction between the qYBE and ZTE that indices for each space are contracted more than twice. More details on the tetrahedron equation and its relation to other quantum group equations can be found, e.g., in [182, 183].

Given the relations between the quasi-classical limit of the quantum Yang–Baxter equation and bi-vector deformations of 10d string backgrounds, it is suggestive to search for similar relations between the tetrahedron equation (in any formulation) and the so-called generalized Yang–Baxter equation (4.5). Unfortunately, this path is not as straightforward as it seems, since it is not known how to define a quasi-classical limit of a tetrahedron equation. One may take the most naive path, for which the Frenkel–Moore equation is best suited and consider $R_{123} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \hbar\, \rho_{123}$. Substituting this into (4.88), we find that the orders $\hbar^0$ and $\hbar^1$ are satisfied trivially, while the order $\hbar^2$ provides

$$\begin{aligned} &[\rho_{123}, \rho_{124}] + [\rho_{123}, \rho_{134}] + [\rho_{124}, \rho_{134}] + [\rho_{123}, \rho_{234}] \\ &\quad + [\rho_{134}, \rho_{234}] + [\rho_{124}, \rho_{234}] = 0 \,. \end{aligned} \tag{4.89}$$

This has the form of a nice generalization of the classical Yang–Baxter equation; however, it cannot be written in the form (4.5) for a general algebra of endomorphisms. To illustrate this, we decompose ρ into a basis $\{t_a\} = \operatorname{bas}\mathfrak{g}$, say,

$$r_{123} = \rho^{abc}\, t_a \otimes t_b \otimes t_c \otimes \mathbf{1} \,, \tag{4.90}$$

assuming $\rho^{abc} = \rho^{[abc]}$ is completely antisymmetric. Now, it is easy to see that in each term in the classical 3-simplex equation (4.89) one obtains expressions of the form

$$a \cdot b \otimes c \cdot d - b \cdot a \otimes d \cdot c \,, \tag{4.91}$$

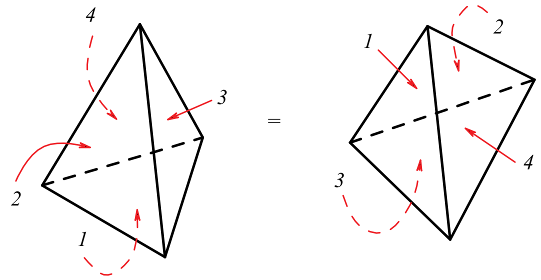

Figure 4. Factorization of the scattering of straight strings depicted in the form of a tetrahedron equation. Labeling scheme is chosen according to Frenkel and Moore, and numbers correspond to faces.

where $a \cdot b$ denotes multiplication in the universal enveloping algebra $U(\mathfrak{g})$. This can be transformed into

$$a \cdot b \otimes c \cdot d - b \cdot a \otimes d \cdot c = [a, b] \otimes \{c, d\} + \{a, b\} \otimes [c, d] \,, \tag{4.92}$$

where $[a, b]$ is the image of the Lie bracket and we formally define $\{a, b\} := a \cdot b + b \cdot a$. Since we are interested in the algebra of Killing vectors, it is not completely clear how to define a symmetric product without introducing a connection. Certainly, this does not prevent us from searching for solutions to other realizations of the algebra $\mathfrak{g}$, e.g., for $\operatorname{Spin}(d)$, the anticommutator is perfectly defined and one may proceed.

A seemingly more fruitful approach is to turn ZTE into the so-called decorated Yang–Baxter equation. For this, suppose that the spaces with labels, say $1, 2, 3$, are considered additional (color) states. Then, the two tetrahedra in Fig. 4 are simply two triangula with additional lines decorating them. Introducing labels α, β, γ instead of $1, 2, 3$, we may rewrite ZTE as

$$R_{\alpha,45} R_{\beta,46} R_{\gamma,56} = R^{-1}_{\alpha\beta\gamma} R_{\gamma,56} R_{\beta,46} R_{\alpha,45} R_{\alpha\beta\gamma} \,. \tag{4.93}$$

Hence, we see the familiar structure of the quantum Yang–Baxter equation, where (i) each R-matrix carries an additional label (is decorated), (ii) the RHS gets twisted by the adjoint action of $R_{\alpha\beta\gamma}$. It is tempting to think that the additional color label corresponds to having loop-space variables; however, a precise realization of this statement is not clear.

5. Conclusions

In this review, we have made an attempt to collect methods aimed at investigating the integrability in string theory as a $1+1$-dimensional sigma model and various observations that hint at a possible generalization of these methods to the theory of membranes. In the main text, we briefly discuss each of the methods and observations in some detail to give a general expression of the corresponding techniques and provide links to original studies, reviews, lectures, and introductory papers. To conclude, let us first recap all this in the form of simple lists.

Let us start with a list of the techniques and observations addressed above that are related to the integrability of the string and to symmetries of its space of vacua, i.e., 10d supergravity backgrounds.

• A $1+1$-dimensional field theory is integrable when its equations of motion can be written in the form of the flatness of a connection; the corresponding Wilson line defines the Lax operator.

• Given a Lax pair, the classical r-matrix can be used to generate Poisson brackets that define an integrable system.

• Integrable deformations of the 2d sigma model are generated by the r-matrix solving the classical Yang–Baxter equation.

• The classical Yang–Baxter equation is a limit of the quantum Yang–Baxter equation describing factorization of the S-matrix for scattering particles.

• Yang–Baxter deformations generate families of classical Drinfeld algebras that stand behind Poisson–Lie T-duality symmetries.

• At the level of supergravity backgrounds, Yang–Baxter deformations are generated by a bi-vector that is the r-matrix dressed by Killing vectors.

• The deformation map has the same form as the open-closed string map, with the bi-vector having the meaning of the noncommutativity parameter.

We see here close connections between the classical Yang–Baxter equation and the integrability of the string, which is in some sense expected, and symmetries of the space of solutions of supergravity field equations. Similar connections have been found in 11d supergravity that provides backgrounds for the membranes. The corresponding list of statements can be composed as follows.

• Families of exceptional Drinfeld algebras standing behind Nambu–Lie U-duality can be generated by generalized Yang–Baxter deformation defined by ρ-tensors satisfying the generalized Yang–Baxter equation.

• At the level of supergravity, generalized Yang–Baxter deformations are generated by tri- and six-vectors that are ρ-tensors dressed by Killing vectors.

• The deformation map has the same form as the open-closed membrane map, with the 3-vector having the meaning of the loop noncommutativity parameter.

• For a given Lax pair, a ρ-tensor can be used to generate Nambu brackets that define a mechanical system.

This list intentionally does not mention speculation on integrability. Indeed, there is no construction for a $1+2$-dimensional theory similar to the Lax–Zakharov–Shabat description of integrability of $1+1$-systems. The results collected in the second list above and their similarity to the first list suggest that the generalized Yang–Baxter equation must have some relation to integrability properties of the membrane. Certainly, the relation is far from obvious; however, there are some observations concerning an algebraic description of membrane dynamics in general and in the context of M-theory that we find particularly useful. They can be combined in the following list.

• The quantum version of the generalized Yang–Baxter equation is not known, but the index structure of the ρ-tensor suggests that it must be the tetrahedron equation that describes factorization of the S-matrix for straight strings.

• A canonical analysis and ADHMN construction for the membrane suggest that natural variables to describe its dynamics could be functions taking values in the algebra of loops.

• Using loops, one is able to introduce a natural ordering on a Wilson surface, presumably defining the Lax operator for the corresponding $1+2$-dimensional system.

• The quasiclassical limit of the tetrahedron equation is not known; however, it can be written as a set of Yang–Baxter equations on decorated quantum R-matrices.

• The natural (Takhtajan) action for a system described by a Nambu 3-bracket has the form of the Wess–Zumino term for the M2-brane ending on an M5-brane.

• A generalization of KP hierarchy that has time variables $t_{m,n}$ parametrized by pairs of indices can be defined using Nambu brackets. Similar variables are found when defining a generalization of Schur polynomials to the case of 3d Young diagrams presumably describing the integrability of $1+2$-dimensional systems [184, 185].

We find the following areas of research the most promising. First, we are naturally interested in finding a way to arrive at the generalized Yang–Baxter equation from the tetrahedron equation, which naturally describes factorization of the S-matrix for straight string scattering. Second, it would be interesting to construct an analogue of the evolution operator in 3d using loop algebra variables, which seem to be natural for describing M2-brane dynamics (or at least the dynamics of its endpoints). Hopefully, more will be reported on these and other related questions in the nearest future.

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References

1. Slavnov N A, arXiv:1804.07350
2. Schäfer-Nameki S *Lett. Math. Phys.* **99** 169 (2012); arXiv:1012.3989
3. Staudacher M *Lett. Math. Phys.* **99** 191 (2012); arXiv:1012.3990
4. Bajnok Z *Lett. Math. Phys.* **99** 299 (2012); arXiv:1012.3995
5. Levkovich-Maslyuk G *J. Phys. A* **49** 323004 (2016); arXiv:1606.02950
6. Bena I, Polchinski J, Roiban R *Phys. Rev. D* **69** 046002 (2004); hep-th/0305116
7. Minahan J A, Zarembo K *J. High Energy Phys.* **2003** (03) 013 (2003); hep-th/0212208
8. Lunin O, Maldacena J *J. High Energy Phys.* **2005** (05) 033 (2005) https://doi.org/10.1088/1126-6708/2005/05/033; hep-th/0502086
9. Leigh R G, Strassler M J *Nucl. Phys. B* **447** 95 (1995); hep-th/9503121
10. Frolov S A, Roiban R, Tseytlin A A *J. High Energy Phys.* **2005** (07) 045 (2005); hep-th/0503192
11. Frolov S *J. High Energy Phys.* **2005** (05) 069 (2005); hep-th/0503201
12. Giataganas D, Pando Zayas L A, Zoubos K *J. High Energy Phys.* **2014** (01) 129 (2014); arXiv:1311.3241
13. Beisert N et al. *Lett. Math. Phys.* **99** 3 (2012); arXiv:1012.3982
14. Dekel A, Oz Y *J. High Energy Phys.* **2011** (08) 004 (2011); arXiv:1106.3446
15. Linardopoulos G, Zarembo K *J. High Energy Phys.* **2021** (05) 203 (2021); arXiv:2102.12381
16. Linardopoulos G *J. High Energy Phys.* **2022** (06) 033 (2022); arXiv:2202.06824
17. Giataganas D, Zoubos K *J. High Energy Phys.* **2017** (10) 042 (2017); arXiv:1707.04033
18. Giataganas D *Phys. Rev. D* **104** 066017 (2021); arXiv:1909.02577
19. Klimcík C *J. High Energy Phys.* **2002** (12) 051 (2002); hep-th/0210095
20. Delduc F, Magro M, Vicedo B *J. High Energy Phys.* **2013** (11) 192 (2013); arXiv:1308.3581
21. Delduc F, Magro M, Vicedo B *Phys. Rev. Lett.* **112** 051601 (2014); arXiv:1309.5850
22. Kawaguchi I, Matsumoto T, Yoshida K *J. High Energy Phys.* **2014** (04) 153 (2014); arXiv:1401.4855

23. van Tongeren S J *J. High Energy Phys.* **2015** (06) 048 (2015); arXiv:1504.05516
24. Hoare B, Seibold F K *J. High Energy Phys.* **2019** (01) 125 (2019); arXiv:1811.07841
25. Arutyunov G, Borsato R, Frolov S *J. High Energy Phys.* **2015** (12) 049 (2015) https://doi.org/10.1007/JHEP12%282015%29049; arXiv:1507.04239
26. Arutyunov G et al. *Nucl. Phys. B* **903** 262 (2016); arXiv:1511.05795
27. Tseytlin A A, Wulff L *J. High Energy Phys.* **2016** (06) 174 (2016) https://doi.org/10.1007/JHEP06%282016%29174; arXiv:1605.04884
28. Orlando D et al. *J. Phys. A* **53** 443001 (2020); arXiv:1912.02553
29. Araujo T et al. *J. Phys. A* **51** 235401 (2018); arXiv:1705.02063
30. Araujo T et al. *Phys. Rev. D* **95** 105006 (2017); arXiv:1702.02861
31. Bakhmatov I et al. *J. High Energy Phys.* **2018** (06) 161 (2018); arXiv:1803.07498
32. Bakhmatov I, Musaev E T *J. High Energy Phys.* **2019** (01) 140 (2019); arXiv:1811.09056
33. Borsato R, Wulff L *J. High Energy Phys.* **2016** (10) 045 (2016); arXiv:1608.03570
34. Bakhmatov I et al. *J. High Energy Phys.* **2019** (08) 126 (2019); arXiv:1906.09052
35. Bakhmatov I, Gubarev K, Musaev E T *J. High Energy Phys.* **2020** (05) 113 (2020); arXiv:2002.01915
36. Gubarev K, Musaev E T *Phys. Rev. D* **103** 066021 (2021); arXiv:2011.11424
37. Malek E, Thompson D C *J. High Energy Phys.* **2020** (04) 058 (2020); arXiv:1911.07833
38. Sakatani Y *Prog. Theor. Exp. Phys.* **2020** 023B08 (2020) https://doi.org/10.1093/ptep/ptz172; arXiv:1911.06320
39. Malek E, Sakatani Y, Thompson D C *J. High Energy Phys.* **2021** (01) 020 (2021); arXiv:2007.08510
40. de Leeuw M, Candu C "Introduction to Integrability FS 2013", ITP Lecture Archive, https://edu.itp.phys.ethz.ch/fs13/int/
41. Retore A L *J. Phys. A* **55** 173001 (2022); arXiv:2109.14280
42. Zamolodchikov A B, Zamolodchikov A B *Ann. Physics* **120** 253 (1979)
43. Zamolodchikov Al B *Nucl. Phys. B* **342** 695 (1990)
44. Yang C N *Phys. Rev. Lett.* **19** 1312 (1967)
45. Baxter R J *Ann. Physics* **70** 193 (1972)
46. Bombardelli D *J. Phys. A* **49** 323003 (2016); arXiv:1606.02949
47. Ryan P "Integrable systems, separation of variables and the Yang–Baxter equation", PhD Thesis (Dublin: School of Mathematics, Trinity College, The Univ. of Dublin, 2021); http://www.tara.tcd.ie/bitstream/handle/2262/97300/Paul_Ryan_PhD_Final.pdf; arXiv:2201.12057
48. Wu F Y *Int. J. Mod. Phys. B* **7** 3737 (1993)
49. Turaev V G *Invent. Math.* **92** 527 (1988)
50. Kauffman L H "Knots, abstract tensors and the Yang–Baxter equation", in *Knots, Topology and Quantum Field Theories. Proc. of the Johns Hopkins Workshop on Current Problems in Particle Theory 13* (Ed. L Lusanna) (Singapore: World Scientific, 1989) p. 179
51. Belavin A A, Drinfel'd V G *Funct. Anal. Its Appl.* **16** 159 (1982); *Funkts. Analiz Ego Prilozh.* **16** (3) 1 (1982)
52. Arnold V I *Mathematical Methods of Classical Mechanics* 2nd ed. (Graduate Texts in Mathematics, Vol. 60) (New York: Springer, 1989) https://doi.org/10.1007/978-1-4757-2063-1; Translated from Russian: *Matematicheskie Metody Klassicheskoi Mekhaniki* 2nd ed. (Moscow: Nauka, 1979)
53. Takhtajan L *Commun. Math. Phys.* **160** 295 (1994); hep-th/9301111
54. Helal M A, El-Eissa H N *Pure Math. Appl.* **7** 263 (1996)
55. Agrawal G *Nonlinear Fiber Optics* 5th ed. (Amsterdam: Elsevier, 2013) https://doi.org/10.1016/C2011-0-00045-5
56. Caudrey P J, Eilbeck J C, Gibbon J D *Nuovo Cimento B* **25** 497 (1975)
57. Aratyn H, in *Proc. of the 8th Jorge Andre Swieca Summer School: Particles and Fields, 5–18 February 1995, Rio de Janeiro, Brazil* (Eds J Barcelos-Neto, S F Novaes, V O Rivelles) (Singapore: World Scientific, 1996) p. 419; hep-th/9503211
58. Vladimirov A A "Lectures on integrable hierarchies and vertex operators", in *2nd Dubna Intern. Advanced School of Theoretical Physics, DIAS – TH, Dubna, 30 Jan. – 7 Feb. 2004*; hep-th/0402097
59. Basu A, Harvey J A *Nucl. Phys. B* **713** 136 (2005); hep-th/0412310
60. Howe P S, Lambert N D, West P C *Nucl. Phys. B* **515** 203 (1998); hep-th/9709014
61. Callan C G (Jr.), Maldacena J M *Nucl. Phys. B* **513** 198 (1998); hep-th/9708147
62. Nahm W *Phys. Lett. B* **90** 413 (1980)
63. Diaconescu D-E *Nucl. Phys. B* **503** 220 (1997); hep-th/9608163
64. Bagger J, Lambert N *Phys. Rev. D* **75** 045020 (2007); hep-th/0611108
65. Gustavsson A *Nucl. Phys. B* **811** 66 (2009); arXiv:0709.1260
66. Aharony O et al. *J. High Energy Phys.* **2008** (10) 091 (2008); arXiv:0806.1218
67. Nambu Y *Phys. Rev. D* **7** 2405 (1973)
68. Chatterjee R *Lett. Math. Phys.* **36** 117 (1996); hep-th/9501141
69. Ho P-M, Matsuo Y *Prog. Theor. Exp. Phys.* **2016** 06A104 (2016) https://doi.org/10.1093/ptep/ptw075; arXiv:1603.09534
70. Guha P, math-ph/9807018
71. Li C et al. *Int. J. Mod. Phys. A* **35** 2050099 (2020)
72. Wang X-L et al. *J. Nonlin. Math. Phys.* **22** 194 (2015)
73. Takasaki K, Takebe T *Rev. Math. Phys.* **7** 743 (1995); hep-th/9405096
74. Takasaki K, Takebe T *Int. J. Mod. Phys. A* **7** (supp01b) 889 (1992); hep-th/9112046
75. van Tongeren S J *J. Phys. A* **47** 433001 (2014); arXiv:1310.4854
76. Sfetsos K *Nucl. Phys. B* **880** 225 (2014); arXiv:1312.4560
77. Hollowood T J, Miramontes J L, Schmidtt D M *J. Phys. A* **47** 495402 (2014); arXiv:1409.1538
78. Zakharov V E, Mikhailov A V *Sov. Phys. JETP* **47** 1017 (1978); *Zh. Eksp. Teor. Fiz.* **74** 1953 (1978)
79. Cherednik I V *Theor. Math. Phys.* **47** 422 (1981); *Teor. Matem. Fiz.* **47** 225 (1981)
80. Klimčík C *J. Math. Phys.* **50** 043508 (2009); arXiv:0802.3518
81. Sakamoto J, Sakatani Y, Yoshida K *Prog. Theor. Exp. Phys.* **2017** 053B07 (2017) https://doi.org/10.1093/ptep/ptx067; arXiv:1703.09213
82. Fernández-Melgarejo J J et al. *Phys. Rev. Lett.* **122** 111602 (2019); arXiv:1811.10600
83. Mück W *J. High Energy Phys.* **2019** (05) 063 (2019); arXiv:1904.06126
84. Hollowood T J, Miramontes J L, Schmidtt D M *J. High Energy Phys.* **2014** (11) 009 (2014); arXiv:1407.2840
85. Balog J et al. *Phys. Lett. B* **324** 403 (1994); hep-th/9307030
86. Evans J M, Hollowood T J *Nucl. Phys. B* **438** 469 (1995); hep-th/9407113
87. Magro M *Lett. Math. Phys.* **99** 149 (2012); arXiv:1012.3988
88. Thompson D C *PoS* **CORFU2018** 099 (2019); arXiv:1904.11561
89. Driezen S "Geometrical approach to integrable and supersymmetric sigma models", PhD Thesis (Brussel: Vrije Univ. Brussel, 2019)
90. Hoare B, Roiban R, Tseytlin A A *J. High Energy Phys.* **2014** (06) 002 (2014); arXiv:1403.5517
91. Lunin O, Roiban R, Tseytlin A A *Nucl. Phys. B* **891** 106 (2015); arXiv:1411.1066
92. Sfetsos K, Siampos K, Thompson D C *Nucl. Phys. B* **899** 489 (2015); arXiv:1506.05784
93. Hoare B, Tseytlin A A *Nucl. Phys. B* **897** 448 (2015); arXiv:1504.07213
94. van Tongeren S J *Nucl. Phys. B* **904** 148 (2016); arXiv:1506.01023
95. Imeroni E *J. High Energy Phys.* **2008** (10) 026 (2008); arXiv:0808.1271
96. Arutyunov G, Frolov S *J. Phys. A* **42** 254003 (2009); arXiv:0901.4937
97. Metsaev R R, Tseytlin A A *Nucl. Phys. B* **533** 109 (1998); hep-th/9805028
98. Borsato R, Wulff L *J. High Energy Phys.* **2018** (08) 027 (2018); arXiv:1806.04083
99. Klimčík C, Ševera P *Phys. Lett. B* **372** 65 (1996); hep-th/9512040
100. de la Ossa X C, Quevedo F *Nucl. Phys. B* **403** 377 (1993); hep-th/9210021
101. Sfetsos K *Fortschr. Phys.* **59** 1149 (2011); arXiv:1105.0537
102. Thompson D C *Fortschr. Phys.* **64** 349 (2016); arXiv:1512.04732
103. Petr I *AIP Conf. Proc.* **1307** 119 (2010)
104. Hassler F *Phys. Lett. B* **807** 135455 (2020); arXiv:1707.08624
105. Blumenhagen R et al. *PoS* **CORFU2016** 128 (2017); arXiv:1703.07347

106. Demulder S, Hassler F, Thompson D C *PoS* **CORFU2018** 113 (2019); arXiv:1904.09992
107. Borsato R, Wulff L *Phys. Rev. Lett.* **125** 201603 (2020); arXiv:2007.07902
108. Hassler F, Rochais T *Fortschr. Phys.* **68** 2000063 (2020); arXiv:2007.07897
109. Borsato R, Driezen S *J. High Energy Phys.* **2021** (05) 180 (2021); arXiv:2102.04498
110. Borsato R, Driezen S, Hassler F *Phys. Lett. B* **823** 136771 (2021); arXiv:2109.06185
111. Musaev E T, Sakatani Y *Phys. Rev. D* **104** 046015 (2021); arXiv:2012.13263
112. Hlavatý L, Šnobl L *Mod. Phys. Lett. A* **17** 429 (2002); hep-th/0110139
113. Šnobl L, Hlavatý L *Int. J. Mod. Phys. A* **17** 4043 (2002); math/0202210
114. Hlavatý L, Šnobl L, math/0202209
115. von Unge R *J. High Energy Phys.* **2002** (07) 014 (2002); hep-th/0205245
116. Sakamoto J, Sakatani Y, Yoshida K *J. Phys. A* **50** 415401 (2017); arXiv:1705.07116
117. Çatal-Özer A, Tunali S *Class. Quantum Grav.* **37** 075003 (2020); arXiv:1906.09053
118. Borsato R, Vilar López A, Wulff L *J. High Energy Phys.* **2020** (07) 103 (2020); arXiv:2003.05867
119. Seiberg N, Witten E *J. High Energy Phys.* **1999** (09) 032 (1999); hep-th/9908142
120. Fradkin E S, Tseytlin A A *Ann. Physics* **162** 31 (1985)
121. Siegel W *Phys. Rev. D* **48** 2826 (1993); hep-th/9305073
122. Siegel W *Phys. Rev. D* **47** 5453 (1993); hep-th/9302036
123. Hohm O, Hull C, Zwiebach B *J. High Energy Phys.* **2010** (08) 008 (2010); arXiv:1006.4823
124. Hohm O, Kwak S K, Zwiebach B *J. High Energy Phys.* **2011** (09) 013 (2011); arXiv:1107.0008
125. Jeon I et al. *Phys. Lett. B* **723** 245 (2013); arXiv:1210.5078
126. Thompson D C *J. High Energy Phys.* **2011** (08) 125 (2011); arXiv:1106.4036
127. Hohm O, Lüst D, Zwiebach B *Fortschr. Phys.* **61** 926 (2013); arXiv:1309.2977
128. Aldazabal G, Marqués D, Núñez C *Class. Quantum Grav.* **30** 163001 (2013); arXiv:1305.1907
129. Geissbühler D et al. *J. High Energy Phys.* **2013** (06) 101 (2013); arXiv:1304.1472
130. Townsend P K "Four lectures on M-Theory", in *1996 Summer School in High Energy Physics and Cosmology, June 10 – July 26, 1996, at the ICTP, in Trieste, Italy*, (The ICTP Ser. in Theoretical Physics, Vol. 13, Eds E Gava et al.) (New York: World Scientific, 1997) p. 385; hep-th/9612121
131. Cremmer E, Julia B *Phys. Lett. B* **80** 48 (1978)
132. Hull C M, Townsend P K *Nucl. Phys. B* **438** 109 (1995); hep-th/9410167
133. Cremmer E et al. *Nucl. Phys. B* **523** 73 (1998); hep-th/9710119
134. Cremmer E et al. *Nucl. Phys. B* **535** 242 (1998); hep-th/9806106
135. Obers N A, Pioline B *Phys. Rep.* **318** 113 (1999); hep-th/9809039
136. Duff M J, Lu J X *Nucl. Phys. B* **347** 394 (1990)
137. Sakatani Y, Uehara S *Prog. Theor. Exp. Phys.* **2020** 073B01 (2020) https://doi.org/10.1093/ptep/ptaa063; arXiv:2001.09983
138. Samtleben H *Class. Quantum Grav.* **25** 214002 (2008); arXiv:0808.4076
139. Musaev E T *Universe* **8** 276 (2022); arXiv:2007.01213
140. Baguet A, Hohm O, Samtleben H *PoS* **CORFU2014** 133 (2015) https://doi.org/10.22323/1.231.0133; arXiv:1506.01065
141. Baguet A "Exceptional field theory and supergravity", PhD Thesis (Lyon: l'École Normale Supérieure, 2017)
142. Hohm O, Samtleben H *PoS* **CORFU2018** 098 (2019) https://doi.org/10.22323/1.347.0098; arXiv:1905.08312
143. Musaev E T *Symmetry* **11** 993 (2019)
144. Berman D S, Blair C *Int. J. Mod. Phys. A* **35** 2030014 (2020); arXiv:2006.09777
145. Bakhmatov I et al. *Phys. Rev. D* **105** L081904 (2022); arXiv:2203.03372
146. Bakhmatov I et al. *Eur. Phys. J. C* **83** 37 (2023); arXiv:2209.01423
147. Berman D S et al. *J. High Energy Phys.* **2002** (02) 012 (2002); hep-th/0109107
148. Myers R *Int. J. Mod. Phys. A* **16** 956 (2001); hep-th/0106178
149. Constable N R, Lambert N D *Phys. Rev. D* **66** 065016 (2002); hep-th/0206243
150. Howe P S, Sezgin E *Phys. Lett. B* **394** 62 (1997); hep-th/9611008
151. Howe P S, Sezgin E, West P C *Phys. Lett. B* **399** 49 (1997); hep-th/9702008
152. Guralnik Z, Ramgoolam S *J. High Energy Phys.* **2001** (02) 032 (2001); hep-th/0101001
153. Gustavsson A *J. High Energy Phys.* **2010** (11) 043 (2010); arXiv:1008.0902
154. Bergshoeff E et al. *Nucl. Phys. B* **590** 173 (2000); hep-th/0005026
155. Bergshoeff E et al. *Phys. Lett. B* **492** 193 (2000); hep-th/0006112
156. Gustavsson A *J. High Energy Phys.* **2008** (04) 083 (2008); arXiv:0802.3456
157. Sämann C *Commun. Math. Phys.* **305** 513 (2011); arXiv:1007.3301
158. Saemann C "Lectures on higher structures in M-theory", in *Workshop on Strings, Membranes and Topological Field Theory, Tohoku University, Sendai, Japan, 5–7 March 2015* (Noncommutative Geometry and Physics, Vol. 4, Eds Y Maeda et al.) (Singapore: World Scientific, 2017) p. 171, https://doi.org/10.1142/9789813144613_0004; arXiv:1609.09815
159. Atiyah M F et al. *Phys. Lett. A* **65** 185 (1978)
160. Ganor O J *Nucl. Phys. B* **489** 95 (1997); hep-th/9605201
161. Gustavsson A *J. High Energy Phys.* **2004** (07) 074 (2004); hep-th/0404150
162. Alekseev A, Chekeres O, Mnev P *J. High Energy Phys.* **2015** (11) 093 (2015); arXiv:1507.06343
163. Pasti P, Sorokin D, Tonin M *Phys. Rev. D* **55** 6292 (1997); hep-th/9611100
164. Bandos I et al. *Phys. Rev. Lett.* **78** 4332 (1997); hep-th/9701149
165. Ko S-L, Sorokin D, Vanichchapongjaroen P *J. High Energy Phys.* **2013** (11) 072 (2013); arXiv:1308.2231
166. Murray M K "An introduction to bundle gerbes", in *The Many Facets of Geometry: A Tribute to Nigel Hitchin* (Eds O Garcia-Prada, J P Bourguignon, S Salamon) (Oxford: Oxford Academic, 2007) p. 237, https://doi.org/10.1093/acprof:oso/9780199534920.003.0012; arXiv:0712.1651
167. Hitchin N J *AMS/IP Stud. Adv. Math.* **23** 151 (2001); math/9907034
168. Alvarez O, Ferreira L A, Sanchez-Guillen J *Int. J. Mod. Phys. A* **24** 1825 (2009); arXiv:0901.1654
169. Freund P G O, Nepomechie R I *Nucl. Phys. B* **199** 482 (1982)
170. Brylinski J-L *Loop Spaces, Characteristic Classes and Geometric Quantization* (Boston, MA: Birkhäuser, 1993)
171. Papageorgakis C, Sämann C *J. High Energy Phys.* **2011** (05) 099 (2011); arXiv:1103.6192
172. Lambert N, Papageorgakis C *J. High Energy Phys.* **2010** (08) 083 (2010); arXiv:1007.2982
173. Alvarez O, Ferreira L A, Sánchez Guillén J *Nucl. Phys. B* **529** 689 (1998); hep-th/9710147
174. Frenkel I, Moore G *Commun. Math. Phys.* **138** 259 (1991); https://projecteuclid.org:443/euclid.cmp/1104202944
175. Zamolodchikov A B *Sov. Sci. Rev. Sect. A Phys. Rev.* **2** 1 (1980)
176. Zamolodchikov A B *Sov. Phys. JETP* **52** 325 (1980); *Zh. Eksp. Teor. Fiz.* **79** 641 (1980)
177. Zamolodchikov A B *Commun. Math. Phys.* **79** 489 (1981); https://projecteuclid.org:443/euclid.cmp/1103909139
178. Bazhanov V V, Stroganov Yu G *Theor. Math. Phys.* **52** 685 (1982); *Teor. Matem. Fiz.* **52** 105 (1982)
179. Maillet J M, Nijhoff F *Phys. Lett. B* **224** 389 (1989)
180. Carter J S, Saito M *Int. J. Mod. Phys. A* **11** 4453 (1996)
181. Hietarinta J *J. Phys. A* **27** 5727 (1994)
182. Talalaev D V *Russ. Math. Surveys* **76** 685 (2021); *Usp. Matem. Nauk* **76** 139 (2021)
183. Isaev A P, arXiv:2206.08902
184. Morozov A, Tselousov N *Phys. Lett. B* **840** 137887 (2023); arXiv:2211.14956
185. Morozov A, Tselousov N, arXiv:2305.12282