METHODOLOGICAL NOTES

Duality of two-dimensional field theory and four-dimensional electrodynamics leading to a finite value of the bare charge

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<u>Abstract.</u> We discuss the holographic duality consisting in the functional coincidence of the spectra of the mean number of photons (or scalar quanta) emitted by a point-like electric (scalar) charge in (3 + 1)-space with the spectra of the mean number of pairs of scalar (spinor) quanta emitted by a point mirror in (1 + 1)-space. Being functions of two variables and functionals of the common trajectory of the charge and the mirror, the spectra differ only by the factor $e^2/\hbar c$ (in Heaviside units). The requirement $e^2/\hbar c = 1$ leads to unique values of the

Received 27 July 2012, revised 30 April 2013 Uspekhi Fizicheskikh Nauk **183** (6) 591–615 (2013) DOI: 10.3367/UFNr.0183.201306c.0591 Translated by S D Danilov; edited by A M Semikhatov point-like charge and its fine structure constant, $e_0 = \pm \sqrt{\hbar c}$, $a_0 = 1/4\pi$, all their properties being as stated by Gell-Mann and Low for a finite bare charge. This requirement follows from the holographic bare charge quantization principle we propose here, according to which the charge and mirror radiations respectively located in four-dimensional space and on its internal two-dimensional surface must have identically coincident spectra. The duality is due to the integral connection of the causal Green's functions for (3 + 1)- and (1 + 1)-spaces and to connections of the current and charge densities in (3 + 1)-space with the scalar products of scalar and spinor massless fields in (1 + 1)-space. We discuss the closeness of the values of the point-like bare charge $e_0 = \sqrt{\hbar c}$, the 'charges' $e_B = 1.077 \sqrt{\hbar c}$ and $e_{\rm L} = 1.073 \sqrt{\hbar c}$ characterizing the shifts $e_{\rm B, L}^2 / 8\pi a$ of the energy of zero-point electromagnetic oscillations in the vacuum by neutral ideally conducting surfaces of a sphere of radius a and a cube of side 2a, and the electron charge e times $\sqrt{4\pi}$. The approximate equality $e_{\rm L} \approx \sqrt{4\pi} e$ means that $a_0 a_{\rm L} \approx a$ is the fine structure constant.

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1. Introduction

1.1 Introductory remarks

Hawking's mechanism of particle emission in the process of black hole formation is analogous to the emission of an ideal 'mirror' accelerated in the vacuum [1, 2]. In turn, there is a close analogy between the emission of photons (in another variant, of scalar quanta) by an accelerated point-like electric (scalar) charge in (3 + 1)-space and the emission of pairs of scalar (spinor) quanta by an accelerated point-like mirror in (1 + 1)-space [3, 4]. All these processes prove to be interconnected. A point-like mirror moving in a (1 + 1)-space implies points of a time-like curve in the (x, t) plane on which the scalar or spinor field satisfies a boundary condition free of any dimensional parameters.

In 1995, Nikishov and the author discovered that the spectrum of photons emitted by a uniformly accelerated charge in (3 + 1)-space, up to the factor $e^2/\hbar c$ (in Heaviside units), coincides with the spectrum of pairs of scalar quanta emitted by a uniformly accelerating mirror in (1 + 1)-space [3]. This correspondence proved to be valid for any other common motion of the charge and mirror. It was shown in [4] that the spectrum of scalar quanta emitted by an accelerated scalar charge in (3 + 1)-space coincides with the spectrum of pairs of spinor quanta emitted by a mirror accelerated analogously in (1 + 1)-space (once again, up to the factor $e^2/\hbar c$, where e is the scalar charge).

Subsequent work by the author was devoted to studying this duality between the emission of individual quanta by a charge accelerated in (3 + 1)-space and pair creation by a mirror accelerated in (1 + 1)-space, and to the causes of why the respective spectra of quanta and pairs coincide. It essentially relied on the interpretation of the Bogoliubov β -coefficient as the amplitude of the source of a pair of particles directed oppositely. It turned out that the boson and fermion pairs emitted by the mirror have the same spins 1 and 0 as photons and scalar quanta emitted by the electric and scalar charges [5].

With quantum exchange effects taken into account, the vacuum–vacuum amplitudes for an accelerated charge and a mirror in (3 + 1)- and (1 + 1)-spaces differ not only by the factor $e^2/\hbar c$ in the charge action functional. The exchange effects, however, do not influence the functionally coincident spectra of mean numbers of quanta and pairs emitted by the charge and mirror [6].

An essential role in the duality of the spectra is played by the connection between causal Green's functions in (3 + 1)and (1 + 1)-spaces [7, 8]. The requirement that the spectra of the charge and the mirror fully coincide leads to the quantization of electric and scalar charges with the common value $e_0 = \sqrt{\hbar c}$. This value exhibits all the properties of a finite bare charge formulated by Gell-Mann and Low [9]. Being small ($\alpha_0 = 1/4\pi \ll 1$), it leaves the electromagnetic interaction weak for all transferred momenta. The duality was generalized to the interactions of a charge and a mirror with fields accompanying them and carrying space-like momenta. These fields define the Bogoliubov α -coefficients and shifts in real parts of actions, i.e., phases of the vacuumvacuum amplitudes.

The structure of this review is as follows. Section 1 presents the basic results demonstrating the holographic connection between the emission spectra of a charge and a mirror; Sections 2 and 3 deal with computational methods

and elaborate on the links between the Bogoliubov coefficients and the current and charge densities; Section 4 discusses most basic physical quantities and connections between them that underlie the duality discussed; Sections 5 and 6 present a brief exposition of the Schwinger source theory and the Bogoliubov coefficient theory. Finally, Section 7 is devoted to the discussion of principal issues and the expression $e_0 = \sqrt{\hbar c}$ for the bare charge.

In this paper, we assume the metric $g_{\alpha\beta} = \text{diag}(1, 1, 1, -1)$, the notation $k^{\alpha} = (\mathbf{k}, k^0)$ for 4-vectors, and the Heaviside units for the charge and the natural system of units $\hbar = c = 1$, except when we need to emphasize the quantum and relativistic meaning of a quantity. We understand 'the charge quantization principle' as the basic physical postulate leading to a specific connection between the charge value and the Planck constant, i.e., the 'charge quantization'. It is the relation $e_0^2 = \hbar c$ for the bare charge.

1.2 Holographic duality of the emission spectra of a point-like charge and mirror and the bare charge

A (1 + 1)-space can be regarded as an internal boundary of a (3 + 1)-space, the outer boundary of which is at infinity. It is assumed that the (1 + 1)-space is endowed with scalar and spinor massless fields, and the emission of pairs of quanta of these fields induced by the boundary condition on a *point-like* 'mirror' moving with acceleration in this (1 + 1)-space is considered within *quantum theory*.

The method of research is the Bogoliubov transformations [10] and the corresponding coefficients $\alpha_{\omega'\omega}$ and $\beta_{\omega'\omega}$ connecting the complete in- and out-systems of wave equation solutions. For a mirror moving with acceleration, the coefficient $\beta_{\omega'\omega} \neq 0$, in- and out-systems are not equivalent, and the mirror emits pairs of quanta with frequencies ω and ω' and the spectrum

$$d\bar{n}_{\omega'\omega}^{B,F} = \left|\beta_{\omega'\omega}^{B,F}\right|^2 \frac{d\omega d\omega'}{\left(2\pi\right)^2} \tag{1.1}$$

for the mean number of pairs. The indices B and F label quantities pertaining to boson (scalar) and fermion (spinor) fields, their quanta forming boson and fermion pairs.

In (3 + 1)-space, with *point-like* electric and scalar charges moving along its internal one-dimensional boundary, the emission of massless quanta by these charges are considered within the *semiclassical* Schwinger source theory [11]; its main tool is the vacuum–vacuum amplitude exp (iW/\hbar) in the presence of a source. The action W is therefore a functional of the electric current density or the scalar charge density.

For a charge moving with acceleration, the imaginary part of the action differs from zero, is positive, and, being divided by the Planck constant \hbar , defines the mean number of quanta \overline{N} emitted by the charge over the entire time:

$$\frac{2}{\hbar} \operatorname{Im} W^{(s)} = \bar{N}^{(s)} = \int \mathrm{d}\bar{n}_k^{(s)} \,. \tag{1.2}$$

The spectrum $d\bar{n}_k^{(s)}$ of the mean number of quanta with spin s and the wave vector $k^{\alpha} = (\mathbf{k}, k^0)$ is defined by Fourier components of the electric current or scalar charge densities,

$$d\bar{n}_{k}^{(1)} = \frac{\left|j_{\alpha}(k)\right|^{2}}{\hbar c} \frac{d^{3}k}{(2\pi)^{3}2k^{0}}, \quad d\bar{n}_{k}^{(0)} = \frac{\left|\rho(k)\right|^{2}}{\hbar c} \frac{d^{3}k}{(2\pi)^{3}2k^{0}}.$$
(1.3)

For charge trajectories lying in the (x, t) plane of the Minkowski (3 + 1)-space, the spectral densities $|j_{\alpha}(k)|^2$ and $|\rho(k)|^2$ depend only on two components $k_{\pm} = k^0 \pm k^1$ of the wave vector but do not depend on the azimuthal angle φ of the projection \mathbf{k}_{\perp} of the vector $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_{\perp}$ on the plane perpendicular to the motion axis.

Writing the invariant number of quantum states as

$$\frac{\mathrm{d}^{3}k}{(2\pi)^{3}2k^{0}} = \frac{\mathrm{d}k_{+}\,\mathrm{d}k_{-}}{(4\pi)^{2}}\frac{\mathrm{d}\varphi}{2\pi}$$

and integrating spectra (1.3) over φ , we obtain

$$d\bar{n}_{k_{+}k_{-}}^{(1)} = \frac{\left|j_{\alpha}(k_{+},k_{-})\right|^{2}}{\hbar c} \frac{dk_{+} dk_{-}}{(4\pi)^{2}} ,$$

$$d\bar{n}_{k_{+}k_{-}}^{(0)} = \frac{\left|\rho(k_{+},k_{-})\right|^{2}}{\hbar c} \frac{dk_{+} dk_{-}}{(4\pi)^{2}} .$$
(1.4)

The duality considered here confirms that spectra (1.4) of photons and scalar quanta emitted by charges in (3 + 1)-space coincide up to the factor $e^2/\hbar c$ with spectrum (1.1) of boson and fermion pairs emitted by a mirror in (1 + 1)-space,

$$d\bar{n}_{k_{+}k_{-}}^{(1)} = \frac{e^{2}}{\hbar c} \ d\bar{n}_{\omega'\omega}^{B}, \qquad d\bar{n}_{k_{+}k_{-}}^{(0)} = \frac{e^{2}}{\hbar c} \ d\bar{n}_{\omega'\omega}^{F}, \tag{1.5}$$

under the condition that the charge and mirror trajectories are identical and the components $k_{\pm} = k^0 \pm k^1$ of the wave 4-vector k^{α} of the quantum emitted by the charge are identified with the doubled frequencies ω and ω' of the quanta of the pair emitted by the mirror:

$$k_+ = 2\omega, \qquad k_- = 2\omega'. \tag{1.6}$$

In other words, the holographic duality states that the number of quantum states in (3 + 1)-space coincides, after integration over φ , with the number of states for a pair of quanta in (1 + 1)-space,

$$\frac{dk_{+} dk_{-}}{(4\pi)^{2}} = \frac{d\omega \, d\omega'}{(2\pi)^{2}} \,, \tag{1.7}$$

while the spectral densities differ only by the factor $e^2/\hbar c$:

$$|j_{\alpha}(k_{+},k_{-})|^{2} = \frac{e^{2}}{\hbar c} |\beta_{\omega'\omega}^{\mathbf{B}}|^{2}, \qquad |\rho(k_{+},k_{-})|^{2} = \frac{e^{2}}{\hbar c} |\beta_{\omega'\omega}^{\mathbf{F}}|^{2}.$$
(1.8)

Because the quanta of the pair are massless, they have the two-dimensional wave vectors (ω, ω) and $(-\omega', \omega')$. Then 2ω and $2\omega'$ in formula (1.6) are the only nonzero 'plus'- and 'minus' components of these vectors and, simultaneously, they are the plus and minus components of the two-dimensional time-like wave vector (k^1, k^0) of the pair as a whole:

$$k^{1} = \omega - \omega', \quad k^{0} = \omega + \omega',$$

$$k_{+} = k^{0} + k^{1} = 2\omega, \quad k_{-} = k^{0} - k^{1} = 2\omega'.$$
(1.9)

In turn, these last coincide with the plus and minus components of the four-dimensional vector k^{α} of a quantum emitted by the charge in (3 + 1)-space. Thus, formulas (1.6), (1.9), and (1.7), relating the 4-vector of a quantum in (3 + 1)space to the wave 2-vectors of the pair and its particles in (1 + 1)-space, have a very transparent physical meaning. We return to them later.

For large values of ω and ω' , the spectral densities $|\beta_{\omega'\omega}^{B}|^{2}$ and $|\beta_{\omega'\omega}^{F}|^{2}$ cease to be different, i.e., cease to depend on the spin of particles forming the pair,

$$|\beta_{\omega'\omega}^{\mathbf{B}}|^2 = |\beta_{\omega'\omega}^{\mathbf{F}}|^2, \quad \omega, \omega' \to \infty.$$
(1.10)

This is demonstrated in Section 3. Hence, the spin of the particles of the pair does not influence the probability of pair emission in this domain of ω and ω' . Nevertheless, as previously, the spectral densities remain functionals (which now become coincident) of the mirror trajectory and functions of ω and ω' . By virtue of Eqns (1.5) and (1.8), it is then natural to assume that the spectral densities $|j_{\alpha}(k_+,k_-)|^2$ and $|\rho(k_+,k_-)|^2$ are also independent of the spin of emitted quanta for large k_{\pm} , i.e., they are the same. This in turn implies that the values of the electric and scalar charge are identical. Just for this reason, they are denoted by the same letter in Eqns (1.5) and (1.8). Because $d\bar{n}_{k_+k_-}^{(s)}$ and $d\bar{n}_{\omega'\omega}^{B,F}$ are the mean values of integer-

Because $d\bar{n}_{k+k-}^{(3)}$ and $d\bar{n}_{\omega'\omega}^{B,F}$ are the mean values of integervalued observables — the number of quanta in (3 + 1)-space and the number of quantum pairs in (1 + 1)-space, which, according to Eqn (1.6), are in the quantum states closely related to each other — it is of interest to trace this relation in more detail for an individual quantum and a pair.

Relation (1.6) between the components k_{\pm} of the wave 4-vector $k^{\alpha} = (\mathbf{k}, k^0)$ of a quantum and the frequencies ω and ω' of quanta in the pair implies that

$$k^{0} = \omega + \omega', \qquad k^{1} = \omega - \omega'.$$
(1.11)

In other words, the time and longitudinal components of the wave 4-vector of the quantum are simultaneously the time and space components of the wave 2-vector of the pair consisting of two massless quanta with wave 2-vectors (ω, ω) and $(-\omega', \omega')$. Such a pair has the mass

$$m = \sqrt{k_0^2 - k_1^2} = k_\perp = 2\sqrt{\omega\omega'}, \qquad (1.12)$$

coinciding with the transverse momentum k_{\perp} of the quantum in (3 + 1)-space. Although the terms of mass and momentum are convenient, the parameters m and k_{\perp} are purely geometrical, with the dimension of inverse length. For each massless quantum with a wave 4-vector $k^{\alpha} = (\mathbf{k}_1 + \mathbf{k}_{\perp}, k^0), k^0 = (k_1^2 + k_{\perp}^2)^{1/2}$ emitted in (3 + 1)-space, we can therefore relate a massive pair emitted in (1 + 1)-space with the wave 2-vector (\mathbf{k}_1, k^0) and the mass $m = k_{\perp}$ equal to the transverse momentum of the quantum in (3 + 1)-space.

This correspondence consists, first of all, in the fact that, independent of the quantity $e^2/\hbar c$, for any coincident trajectories of the charge and the mirror, the probability of emitting a quantum in a state k_+ , k_- by the charge equals the probability of emitting a pair of quanta in the state ω , ω' by the mirror:

$$\frac{\mathrm{d}\bar{n}_{k_{+}k_{-}}^{(1,0)}}{\int \mathrm{d}\bar{n}_{k_{+}k_{-}}^{(1,0)}} = \frac{\mathrm{d}\bar{n}_{\omega'\omega}^{\mathrm{B},\mathrm{F}}}{\int \mathrm{d}\bar{n}_{\omega'\omega}^{\mathrm{B},\mathrm{F}}}, \quad \text{if} \quad k_{+} = 2\omega, \quad k_{-} = 2\omega'.$$
(1.13)

It should be kept in mind that the components k_{\pm} of momenta of the quantum and the pair are identical. The parameter $e^2/\hbar c$ in the left-hand side has dropped out and the probabilities of emitting quanta with spins 1 and 0 and of emitting boson and fermion pairs have become identical as functions of two variables $k_{+} = 2\omega$ and $k_{-} = 2\omega'$ and the

functionals of identical trajectories of the charge and mirror. But the mean values $d\bar{n}_{k_+k_-}^{(1,0)}$ and $d\bar{n}_{\omega'\omega}^{B,F}$ of *integer-valued* observables—the number of quanta in the state k_+, k_- and the number of pairs of quanta in the state ω , ω' , contained in equal-phase volumes (1.7)—are equal only if $e^2/\hbar c = 1$.

In this case, a one-to-one correspondence exists between the quanta with components k_+ , k_- in the phase volume $dk_+ dk_-/(4\pi)^2$ emitted by the charge and quantum pairs with frequencies ω and ω' in the phase volume $d\omega d\omega'/(2\pi)^2$ emitted by the mirror. This correspondence can be considered holographic, because the one-dimensional spatial x axis along which the mirror and charge move and the pairs propagate, can be viewed as an internal boundary of the three-dimensional space hosting the propagating quanta emitted by the charge.

Naturally, the total mean number of emitted quanta in the case $e^2/\hbar c = 1$ is equal to that for emitted pairs,

$$\bar{N}^{(1,0)} = \frac{e^2}{\hbar c} \left. \bar{N}^{\mathbf{B},\mathbf{F}} \right|_{e^2/\hbar c = 1} = \bar{N}^{\mathbf{B},\mathbf{F}}.$$
(1.14)

For any other value of $e^2/\hbar c$, equality (1.14) would be violated despite the coincidence in the geometrical mechanisms of creation of quanta and pairs of quanta, manifested in the coincidence of probability distributions (1.13) for any shared trajectory of the charge and the mirror.

The spectrum of pairs of quanta $d\bar{n}^{\rm B,F}_{\omega'\omega}$ is purely geometrical, i.e., defined by a dimensionless function of quantities with dimensions of powers of length and time, but the spectrum of quanta $d\bar{n}_{k,k-}^{(1,0)}$ differs from it by the extra dimensionless factor $e^2/\hbar c$ containing quantities e^2 and \hbar of tion and becomes geometrical. For $e^2/\hbar c \neq 1$, this factor and the spectrum $d\bar{n}_{k+k_-}^{(1,0)}$ contain information of a nongeometrical character that is lacking in the spectrum $d\bar{n}_{\omega'\omega}^{B,F}$; the holo-graphic principle is violated. Thus, a value $e^2/\hbar c < 1$ would imply screening of the bare charge.

The holographic principle demands that the information contained in the spectra $d\bar{n}_{k_{\pm}k_{-}}^{(1,0)}$ and $d\bar{n}_{\omega'\omega}^{B,F}$ be coincident, which is equivalent to quantizing the bare charge, i.e., fixing its relation to the Planck constant. Such a coincidence is possible owing to the holographic duality of the *semiclassical* description of the emission of quanta by a point-like charge in (3+1)-space and the *quantum* description of the emission of pairs of quanta by a point-like mirror in (1 + 1)-space.

Because the charges and the mirror are supposed to be point-like, the quantized value of the charge obtained in this manner must be related to the bare charge, not screened by the vacuum polarization, which we denote by e_0 . Therefore, $e_0^2 = \hbar c$, $e_0 = \pm \sqrt{\hbar c}$, while the value of the corresponding fine structure constant is

$$\alpha_0 = \frac{e_0^2}{4\pi\hbar c} = \frac{1}{4\pi} \,. \tag{1.15}$$

It is identical for the bare electric and scalar charges and is small compared to unity.

1.3 Gell-Mann and Low on a point-like bare charge

The result obtained above agrees with the asymptotic behavior of the effective coupling constant $\alpha(k^2/m^2, \alpha)$ described in the well-known work of Gell-Mann and Low [9] as variant (b).

This work shows that the effective parameter of interaction ('running constant', effective charge, ...), defined by the product

$$\alpha d_R\left(\frac{k^2}{m^2},\,\alpha\right) \tag{1.16}$$

of the fine structure constant $\alpha = e^2/4\pi\hbar c$ and the renormalized photon propagator d_R , for large values of transferred momenta, becomes a function of a single argument:

$$\alpha\left(\frac{k^2}{m^2},\,\alpha\right) \equiv \alpha d_R^{\rm as}\left(\frac{k^2}{m^2},\,\alpha\right) = F\left(\frac{k^2}{m^2}\,\phi(\alpha)\right), \quad k^2 \gg m^2$$
(1.17)

This implies that for $k^2 \gg m^2$, the functional form of $\alpha(k^2/m^2, \alpha)$ ceases to depend on the value of the fine structure constant, which, in this case, only enters the scale factor $\phi(\alpha)$ that sets the scale of momenta.

This result is obtained by exploring the function $d_R(k^2/m^2, \alpha)$ by the perturbation method. In the asymptotic domain $x \equiv k^2/m^2 \to \infty$, the function $d_R(x, \alpha)$ is a double series in powers of α and $\ln x$ with finite numerical coefficients. The convergence of this series remains unknown. But it is assumed that the series for $d_R(x, \alpha)$ defines a function that also satisfies the same functional relations as derived by the authors for this series.

The authors discuss two possible behavior variants for the function $F((k^2/m^2)\phi(\alpha))$ as $k^2/m^2 \to \infty$. (a) If $F((k^2/m^2)\phi) \to \infty$ as $k^2/m^2 \to \infty$, then e_0^2 (the

square of the bare, unscreened charge) is infinite and the singularity at the center of the charge distribution is stronger than $\delta(\mathbf{x})$, which would correspond to a finite point-like charge. The perturbation theory points to this result.

(b) If $F((k^2/m^2)\phi)$ tends to a finite value $\alpha_0 = e_0^2/4\pi\hbar c$ as $k^2/m^2 \rightarrow \infty$, then this quantity should not depend on the fine structure constant α and should exceed α . In this case, at very small distances, the charge distribution is described by a spatial delta-function $e_0\delta(\mathbf{x})$ with a finite value of the unscreened bare charge e_0 .

To facilitate the comparison of Eqn (1.17) with a perturbative series, the authors used its alternative form,

$$\ln \frac{k^2}{m^2} = \int_{\alpha(1,\alpha)}^{\alpha(x,\alpha)} \frac{\mathrm{d}z}{\psi(z)} , \qquad x = \frac{k^2}{m^2} \gg 1 , \qquad (1.18)$$

in which the main role is delegated to the Gell-Mann-Low function $\psi(z)$. It is invariant under a multiplicative change of the argument of F in representation (1.17). The function $\phi(\alpha)$ is defined up to a multiplicative constant, which does not affect the effective charge αd_R^{as} , but does affect the function $F(x\phi)$. Expressing its argument in terms of the inverse function,

$$x\phi(\alpha) = \tilde{F}(\alpha(x,\alpha)), \qquad (1.19)$$

and differentiating the logarithm of the expression obtained over x, we eliminate the multiplicative arbitrariness in $\phi(\alpha)$ and $\dot{F}(z)$ and arrive at the differential form of the Gell-Mann– Low equation

$$x \frac{\partial \alpha(x,\alpha)}{\partial x} = \psi(\alpha(x,\alpha)), \qquad (1.20)$$

where

$$\frac{1}{\nu(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \left[\ln \tilde{F}(z) \right]. \tag{1.21}$$

According to the perturbation theory, the function $\psi(z)$ behaves at small z as [12]

$$\psi(z) = \frac{z^2}{3\pi} + \frac{z^3}{4\pi^2} + \left[\zeta(3) - \frac{101}{96}\right] \frac{z^4}{3\pi^3} + \dots, \quad \zeta(3) = 1.202\dots$$
(1.22)

If $\psi(z)$ is positive for all $z > \alpha(1, \alpha)$, then the effective coupling parameter $\alpha(x, \alpha)$ tends to infinity, either at $x \to \infty$, if the integral $\int_{\alpha(1,z)}^{\infty} dz/\psi(z)$ diverges at its upper limit, or at $x \to x_0 = \exp[\int_{\alpha(1,\alpha)}^{\infty} dz/\psi(z)]$, if the integral converges. In the latter case, $\alpha(x, \alpha)$ has a Landau pole at $x = x_0 \sim \exp(3\pi/\alpha)$ [13].

In both cases, the singularity of $\alpha(x, \alpha)$ lies in the far asymptotic domain where all the terms in the perturbative series for $\alpha d_R(x, \alpha)$ are comparable. A rigorous analysis of the full expression for $\alpha d_R(x, \alpha)$ in this domain is lacking. For this reason, the behavior of $\psi(z)$ at large z leading to large values of $\alpha(x, \alpha)$ does not find sufficient support in the framework of the existing theory. This is the conclusion in Refs [14–17].

If the function $\psi(z)$ nevertheless makes sense, then the other possibility is that $\psi(z)$ vanishes at some finite point $z = \alpha_0 > \alpha(1, \alpha)$. In this case, if the integral $\int_{\alpha(1,\alpha)}^{z} dz/\psi(z)$ diverges at $z \to \alpha_0$, then $\alpha(x, \alpha) \to \alpha_0$ as $x \to \infty$. Here, α_0 is the bare fine structure constant.

Being a root of the equation $\psi(z) = 0$, α_0 has the following properties, formulated by Gell-Mann and Low for a finite bare charge:

(1) α_0 is independent of α ,

(2) $\alpha_0 > \alpha$,

(3) α_0 defines the quantity $e_0 = \pm \sqrt{4\pi\alpha_0\hbar c}$ of the unscreened point-like charge distributed as $e_0\delta(\mathbf{x})$ in space.

In our approach, we assume a point-like charge moving with acceleration along a time-like trajectory and, for α_0 , derive the finite value $\alpha_0 = 1/4\pi$ that satisfies properties (1) and (2) and, additionally, remains small: $\alpha_0 \approx 0.08$. This implies that the electromagnetic interaction remains weak for all transferred momenta, up to the unification with other, comparably weak interactions.

The relation to the analysis of Gell-Mann and Low is not accidental. In essence, Gell-Mann and Low consider the interaction between two ultrarelativistic charges in their head-on collision, when in the region of their closest proximity they exchange a quantum with a giant space-like momentum and change velocities to opposite ones. In this region, the interaction between charges is described by the function $\alpha d_R(x, \alpha)$, which accounts for all radiation corrections as $x = k^2/m^2 \rightarrow \infty$, while the charges proper, in agreement with variant (b), become point-like. This implies that each of them moves along a time-like trajectory with a very large acceleration and has the unscreened charge e_0 .

The radiation spectrum of one of such charges is described by the Schwinger source theory. It is contained in the imaginary part of the self-action W defining the vacuum– vacuum amplitude exp (iW/\hbar) due to a classical source. This spectrum shows a holographic link to the spectrum of pairs of quanta emitted in (1 + 1)-space by a point-like mirror moving along the same trajectory as the charge. With its dimension less by two, this space is internal to the (3 + 1)-space hosting the charge radiation, which is the rationale to call the relation between the spectra holographic.

2. Bogoliubov coefficients

for fields with spin 0 and 1/2 in (1+1)-space with a mirror boundary

2.1 Bogoliubov coefficients

for a scalar field and boundary conditions

In problems with moving mirrors, the complete in-system $\{\phi_{\text{in}\,\omega'}, \phi^*_{\text{in}\,\omega'}\}$ and the complete out-system $\{\phi_{\text{out}\,\omega}, \phi^*_{\text{out}\,\omega}\}$ of wave equation solutions are standardly used [18, 19]. For a massless scalar field ϕ satisfying the wave equation in (1 + 1)-space,

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0, \qquad u = t - x, \qquad v = t + x, \tag{2.1}$$

they are given by

$$\phi_{\mathrm{in}\,\omega'}(u,v) = \frac{1}{\sqrt{2\omega'}} \Big[\exp\left(-\mathrm{i}\omega'v\right) - \exp\left(-\mathrm{i}\omega'f(u)\right], \quad (2.2)$$

$$\phi_{\operatorname{out}\omega}(u,v) = \frac{1}{\sqrt{2\omega}} \left[\exp\left(-\mathrm{i}\omega \,g(v)\right) - \exp\left(-\mathrm{i}\omega u\right) \right]. \tag{2.3}$$

Each consists of waves incident on and reflected from the mirror. In agreement with the 'in' and 'out' indices, it is assumed that the analysis is carried out in the half-plane (x, t) to the right of the mirror. The in- and out-solutions are taken to be monochromatic with respective frequencies ω' and ω in the remote past and future. These solutions satisfy the zero boundary condition

$$\phi(u,v)\Big|_{\rm traj} = 0 \tag{2.4}$$

on the trajectory of a point-like mirror. In the (u, v) plane, the trajectory is described by any of the mutually inverse functions

$$v_{\text{traj}} = f(u), \quad u_{\text{traj}} = g(v); \quad f(g(v)) = v.$$
 (2.5)

Accordingly, the subscript traj in Eqn (2.4) implies that the variables u and v are related by (2.5).

We note that the foregoing is also valid for the field to the left of the mirror on exchanging the indices in and out, but we limit ourself to processes in the field on the right of the mirror.

Any solution of the wave equation with the zero boundary condition on the mirror can be expanded in both in- and outsystems. The Bogoliubov coefficients α and β arise in the expansion of one of these system with respect to the other:

$$\phi_{\text{out}\,\omega} = \int_0^\infty \frac{\mathrm{d}\omega'}{2\pi} \left(\alpha_{\omega'\omega} \,\phi_{\text{in}\,\omega'} + \beta_{\omega'\omega} \,\phi_{\text{in}\,\omega'}^* \right), \tag{2.6}$$

$$\phi_{\mathrm{in}\,\omega'} = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \left(\alpha^*_{\omega'\omega} \,\phi_{\mathrm{out}\,\omega} - \beta_{\omega'\omega} \,\phi^*_{\mathrm{out}\,\omega} \right). \tag{2.7}$$

They are given by the scalar products

$$\alpha^{\mathbf{B}}_{\omega'\omega} = \mathbf{i} \int \phi^*_{\mathbf{i} \mathbf{n}\,\omega'} \, \frac{\partial}{\partial t} \, \phi_{\mathbf{out}\,\omega} \, \mathrm{d}x \,, \tag{2.8}$$

$$\beta_{\omega'\omega}^{\mathbf{B}*} = \mathbf{i} \int \phi_{\mathrm{in}\,\omega'}^* \frac{\partial}{\partial t} \phi_{\mathrm{out}\,\omega}^* \,\mathrm{d}x \,. \tag{2.9}$$

The superscript B indicates the coefficients belonging to a scalar (boson) field. From the orthogonality and normalization conditions for both sets, it follows that

$$\int_{0}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \left(\alpha_{\omega'\omega} \alpha_{\omega'\omega''}^{*} - \beta_{\omega'\omega} \beta_{\omega'\omega''}^{*} \right) = 2\pi \delta(\omega - \omega'') ,$$

$$\int_{0}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} \left(\alpha_{\omega'\omega} \beta_{\omega'\omega''} - \beta_{\omega'\omega} \alpha_{\omega'\omega''} \right) = 0 .$$
(2.10)

The explicit expressions for functions in the in- and out-sets allow writing the Bogoliubov coefficients in terms of the Fourier transforms:

$$\alpha^{\mathbf{B}}_{\omega'\omega}, \beta^{\mathbf{B}*}_{\omega'\omega} = \pm \sqrt{\frac{\omega}{\omega'}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left(\mp \mathrm{i}\omega u + \mathrm{i}\omega' f(u)\right) \qquad (2.11)$$

$$= \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\infty} \mathrm{d}v \exp\left(\mathrm{i}\omega' v \mp \mathrm{i}\omega g(v)\right).$$
 (2.12)

Here, the upper sign is related to α and the lower to β^* . It can be easily seen that these expressions are Lorentz-invariant, as it should be for scalar products of wave equation solutions.

When the mirror speed β is constant, the in- and outsystems are still different, but in this case

$$\beta_{\omega'\omega} = 0, \qquad \alpha_{\omega'\omega} = 2\pi\delta \left(D(\beta)\omega' - D(-\beta)\omega \right),$$

$$D(\beta) = \sqrt{\frac{1+\beta}{1-\beta}} = e^{\alpha}, \qquad \beta = \tanh \alpha.$$
(2.13)

The Doppler factors $D(\pm\beta)$ relate the frequencies ω' and ω of the incident and reflected waves to the frequencies of these waves $D(\beta)\omega'$ and $D(-\beta)\omega$ in the system comoving with the mirror. For $\beta = 0$, the in- and out-systems coincide.

Scalar products (2.8) and (2.9) for functions of only in- or only out-sets reduce to expressions (2.13) taken at $\beta = 0$, i.e., transform to the orthogonality and normalization conditions for the set. In this case, the frequencies ω' and ω acquire the same covariance.

We note that if the boundary condition on the mirror $\phi|_{\text{traj}} = 0$ is replaced with the condition that the derivative of field ϕ along the normal n^{α} to the mirror world line vanish,

$$n^{\alpha} \left. \frac{\partial \phi}{\partial x^{\alpha}} \right|_{\text{traj}} \equiv \sqrt{f'(u)} \left. \frac{\partial \phi}{\partial v} - \sqrt{g'(v)} \left. \frac{\partial \phi}{\partial u} \right|_{\text{traj}} = 0, \quad (2.14)$$

then the signs of the second terms in the in- and out-solutions (2.2) and (2.3), describing waves propagating to the right, would change to the opposite. But the Bogoliubov coefficients remain the same, implying that the emitted spectrum is insensitive to the change in the mirror boundary condition of that kind. We can assert that the Bogoliubov coefficient is degenerate with respect to replacing Dirichlet boundary condition (2.4) with the Neumann one in (2.14).

B M Barbashov¹ has drawn our attention to the fact that the general solution of the wave equation satisfying boundary condition (2.4) can be written in the form

$$\phi(u, v) = F(v) - F(f(u))$$
 or $G(g(v)) - G(u)$, (2.15)

where F(v) and G(u) are arbitrary functions of their arguments. They can be associated with the incident and

¹ A remark at a seminar in Dubna.

reflected waves in the in- and out-solutions. Monochromatic plane waves are used in Eqns (2.2) and (2.3) as *F* and *G*.

It can easily be shown that the general solution of the wave equation satisfying boundary condition (2.14) is

$$\phi(u,v) = F(v) + F(f(u))$$
 or $G(g(v)) + G(u)$. (2.16)

2.2 Bogoliubov coefficients for a spinor field and the boundary condition

The complete in- and out-systems $\{\psi_{in\omega'}, \psi_{in\omega'}^*\}$ and $\{\psi_{out\omega}, \psi_{out\omega}^*\}$ of solutions of the massless Dirac equation

$$\begin{pmatrix} \gamma^0 \frac{\partial}{\partial t} + \gamma^3 \frac{\partial}{\partial x} \end{pmatrix} \psi = 0,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -\sigma_3\\ \sigma_3 & 0 \end{pmatrix},$$

$$(2.17)$$

were obtained in Ref. [4]. In the so-called spinor representation [20], solutions with spin projections on the x axis equal to $s = \pm 1/2$ are given by

$$\psi_{\mathrm{in}\,\omega'+1/2} = \begin{pmatrix} 0\\\eta_1 \end{pmatrix} \exp\left(-\mathrm{i}\omega'v\right) + \begin{pmatrix} \xi_1\\0 \end{pmatrix} \sqrt{f'(u)} \exp\left(-\mathrm{i}\omega'f(u)\right),$$
(2.18a)

 $\psi_{\operatorname{in}\omega'-1/2}$

$$= \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix} \exp\left(-\mathrm{i}\omega' v\right) + \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \sqrt{f'(u)} \exp\left(-\mathrm{i}\omega' f(u)\right),$$
(2.18b)

 $\psi_{\text{out}\,\omega+1/2}$

$$= \begin{pmatrix} 0\\\eta_1 \end{pmatrix} \sqrt{g'(v)} \exp\left(-i\omega g(v)\right) + \begin{pmatrix} \xi_1\\0 \end{pmatrix} \exp\left(-i\omega u\right),$$
(2.19a)

 $\psi_{\text{out}\,\omega-1/2}$

$$= \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix} \sqrt{g'(v)} \exp\left(-i\omega g(v)\right) + \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \exp\left(-i\omega u\right).$$
(2.19b)

Here,

$$\xi_1 = \eta_1 = \begin{pmatrix} 1\\0 \end{pmatrix}, \quad \xi_2 = \eta_2 = \begin{pmatrix} 0\\1 \end{pmatrix}$$
(2.20)

are two-component spinors corresponding to spin projections $\pm 1/2$. The bispinors $\psi_{\omega s}$ are eigenfunctions of the matrix Σ_3 ,

$$\Sigma_3 \psi_s = 2s \psi_s, \quad \Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$
 (2.21)

The operator Σ_3 is conserved: it does not depend on time and commutes with the Hamiltonian of the Dirac equation in (1 + 1)-space.

The difference between the spinors ξ and η is that under the Lorentz transformation with the speed $v = \tanh \varphi$ along the x axis, they are transformed by mutually inverse matrices,

$$B = \exp\left(-\frac{\varphi}{2}\sigma_3\right), \quad B^{-1} = \exp\left(\frac{\varphi}{2}\sigma_3\right).$$
 (2.22)

This implies that

$$B\xi_{1,2} = \exp\left(\mp\frac{\varphi}{2}\right)\xi_{1,2}, \qquad B^{-1}\eta_{1,2} = \exp\left(\pm\frac{\varphi}{2}\right)\eta_{1,2},$$
(2.23)

and that the bispinors ψ_s are transformed by the 4 \times 4 matrix

$$\left(\begin{array}{cc} B & 0\\ 0 & B^{-1} \end{array}\right).$$

Under the spatial reflection, the spinors ξ and η transform into each other, preserving the spin projections,

$$\xi_{1,2} \rightleftharpoons \eta_{1,2}$$

whereas bispinors are transformed by the matrix γ^0 (see Ref. [20], §§ 19–21). As a result, under a reflection in space (which amounts to the change

$$u \rightleftharpoons v, \quad f(u) \rightleftharpoons g(v), \quad \omega \rightleftharpoons \omega', \quad \xi_{1,2} \rightleftarrows \eta_{1,2},$$

and the action of γ^0), the in and out bispinor solutions transform into each other,

 $\psi_{\operatorname{in}\omega'\pm 1/2} \rightleftharpoons \psi_{\operatorname{out}\omega\pm 1/2}$,

without a change in the spin projection.

Each of the above in (out)-solutions in the (x, t) plane to the right of the mirror world line describes a spinor field with a definite spin projection, consisting of a monochromatic (nonmonochromatic) wave incident on the mirror and nonmonochromatic (monochromatic) wave reflected from it.

Such solutions are uniquely defined by prescribing a monochromatic wave on the characteristic $u = u_R^- \to -\infty$ in the remote past (on the characteristic $v = v_R^+ \to \infty$ in the far future), the phase coincidence for the incident and reflected waves on the mirror, and the relation $j^1 = \dot{x}(t) j^0$ between the space and time components of the current density, where $\dot{x}(t)$ is the mirror speed. This implies that on the mirror trajectory, the current is present only along the trajectory, and its component in the direction perpendicular to the trajectory vanishes:

$$n_{\alpha}\bar{\psi}\gamma^{\alpha}\psi\Big|_{\rm traj}=0.$$
 (2.24)

The in- and out-solutions satisfy the orthogonality and normalization conditions

$$\int dx \psi^{+}_{in\,\omega''s''}(x,t) \psi_{in\,\omega s}(x,t) = 2\pi \delta(\omega''-\omega) \delta_{s''s},$$

$$\int dx \psi^{+}_{in\,\omega''s''}(x,t) \psi^{*}_{in\,\omega s}(x,t) = 0,$$
(2.25)

and similarly for the out-solutions.

The Bogoliubov coefficients are defined by the scalar products

$$\alpha^{\rm F}_{\omega's',\,\omega s} = \int \mathrm{d}x \,\psi^{\,+}_{\mathrm{in}\,\omega's'}(x,t) \,\psi_{\mathrm{out}\,\omega s}(x,t) \,, \qquad (2.26)$$

$$\beta_{\omega's',\omega s}^{F*} = \int \mathrm{d}x \,\psi_{\mathrm{in}\,\omega's'}^+(x,t) \,\psi_{\mathrm{out}\,\omega s}^*(x,t) \,. \tag{2.27}$$

It can be easily shown that they reduce to the expressions

$$\alpha^{\rm F}_{\omega'\omega}, \beta^{\rm F*}_{\omega'\omega} = \int_{-\infty}^{\infty} \mathrm{d}u \sqrt{f'(u)} \exp\left(\mp \mathrm{i}\omega u + \mathrm{i}\omega' f(u)\right) \qquad (2.28)$$

$$= \int_{-\infty}^{\infty} dv \sqrt{g'(v)} \exp\left(i\omega' v \mp i\omega g(v)\right), \qquad (2.29)$$

diagonal with respect to the spin projection and independent of it. For this reason, in the final expressions (2.28) and (2.29), the spin projection indices are dropped. However, to stress the difference between the Bogoliubov coefficients for Bose and Fermi fields, we have supplied them with superscripts B and F. It follows from comparing Eqns (2.28) and (2.29) with (2.11) and (2.12) that this difference amounts to replacing the functions $\sqrt{f'(u)}$ and $\sqrt{g'(v)}$ by the factors $\pm \sqrt{\omega/\omega'}$ and $\sqrt{\omega'/\omega}$, which behave similarly under Lorentz transformations.

We note that instead of solutions $\psi_{\omega s}$ with a definite projection $s = \pm 1/2$ of spin 1/2, we could use solutions $\psi_{\omega \lambda}$ with a definite chirality $\lambda = \pm 1$, which are eigenfunctions of the matrix γ^5 . The operator γ^5 is conserved: it does not depend on time and commutes with the Hamiltonian of the massless Dirac equation. The solutions $\psi_{\omega \lambda}$ are related to $\psi_{\omega s}$ as

$$\psi_{\pm 1} = \frac{1}{2} (1 \pm \gamma^5) (\psi_{\pm 1/2} + \psi_{\pm 1/2}), \qquad (2.30)$$

$$\gamma^5 \psi_{\pm 1} = \pm \psi_{\pm 1}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

They satisfy the same conditions of orthogonality and normalization, satisfy the same boundary condition (2.24), and lead to the same Bogoliubov coefficients.

We write the in- and out-solutions with definite chirality:

$$\psi_{\mathrm{in}\,\omega'+} = \begin{pmatrix} 0\\\eta_1 \end{pmatrix} \exp\left(-\mathrm{i}\omega'v\right) + \begin{pmatrix} 0\\\eta_2 \end{pmatrix} \sqrt{f'(u)} \exp\left(-\mathrm{i}\omega'f(u)\right),$$
(2.31a)

$$\psi_{\mathrm{in}\,\omega'-} = \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix} \exp\left(-\mathrm{i}\omega'v\right) + \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \sqrt{f'(u)} \exp\left(-\mathrm{i}\omega'f(u)\right)$$
(2.31b)

$$\psi_{\text{out}\,\omega+} = \begin{pmatrix} 0\\\eta_1 \end{pmatrix} \sqrt{g'(v)} \exp\left(-\mathrm{i}\omega g(v)\right) + \begin{pmatrix} 0\\\eta_2 \end{pmatrix} \exp\left(-\mathrm{i}\omega u\right),$$
(2.32a)

$$\psi_{\text{out}\,\omega-} = \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix} \sqrt{g'(v)} \exp\left(-i\omega g(v)\right) + \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \exp\left(-i\omega u\right).$$
(2.32b)

It follows from these solutions that for waves of positive chirality, the spin of incident and reflected waves is directed oppositely to the momentum, and is directed along it for waves with negative chirality. Such chiralities and respective helicities are called left and right, with $\gamma^5 \psi_{L,R} = \pm \psi_{L,R}$. Accordingly, the sign of helicity (the spin projection on the momentum direction) is opposite to the sign of chirality.

Passing to the coordinate system with a positive velocity $v = \tanh \varphi$ along the x axis amplifies the wave incident on the mirror by the factor $e^{\varphi/2}$ and reduces the reflected wave by the same factor, irrespective of whether it is an in- or out-solution. But this transformation does not change the Lorentz-invariant Bogoliubov coefficients, which are defined by scalar products of solutions.

2.3 Behavior of the spin of a wave under reflection from a mirror and the invariance of the Bogoliubov coefficients

It can be readily seen that in (out)-solutions ψ_{\pm} with chirality ± 1 differ from in (out)-solutions $\psi_{\pm 1/2}$ with the spin projection $\pm 1/2$ by the permutation of reflected waves, i.e., by the permutation

$$\begin{pmatrix} 0\\\eta_2 \end{pmatrix} \rightleftharpoons \begin{pmatrix} \xi_1\\0 \end{pmatrix}$$

of bispinor coefficients at these waves. Such a permutation changes the sign of the spin projection for the reflected waves, preserving their transformation properties under Lorentz transformations [see Eqns (2.22) and (2.23)]. It can be argued that the system of solutions (2.18) and (2.19) serves as a mirror preserving the wave spin projection under reflection, whereas systems (2.31) and (2.32) serve as a mirror preserving helicity under reflection, i.e., inverting the spin projection. In both cases, the transformation properties of the reflected wave under Lorentz transformations are preserved and are opposite to those of the incident wave.

Spatial reflection of helical solutions is equivalent to the permutation

 $\psi_{\operatorname{in}\omega'\pm} \rightleftharpoons \psi_{\operatorname{out}\omega\mp},$

with the change of helicity to the opposite value.

As follows from the explicit expressions for the Bogoliubov coefficients, they are defined by products of waves incident on the mirror or reflected by the mirror, which enter the in- or out-solutions. Such products are identical for the in- and out-solutions characterized by the spin projection, as well as for solutions characterized by helicity [cf. Eqns (2.18) and (2.19) with (2.31) and (2.32)]. This explains the invariance of the Bogoliubov coefficients in passing from a system of solutions characterized by the spin projection to the one characterized by helicity. In other words, the Bogoliubov coefficients are degenerate with respect to the spin behavior under wave reflection from the mirror. They are diagonal with respect to both the spin projection and helicity, and do not depend on their values.

3. Relation of current and charge densities to the Bogoliubov coefficients of scalar and spinor fields

3.1 Actions and spectra of the number of quanta of charged sources in (3+1)-space

The action W defining the vacuum-vacuum amplitude $\exp(iW/\hbar)$ in the presence of a source, albeit considered a classical quantity, nevertheless has a direct relation to quantum theory, similarly to the amplitude. In particular, its doubled imaginary part divided by the Planck constant is equal to the total number of quanta emitted by the source over the entire time. We discuss the information contained in the amplitude $\exp(iW/\hbar)$ in Section 5 in more detail.

For a real-valued vector source with the current density $j^{\alpha}(x)$, the action is

$$W^{(1)} = \frac{1}{2c} \int d^4x \ d^4x' \ j_{\alpha}(x) \ \Delta_4^f(x - x', \mu) \ j^{\alpha}(x') \ . \tag{3.1}$$

Accordingly, the total number of emitted quanta is given by

$$\bar{N}^{(1)} = \frac{2}{\hbar} \operatorname{Im} W^{(1)}$$

$$= \frac{1}{\hbar c} \int d^4 x \, d^4 x' \, j_{\alpha}(x) \operatorname{Im} \varDelta_4^f(x - x', \mu) \, j^{\alpha}(x')$$

$$= \frac{1}{\hbar c} \int d\omega_k \, \left| j_{\alpha}(k) \right|^2.$$
(3.2)

This relativistically invariant and positive result follows directly from the invariant representation of the causal function

$$\begin{aligned} & \Delta_4^f(z,\mu) = i \int d\omega_k \, \exp\left(i\mathbf{k}z - ik^0 |z^0|\right), \quad z = x - x', \quad (3.3) \\ & d\omega_k = \frac{d^3k}{(2\pi)^3 2k^0}, \qquad k^0 = \sqrt{\mathbf{k}^2 + \mu^2}. \end{aligned}$$

Indeed, the symbol of the imaginary part allows removing the modulus of the coordinate z^0 in representation (3.3), whence

$$\operatorname{Im} \Delta_4^f(z,\mu) = \operatorname{Re} \int d\omega_k \, \exp\left(ik_{\alpha} z^{\alpha}\right), \quad z = x - x', \quad (3.4)$$

and the subsequent integration over x and x' leads to the expression above. The spectral density

$$|j_{\alpha}(k)|^{2} \equiv |\mathbf{j}(k)|^{2} - |j_{0}(k)|^{2} > 0,$$
 (3.5)

is Lorentz invariant and positive because, for a time-like or isotropic k^{α} , the 4-vector

$$j_{\alpha}(k) = \int \mathrm{d}^4 x \, j_{\alpha}(x) \exp\left(-\mathrm{i}kx\right) \tag{3.6}$$

is space-like owing to the conservation of the current:

$$k^{\alpha}j_{\alpha}(k) = 0. \tag{3.7}$$

Therefore, $\operatorname{Im} W^{(1)} > 0$.

The parameter μ with the dimension of inverse length is kept to remove the infrared divergence if it occurs, and is assumed to be infinitesimal.

For a real-valued scalar source with the charge density $\rho(x)$, we must replace $j_{\alpha}(x) \rightarrow \rho(x)$ in the formulas presented above. Then the total number of emitted scalar quanta is

$$\bar{N}^{(0)} = \frac{2}{\hbar} \operatorname{Im} W^{(0)}$$

= $\frac{1}{\hbar c} \int d^4 x \, d^4 x' \, \rho(x) \operatorname{Im} \Delta_4^f(x - x', \mu) \, \rho(x')$
= $\frac{1}{\hbar c} \int d\omega_k \, |\rho(k)|^2$. (3.8)

Here, too, $\operatorname{Im} W^{(0)} > 0$. Hence, the vacuum persistence probabilities $\exp\left(-2\operatorname{Im} W^{(1,0)}/\hbar\right)$ in the presence of a radiating vector or scalar source are less than unity.

The integrands in Eqns (3.2) and (3.8) are the spectra of the mean numbers of quanta with spin s = 1 and 0 given in the Introduction [formula (1.3)].

For point-like electric and scalar sources moving along a trajectory $x_{\alpha}(\tau)$ in the Minkowski (3 + 1)-space, the current and charge densities and their Fourier components are defined by the formulas

$$j_{\alpha}(x), \rho(x) = e \int_{-\infty}^{\infty} \mathrm{d}\tau \left\{ \dot{x}_{\alpha}(\tau), 1 \right\} \delta_4 \left(x - x(\tau) \right), \qquad (3.9)$$

$$j_{\alpha}(k), \rho(k) = e \int_{-\infty}^{\infty} \mathrm{d}\tau \left\{ \dot{x}_{\alpha}(\tau), 1 \right\} \exp\left(-\mathrm{i}kx(\tau)\right). \quad (3.10)$$

In agreement with the adopted expressions for the action W and propagator Δ_4^f , it is assumed that e is the electric and the scalar charge in Heaviside units. In this case, the mean number of quanta emitted by the charge over the entire trajectory can be written as

$$\bar{N}^{(1,0)} = \frac{2 \operatorname{Im} W^{(1,0)}}{\hbar}$$
$$= \frac{e^2}{\hbar c} \iint d\tau \, d\tau' \left\{ \dot{x}_{\alpha}(\tau) \, \dot{x}^{\alpha}(\tau'), 1 \right\} \operatorname{Im} \Delta_4^f(z,\mu) \,,$$
$$z^{\alpha} = x^{\alpha}(\tau) - x^{\alpha}(\tau') \,.$$
(3.11)

If the charge trajectory lies in the (x, t) plane, i.e., in (1 + 1)-space, integral (3.10) involves only the zeroth and first components of the vectors $x^{\alpha}(\tau)$ and k^{α} . However, k^{α} remains a 4-vector because the quantum propagates in the (3 + 1)-space. Accordingly, $j_{\alpha}(k)$ and $\rho(k)$ depend only on two independent variables, and $k_{\pm} = k^0 \pm k^1$, where $k^0 = \sqrt{\mathbf{k}^2 + \mu^2}$, can be conveniently taken as these. For $\bar{N}^{(1,0)}$ and the spectra $d\bar{n}_{k,k_{-}}^{(1,0)}$, writing the invariant measure $d\omega_k$ in terms of k_{\pm} and the azimuthal angle φ , as was done in the Introduction, we obtain the expressions

$$\bar{N}^{(1,0)} = \int d\bar{n}^{(1,0)}_{k_{+}k_{-}},$$

$$d\bar{n}^{(1,0)}_{k_{+}k_{-}} = \left\{ \left| j_{\alpha}(k_{+},k_{-}) \right|^{2}, \left| \rho(k_{+},k_{-}) \right|^{2} \right\} \frac{dk_{+} dk_{-}}{(4\pi)^{2}},$$
(3.12)

given in the Introduction [formula (1.4)].

3.2 Symmetry of linear relations between the Bogoliubov coefficients and densities of current and charge

From the secondary-quantized theory, it follows that the absolute amplitude $\langle \text{out } \omega'' \omega | \text{in} \rangle$ of pair creation and the amplitude $\langle \text{out } \omega'' | \omega' \text{in} \rangle$ of single-particle scattering by a point-like mirror in (1 + 1)-space are related as

$$\langle \operatorname{out} \omega'' \omega | \operatorname{in} \rangle = -\sum_{\omega'} \langle \operatorname{out} \omega'' | \omega' \operatorname{in} \rangle \beta^*_{\omega' \omega}.$$
 (3.13)

This formula allows interpreting $\beta_{\omega'\omega}^*$ as the amplitude of a source of a pair of massless particles potentially emitted to the right and to the left with respective frequencies ω and ω' [5]. The particle with the frequency ω freely moves to the right, and the particle with frequency ω' propagates to the left only for a certain time interval, being then reflected by the mirror and emitted in the right direction with a changed frequency ω'' (see Fig. 1). In this case, in the time interval between the creation of the pair and reflection of the left particle, we are dealing with a virtual pair with the energy k^0 , momentum k^1 ,



Figure 1. Creation of a pair of massless oppositely moving particles with frequencies ω and ω' by an accelerated mirror.

and mass m:

$$k^{0} = \omega + \omega', \quad k^{1} = \omega - \omega', \quad m = \sqrt{-k^{2}} = 2\sqrt{\omega\omega'}.$$

(3.14)

In addition to this time-like polar 2-vector k^{α} , a space-like axial 2-vector q^{α} constructed using the antisymmetric unit tensor $\varepsilon_{\alpha\beta}$ and the vector k^{β} is also very important:

$$q_{\alpha} = \varepsilon_{\alpha\beta} k^{\beta}, \quad q^{0} = -k^{1} = -\omega + \omega',$$

$$q^{1} = -k^{0} = -\omega - \omega' < 0.$$
(3.15)

In terms of the vectors k^{α} and q^{α} and the current and charge densities in (3.10), the duality discussed here between the descriptions of processes in (3 + 1)- and (1 + 1)-spaces, as well as the symmetry between the coefficients $\alpha_{\omega'\omega}$ and $\beta^*_{\omega'\omega}$, is expressed in the most compact way.

The radiation of a boson pair in (1 + 1)-space and a quantum with spin s = 1 in (3 + 1)-space are

$$e\beta^{\mathbf{B}*}_{\omega'\omega} = -\frac{q_{\alpha}j^{\alpha}(k)}{\sqrt{k_+k_-}}, \quad e\alpha^{\mathbf{B}}_{\omega'\omega} = -\frac{k_{\alpha}j^{\alpha}(q)}{\sqrt{k_+k_-}}.$$
 (3.16)

The radiation of a fermion pair in (1 + 1)-space and a quantum with the spin s = 0 in (3 + 1)-space are

$$e\beta_{\omega'\omega}^{\mathrm{F}*} = \rho(k), \qquad e\alpha_{\omega'\omega}^{\mathrm{F}} = \rho(q).$$
(3.17)

The current densities $j^{\alpha}(k)$ and $j^{\alpha}(q)$ and charge densities $\rho(k)$ and $\rho(q)$ involved here are defined by formula (3.10) as functionals of the trajectory $x^{\alpha}(\tau)$ on the (x, t) plane, i.e., in the (1 + 1)-space, where they become functions of the two-dimensional vectors k^{α} and q^{α} with components (3.14) and (3.15). It can be shown that $j^{\alpha}(k)$ and $j^{\alpha}(q)$ are space-like and time-like polar 2-vectors in (1 + 1)-space if k^{α} and q^{α} are respectively time-like and space-like 2-vectors.

In the vacuum of massless scalar or spinor fields, the boundary condition on the mirror triggers the appearance of vector or scalar perturbation waves bilinear in massless fields. There are two types of such waves: (1) waves of the amplitude $\beta^*_{\omega'\omega}$ with the time-like momentum k^{α} and a positive frequency, carrying energy away from the mirror;

(2) waves of the amplitude $\alpha_{\omega'\omega}$ with the space-like momentum q^{α} directed to the left, toward the mirror, and accompanying it as it moves.

The waves with space-like momenta appear even when the mirror is at rest or moves uniformly (the Casimir effect), whereas waves with time-like momenta appear only for an accelerated mirror.

The pairs of Bose (Fermi) particles have spin 1 (0) because their source is the current density vector (scalar of charge density) (see Eqns (3.16) and (3.17) and also Ref. [11] or problem 12.15 in Ref. [21]).

3.3 Coincidence of the spin of pairs emitted

by mirrors and the spin of quanta emitted by charges

We consider the relation of expressions (3.16) and (3.17) to the initial formulas (2.11), (2.12), (2.28), and (2.29) for the Bogoliubov coefficients, making the transformation in the latter from the integration variables u and v to the coordinates of points on the mirror trajectory at propertime instants τ and τ' :

$$u = x_{-}(\tau), \quad f(u) = x_{+}(\tau);$$

$$v = x_{+}(\tau'), \quad g(v) = x_{-}(\tau').$$
(3.18)

The pairs of expressions for $\beta_{\omega'\omega}^{B*}$ and $\alpha_{\omega'\omega}^{B}$ are then written in terms of \pm components of 2-vectors $u^{\alpha}(k)$ and $u^{\alpha}(q)$, differing only in the physical meaning of their vector arguments k^{α} and q^{α} :

$$\beta_{\omega'\omega}^{\mathbf{B}*} = \frac{k_{-}}{\sqrt{k_{+}k_{-}}} u_{+}(k) = -\frac{k_{+}}{\sqrt{k_{+}k_{-}}} u_{-}(k), \qquad (3.19)$$

$$\alpha^{\rm B}_{\omega'\omega} = \frac{q_-}{\sqrt{k_+k_-}} \, u_+(q) = -\frac{q_+}{\sqrt{k_+k_-}} \, u_-(q) \,. \tag{3.20}$$

The vectors

$$u^{\alpha}(k) = \int d\tau \, \dot{x}^{\alpha}(\tau) \exp\left(-ikx(\tau)\right),$$

$$u^{\alpha}(q) = \int d\tau \, \dot{x}^{\alpha}(\tau) \exp\left(-iqx(\tau)\right)$$
(3.21)

differ only by the absence of the factor e (the charge in Heaviside units) from the current densities

$$j^{\alpha}(k) = eu^{\alpha}(k), \quad j^{\alpha}(q) = eu^{\alpha}(q)$$
(3.22)

of a point-like source moving along the same trajectory as the mirror. For the β coefficient, the argument of the current is the time-like polar vector k^{α} with components (3.14), and for the α coefficient, it is the space-like axial vector q^{α} with components (3.15).

We note that double representations (3.19) and (3.20) for the coefficients β^* and α are related by the current conservation law

$$k_{-}u_{+}(k) + k_{+}u_{-}(k) = -2k_{\alpha}u^{\alpha}(k) = 0, \qquad (3.23)$$

and similarly for $u^{\alpha}(q)$:

$$q_{-}u_{+}(q) + q_{+}u_{-}(q) = -2q_{\alpha}u^{\alpha}(q) = 0.$$
(3.24)

Hence, the terms compensating each other in these conservation laws have the physical meaning of the Bogoliubov coefficients $\pm \beta^{B*}_{\omega'\omega}$ and $\pm \alpha^{B}_{\omega'\omega}$.

Because $k_{\pm} = \mp q_{\pm}$, half the sums of the two expressions for the Bogoliubov coefficients in Eqns (3.19) and (3.20) (times the charge) are the products of unit vectors

$$-\frac{q_{\alpha}}{\sqrt{k_+k_-}}, \quad -\frac{k_{\alpha}}{\sqrt{k_+k_-}} \tag{3.25}$$

and respective current density vectors $j^{\alpha}(k)$ and $j^{\alpha}(q)$ [see Eqn (3.16)].

Because $j^{\alpha}(k)$ and $j^{\alpha}(q)$ are space-like and time-like polar vectors in (1 + 1)-space and their arguments k^{α} and q^{α} are time-like and space-like vectors, $\beta^{B*}_{\omega'\omega}$ is a pseudoscalar contracted from space-like polar and axial vectors and $\alpha^{B}_{\omega'\omega}$ a scalar contracted from time-like polar vectors.

Expressions (2.28) and (2.29) for the β^{F*} and α^{F} coefficients of a fermion field with the same change of variables (3.18) reduce to scalars u(k) and u(q), differing only in the physical meaning of their arguments k^{α} and q^{α} :

$$\beta_{\omega'\omega}^{F*} = u(k) = \int d\tau \exp\left(-ikx(\tau)\right),$$

$$\alpha_{\omega'\omega}^{F} = u(q) = \int d\tau \exp\left(-iqx(\tau)\right).$$
(3.26)

They differ only by the absence of the factor e from the Fourier transforms

$$\rho(k) = eu(k), \qquad \rho(q) = eu(q) \tag{3.27}$$

of the density of the scalar charge moving along the same trajectory as the mirror [see Eqn (3.17)].

The fact that the amplitude $\beta_{\omega'\omega}^{B*}$ of the source of the virtual boson pair is defined by the current $j^{\alpha}(k_+, k_-)$ and the amplitude $\beta_{\omega'\omega}^{F*}$ of the virtual fermion pair by the scalar $\rho(k_+, k_-)$ implies that the spin of the boson pair is 1, whereas it is 0 for the fermion pair. Thus, the coincidence of spectra of mirror radiation in (1 + 1)-space and charge radiation in (3 + 1)-space can be explained by the coincidence of the spin of the pair emitted by the mirror with the spin of a quantum emitted by the charge [5]. In this respect, we note that in scalar product (3.16) defining $\beta_{\omega'\omega}^{B*}$, the space-like pseudovector $-q_{\alpha}/\sqrt{k_+k_-}$ is orthogonal to the pair 2-momentum, has unit length, and, in the proper system of the pair, has only a space-like component, the same as the current vector $j^{\alpha}(k)$. It can be treated as the polarization vector or the spin of the boson pair.

To conclude Sections 3.2 and 3.3, we note that on the level of linear relations, formulas (3.16), (3.17) and (3.19), (3.20), (3.26), (3.27) demonstrate the duality of the semiclassical description of the radiation of quanta by a point-like charge in (3 + 1)-space and the quantum description of the radiation of pairs of quanta by a point-like mirror in (1 + 1)-space.

3.4 Functional coincidence of the spectra of boson and fermion pairs emitted by mirrors with the spectra of quanta emitted by electric and scalar charges

The mean number of Bose or Fermi quanta emitted by the mirror at a frequency ω in the interval d ω is given by

$$d\bar{n}_{\omega}^{B,F} = \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{d\omega'}{2\pi} \left| \beta_{\omega'\omega}^{B,F} \right|^{2}, \qquad (3.28)$$

where the integral is the mean value of the operator $N_{\text{out}\,\omega} = a_{\text{out}\,\omega}^+ a_{\text{out}\,\omega}$ for the number of out-particles of the frequency ω in the vacuum of in-particles:

$$\langle \mathrm{in}|a_{\mathrm{out}\,\omega}^{+}a_{\mathrm{out}\,\omega}|\mathrm{in}\rangle = \int_{0}^{\infty} \frac{\mathrm{d}\omega'}{2\pi} |\beta_{\omega'\omega}|^{2} \,. \tag{3.29}$$

Because particles appear in pairs, the number of pairs is that of the particles if the particle and the antiparticle are different, and half of that if the particle and the antiparticle are identical. We consider the first case, where the total mean number of emitted pairs is

$$\bar{N}^{B,F} = \iint_{0}^{\infty} \frac{\mathrm{d}\omega \,\mathrm{d}\omega'}{\left(2\pi\right)^{2}} \left|\beta_{\omega'\omega}^{B,F}\right|^{2}.$$
(3.30)

Such a physical interpretation of $d\bar{n}_{\omega}^{B,F}$ and $\bar{N}^{B,F}$ follows from the secondary quantization of the fields ϕ and ψ , when the coefficients of the field expansion in plane waves with positive and negative frequencies are interpreted as the operators of particle absorption and antiparticle creation. The secondary-quantized theory allows constructing all possible amplitudes of many-particle creation, annihilation, and scattering with the help of the Bogoliubov coefficients [5, 19, 22].

The Bogoliubov coefficients and the number of states $d\omega d\omega'/(2\pi)^2$ are Lorentz invariant. Accordingly, the spectrum $d\bar{n}_{\omega}$ of the mean number of emitted quanta and the total mean number of quanta are relativistically invariant.

We now demonstrate that the expressions for $\bar{N}^{(1)}$ and $\bar{N}^{(0)}$ in Eqns (3.12) and (1.4) differ only by the factors $e^2/\hbar c$ from the total mean numbers of boson and fermion pairs

$$\bar{N}^{B,F} = \iint_{0}^{\infty} \frac{\mathrm{d}\omega \,\mathrm{d}\omega'}{(2\pi)^{2}} \,\left|\beta_{\omega'\omega}^{B,F}\right|^{2} \tag{3.31}$$

emitted by a point-like mirror in (1 + 1)-space. We first consider the squared modulus

$$\left|\beta^{\,\mathrm{B}}_{\omega'\omega}\right|^2 = \beta^{\,\mathrm{B}}_{\omega'\omega}\,\beta^{\,\mathrm{B}*}_{\omega'\omega}$$

using both expressions given in Eqn (3.19) for the factors. Then

$$\left|\beta_{\omega'\omega}^{\mathbf{B}}\right|^{2} = -\frac{1}{2}(u_{+}u_{-}^{*} + u_{-}u_{+}^{*}) = u_{\alpha}(k) u^{\alpha*}(k) = \left|u_{\alpha}(k)\right|^{2}.$$
(3.32)

According Eqn (3.26), the squared modulus of the Bogoliubov coefficient for the Fermi field is just

$$\left|\beta_{\omega'\omega}^{\mathrm{F}}\right|^{2} = \left|u(k)\right|^{2}.$$
(3.33)

The time-like 2-vector (k^1, k^0) in Eqns (3.32) and (3.33) is related to ω and ω' by formulas (3.14), i.e.,

$$k^0 = \omega + \omega', \quad k^1 = \omega - \omega',$$

or, equivalently,

$$k_+=2\omega\,,\qquad k_-=2\omega\,'\,.$$

Then

$$\frac{\mathrm{d}\omega\,\mathrm{d}\omega^{\,\prime}}{\left(2\pi\right)^2} = \frac{\mathrm{d}k_+\,\mathrm{d}k_-}{\left(4\pi\right)^2}$$

and

$$\bar{N}^{B,F} = \iint_{0}^{\infty} \frac{\mathrm{d}k_{+} \,\mathrm{d}k_{-}}{\left(4\pi\right)^{2}} \left\{ \left| u_{\alpha}(k) \right|^{2}, \left| u(k) \right|^{2} \right\}.$$
(3.34)

Because $j_{\alpha}(k) = eu_{\alpha}(k)$ and $\rho(k) = eu(k)$, in accordance with Eqn (3.12), we find the relation

$$\bar{N}^{(1,0)} = \frac{e^2}{\hbar c} \,\bar{N}^{\mathrm{B},\mathrm{F}} \,, \qquad \mathrm{d}\bar{n}^{(1,0)}_{k_+k_-} = \frac{e^2}{\hbar c} \,\mathrm{d}\bar{n}^{\mathrm{B},\mathrm{F}}_{k_+k_-} \tag{3.35}$$

between the total mean numbers of quanta with spin 1 and 0 emitted by charges in (3 + 1)-space and the boson and fermion pairs emitted by the mirror in (1 + 1)-space, as well as between their spectra.

3.5 Coincidence of the spectra of boson and fermion pairs at high frequencies

We consider the spectra of boson and fermion pairs using the original representations (2.11) and (2.28) for $\beta_{\omega'\omega}^{B*}$ and $\beta_{\omega'\omega}^{F*}$:

$$d\bar{n}_{k_{+}k_{-}}^{B} = \left| \sqrt{\frac{k_{+}}{k_{-}}} \int_{-\infty}^{\infty} du \exp\left[\frac{i}{2} \left(k_{+}u + k_{-}f(u)\right)\right] \right|^{2} \frac{dk_{+} dk_{-}}{(4\pi)^{2}},$$
(3.36)
$$d\bar{n}_{k_{+}k_{-}}^{F} = \left| \int_{-\infty}^{\infty} du \sqrt{f'(u)} \exp\left[\frac{i}{2} \left(k_{+}u + k_{-}f(u)\right)\right] \right|^{2} \frac{dk_{+} dk_{-}}{(4\pi)^{2}}$$
(3.37)

As can be seen, these expressions are essentially different. But for a sufficiently smooth mirror trajectory and large values of k_{\pm} , when the integrals in Eqns (3.36) and (3.37) can be computed by the steepest descent method, these spectra do coincide. Indeed, in this case, the stationary point $u = u_0$ satisfies the equation

$$f'(u_0) = -\frac{k_+}{k_-}$$

and lies in the complex plane u because f'(u) > 0 on the real axis due to the time-like character of the trajectory. Therefore, for large k_{\pm} , the spectra have the common exponential asymptotic behavior

$$d\bar{n}_{k_{+}k_{-}}^{B} = d\bar{n}_{k_{+}k_{-}}^{F}$$
$$= \frac{k_{+}}{k_{-}^{2} |f''(u_{0})|} \exp\left[-\operatorname{Im}\left(k_{+}u_{0} + k_{-}f(u_{0})\right)\right] \frac{dk_{+} dk_{-}}{4\pi}.$$
(3.38)

This asymptotic behavior agrees with the theorem according to which the Fourier component of a smooth function in the high-frequency domain decays faster than any negative integer power of frequency [23]. (For high-frequency spectrum asymptotic expressions in electrodynamics, see Ref. [24].)

Additionally, the coincidence of the spectra of boson and fermion pairs in the range of high frequencies ω and ω' of quanta with spin 0 and 1/2 can be viewed as a manifestation of a peculiar supersymmetry. The spectra coincide, remaining functionals (which are now identical) of the mirror trajectory and functions of two variables.

Because the spectra $d\bar{n}_{k+k_{-}}^{B}$ and $d\bar{n}_{k+k_{-}}^{F}$ of boson and fermion pairs emitted by the mirror are purely geometrical quantities and coincide at $k_{\pm} \rightarrow \infty$, it follows by virtue of the discovered symmetry that

$$\mathrm{d}\bar{n}^{(1)}_{k_{+}k_{-}} = \frac{e^2}{\hbar c} \,\mathrm{d}\bar{n}^{\mathrm{B}}_{k_{+}k_{-}} \,, \qquad \mathrm{d}\bar{n}^{(0)}_{k_{+}k_{-}} = \frac{e^2}{\hbar c} \,\mathrm{d}\bar{n}^{\mathrm{F}}_{k_{+}k_{-}} \,,$$

and hence it is natural to suppose that the spectra $d\bar{n}_{k_+k_-}^{(1)}$ and $d\bar{n}_{k_+k_-}^{(0)}$ for large values of k_\pm are independent of the spin of the emitted quanta, i.e., are identical. This means that the electric and scalar charge values are also equal.

4. Holographic principle of bare charge quantization

4.1 Relation between causal Green's functions in d- and (d-2)-dimensional spaces

The principal basis for the duality discussed here is the relation between causal functions in d- and (d-2)-dimensional Minkowski spaces:

$$\Delta_d^f(x,\mu) = \frac{1}{4\pi} \int_{\mu^2}^{\infty} \mathrm{d}m^2 \, \Delta_{d-2}^f(x,m) \,. \tag{4.1}$$

Because $\Delta_d^f(x,\mu)$ satisfies the inhomogeneous wave equation

$$(-\partial_{\alpha}^{2} + \mu^{2}) \Delta_{d}^{f}(x,\mu) = \delta_{d}(x), \qquad (4.2)$$

where ∂_{α}^2 is the Lorentz square of the *d*-dimensional vector $\partial_{\alpha} = \partial/\partial x^{\alpha}$ and $\delta_d(x)$ is the *d*-dimensional Dirac delta function, relation (4.1) involves not only integration over m^2 but also an analytic continuation of the right-hand side in the argument x^2 from the (d-2)-dimensional domain into the *d*-dimensional one.

The solution of Eqn (4.2) can be written in terms of the MacDonald function

$$\Delta_d^f(x,\mu) = \frac{i(\mu\sqrt{x^2})^{\nu}}{(2\pi)^{\nu+1}x^{2\nu}} K_{\nu}(\mu\sqrt{x^2}), \quad \nu = \frac{d-2}{2}, \quad (4.3)$$

if $x^2 = \mathbf{x}^2 - x_0^2 > 0$, and analytically continued through the upper half-plane of complex x^2 to the semiaxis $x^2 < 0$, where it can be conveniently written in terms of the Hankel function:

$$\Delta_{d}^{f}(x,\mu) = \frac{\exp\left(-i\pi\nu\right)\left(\mu\sqrt{-x^{2}-i\varepsilon}\right)^{\nu}}{4(2\pi)^{\nu}(-x^{2}-i\varepsilon)^{\nu}} H_{\nu}^{(2)}\left(\mu\sqrt{-x^{2}-i\varepsilon}\right)$$
(4.4)

It can be easily seen that integral relation (4.1) follows from the well-known differential relation

$$-[z^{\nu}K_{\nu}(z)]' = z^{\nu}K_{\nu-1}(z)$$
(4.5)

between MacDonald functions and from the vanishing of $z^{\nu}K_{\nu}(z)$ at infinity (see Ref. [25]).

Of relevance for us is the relation between the causal function in four-dimensional (d = 4) and two-dimensional (d-2=2) spaces. We emphasize two points.

(1) The link between the propagation processes in the two spaces of different dimensions is not arbitrary, but is uniquely defined mathematically by formula (4.1). (2) It is essential that the function $\Delta_4^f(x,\mu)$ in the left-hand

side of Eqn (4.1) describes the propagation of a particle with a

mass μ that takes one arbitrary, for example, infinitely small value, while the function $\Delta_2^f(x,m)$ in the right-hand side of Eqn (4.1) describes the propagation of a 'particle' with the mass *m* taking all possible values in the interval $\mu \leq m < \infty$. The role of such a 'particle' is played by the pair of two massless oppositely moving particles, formed via the change in the energy-momentum of the vacuum field fluctuations in (1+1)-space brought about by a mirror moving with acceleration. These particles should be treated as massless, otherwise Eqn (4.1) would not contain the only free parameter μ .

Thus, Eqn (4.1) not only relates the propagators of a quantum and a pair in spaces of dimensions d = 4 and d = 2but also points to the massless character of particles forming the pair. The masslessness of particles forming a massive pair in (1 + 1)-space is the principal, purely geometrical aspect of the duality discussed here.

4.2 Once again on the functional coincidence of the spectra of mean numbers of quanta and pairs emitted by the charge

and mirror in spaces with d = 4 and d = 2

We consider the mean number of quanta emitted by the charge over its entire trajectory that entirely lies in the (x, t)plane. For such trajectories, if we use the relation between the causal functions in (3 + 1)- and (1 + 1)-spaces in Eqn (3.11),

$$\begin{aligned} \Delta_4^f(z,\mu) &= \frac{1}{4\pi} \int_{\mu^2}^{\infty} \mathrm{d}m^2 \, \Delta_2^f(z,m) \,, \\ \Delta_2^f(z,m) &= \mathrm{i} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{4\pi} \exp\left[\mathrm{i}m(z^1 \sinh\theta - |z^0|\cosh\theta)\right] \,, \end{aligned} \tag{4.6}$$

and use the above representation for Δ_2^f with the twodimensional vector $z^{\alpha} = x^{\alpha}(\tau) - x^{\alpha}(\tau')$, then we can replace Im Δ_4^f in Eqn (3.11) with the expression

$$\operatorname{Im} \Delta_{4}^{f}(z,\mu) = \operatorname{Re} \int_{\mu^{2}}^{\infty} \frac{\mathrm{d}m^{2}}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{4\pi} \exp\left(\mathrm{i}kz\right)$$
(4.7)

and integrate over τ , τ' . Then $\bar{N}^{(1,0)}$ becomes

$$\bar{N}^{(1,0)} = \frac{e^2}{\hbar c} \int_{\mu^2}^{\infty} \frac{\mathrm{d}m^2}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{4\pi} \left\{ \left| u_{\alpha}(k) \right|^2, \left| u(k) \right|^2 \right\}, \qquad (4.8)$$

where the 2-vector $u_{\alpha}(k)$ and the scalar u(k) are defined in Eqns (3.21) and (3.26), and the wave 2-vector (k^1, k^0) of a particle of mass *m* appearing in (1 + 1)-space is related to the wave 4-vector $k^{\alpha} = (\mathbf{k}_1 + \mathbf{k}_{\perp}, k^0 = (k_1^2 + k_{\perp}^2 + \mu^2)^{1/2})$ of the quantum in (3 + 1)-space by

$$k^{1} = m \sinh \theta$$
, $k^{0} = m \cosh \theta$, $m = \sqrt{k_{\perp}^{2} + \mu^{2}}$. (4.9)

We note that if *m* is the particle mass, then $k^1/k^0 = \tanh \theta$ is its velocity and θ is its rapidity. This physical meaning of the parameters m and θ is dictated by the relation of the propagators in (3 + 1)- and (1 + 1)-spaces.

Moreover, according to Eqn (4.9), k^1 and k^0 are the difference and the sum of two frequencies defined below in Eqn (4.11), i.e., the particle with a mass *m* can be considered a pair of massless particles flying apart in opposite directions. On the other hand, for the mean numbers

$$\bar{N}^{\mathrm{B,F}} = \iint_{0}^{\infty} \frac{\mathrm{d}\omega \,\mathrm{d}\omega'}{\left(2\pi\right)^{2}} \left|\beta_{\omega'\omega}^{\mathrm{B,F}}\right|^{2} \tag{4.10}$$

of boson and fermion pairs, changing the variables as

$$\omega = \frac{1}{2} m \exp \theta, \quad \omega' = \frac{1}{2} m \exp (-\theta), \quad d\omega d\omega' = \frac{1}{4} dm^2 d\theta,$$
(4.11)

we can obtain exactly the same expression as in Eqn (4.8), but without the factor $e^2/\hbar c$:

$$\bar{N}^{B,F} = \int_{0}^{\infty} \frac{\mathrm{d}m^{2}}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{4\pi} \left\{ \left| u_{\alpha}(k) \right|^{2}, \left| u(k) \right|^{2} \right\},$$

$$\bar{N}^{(1,0)} = \frac{e^{2}}{\hbar c} \bar{N}^{B,F}.$$
(4.12)

Here, the 2-vector (k^1, k^0) is related to the frequencies ω and ω' of the quanta forming the pair by

$$k^{1} = \omega - \omega' = m \sinh \theta, \qquad k^{0} = \omega + \omega' = m \cosh \theta,$$

$$m = 2\sqrt{\omega\omega'} = \sqrt{k_{0}^{2} - k_{1}^{2}}.$$
(4.13)

Expression (4.12) coincides with the previous expression for $\overline{N}^{B,F}$ in Eqn (3.34); instead of ω and ω' , that expression involved the variables $k_{+} = 2\omega$ and $k_{-} = 2\omega'$, which are equivalent to the variables *m* and θ because $k_{\pm} = m \exp(\pm \theta)$ according to Eqn (4.13).

While formula (4.9) relates the wave 2-vector (k^1, k^0) of a massive particle appearing in (1 + 1)-space to the wave 4-vector of a quantum emitted by the charge in (3 + 1)-space, formula (4.13) gives it the interpretation of the wave vector of a pair of massless particles with frequencies ω and ω' moving oppositely. The pair has the mass *m* in the continuum $\mu \leq m < \infty$ and the velocity $\tanh \theta$ in the interval (-1, 1) and owes its existence to a point-like accelerated mirror in (1 + 1)-space.

Thus, in formula (4.8) for $\bar{N}^{(1,0)}$, the variables *m* and θ appeared as the result of the transition from Δ_4^f to Δ_2^f and have the physical meaning of the mass and rapidity of a particle in (1 + 1)-space. At the same time, the current density $u_{\alpha}(k)$ and the charge density u(k), as well as the factor $e^2/\hbar c$, entered this formula as the Fourier transforms of the sources $j_{\alpha}(x)$ and $\rho(x)$ contained in the actions $W^{(1,0)}$.

These same densities $u_{\alpha}(k)$ and u(k) appeared in formula (4.12) for $\overline{N}^{B,F}$ because they enter the relativistically invariant structure of scalar products—the Bogoliubov coefficients $\beta_{\omega'\omega}^{B}$ and $\beta_{\omega'\omega}^{F}$ for scalar and spinor fields [Eqns (2.8) with (2.9) and (2.26) with (2.27)]. Roughly speaking, the presence of the derivative $\overrightarrow{\partial}_{t}$ in scalar products (2.8) and (2.9) implies their proportionality to the currents $u_{\alpha}(k)$ and $u_{\alpha}(q)$, and its absence in scalar products (2.26) and (2.27) relates them to the scalars u(k) and u(q). And the variables *m* and θ emerged in Eqn (4.12) owing to the replacement of the frequencies of quanta in the pair by its mass and rapidity.

To summarize, at the core of the duality discussed here are the geometrical links between

(a) the causal Green's functions in (3 + 1)- and (1 + 1)-spaces;

(b) the current and charge densities and scalar products of scalar and spinor fields in the same spaces.

It would be wrong to think that every time the charge and mirror pass the same part of their common trajectory, the radiation of a quantum with the 4-momentum $k^{\alpha} = (\mathbf{k}, k^0)$, $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_{\perp}$, $k^0 = (k_1^2 + k_{\perp}^2)^{1/2}$, by the charge in (3 + 1)space is accompanied by the radiation of a pair with the 2-momentum (\mathbf{k}_1, k^0) and mass $m = k_{\perp}$ by the mirror in (1 + 1)-space. It would be so if there had been a physical relation combining the quantum radiation in (3 + 1)-space and pair radiation in (1 + 1)-space into a single event, and if a theory had existed comprising, in particular, the Schwinger source theory and the theory of Bogoliubov transformations for these spaces. In such a theory, the condition $e^2/\hbar c = 1$ would be maintained automatically.

However, if $e^2/\hbar c = 1$, then even in the absence of such a physical link and a theory, given a large number of passes over the trajectory, the mean numbers of quanta and pairs emitted by the charge and the mirror from identical intervals of the trajectory and having uniquely related quantum numbers are arbitrarily close. In this case, there is an informational, holographic link between the processes of radiation in 4- and 2-dimensional spaces. This link implies the coincidence of the spectra of *mean numbers* of quanta and pairs emitted by the charge and the mirror from their entire common trajectory. They coincide as functions of two variables and functionals of the trajectory.

The duality addressed here, relating the classical and quantum theories in the Minkowski spaces of 4 and 2 dimensions, in a certain sense resembles the duality of classical and quantum descriptions in spaces of adjacent dimensions announced by 't Hooft [26] and Susskind [27] as the holographic principle. Such a duality was indeed discovered by Gubser, Klebanov, and Polyakov [28] and Maldacena [29] for various types of semiclassical supergravities in an anti-de Sitter space and quantum conformal theories at the boundary of this space. It seems likely, at least in our case, that the reason for such dualities can be the correspondence between an individual particle in the higherdimensional space and a pair of particles in the lowerdimensional space. Describing a larger number of particles in the lower-dimensional space calls for accounting for quantum mechanical interference effects.

5. Source theory and spectra of the mean number of quanta emitted by charges

We trace how the quantum spectrum $d\bar{n}_k$ for the mean number of emitted quanta and its relation $d\mathcal{E}_k = \hbar\omega d\bar{n}_k$ to the classical spectrum $d\mathcal{E}_k$ of mean radiated energy emerge in the framework of the source theory developed by Schwinger in monograph [11]. In this theory, the full description of the particle emission and absorption processes is furnished by the vacuum–vacuum amplitude in the presence of a source *S*,

$$\langle 0_+|0_-\rangle^S = \exp\left(\frac{\mathrm{i}}{\hbar} W(S)\right).$$
 (5.1)

In this case, the doubled imaginary part of the action W divided by the Planck constant \hbar equals the mean number of particles \overline{N} created by the source over the total time. In turn, \overline{N} represents the integral of the spectrum $d\overline{n}_k$ of the mean number of emitted quanta,

$$\bar{N} = \frac{2 \operatorname{Im} W}{\hbar} = \int \mathrm{d}\bar{n}_k \,, \tag{5.2}$$

each carrying away the momentum $\hbar \mathbf{k}$ and energy $\hbar \omega$ determined by the wave vector $k^{\alpha} = (\mathbf{k}, k^0), k^0 = \omega/c$.

This is a direct relation between the very important semiclassical quantity and the most important object of quantum physics, the quanta of a source field.

The semiclassical aspect of the vacuum amplitude resides in the fact that the source motion is regarded as given, i.e., the back reaction of emission or absorption of quanta on the source as well as the interaction between the quanta are neglected. This approximation resembles that used in computing the black-body spectrum: the temperature of the photon gas in a black-body cavity is regarded as given, independent of radiation or absorption of quanta, and the interaction between the quanta is neglected.

In the exposition below, the reader is advised to pay attention to the appearance of integer numbers $n_k = 0$, $1, 2, \ldots$ —the numbers of quanta filling the state with the momentum $k^{\alpha} = (\mathbf{k}, k^0)$ —and to the procedure by which they are converted to the average occupation numbers \bar{n}_k and the spectral distribution $d\bar{n}_k$ for the mean number of quanta. The relation between the spectrum of the mean number of emitted quanta and the spectrum of mean radiation energy, mentioned above, relies on the idea that the energy $\hbar\omega$ is associated with a quantum.

5.1 Vacuum amplitude of a source of spin-0 particles

The action W is quadratic in sources, and for spin-0 particles is given by

$$W(\rho) = \frac{1}{2c} \int d^4x \, d^4x' \, \rho(x) \, \varDelta^f(x - x', \mu) \, \rho(x')$$
(5.3)

with the scalar function $\rho(x)$, the scalar charge density, as the source. The dimension of $\rho(x)$ is $e \text{ cm}^{-3}$. The dimension of the scalar charge e coincides with that of the electric charge, $erg^{1/2} \text{ cm}^{1/2}$.

The causal function for the propagation of a field radiated by the source satisfies the equation

$$(-\partial_{\alpha}^{2} + \mu^{2}) \Delta^{f}(x - x') = \delta(x - x').$$
(5.4)

It is relativistically invariant, symmetric,

$$\Delta^{f}(x - x') = i \int d\omega_{k} \exp\left[i\mathbf{k}(\mathbf{x} - \mathbf{x}') - ik^{0}|x^{0} - x'^{0}|\right],$$

$$d\omega_{k} = \frac{d^{3}k}{(2\pi)^{3}2k^{0}}, \qquad k^{0} = \frac{\omega}{c} = \sqrt{\mathbf{k}^{2} + \mu^{2}},$$

(5.5)

and contains the Planck constant only in the parameter $\mu = mc/\hbar$ with the meaning of the inverse Compton length for the field quanta. The space and time components of the wave vector $k^{\alpha} = (\mathbf{k}, k^0)$ have the dimension of inverse length, and the 4-vector coordinates are $x^{\alpha} = (\mathbf{x}, x^0 = ct)$. The dimension of the invariant measure $d\omega_k$ is the inverse length squared.

We suppose that the source $\rho(x)$ is composed of the radiating $\rho_2(x)$ and absorbing $\rho_1(x)$ sources,

$$\rho(x) = \rho_1(x) + \rho_2(x) \,,$$

occupying finite space-time domains in which the absorption by the source $\rho_1(x)$ occurs after the radiation process by the source $\rho_2(x)$ is completed. In this case, the amplitude

$$\langle 0_+ | 0_- \rangle^{\rho}$$
 can be written in the form

$$\begin{aligned} &0_{+}|0_{-}\rangle^{\rho} = \langle 0_{+}|0_{-}\rangle^{\rho_{1}} \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar c} \int \mathrm{d}^{4}x \, \mathrm{d}^{4}x' \, \rho_{1}(x) \, \varDelta^{f}(x-x',\mu) \, \rho_{2}(x')\right] \langle 0_{+}|0_{-}\rangle^{\rho_{2}} \,, \end{aligned}$$
(5.6)

using the symmetry of the argument of the exponential under the exchange $\rho_1(x) \rightleftharpoons \rho_2(x)$ and the symmetry of the causal function. In what follows, the factor $1/\hbar c$ in the argument of the exponential function is typically set to unity, i.e., the system of units with $\hbar, c = 1$ is used.

In accordance with the causal position of sources, the argument can be written as a sum over discrete values of the wave vector,

$$i \int d^{4}x \, d^{4}x' \,\rho_{1}(x) \,\Delta^{f}(x-x') \,\rho_{2}(x')$$

= $i \int d^{4}x \, d^{4}x' \,\rho_{1}(x) \left[i \int d\omega_{k} \exp\left(ik(x-x')\right) \right] \rho_{2}(x')$
= $\sum_{k} i \rho_{1k}^{*} i \rho_{2k}$, (5.7)

if we use the definitions

$$\rho_k = \sqrt{\mathrm{d}\omega_k} \ \rho(k) , \qquad \rho(k) = \int \mathrm{d}^4 x \ \rho(x) \exp\left(-\mathrm{i}kx\right) , \quad (5.8)$$
$$\rho^*(k) = \rho(-k)$$

and suppose that each discrete value of the wave vector is within the corresponding interval d^3k . In that case, the exponential in Eqn (5.6) can be rearranged into a product of exponentials associated with individual values of k,

$$\exp\left[\sum_{k} \mathrm{i}\rho_{1k}^{*} \mathrm{i}\rho_{2k}\right] = \prod_{k} \exp\left[\mathrm{i}\rho_{1k}^{*} \mathrm{i}\rho_{2k}\right]$$
$$= \prod_{k} \sum_{n_{k}=0}^{\infty} \frac{(\mathrm{i}\rho_{1k}^{*})^{n_{k}}}{\sqrt{n_{k}!}} \frac{(\mathrm{i}\rho_{2k})^{n_{k}}}{\sqrt{n_{k}!}} , \qquad (5.9)$$

or into the product of the corresponding expansions of these functions. We note that the dimension of ρ_k is that of the charge *e*, and hence the exponent in Eqn (5.6) containing the factor $1/\hbar c$ is dimensionless.

Here, for the first time, each k becomes associated with an integer $n_k = 0, 1, 2, ...$, which can be treated as the number of particles in a state with the wave number k lying in the interval d^3k . In other words, these integers are the occupation numbers of different states k.

Expansion (5.9) derived above allows writing vacuum amplitude (5.1) for the causally ordered source pair as

$$\langle 0_{+}|0_{-}\rangle^{\rho} = \sum_{\{n_{k}\}} \langle 0_{+}|\{n_{k}\}\rangle^{\rho_{1}} \langle \{n_{k}\}|0_{-}\rangle^{\rho_{2}},$$
 (5.10)

with the following expressions for the many-particle creation and absorption amplitudes:

$$\langle \{n_k\} | 0_- \rangle^{\rho} = \langle 0_+ | 0_- \rangle^{\rho} \prod_k \frac{(i\rho_k)^{n_k}}{\sqrt{n_k!}},$$
 (5.11)

$$\langle 0_+ | \{n_k\} \rangle^{\rho} = \langle 0_+ | 0_- \rangle^{\rho} \prod_k \frac{(i\rho_k^*)^{n_k}}{\sqrt{n_k!}} .$$
 (5.12)

The symbol $\{n_k\}$ denotes the set of integer occupation numbers for all states k characterizing the many-particle amplitudes. In other words, $\{n_k\}$ are $n_{k_1}, n_{k_2}, n_{k_3}, \ldots$, where k_1, k_2, k_3, \ldots are the wave vectors involved in the product \prod_k . Accordingly, \sum_{n_k} denotes the sum over integers $n_{k_1}, n_{k_2}, n_{k_3}, \ldots$, each taking values $0, 1, 2, \ldots$

The fact that the n_k are occupation numbers of the states with momentum k is confirmed by the transformation law of many-particle amplitudes under source translations in space-time. Namely, after replacing $\rho(x)$ with $\rho(x+X)$, the Fourier component of the source and probability amplitudes of creation and absorption acquire the phase factors

$$\rho(k) \to \exp(ikX) \rho(k),$$

$$\langle \{n_k\} | 0_- \rangle^{\rho} \to \exp(iKX) \langle \{n_k\} | 0_- \rangle^{\rho},$$

$$\langle 0_+ | \{n_k\} \rangle^{\rho} \to \langle 0_+ | \{n_k\} \rangle^{\rho} \exp(-iKX).$$
(5.13)

Their phase is proportional to the total wave 4-vector

$$K^{\mu} = \sum_{k} n_k \, k^{\mu} \tag{5.14}$$

of the many-particle state $\{n_k\}$. Each value of the momentum in this sum is multiplied by the number of particles with this momentum.

The sum of absolute probabilities of creating particles in all possible states by the source ρ should be equal to unity. For amplitude (5.11), this implies that

$$\sum_{\{n_k\}} \left| \left\langle \{n_k\} \left| 0_- \right\rangle^{\rho} \right|^2 = \left| \left\langle 0_+ \left| 0_- \right\rangle^{\rho} \right|^2 \exp\left[\sum_k \left| \rho_k \right|^2 \right] = 1. \quad (5.15)$$

But according to the initial expression (5.1) for the vacuum amplitude,

$$\begin{aligned} \left| \langle 0_{+} | 0_{-} \rangle^{\rho} \right|^{2} &= \exp\left(-\frac{2}{\hbar} \operatorname{Im} W\right) \\ &= \exp\left[-\int d^{4}x \, d^{4}x' \, \rho(x) \operatorname{Im} \varDelta^{f}(x-x') \, \rho(x')\right] \\ &= \exp\left[-\int d^{4}x \, d^{4}x' \, \rho(x) \int d\omega_{k} \, \exp\left(ik(x-x')\right) \, \rho(x')\right] \\ &= \exp\left[-\sum_{k} \left|\rho_{k}\right|^{2}\right], \end{aligned}$$
(5.16)

and therefore completeness condition (5.15) is indeed satisfied. Hence, up to a phase factor,

$$\langle 0_{+}|0_{-}\rangle^{\rho} = \exp\left(-\frac{1}{2}\sum_{k}|\rho_{k}|^{2}\right),$$
 (5.17)

and then, in agreement with Eqn (5.11),

$$\langle \{n_k\} | \mathbf{0}_- \rangle^{\rho} = \prod_k \frac{(\mathbf{i}\rho_k)^{n_k}}{\sqrt{n_k!}} \exp\left(-\frac{1}{2} |\rho_k|^2\right).$$
(5.18)

This is the final expression for the probability amplitude for the many-particle state $\{n_k\}$ created by the source ρ .

Obviously, each factor in this amplitude is the probability amplitude of emitting n_k quanta with momentum k in the interval d^3k :

$$\langle n_k | 0_- \rangle^{\rho} = \frac{(i\rho_k)^{n_k}}{\sqrt{n_k!}} \exp\left(-\frac{1}{2} |\rho_k|^2\right),$$
 (5.19)

and its modulus squared is the probability of radiating n_k quanta in the mode k:

$$|\langle n_k | 0_- \rangle^{\rho}|^2 = \frac{|\rho_k|^{2n_k}}{n_k!} \exp\left(-|\rho_k|^2\right).$$
 (5.20)

This probability is in effect the Poisson distribution with the mean number of quanta emitted in the mode k equal to $\bar{n}_k = |\rho_k|^2$.

5.2 Relation to the secondary quantization method

We now turn to the method of secondary quantization [30, 31] and to single-mode coherent states $|\alpha\rangle$, which are eigenstates of the absorption operator *a* with a complex eigenvalue α : $a|\alpha\rangle = \alpha |\alpha\rangle$ [32–34]. The operator *a* and its adjoint a^+ act in the space of occupation numbers, decreasing and increasing the number of particles in states with a definite number by unity,

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle, \quad (5.21)$$

and commuting as

$$aa^+ - a^+a = 1. (5.22)$$

The state $|\alpha\rangle$ can be expanded in *n*-quantum states,

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$
 (5.23)

which are eigenstates of the operator a^+a of the number of quanta with the eigenvalue $n: a^+a|n\rangle = n|n\rangle$. The probability amplitude of finding *n* quanta in the state $|\alpha\rangle$ is

$$\langle n|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{\sqrt{n!}},$$
 (5.24)

and the probability itself is given by the Poisson distribution

$$w(n) = \frac{\bar{n}^n}{n!} \exp\left(-\bar{n}\right) \tag{5.25}$$

with the mean number of quanta

$$\bar{n} = \langle \alpha | a^+ a | \alpha \rangle = |\alpha|^2 \tag{5.26}$$

in the mode.

The above relations demonstrate an interesting property of coherent states. Because such a state is characterized by an indefinite number of quanta, the disappearance of one of them does not, in fact, change the state, and just multiplies it by a factor.

Hence, the amplitude $\langle n_k | 0_- \rangle^{\rho}$ coincides with the amplitude $\langle n | \alpha \rangle$ if we identify the complex number $i\rho_k$ with α and the integer number n_k with n; in this case, $|\rho_k|^2$ acquires the physical meaning of the mean number of particles created by the source ρ in the state (mode) k:

$$\bar{n}_{k} = |\rho_{k}|^{2} = \frac{\mathrm{d}^{3}k}{(2\pi)^{3}2k^{0}} |\rho(k)|^{2} = \mathrm{d}\bar{n}_{k} \,.$$
(5.27)

The mean number of particles created by the source in all states is then equal, according to Eqn (5.16), to twice the imaginary part of the action in units of \hbar ,

$$\bar{N}^{(0)} = \sum_{k} n_{k} = \sum_{k} \bar{n}_{k} = \frac{1}{\hbar c} \sum_{k} |\rho_{k}|^{2}$$
$$= \frac{1}{\hbar c} \int d\omega_{k} |\rho(k)|^{2} = \frac{2}{\hbar} \operatorname{Im} W^{(0)}.$$
(5.28)

The superscript appearing here at \overline{N} and W indicates the spin of the emitted quanta. We have already used these formulas in the Introduction.

Analogously, the mean value of the 4-momentum emitted by the source is

$$\bar{K}^{\mu} \equiv \overline{\sum_{k} k^{\mu} n_{k}} = \sum_{k} k^{\mu} \bar{n}_{k} = \int \mathrm{d}\omega_{k} \left| \rho(k) \right|^{2} k^{\mu}.$$
(5.29)

5.3 Vacuum amplitude of the source of spin-1 particles

For massless spin-1 particles—photons—the source is a conserved 4-vector of current density $j^{\alpha}(x)$, and the vacuum–vacuum amplitude is given by

$$\langle 0_{+}|0_{-}\rangle^{j} = \exp\left(\frac{\mathrm{i}}{\hbar} W(j)\right)$$

= $\exp\left[\frac{\mathrm{i}}{2\hbar c} \int \mathrm{d}^{4}x \, \mathrm{d}^{4}x' \, j^{\alpha}(x) \, \varDelta^{f}(x-x',\mu) \, j_{\alpha}(x')\right],$ (5.30)
 $\partial_{\alpha} j^{\alpha}(x) = 0.$

In Δ^{f} , we keep the infinitesimal mass parameter, which is convenient for eliminating the infrared divergence if it appears. We note that the dimension of $j^{\alpha}(x)$ coincides with that of $\rho(x)$, since the electric and scalar charges have the same dimension.

Similarly to the vacuum amplitude for a scalar source, amplitude (5.30) describes the system of vector sources of an arbitrary intensity, if there is no interaction between particles and the radiation and absorption of individual particles do not affect the source properties (classical sources).

The action for the vector source differs from its scalar counterpart by the replacement of $\rho(x)$ with $j^{\alpha}(x)$. Using, instead of Eqn (5.8), the definitions

$$j_k^{\alpha} = \sqrt{d\omega_k} j^{\alpha}(k), \quad j^{\alpha}(k) = \int d^4 x \exp\left(-ikx\right) j^{\alpha}(x),$$

$$j^{\alpha*}(k) = j^{\alpha}(-k)$$
(5.31)

for the vacuum amplitude with the causal sequence of sources j_1^{α} and j_2^{α} , we obtain

$$\langle 0_{+}|0_{-}\rangle^{j} = \langle 0_{+}|0_{-}\rangle^{j_{1}} \exp\left[\sum_{k} i j_{1k}^{\alpha*} g_{\alpha\beta} i j_{2k}^{\beta}\right] \langle 0_{+}|0_{-}\rangle^{j_{2}}.$$
(5.32)

We introduce a quadruple of mutually orthogonal unit 4-vectors $e_{k\lambda}^{\alpha}$ numbered by $\lambda = 0, 1, 2, 3$ and directed along the time and three spatial axes of a special coordinate system with its axis 3 aligned with the vector **k** (i.e., $e_{k\lambda}^{\alpha} = \delta_{\lambda}^{\alpha}$ in this system). In an arbitrary coordinate system, the metric tensor $g^{\alpha\beta}$ can then be represented in the form

$$g^{\alpha\beta} = \sum_{\lambda=1,2} e^{\alpha}_{k\lambda} e^{\beta}_{k\lambda} + e^{\alpha}_{k3} e^{\beta}_{k3} - e^{\alpha}_{k0} e^{\beta}_{k0} \,. \tag{5.33}$$

From the current conservation condition $k_{\alpha} j^{\alpha}(k) = 0$ and the isotropy of the photon 4-momentum $k^2 = 0$, it follows that $j^3 = j^0$ in the special coordinate system mentioned above. Therefore, in an arbitrary system,

$$e_{k3\alpha} j^{\alpha}(k) = -e_{k0\alpha} j^{\alpha}(k), ,$$

and then

$$\sum_{k} j_{1k}^{\alpha *} g_{\alpha\beta} j_{2k}^{\beta} = \sum_{k\lambda=1,2} j_{1k\lambda}^{*} j_{2k\lambda} , \qquad (5.34)$$

where $j_{k\lambda} = \sqrt{\mathrm{d}\omega_k} \, j^{\alpha}(k) e_{k\lambda\alpha}, \, \lambda = 1, 2.$

As a result, we express the amplitudes of many-photon creation and absorption by the current j as

$$\left\langle \{n_{k\lambda}\} \left| 0_{-} \right\rangle^{j} = \left\langle 0_{+} \right| 0_{-} \right\rangle^{j} \prod_{k\lambda} \frac{\left(\mathrm{i} j_{k\lambda}\right)^{n_{k\lambda}}}{\sqrt{n_{k\lambda}!}} , \qquad (5.35)$$

$$\langle 0_+ | \{ n_{k\lambda} \} \rangle^j = \langle 0_+ | 0_- \rangle^j \prod_{k\lambda} \frac{(i j_{k\lambda}^*)^{n_{k\lambda}}}{\sqrt{n_{k\lambda}!}} .$$
 (5.36)

These amplitudes for a vector source differ from the respective expressions for a scalar source by the replacement in the latter of ρ with j, ρ_k with $j_{k\lambda}$, and n_k with $n_{k\lambda}$, because the state of a photon, in contrast to the state of a scalar quantum, is characterized not only by its momentum but also by the transverse polarization that takes two independent values.

For the vacuum persistence probability in the presence of a vector source, using the original expression (5.30), we obtain the amplitude

$$\left| \langle 0_+ | 0_- \rangle^j \right|^2 = \exp\left(-\frac{2}{\hbar} \operatorname{Im} W^{(1)} \right), \tag{5.37}$$

where, in analogy with Eqn (5.16),

$$\frac{2}{\hbar} \operatorname{Im} W^{(1)} = \frac{1}{\hbar c} \int d\omega_k \left| j_{\alpha}(k) \right|^2$$
$$= \frac{1}{\hbar c} \int d\omega_k \sum_{\lambda=1,2} \left| j_{\alpha}(k) e_{k\lambda}^{\alpha} \right|^2 = \sum_{k\lambda} \bar{n}_{k\lambda}^{(1)} = \bar{N}^{(1)}. \quad (5.38)$$

The superscript at W and \overline{N} denotes the spin of the emitted quanta. We stress that the doubled imaginary part of the action contains the contribution from both polarizations. It is given by the sum of *mean* occupation numbers of states $k\lambda$, not of integer numbers $n_{k\lambda}$ involved in many-particle amplitudes.

The mean value of the 4-momentum of radiation is, obviously,

$$\bar{K}^{\mu} = \frac{1}{\hbar c} \int \mathrm{d}\omega_k \left| j_{\alpha}(k) \right|^2 k^{\mu} = \sum_{k\lambda} \bar{n}_{k\lambda} k^{\mu} \,. \tag{5.39}$$

It is also determined by the mean occupation numbers of states $k\lambda$, and therefore radically differs from the wave vector of the many-particle state [see Eqn (5.14)].

To conclude this section, we stress certain aspects of the semiclassical source theory.

(1) Despite the classical character of the action W, its doubled imaginary part, divided by \hbar , in accordance with Eqns (5.28) and (5.38), has the physical meaning of the mean number of quanta emitted by a scalar or vector source. This

establishes an instructive relation between the action and the secondary-quantized field theory, in which the state occupation numbers play the role of independent variables. The presence of a classical source in the vacuum makes the vacuum state $|0_{-}\rangle^{\rho}$ or $|0_{-}\rangle^{j}$ an analog of the coherent state $|\alpha\rangle$, which is an eigenstate of the absorption operator in the secondary quantization method.

(2) The mean value of the number of particles formed by a source in the vacuum has no bounds and need not be small.

(3) The Poisson distribution implies that quanta are emitted independently. Moreover, the particles are emitted independently not only inside the same interval of momenta $\Delta = d^3k$ but also in two different momenta intervals $\Delta_1 = d^3k_1$ and $\Delta_2 = d^3k_2$. Indeed, in the combined interval $\Delta = \Delta_1 + \Delta_2$, the mean number of quanta is $\bar{n}_{\Delta} = \bar{n}_{\Delta_1} + \bar{n}_{\Delta_2}$, and the distribution over the number of emitted quanta, by the addition theorem for Poisson distributions, is also Poisson,

$$w_{\Delta}(n) = \sum_{m=0}^{n} w_{\Delta_{1}}(m) w_{\Delta_{2}}(n-m) = \frac{(\bar{n}_{\Delta})^{n}}{n!} \exp(-\bar{n}_{\Delta}), \quad n \equiv n_{\Delta}$$
(5.40)

The interval Δ can be expanded over the entire range of momenta. Then $n_{\Delta} = N$ and $\bar{n}_{\Delta} = \bar{N}$, and the probability of radiation of N particles into all possible states by a source is

$$w(N) = \frac{\bar{N}^N}{N!} \exp\left(-\bar{N}\right).$$
(5.41)

The Poisson distribution implies the following relation between the mean squared fluctuation of the number of particles and the mean number of particles:

$$\overline{\left(N-\bar{N}\right)^2} = \overline{N^2} - \bar{N}^2 = \bar{N}.$$
(5.42)

6. Bogoliubov transformation: the quantum theory of point-like mirror radiation

6.1 Radiation of pairs

of identical particle and antiparticle

For a consistent description of the quantized wave field existing both to the right and to the left from a point-like mirror and satisfying a common boundary condition on the mirror, it is convenient to use the two complete sets $\{\phi_{out\omega}, \phi_{out\omega}^*\}$ and $\{\phi_{in\omega'}, \phi_{in\omega'}^*\}$ of wave equation solutions, written for Bose and Fermi fields in Section 2 (see also Refs [3, 4]). In the right Minkowski half-plane, these two solutions have the physical meaning of out- and in-sets and satisfy the boundary conditions on the mirror; they can be smoothly continued into the left half-plane without changing their functional form. However, in the left half-plane, these sets acquire the meaning of in- and out- sets, and have to be denoted there as $\{\phi_{in\omega}, \phi_{in\omega}^*\}$ and $\{\phi_{out\omega'}, \phi_{out\omega'}^*\}$.

In reality, each such solution is uniquely characterized by the frequency ω or ω' of its monochromatic component propagating to the right or to the left and the condition on the mirror. Under the Lorentz transformation with the velocity β along the x axis, the frequencies ω and ω' transform into $\tilde{\omega}$ and $\tilde{\omega}'$ by mutually inverse laws,

$$\tilde{\omega} = D^{-1}(\beta)\omega, \quad \tilde{\omega}' = D(\beta)\omega', \quad D(\beta) = \sqrt{\frac{1+\beta}{1-\beta}}, \quad (6.1)$$

where $D(\beta)$ is the Doppler factor. Therefore, ω and ω' have opposite covariance. In what follows, the frequencies transforming as ω are labeled by an even number of primes, and those transforming as ω' by an odd number of primes. In this case, the index 'in' or 'out', in addition to the frequency index, simply points to the side of the Minkowski plane where the solution is considered.

We write the expansion of solutions of the first set with respect to the solutions of the second and the inverse expansion (in the right half-plane) in the form

$$\phi_{\operatorname{out}\omega} = \alpha_{\omega'\omega}\phi_{\operatorname{in}\omega'} + \beta_{\omega'\omega}\phi^*_{\operatorname{in}\omega'}, \qquad (6.2)$$

$$\phi_{\mathrm{in}\,\omega'} = \alpha^*_{\omega'\omega} \phi_{\mathrm{out}\,\omega} \mp \beta_{\omega'\omega} \phi^*_{\mathrm{out}\,\omega} \,, \tag{6.3}$$

or, if we resort to matrix notation,

$$\begin{pmatrix} \phi_{\text{out}} \\ \phi_{\text{out}}^* \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \beta^+ & \alpha^+ \end{pmatrix} \begin{pmatrix} \phi_{\text{in}} \\ \phi_{\text{in}}^* \end{pmatrix},$$

$$\begin{pmatrix} \phi_{\text{in}} \\ \phi_{\text{in}}^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \mp \beta \\ \mp \beta^* & \alpha \end{pmatrix} \begin{pmatrix} \phi_{\text{out}} \\ \phi_{\text{out}}^* \end{pmatrix}.$$

$$(6.4)$$

Here and below, the upper and lower signs correspond to Bose and Fermi fields, which are now denoted by the same letter ϕ . Because of the orthogonality and normalization of the solutions in both sets, the matrices in Eqns (6.4) are the inverse of each other. This implies that the Bogoliubov coefficients satisfy four independent matrix relations

$$\alpha^{+}\alpha \mp \beta^{+}\beta = 1, \qquad \beta^{+}\alpha^{*} \mp \alpha^{+}\beta^{*} = 0,$$

$$\alpha\alpha^{+} \mp \beta^{*}\tilde{\beta} = 1, \qquad \alpha\beta^{+} \mp \beta^{*}\tilde{\alpha} = 0.$$
(6.5)

In the left half-plane, relations (6.2)–(6.4) are preserved, but the new physical interpretation requires exchanging the indices in \rightleftharpoons out in the functions, which is equivalent to the change

$$\alpha \to \alpha^+, \qquad \beta \to \mp \tilde{\beta} \,.$$
 (6.6)

For the quantized field in the right half-plane, the relation between the in and out creation and absorption operators a^+ and a is given by the Bogoliubov transformations

$$\begin{pmatrix} a_{\rm in} \\ a_{\rm in}^+ \end{pmatrix} = \begin{pmatrix} \alpha & \beta^* \\ \beta & a^* \end{pmatrix} \begin{pmatrix} a_{\rm out} \\ a_{\rm out}^+ \end{pmatrix},$$

$$\begin{pmatrix} a_{\rm out} \\ a_{\rm out}^+ \end{pmatrix} = \begin{pmatrix} \alpha^+ & \mp \beta^+ \\ \mp \tilde{\beta} & \tilde{a} \end{pmatrix} \begin{pmatrix} a_{\rm in} \\ a_{\rm in}^+ \end{pmatrix}.$$

$$(6.7)$$

For the field in the left half-plane, the subscripts at the operators a and a^+ in transformations (6.7) should be exchanged, in \rightleftharpoons out. This is once again equivalent to the replacement in Eqn (6.6).

Following the work of DeWitt [19] and his notation, we write the vector of the vacuum field state in the remote past as the expansion with respect to vectors of *n*-particle field states in the far future:

$$|\text{in}\rangle = \exp(\text{i}W) \sum_{n=0}^{\infty} \frac{\text{i}^{n/2}}{n!} \sum_{i_1 i_2 \dots i_n} V_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n \text{ out}\rangle.$$
 (6.8)

In our case, the quantum numbers $i_1i_2...i_n$ of out-states of individual particles should be understood as frequencies transforming as ω or as ω' , if we are respectively dealing with the field to the right or to the left of the mirror.

Using the equation $a_{in}|in\rangle = 0$, transformations (6.7), and expansion (6.8), it can be easily shown [19, 22] that the relative amplitudes $V_{i_1i_2...i_n}$ of creating *n* particles vanish for odd *n*, and can be expressed in terms of the particle pair creation amplitude for even *n*:

$$V_{i_1i_2...i_n} = \sum_p \delta_p V_{i_1i_2} V_{i_3i_4} \dots V_{i_{n-1}i_n} \,. \tag{6.9}$$

Here, \sum_{p} stands for the summation over $n!/2^{n/2}(n/2)!$ different pairings of indices $i_1i_2...i_n$, and $\delta_p = 1$ for bosons and $\delta_p = \pm 1$ for fermions, for even or odd parity of the permutation leading to a given pairing. The creation amplitudes for a particle pair with frequencies ω'' and ω in the right domain and frequencies ω''' and ω' in the left domain are

$$V_{\omega''\omega} = i(\alpha^{-1}\beta^{*})_{\omega''\omega}, \qquad V_{\omega'''\omega'} = -i(\beta\alpha^{-1})^{*}_{\omega'''\omega'}.$$
(6.10)

They are related to each other by transformation (6.6) and are symmetric for a Bose field and antisymmetric for a Fermi field, as follows from Eqn (6.5).

The above number of terms in amplitude (6.9) arises due to its symmetrization (antisymmetrization) and equals the number n! of permutations in its indices reduced by $2^{n/2}$ times owing to the already existing symmetry (antisymmetry) of two-particle amplitudes and by (n/2)! times because exchanging these amplitudes does not matter.

The particle creation in pairs owes its existence to the linearity of the Bogoliubov transformations in the operators a and a^+ . The operator a_{in} acting on an *n*-particle out-state transforms it into a superposition of (n-1)-particle and (n+1)-particle out-states. Therefore, in the expansion of the zeroth vector $a_{in}|in\rangle$ in *n*-particle out-states, the zero expansion coefficients express a linear dependence between the amplitudes of (n+1)- and (n-1)-particle creation. Because $n \ge 0$, the one-particle creation amplitude V_{i_1} vanishes, and together with it, all amplitudes of creation of an odd number of particles also vanish.

The absolute *n*-particle creation amplitudes are defined and related to the relative amplitudes by

$$\langle \operatorname{out} i_1 i_2 \dots i_n | \operatorname{in} \rangle \equiv \langle \operatorname{out} | a_{\operatorname{out} i_n} \dots a_{\operatorname{out} i_2} a_{\operatorname{out} i_1} | \operatorname{in} \rangle$$

= exp (iW) i^{n/2} V_{i1i2...in}. (6.11)

The vacuum–vacuum amplitude $\langle \text{out}|\text{in} \rangle = \exp(\text{i}W)$ is defined up to a phase factor by the equality of the full transition probability from the initial vacuum state to unity:

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \dots i_n} |\langle \text{out } i_1 i_2 \dots i_n | \text{in} \rangle|^2$$

= exp (-2 Im W) $\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \dots i_n} |V_{i_2 i_2 \dots i_n}|^2$. (6.12)

The sum of relative probabilities

$$q_n = \frac{1}{n!} \sum_{i_1 i_2 \dots i_n} \left| V_{i_1 i_2 \dots i_n} \right|^2$$
(6.13)

of the creation of *n* particles (or n/2 pairs) in the left-hand side of Eqn (6.12) is referred to as the partition function in what follows. It can be shown that in the case considered, when pairs consist of identical particles and antiparticles, the partition function is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \dots i_n} |V_{i_1 i_2 \dots i_n}|^2 = \det (1 \mp M)^{\mp 1/2}$$

= $\exp (\mp \frac{1}{2} \operatorname{tr} \ln (1 \mp M)),$ (6.14)

where $M = VV^+$ is the Hermitian positive semi-definite matrix formed of matrices (6.10). In particular, the first four terms in the partition function, defined by the relative amplitudes

1,
$$V_{i_{1}i_{2}}$$
, $V_{i_{1}i_{2}}V_{i_{3}i_{4}} \pm V_{i_{1}i_{3}}V_{i_{2}i_{4}} + V_{i_{1}i_{4}}V_{i_{2}i_{3}}$, (6.15)
 $V_{i_{1}i_{2}}V_{i_{3}i_{4}}V_{i_{5}i_{6}} \pm V_{i_{1}i_{2}}V_{i_{3}i_{5}}V_{i_{4}i_{6}} + V_{i_{1}i_{2}}V_{i_{3}i_{6}}V_{i_{4}i_{5}}$
 $\pm V_{i_{1}i_{3}}V_{i_{2}i_{4}}V_{i_{5}i_{6}} + V_{i_{1}i_{3}}V_{i_{2}i_{5}}V_{i_{4}i_{6}} \pm V_{i_{1}i_{3}}V_{i_{2}i_{6}}V_{i_{4}i_{5}}$
 $+ V_{i_{1}i_{4}}V_{i_{2}i_{3}}V_{i_{5}i_{6}} \pm V_{i_{1}i_{4}}V_{i_{2}i_{5}}V_{i_{3}i_{6}} + V_{i_{1}i_{4}}V_{i_{2}i_{6}}V_{i_{3}i_{5}}$
 $\pm V_{i_{1}i_{5}}V_{i_{2}i_{3}}V_{i_{4}i_{6}} + V_{i_{1}i_{5}}V_{i_{2}i_{4}}V_{i_{3}i_{6}} \pm V_{i_{1}i_{5}}V_{i_{2}i_{6}}V_{i_{3}i_{4}}$
 $+ V_{i_{1}i_{6}}V_{i_{2}i_{3}}V_{i_{4}i_{5}} \pm V_{i_{1}i_{6}}V_{i_{2}i_{4}}V_{i_{3}i_{5}} + V_{i_{1}i_{6}}V_{i_{2}i_{5}}V_{i_{3}i_{4}}$

and formula (6.13), are equal to

$$q_{0} = 1, \quad q_{2} = \frac{1}{2} \operatorname{tr} M, \quad q_{4} = \frac{1}{8} (\operatorname{tr} M)^{2} \pm \frac{1}{4} \operatorname{tr} M^{2},$$

$$q_{6} = \frac{1}{48} (\operatorname{tr} M)^{3} \pm \frac{1}{8} \operatorname{tr} M \operatorname{tr} M^{2} + \frac{1}{6} \operatorname{tr} M^{3}.$$
(6.16)

The first term in q_6 describes the independent radiation of three pairs, the second describes the interference of two pairs and the independent radiation of one pair, and the third describes the interference of three pairs.

Generally, tr M^k describes the interference of k pairs if $k \ge 2$, and $(\text{tr } M)^k$ describes the independent radiation of k pairs.

The absolute probability for *n* pairs to form is $p_{2n} = p_0 q_{2n}$, where p_0 is the vacuum persistence probability,

$$p_0 = \exp(-2 \operatorname{Im} W), \quad 2 \operatorname{Im} W = \mp \frac{1}{2} \operatorname{tr} \ln(1 \mp M). \quad (6.17)$$

Because the relative probability $q_{2n}(M)$ of the creation of *n* pairs is a homogeneous function of degree *n*, $q_{2n}(\lambda M) = \lambda^n q_{2n}(M)$, the mean number of pairs can be conveniently found from the formula

$$\bar{n} = \sum_{n=0}^{\infty} np_{2n} = p_0 \lambda \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \lambda^n q_{2n}(M) \Big|_{\lambda=1}$$
$$= \lambda \frac{\partial}{\partial \lambda} 2 \operatorname{Im} W(\lambda M) \Big|_{\lambda=1} = \frac{1}{2} \operatorname{tr} \frac{M}{1 \mp M}.$$
(6.18)

In the right and left domains, the matrices M are different,

$$I = VV^{+} = \begin{cases} \beta^{+}\beta(1 \pm \beta^{+}\beta)^{-1}, & (6.19) \\ \ddots & \ddots & \vdots \end{cases}$$

$$M = VV^{+} = \begin{cases} \gamma & \gamma & \gamma \\ \beta^{*} \tilde{\beta} (1 \pm \beta^{*} \tilde{\beta})^{-1}, \end{cases}$$
(6.20)

but are linked to each other by transformation (6.6). However, the positive-definite quantities tr M^n , n = 1, 2,..., are invariants of this transformation. Accordingly, the above full probabilities p_0 of vacuum persistence, the creation of *n* pairs p_{2n} , and the mean number of pairs \bar{n} are identical for the right and left domains. In particular, the quantities

$$p_0 = \exp(-2 \operatorname{Im} W), \quad 2 \operatorname{Im} W = \pm \frac{1}{2} \operatorname{tr} \ln(1 \pm \beta^+ \beta),$$

(6.21)

$$p_2 = \exp(-2\operatorname{Im} W) \frac{1}{2} \operatorname{tr} \beta^+ \beta (1 \pm \beta^+ \beta)^{-1}, \qquad (6.22)$$

$$\bar{n} = \frac{1}{2} \operatorname{tr} \beta^{+} \beta \tag{6.23}$$

are preserved under transformation (6.6) or $\beta^+\beta \rightarrow \beta\beta^+$. We note the simplicity of the formula for \bar{n} and its difference from 2 Im *W*. We discuss this at the end of this section.

Nevertheless, the frequency distributions of probabilities and the mean particle number do not have the left–right symmetry. For example, the creation probability of a pair with one particle of definite frequency and the other of an arbitrary frequency is

$$p_{2\omega} = \exp\left(-2\operatorname{Im} W\right) \left(\frac{\beta^{+}\beta}{1\pm\beta^{+}\beta}\right)_{\omega\omega}$$
(6.24)

for the right domain and

$$p_{2\omega'} = \exp\left(-2\operatorname{Im} W\right) \left(\frac{\beta\beta^+}{1\pm\beta\beta^+}\right)_{\omega'\omega'} \tag{6.25}$$

for the left domain. The frequency distributions of the mean number of particles radiated by the mirror respectively to the right and to the left are also functionally different:

$$N_{\omega} = (\beta^{+}\beta)_{\omega\omega}, \qquad N_{\omega'} = (\beta\beta^{+})_{\omega'\omega'}.$$
(6.26)

Along with amplitudes (6.11) of particle formation by the mirror from the vacuum, we need to consider the amplitudes of one-particle scattering by the mirror,

$$\langle \operatorname{out} \omega | \omega' \operatorname{in} \rangle = \langle \operatorname{out} | a_{\operatorname{out} \omega} a_{\operatorname{in} \omega'}^+ | \operatorname{in} \rangle = \exp\left(\operatorname{i} W\right) \alpha_{\omega \omega'}^{-1}, \quad (6.27)$$

$$\langle \operatorname{out} \omega' | \omega \operatorname{in} \rangle = \langle \operatorname{out} | a_{\operatorname{out} \omega'} a_{\operatorname{in} \omega}^+ | \operatorname{in} \rangle = \exp(\operatorname{i} W) \alpha_{\omega \omega'}^{-1*}$$
 (6.28)

respectively for the right and left domains. These amplitudes differ only by their phases. Needless to say, they are connected with each other by transformations (6.6), but we are interested in their connection with the corresponding pair creation amplitudes

$$\langle \operatorname{out} \omega'' \omega | \operatorname{in} \rangle = - \exp\left(\mathrm{i} W\right) (\alpha^{-1} \beta^*)_{\omega'' \omega}$$
$$= -\sum_{\omega'} \langle \operatorname{out} \omega'' | \omega' \operatorname{in} \rangle \beta^*_{\omega' \omega}, \qquad (6.29)$$

$$\langle \operatorname{out} \omega' \omega'' | \operatorname{in} \rangle = \exp\left(\operatorname{i} W\right) (\beta \alpha^{-1})^*_{\omega' \omega'''}$$
$$= \sum_{\omega} \beta^*_{\omega' \omega} \langle \operatorname{out} \omega''' | \omega \operatorname{in} \rangle.$$
(6.30)

Because the amplitudes of pair creation and amplitudes of one-particle scattering are, in principle, experimentally measurable by the respective probabilities, relations (6.29) and (6.30) enable the quantity $\beta^*_{\omega'\omega}$ to be experimentally measured. Moreover, these relationships allow considering $\beta^*_{\omega'\omega}$ the amplitude of the source of a pair of particles

potentially emitted to the right and to the left with respective frequencies ω and ω' . While a particle with the frequency ω actually propagates to the right, a particle with the frequency ω' does not fly to the left, but experiences total internal reflection and in reality is emitted to the right, but with the modified frequency ω'' . Conversely, if a particle with the frequency ω is indeed flying to the left, the particle with the frequency ω cannot fly to the right, experiences total internal reflection, and is actually emitted to the left with another frequency ω''' .

For fermions, the amplitude $\beta_{\omega'\omega}^{\rm F}$ is diagonal with respect to spin projection of in- and out-waves (see Section 2 and Ref. [4]). But one of the waves forming $\beta_{\omega'\omega}^{\rm F}$ has a negative frequency and therefore describes an antiparticle with the frequency and spin projection opposite in sign to the frequency and spin of this wave (see § 26 in Ref. [20] or § 9 of chapter 2 in Ref. [35]). Therefore, the spin of a pair of forming fermions equals zero. This is confirmed by the scalar character of two identical integrals in Eqns (2.28) and (2.29), in which $du \sqrt{f'(u)}$ and $dv \sqrt{g'(v)}$ are elements of the proper time $d\tau$, and their coincidence,

$$\beta_{\omega'\omega}^{F*} = \frac{1}{e} \rho(k_+, k_-), \qquad (6.31)$$

with the Fourier component of the scalar charge density in (3 + 1)-space.

The amplitude $\beta_{\omega'\omega}^{B*}$ of the boson pair source, according to Eqns (2.11) and (2.12), is expressed linearly in terms of the Fourier components $j_{\pm}(k)$ of the electric charge current density in (3 + 1)-space,

$$\beta_{\omega'\omega}^{B*} = -\sqrt{\frac{k_+}{k_-}} \frac{j_-}{e} = \sqrt{\frac{k_-}{k_+}} \frac{j_+}{e}, \qquad (6.32)$$

$$j_{-} = e \int_{-\infty}^{\infty} du \exp\left[\frac{1}{2}(k_{+}u + k_{-}f(u))\right],$$

$$j_{+} = e \int_{-\infty}^{\infty} dv \exp\left[\frac{i}{2}(k_{-}v + k_{+}g(v))\right]$$
(6.33)

(see also Eqn (3.19) and formulas (43) and (44) in Ref. [3]). The last equality in Eqn (6.32) is nothing but the transversality condition for the current, $k_+j_- + k_-j_+ = 0$. From Eqn (6.32), it also follows that $\beta_{\omega'\omega}^{B}$ is a pseudoscalar, because under reflection, $k_{\pm} \rightarrow k_{\mp}$, $j_{\pm} \rightarrow j_{\mp}$, and β^{B} changes its sign. The vector $j_{\alpha}(k)$ is space-like and in the system where $k_+ = k_-$ (or $\omega = \omega'$) has only a space-like component, equal exactly to $e\beta_{\omega'\omega}^{B}$. In covariant form,

$$e\beta^{\mathbf{B}*}_{\omega'\omega} = rac{arepsilon_{lphaeta} k^{lpha} j^{eta}}{\sqrt{k_+k_-}} \; .$$

Hence, the source of the boson pair is conserved current vector (6.33), and this implies that its spin is 1 [11].

The fact that the spin of the boson pair is 1 and that of the fermion pair is 0 is essential for understanding the coincidence of the mirror and charge spectra.

If $\beta_{\omega'\omega}^*$ is small, i.e., the mean number of emitted quanta is small, then, as can be easily obtained from formulas (2.11) and (2.28),

$$\alpha_{\omega'\omega} \approx 2\pi\delta(\tilde{\omega}' - \tilde{\omega}), \qquad \alpha_{\omega\omega'}^{-1} \approx 2\pi\delta(\tilde{\omega} - \tilde{\omega}'), \qquad (6.34)$$

where $\tilde{\omega}$ and $\tilde{\omega}'$ are related to ω and ω' by transformation (6.1) in which β is the mirror effective speed on the interval of

radiation. In this approximation, amplitudes (6.29) and (6.30) of the emission of a pair of particles with frequencies ω and ω'' to the right and a pair of particles with frequencies ω' and ω''' to the left are respectively given by

$$\langle \operatorname{out} \omega'' \omega | \operatorname{in} \rangle \approx - \exp(\operatorname{i} W) D^{-1}(\beta) \beta^*_{\omega'\omega}, \quad \omega' = D^{-2}(\beta) \omega'',$$

(6.35)

 $\langle \operatorname{out} \omega' \omega''' | \operatorname{in} \rangle \approx \exp(\mathrm{i}W) D(\beta) \beta^*_{\omega'\omega}, \quad \omega = D^2(\beta) \omega'''.$ (6.36)

These formulas, with the relation between the frequencies of waves incident on the mirror and reflected from it, confirm the interpretation of $\beta^*_{\omega'\omega}$ given above.

We now discuss the interference effects accompanying the creation of Bose and Fermi particles. These effects become most pronounced when the matrices M for bosons and fermions satisfy the conditions

$$\mp \frac{1}{2} \operatorname{tr} \ln \left(1 \mp M \right) = \mp \ln \left(1 \mp \frac{1}{2} \operatorname{tr} M \right),$$

i.e.,

$$\frac{1}{2} \operatorname{tr} M^n = \left(\frac{1}{2} \operatorname{tr} M\right)^n, \quad n = 2, 3, \dots$$
 (6.37)

Partition function (6.14) for Bose and Fermi particles then reduces to the respective expressions

$$\frac{1}{1-(1/2) \operatorname{tr} M}$$
 and $1+\frac{1}{2} \operatorname{tr} M$. (6.38)

This implies that the creation probabilities of n boson pairs form a geometric progression,

$$p_{2n}^{B} = p_0^{B} q_2^{Bn}, \quad p_0^{B} = 1 - \frac{1}{2} \operatorname{tr} M, \quad q_2^{B} = \frac{1}{2} \operatorname{tr} M, \quad (6.39)$$

and the probabilities of radiating two or more fermion pairs vanish, i.e., only a single fermion pair can be created:

$$p_0^{\rm F} = \left(1 + \frac{1}{2}\operatorname{tr} M\right)^{-1}, \quad p_2^{\rm F} = p_0 \frac{1}{2}\operatorname{tr} M, \quad p_{2n}^{\rm F} = 0, \quad n \ge 2.$$
(6.40)

In other words, conditions (6.37) imply the most constructive interference for bosons and the most destructive interference for fermions. In these cases, the mean squared fluctuation of the number of boson pairs is always larger than \bar{n}^{B} , and that of fermion pairs is less than \bar{n}^{F} , being equal to $\bar{n}(1 \pm \bar{n})$ for fermion pairs, where

$$0 < \bar{n}^{B} = \frac{(1/2) \operatorname{tr} M}{1 - (1/2) \operatorname{tr} M} = \frac{1}{2} \operatorname{tr} (\beta^{+}\beta)^{B} < \infty,$$

$$0 < \bar{n}^{F} = \frac{(1/2) \operatorname{tr} M}{1 + (1/2) \operatorname{tr} M} = \frac{1}{2} \operatorname{tr} (\beta^{+}\beta)^{F} < 1.$$
(6.41)

Less interesting is the case where the interference effects can be neglected:

$$\operatorname{tr} M^k \ll \operatorname{tr} M, 1; \quad k \ge 2, \tag{6.42}$$

or, in the language of matrices β and β^+ ,

$$\operatorname{tr} \left(\beta^+\beta\right)^k \ll \operatorname{tr} \beta^+\beta, \ 1; \quad k \ge 2.$$

In this case, the probability distribution for the number of created pairs coincides with the Poisson distribution

$$p_{2n} = e^{-\bar{n}} \frac{(\bar{n})^n}{n!}, \quad \bar{n} = \frac{1}{2} \operatorname{tr} \beta^+ \beta.$$
 (6.43)

For this distribution, the squared fluctuation of the number of pairs is

$$\overline{\left(n-\bar{n}\right)^2}=\bar{n}$$

in contrast to $\bar{n}(1 \pm \bar{n})$ for geometrical distribution (6.39) and Bernoulli distribution (6.40).

6.2 Radiation of pairs

of nonidentical particle and antiparticle

When the particle and the antiparticle composing a pair are not identical (*ab*-pairs), the direct and inverse Bogoliubov transformations (6.7) are replaced with

$$\begin{pmatrix} a_{\rm in} \\ b_{\rm in}^+ \end{pmatrix} = \begin{pmatrix} \alpha_{aa} & \beta_{ab}^* \\ \beta_{ba} & \alpha_{bb}^* \end{pmatrix} \begin{pmatrix} a_{\rm out} \\ b_{\rm out}^+ \end{pmatrix},$$

$$\begin{pmatrix} a_{\rm out} \\ b_{\rm out}^+ \end{pmatrix} = \begin{pmatrix} \alpha_{aa}^+ & \mp \beta_{ba}^+ \\ \mp \beta_{ab} & \tilde{\alpha}_{bb} \end{pmatrix} \begin{pmatrix} a_{\rm in} \\ b_{\rm in}^+ \end{pmatrix}.$$

$$(6.44)$$

These transformations contain four matrices α_{aa} , α_{bb} , β_{ab} , and β_{ba} instead of two, which satisfy six relations instead of four relations (6.5):

$$\begin{aligned} &\alpha_{aa}^{+}\alpha_{aa} \mp \beta_{ba}^{+}\beta_{ba} = 1 , \qquad \alpha_{bb}^{+}\alpha_{bb} \mp \beta_{ab}^{+}\beta_{ab} = 1 , \\ &\beta_{ba}^{+}\alpha_{bb}^{*} \mp \alpha_{aa}^{+}\beta_{ab}^{*} = 0 , \qquad \alpha_{aa}\alpha_{aa}^{+} \mp \beta_{ab}^{*}\tilde{\beta}_{ab} = 1 , \\ &\alpha_{bb}\alpha_{bb}^{+} \mp \beta_{ba}^{*}\tilde{\beta}_{ba} = 1 , \qquad \alpha_{aa}\beta_{ba}^{+} \mp \beta_{ab}^{*}\tilde{\alpha}_{bb} = 0 . \end{aligned}$$

$$(6.45)$$

We note that these relations can still be rewritten in form (6.5) if α and β are regarded as 2 × 2 matrices composed of the four matrices mentioned above:

$$\alpha = \begin{pmatrix} \alpha_{aa} & 0\\ 0 & \alpha_{bb} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \beta_{ab}\\ \beta_{ba} & 0 \end{pmatrix}.$$
(6.46)

As can be seen from Eqn (6.44), the in \rightleftharpoons out permutation is now equivalent to the change

$$\begin{aligned} \alpha_{aa} &\to \alpha_{aa}^{+}, \qquad a_{bb} \to \alpha_{bb}^{+}, \\ \beta_{ab} &\to \mp \tilde{\beta}_{ba}, \qquad \beta_{ba} \to \mp \tilde{\beta}_{ab}, \end{aligned}$$
(6.47)

which can be written in form (6.6) if α and β are the matrices in Eqn (6.46).

Using expansions like (6.8) for the in-vacuum state and the equations $a_{in}|in\rangle = b_{in}|in\rangle = 0$, it can be shown that all the amplitudes of the emission of an odd number of particles are equal to zero, while the creation amplitudes for an even number of particles are given by the products of the creation amplitudes for *ab*-pairs,

$$V^{ab}_{\omega''\omega} = i(\alpha^{-1}_{aa}\beta^*_{ab})_{\omega''\omega},$$

$$V^{ab}_{\omega'''\omega'} = -i(\beta_{ab}\alpha^{-1}_{bb})^*_{\omega'''\omega'},$$
(6.48)

respectively for the right and left domains. As follows from Eqns (6.45), amplitudes (6.48) have the property of Bose symmetry or Fermi antisymmetry,

$$V^{ab}_{\omega''\omega} = \pm V^{ba}_{\omega\omega''} \equiv \pm i(\alpha_{bb}^{-1}\beta_{ba}^*)_{\omega\omega''},$$

$$V^{ab}_{\omega''\omega'} = \pm V^{ba}_{\omega'\omega'''} \equiv \mp i(\beta_{ba}\alpha_{aa}^{-1})^*_{\omega'\omega'''}.$$
(6.49)

Therefore, the amplitude of *ab*-pair creation can be denoted as $V_{i_1i_2}$, where the index i_1 characterizes the state of the particle and i_2 characterizes the state of the antiparticle. The creation of two *ab*-pairs is described by the amplitude

$$V_{i_1 i_2 i_3 i_4} = V_{i_1 i_2} V_{i_3 i_4} \pm V_{i_3 i_2} V_{i_1 i_4} , \qquad (6.50)$$

which is symmetric (antisymmetric) with respect to the states i_1 and i_3 of particles and, separately, with respect to the states i_2 and i_4 of antiparticles. We also write the amplitude of the creation of three pairs,

$$V_{i_{1}i_{2}...i_{6}} = V_{i_{1}i_{2}}V_{i_{3}i_{4}}V_{i_{5}i_{6}} \pm V_{i_{3}i_{2}}V_{i_{1}i_{4}}V_{i_{5}i_{6}} + V_{i_{3}i_{2}}V_{i_{5}i_{4}}V_{i_{1}i_{6}}$$

$$\pm V_{i_{1}i_{2}}V_{i_{5}i_{4}}V_{i_{3}i_{6}} + V_{i_{5}i_{2}}V_{i_{1}i_{4}}V_{i_{3}i_{6}} \pm V_{i_{5}i_{2}}V_{i_{3}i_{4}}V_{i_{1}i_{6}}.$$
 (6.51)

In general, the amplitude for n/2 pairs to be created takes the form

$$V_{i_1 i_2 \dots i_n} = \sum_p \delta_p V_{i_1 i_2} V_{i_3 i_4} \dots V_{i_{n-1} i_n}, \qquad (6.52)$$

where the sum is taken over all (n/2)! terms that differ by the permutation in odd indices (or by the permutation of even indices, which is equivalent), and, in the case of fermions, $\delta_p = \pm 1$ for even and odd permutations, whereas $\delta_p = 1$ for bosons. The amplitude $V_{i_1i_2...i_n}$ is then symmetric (antisymmetric) with respect to both the states of particles $i_1i_3...i_{n-1}$ and the states of antiparticles $i_2i_4...i_n$.

The relative probability

$$q_n = \frac{1}{(n/2)!(n/2)!} \sum_{i_1 i_2 \dots i_n} |V_{i_1 i_2 \dots i_n}|^2$$
(6.53)

of the creation of n/2 pairs composed of nonidentical particles and antiparticles contains the factor 1/(n/2)!(n/2)!, which, together with the symmetry (antisymmetry) of the amplitude $V_{i_1i_2...i_n}$ in even and odd indices separately, allows summing over the states of particles and antiparticles by assuming the ranges of the quantum numbers of these states to be independent. Without this factor, the sum over $i_1i_2...i_n$ would include only physically different states. In our case, for example, this would imply that the frequencies of particles satisfy the condition $\omega_1 \ge \omega_3 \ge ... \ge \omega_{n-1}$, and that the frequencies of antiparticles satisfy the condition $\omega_2 \ge \omega_4 \ge ... \ge \omega_n$.

Based on the relative amplitudes written above, it is straightforward to construct the first four terms of the partition function:

$$q_{0} = 1, \quad q_{2} = \operatorname{tr} M, \quad q_{4} = \frac{1}{2} (\operatorname{tr} M)^{2} \pm \frac{1}{2} \operatorname{tr} M^{2},$$

$$q_{6} = \frac{1}{6} (\operatorname{tr} M)^{3} \pm \frac{1}{2} \operatorname{tr} M \operatorname{tr} M^{2} + \frac{1}{3} \operatorname{tr} M^{3}.$$
(6.54)

For the full partition function, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(n/2)!(n/2)!} \sum_{i_1 i_2 \dots i_n} |V_{i_1 i_2 \dots i_n}|^2$$

= det $(1 \mp M)^{\mp 1}$ = exp $(\mp \operatorname{tr} \ln (1 \mp M))$. (6.55)

Here, as in Eqn (6.14), $M = VV^+$ is a Hermitian positivesemidefinite matrix. It is given by formulas (6.19) and (6.20), in which β is respectively understood as β_{ba} and β_{ab} . In the same way as before, the absolute probabilities of the creation of *n* pairs of nonidentical particles and antiparticles are equal to $p_{2n} = p_0q_{2n}$, where p_0 is the vacuum persistence probability:

$$p_0 = \exp(-2 \operatorname{Im} W),$$

$$2 \operatorname{Im} W = \mp \operatorname{tr} \ln(1 \mp M) = \pm \operatorname{tr} \ln(1 \pm \beta^+ \beta).$$
(6.56)

The mean number of pairs, computed in agreement with rule (6.18), is

$$\bar{n} = \operatorname{tr} \frac{M}{1 \mp M} = \operatorname{tr} \beta^{+} \beta \,. \tag{6.57}$$

As is apparent, these formulas differ from formulas (6.17) and (6.18) for the creation of pairs consisting of identical particles by the appearance of tr instead of (1/2) tr. By virtue of the $a \rightleftharpoons b$ symmetry in the matrices under the trace, β can be either β_{ba} or β_{ab} . It can be readily seen that this rule connects all formulas for the integral characteristics related to the creation of identical particles with formulas for corresponding characteristics for the creation of *ab*-pairs. Therefore, in order to obtain the corresponding expressions for the creation of *ab*-pairs from formulas (6.21)–(6.23) and (6.37)–(6.43), it suffices to replace (1/2) tr in these formulas by tr and treat β as β_{ba} or β_{ab} .

As regards the spectral characteristics, for example, given by formulas (6.24)–(6.26), they do not experience transformations in the case considered if β is understood as β_{ba} (β_{ab}) for the spectrum of particles (antiparticles) emitted to the right and as β_{ab} (β_{ba}) for the spectrum of particles (antiparticles) emitted to the left. Indeed, for the differential probability $p_{2\omega}$ given by Eqn (6.24), the original integral

$$p_{2\omega} = \int_0^\infty \frac{\mathrm{d}\omega''}{2\pi} \left| \langle \operatorname{out} \omega \omega'' | \operatorname{in} \rangle \right|^2 \tag{6.58}$$

represents it as the sum of probabilities of physically different events, independent of whether the particles are identical. The total probability of pair formation p_2 as the sum of probabilities of physically different events for identical particles is given by the integral

$$p_2 = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \int_0^\omega \frac{\mathrm{d}\omega''}{2\pi} \left| \left\langle \operatorname{out} \omega \omega'' | \operatorname{in} \right\rangle \right|^2 = \frac{1}{2} \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} p_{2\omega} \quad (6.59)$$

because, in this case, the states differ only by values of larger ω and smaller ω'' frequencies of two identical particles. But for an *ab*-pair,

$$p_2 = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \int_0^\infty \frac{\mathrm{d}\omega''}{2\pi} \left| \langle \operatorname{out} \omega \omega'' | \operatorname{in} \rangle \right|^2 = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} p_{2\omega} \qquad (6.60)$$

because the states differ by mutually unrelated frequency values ω'' and ω of the particle and the antiparticle, which in turn differ in the way they interact with counters.

Returning to the *ab*-pair creation amplitude,

$$\langle \operatorname{out}|b_{\operatorname{out}\omega''}a_{\operatorname{out}\omega}|\operatorname{in}\rangle \equiv \langle \operatorname{out}\omega\omega''|\operatorname{in}\rangle$$
$$= -\exp\left(\mathrm{i}W\right)(\alpha_{aa}^{-1}\beta_{ab}^*)_{\omega\omega''} = \mp\exp\left(\mathrm{i}W\right)(\alpha_{bb}^{-1}\beta_{ba}^*)_{\omega''\omega},$$
(6.61)

we note that it reduces to the product of the amplitude β_{ab}^* or β_{ba}^* of the source of oppositely directed *a*, *b*-particles and the

backscattering amplitude α_{aa}^{-1} or α_{bb}^{-1} for one of them, as the result of which both particles in the pair move in the same direction. Symmetry (6.49) does not allow distinguishing which particle of the *ab*-pair experiences backscattering.

7. Discussion

We concentrate our attention on several important points underlying the observed duality that leads to quantization of the bare charge.

(1) For charge quantization, the existence of physical quantities (the spectra of the number of quanta emitted by a *charge* and the number of pairs emitted by a *neutral mirror*) with an identical quantum meaning of their observables turned out to be of principal significance. For any common trajectory of the charge and mirror, these spectra differ only by the factor $e^2/\hbar c$,

$$d\bar{n}_{k_{+}k_{-}}^{(1,0)} = \frac{e^2}{\hbar c} \, d\bar{n}_{\omega\omega'}^{B,F} \,. \tag{7.1}$$

The requirement that the spectra of mean values of integer observables — quanta and pairs — coincide identically fixes the relation between the charge squared and the Planck constant, $e_0^2 = \hbar c$.

(2) The point-like character of the charge and mirror implies that the charge is considered at very small distances and is unscreened by vacuum polarization, such that e_0 is its unscreened value, and the mirror is characterized by a boundary condition that does not contain dimensional parameters.

(3) The relation between the radiation of quanta in (3 + 1)-space and pairs of quanta in (1 + 1)-space is not unexpected. It is dictated by the integral connection between the causal Green's functions in these spaces, which underlies the holographic principle of charge quantization.

(4) The relation between Green's functions implies the relation between the wave 4-vector of a quantum emitted by the charge and the wave 2-vector of a massive pair emitted by the mirror. In turn, the 2-vector of the pair is defined by the frequencies of massless quanta making up the pair.

(5) Quanta emitted in (3 + 1)-space by the vector and scalar sources $j_{\alpha}(x)$ and $\rho(x)$ have the respective spin 1 and 0. The pairs of quanta of scalar and spinor fields emitted by the mirror in (1 + 1)-space have the respective spin 1 and 0, because the amplitudes of their sources ($\beta^{B,F}$, which are the Bogoliubov coefficients) are proportional to the vector $u_{\alpha}(k)$ and scalar u(k) owing to the structure of corresponding scalar products. The relation between the densities of the current $j_{\alpha}(k)$ and charge $\rho(k)$ and the Bogoliubov coefficients $\beta_{\omega'\omega}^{B}$ and $\beta_{\omega'\omega}^{F}$ is responsible for the functional coincidence of the spectra of photons and scalar quanta emitted by charges with the spectra of boson and fermion pairs emitted by mirrors.

(6) We draw attention to the difference between the action $W^{(1,0)}$, which defines the vacuum–vacuum amplitude in the presence of a charged source, and the action $W^{B,F}$, which defines the vacuum–vacuum amplitude in the presence of a mirror. Twice the imaginary part of the first is directly related to the mean number of emitted quanta,

$$\frac{2}{\hbar} \operatorname{Im} W^{(1,0)} = \bar{N}^{(1,0)} = \frac{1}{\hbar c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \left\{ \left| j_{\alpha}(k) \right|^2, \left| \rho(k) \right|^2 \right\},\tag{7.2}$$

while twice the imaginary part of the second differs from the mean number of emitted pairs

$$\bar{\mathbf{V}}^{\mathbf{B},\mathbf{F}} = \operatorname{tr}\left(\beta^{+}\beta\right)^{\mathbf{B},\mathbf{F}} \equiv \iint_{0}^{\infty} \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\left(2\pi\right)^{2}} \left|\beta_{\omega'\omega}^{\mathbf{B},\mathbf{F}}\right|^{2}$$
(7.3)

and is directly related to it only for small tr $(\beta^+\beta)^2$ compared with tr $\beta^+\beta$,

$$2 \operatorname{Im} W^{B,F} = \pm \operatorname{tr} \ln \left(1 \pm \beta^{+} \beta \right)^{B,F} \Big|_{\beta^{+} \beta \ll 1}$$
$$= \operatorname{tr} \left(\beta^{+} \beta \right)^{B,F} \mp \frac{1}{2} \operatorname{tr} \left(\beta^{+} \beta \beta^{+} \beta \right)^{B,F} + \dots \leq \bar{N}^{B,F}. \quad (7.4)$$

Quantum mechanical exchange effects of attraction and repulsion for respective Bose and Fermi particles lead to a difference between $2 \text{ Im } W^{B,F}$ and $\bar{N}^{B,F}$. And yet, the exact expressions for $\bar{N}^{(1,0)}$ and $\bar{N}^{B,F}$ are surprisingly simple, and just they are linked with the duality discussed here [see Eqns (7.1)–(7.3)].

Representation (7.4) is analogous to the virial expansion of the pressure of an ideal Bose or Fermi gas in powers of the degeneration parameter—the mean number of particles in the 3-volume defined by the thermal de Broglie wavelength (see § 56 in Ref. [36]).

Another example, more tightly related to Eqn (7.4), is furnished by the doubled imaginary part of the action defining the vacuum–vacuum amplitude $\exp(iW/\hbar)$ in the presence of a constant electric field ε that creates pairs. According to [37, 38],

$$\frac{2 \operatorname{Im} W}{\hbar} = \pm \sum_{r} \int \frac{\mathrm{d}^{3} p V}{(2\pi\hbar)^{3}} \ln\left(1 \pm \bar{n}_{p}\right),$$

$$\bar{n}_{p} = \exp\left(-\pi \frac{m^{2} + p_{\perp}^{2}}{|e\varepsilon|}\right),$$
(7.5)

where \bar{n}_p is the mean number of pairs formed by the field with a particle (or antiparticle) in a state with given momentum and spin projections $p = \mathbf{p}, r$; the upper and lower signs respectively correspond to boson and fermion pairs composed of oppositely charged massive particles and antiparticles with spin s = 0 or s = 1/2. The distribution \bar{n}_p is degenerate with respect to the spin projection rand the momentum component p_{\parallel} is directed along the field with multiplicities 2s + 1 and $L_{\parallel}\Delta p_{\parallel}/2\pi\hbar$, where $\Delta p_{\parallel} = e\varepsilon T$ (see Refs [37–39]). Integration over \mathbf{p} leads to a series for the imaginary part of the Heisenberg–Euler Lagrange function [40, 41]:

$$2\operatorname{Im}\mathscr{L} = (2s+1)\frac{(e\varepsilon)^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(\mp 1)^{n+1}}{n^2}\exp\left(-\frac{\pi n}{\beta}\right), \quad \beta = \frac{|e\varepsilon|}{m^2}.$$
(7.6)

Its first term is the mean number of pairs in a unit 4-volume. The next terms are the quantum mechanical exchange corrections describing the Bose attraction or Fermi repulsion of particles for a given mean 4-density of their number. They emerge because of the coherent creation of two or more pairs in the same 4-volume of pair formation (see Refs [42, 43]).

In all the examples presented above, 2 Im $W^{B,F}$ has just the same functional dependence on the mean occupation numbers $\bar{n}_k^{B,F}$ as the thermodynamic potentials of Bose and Fermi gases:

$$\Omega^{\mathbf{B},\mathbf{F}} = \mp T \sum_{k} \ln\left(1 \pm \bar{n}_{k}^{\mathbf{B},\mathbf{F}}\right).$$

The value found for the bare charge $e_0 = \pm \sqrt{\hbar c}$ and the corresponding value of the fine structure constant $\alpha_0 = 1/4\pi$ have the properties mentioned by Gell-Mann and Low for a finite bare charge [see variant (b) in their work [9] and also Section 1]. These properties are indicative of a purely geometrical origin of the value of α_0 obtained, which is natural, because for computing spectra we used solutions of wave equations for massless fields, and the trajectory of point-like charges and mirrors was described by time-like curves.

We note in passing that Dirac, discussing the value of the fine structure constant α and the proton-to-electron mass ratio at the University of New South Wales, remarked that "One may expect these numbers to occur as being built up from 4π s and other simple numbers like that" [44]. One would be startled by Dirac's amazing intuition had he meant not the value of α but that of the bare fine structure constant α_0 , which, in accordance with the discussed duality, is precisely equal to $1/4\pi$. The difference between these two quantities is due to vacuum polarization by a point-like electric charge, and their ratio $\alpha_0/\alpha = 10.90...$ is the vacuum dielectric permittivity.

The value $e_0 = \sqrt{\hbar c}$ is so significant that it is highly desirable to find an alternative variant of substantiating and computing it. In this respect, it seems natural to turn to the old suggestion by Casimir on finding α .

As is well known, a purely electromagnetic classical model of the electron is impossible—stability of the charge distribution requires adding the so-called Poincaré tension. In 1953, Casimir, having computed the attraction of two ideally conducting parallel plates caused by a reduction in the electromagnetic energy of zero-point oscillations between them, proposed an electromagnetic model of the electron [45] in which the Poincaré tension occurs at the expense of the energy of zero-point oscillations.

According to this model, the electron is regarded as a conducting spherical shell with the surface charge e. The electrostatic energy of such a charge is

$$E = \frac{e^2}{8\pi a} , \qquad (7.7)$$

if a is the sphere radius and the charge e is measured in Heaviside units. Casimir supposed that this energy of Coulomb repulsion can be compensated by a negative shift in the energy of quantum fluctuations of the electromagnetic field in the vacuum arising because of the interaction of fluctuations with the conducting shell and leading to attracting forces.

A concrete derivation of the change in the energy of zeropoint oscillations caused by the presence of a neutral ideally conducting shell was carried out by Boyer [46]. He showed that the energy shift is not negative, as for the plates, but positive, and is

$$E_{\rm B} = \frac{0.09\hbar c}{2a} \,. \tag{7.8}$$

Four years later, Davis [47] confirmed the sign of E_B and computed the numerical factor (0.09) to three significant digits, as 0.0924. In 1978, Balian and Duplantier [48] also obtained a positive coefficient with the same three significant digits. In the same year, Milton, DeRaad, and Schwinger [49] found a positive coefficient to five significant digits, 0.092353 (also see their work [50]). Therefore, the Casimir forces cannot play the role of the Poincaré tension, and the electron does not have the exotic structure proposed by Casimir.

If the sign of the coefficient had been negative, the value of the charge deduced from its stability condition

$$E + E_{\rm B} = 0 \tag{7.9}$$

would have been defined by the corresponding 'fine structure constant'

$$\alpha_{\rm B} = \frac{e_{\rm B}^2}{4\pi\hbar c} \approx 0.0924\,,\tag{7.10}$$

which is about 13 times larger than the actual fine structure constant $\alpha = 1/137$.

However, the energy E_B found by Boyer is very important in and of itself: it is the energy of the interaction of a neutral ideally conducting spherical shell with electromagnetic field fluctuations in the vacuum. It is defined by solutions of the purely geometric Maxwell equations for the field inside and outside an ideally conducting shell and boundary conditions on the shell, which *do not contain any dimensional parameters* except the shell radius.

A quantized electromagnetic field is represented as a system of independent harmonic oscillators, whose frequencies ω_k are defined by solutions of the Maxwell equations. Such a system, even in its ground state, has the nonzero energy

$$E = \sum_{k} \frac{1}{2} \hbar \omega_k \tag{7.11}$$

called the zero-point energy. In the absence of a conducting sphere, the frequencies ω_k are determined by boundary conditions on a sphere of a very large radius $R \ge a$. The presence of a conducting sphere of radius *a* inside that sphere modifies the energy *E* by a finite amount $E_{\rm B}$ in (7.8) as a result of a substantial change in frequencies $\omega_k \sim c/a$ by the boundary conditions on the conducting sphere. High frequencies $\omega_k \ge c/a$ are not affected by these conditions and in the limit $R/a \rightarrow \infty$ do not contribute to the difference $E_{\rm B} = E_{\rm sph} - E$ in the zero-point energies with or without a sphere of radius *a*.

As a result, the energy $E_{\rm B}$ coincides in form with the Coulomb energy of a sphere of radius *a* and with the charge

$$e_{\rm B} = \sqrt{4\pi \, 0.0924 \, \hbar c} = 1.077 \sqrt{\hbar c} \,. \tag{7.12}$$

Its value characterizes the strength of interaction between a conducting sphere and quantum fluctuations of the electromagnetic field in the vacuum. It is defined by the product $\hbar c$ of world constants, being a result of purely geometrical quantization, and *does not depend on the radius of the sphere*, which can be arbitrarily small. It is close to the value of the bare charge $e_0 = \sqrt{\hbar c}$ derived from the coincidence between the spectra of mean numbers of quanta emitted by a point-like charge in (3 + 1)-space and pairs emitted by a point-like mirror in (1 + 1)-space.

The difference between e_B and e_0 may stem from the difference in geometry (and topology) of a conducting sphere and a point-like mirror, while their closeness comes from the fact that the interactions of a conducting sphere and a point-like mirror with fluctuations of respective fields in the vacua of (3 + 1)- and (1 + 1)-spaces do indeed define, albeit with

different degrees of approximation, a very important quantum quantity, the value of the *bare* charge.

The approach to quantization based on the coincidence of the radiation spectra for a point-like charge and a mirror is manifestly relativistically invariant, but the approach based on the coincidence of the Coulomb and Casimir energies of charged and neutral conducting shells depends on the geometry of these shells. Therefore, if the sphere is replaced with the surface of a cube with the edge 2a (equal to the diameter of the sphere), the Casimir energy $E_{\rm B}$ is replaced with the energy

$$E_{\rm L} = \frac{0.0916\,\hbar c}{2a}\,,\qquad \alpha_{\rm L} = \frac{e_{\rm L}^2}{4\pi\hbar c} \approx 0.0916\,,$$
 (7.13)

found by Lukosh [51]. It differs from $E_{\rm B}$ by less than 1%. The Casimir energy of other compact shells is considered in Ref. [52].

Curiously, the products²

$$\alpha_0 \alpha_{\rm B} = \frac{1}{136.069} \text{ and } \alpha_0 \alpha_{\rm L} = \frac{1}{137.101}$$
 (7.14)

of purely geometrical constants differ only slightly from the fine structure constant $\alpha = 1/137.036$, which is bounded by them, $\alpha_0 \alpha_L < \alpha < \alpha_0 \alpha_B$. Could it imply that there is a regular (or semiregular) polyhedron, whose ideally conducting surface shifts the energy of electromagnetic vacuum fluctuations by the amount

$$E_{\Gamma} = \frac{\alpha_{\Gamma} \hbar c}{2a} , \qquad (7.15)$$

where the parameter α_{Γ} is just $4\pi\alpha$, i.e., $\alpha_{0}\alpha_{\Gamma} = \alpha$, and *a* is the radius of a sphere inscribed into the polyhedron? The symmetry of such a surface would be of immense interest.

In that case, the quantity α_{Γ} equal to the ratio α/α_0 of squares of physical and bare charges would be the inverse value of the vacuum dielectric permittivity. The values α_B and α_L obtained by Boyer and Lukosh could be considered approximate reciprocals of the vacuum dielectric permittivity.

To conclude, we write the analytic expression for $\alpha_0 \alpha_L$ that differs numerically from the experimental α by less than 0.05%:

$$\alpha_0 \alpha_{\rm L} = \frac{1}{4\pi} \frac{\pi}{16} \left[1 - \frac{1}{\pi^3} \sum_{m_1, m_2, m_3 = -\infty}^{\infty} (m_1^2 + m_2^2 + m_3^2)^{-2} \right].$$
(7.16)

The value of the sum appearing here (Epstein's zeta function) is 16.53231596... [51].

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