rule out this possibility. In early reports on CDW dynamics, NDR has also been observed in $NbSe_3$ crystals at low temperatures (45 K) accompanied by unusually large 1/f noise [17]. At even lower temperatures, these samples show switching events near threshold. The measurements [17] are performed with voltage probes spaced at macroscopic distances ($L \gg 100\ \mu m$). For $L > 5\ \mu m$, we have never seen NDR nor switching events in our TaS_3 crystals in the temperature range studied ($T > 90$ K).

We believe that the N(D)R we observe, originates from the response to local CDW deformations. Assuming that there are a few strong pinning centers or line dislocations in our crystal, strong deformations in the strain profile may be present around these regions. Strain leads to a shift of the chemical potential and this shift can be either up or down depending on the direction of sliding. For a semiconducting CDW with the Fermi level in the middle of the gap, a shift of the chemical potential leads to an increase of quasi-particles (either electrons or holes) and consequently to a decrease of the quasi-particle resistance. Such a resistance decrease may then produce regions with NDR as argued previously by Latyshev et al. [18] to explain NDR in their data on partly irradiated macroscopic samples. However, this reasoning cannot be used to explain a negative resistance: the decrease of the quasi-particle resistance cannot change the sign of the resistance. Most likely, nonequilibrium processes between the CDW and the quasi-particles must also play a role in this new phenomenon as well. No theory is available as yet.

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References

1. Grüner G *Density Waves in Solids* (Reading, Mass.: Addison-Wesley, 1994)
2. Peierls R, in *Quantum Theory of Solids* (Oxford: Oxford Univ. Press, 1955) p. 108
3. Mantel O C et al. *Synthetic Metals* **103** 2612 (1999)
4. Gor'kov L P *Pis'ma Zh. Eksp. Teor. Fiz.* **38** 76 (1983) [*JETP Lett.* **38** 87 (1983)]
5. Ong N P, Maki K *Phys. Rev. B* **32** 6582 (1985)
6. Itkis M E, Emerling B M, Brill J W *Phys. Rev. B* **52** R11545 (1995)
7. DiCarlo D et al. *Phys. Rev. Lett.* **70** 845 (1993)
8. Requardt H et al. *Phys. Rev. Lett.* **80** 5631 (1998)
9. Gill J C *Physica B* **143** 49 (1986)
10. Borodin D V, Zaitsev-Zotov S V, Nad' F Ya *Zh. Eksp. Teor. Fiz.* **93** 1394 (1987) [*Sov. Phys. JETP* **66** 793 (1987)]
11. Gill J C *Phys. Rev. Lett.* **70** 331 (1993)
12. Maher M P et al. *Phys. Rev. B* **52** 13850 (1995)
13. Lemay S G et al. *Phys. Rev. B* **57** 12781 (1998)
14. McCarten J et al. *Phys. Rev. B* **46** 4456 (1992)
15. Ramakrishna S et al. *Phys. Rev. Lett.* **68** 2066 (1992); Ramakrishna S *Phys. Rev. B* **48** 5025 (1993)
16. Mantel O C et al. *Phys. Rev. Lett.* **84** 538 (2000)
17. See, e.g., Hall R P, Sherwin M, Zettl A *Phys. Rev. Lett.* **52** 2293 (1984)
18. Latyshev Yu I, Pogotovsky N P, Artemenko S N *Synthetic Metals* **29** F415 (1989)

Mesoscopic physics on graphs

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Abstract. This report is a summary of recent work on the properties of phase coherent diffusive conductors, especially in the geometry of networks — also called graphs — made of quasi-1D diffusive wires. These properties are written as a function of the spectral determinant of the diffusion equation (the product of its eigenvalues). For a network with N nodes, this spectral determinant is related to the determinant of an $N \times N$ matrix which describes the connectivity of the network. I also consider the transmission through networks made of 1D ballistic wires and show how the transmission coefficient can be written in terms of an $N \times N$ matrix very similar to the above one. Finally I present a few considerations on the relation between the magnetism of noninteracting systems and the magnetism of interacting diffusive systems.

1. Return probability

Transport and thermodynamic properties of phase coherent disordered conductors can be described in a simple unified way: all can be related to the classical return probability $P(t)$ for a diffusive particle. I consider diffusive conductors, for which the mean free path l_e is much larger than the distance between electrons: $k_F l_e \gg 1$. The return probability has two components, a purely classical one (diffuson) and an interference term which results from interferences between pairs of time-reversed trajectories (Cooperon). The classical term is solution of the differential equation

$$[-i\omega - D\Delta]P_{cl}(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')\,, \tag{1}$$

and the interference term is solution of the equation [1]:

$$\left[\gamma - i\omega - D\left(\boldsymbol{\nabla} + \frac{2ie\mathbf{A}}{\hbar c}\right)^2\right] P_{int}(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')\,, \tag{2}$$

whose solution has to be taken at $\mathbf{r}' = \mathbf{r}$. D is the diffusion coefficient. The scattering rate $\gamma = 1/\tau_\phi = D/L_\phi^2$ describes the breaking of phase coherence. L_ϕ is the phase coherence length and τ_ϕ is the phase coherence time. Finally, the space integrated (dimensionless) return probability is defined as

$$P(t) = \int P(\mathbf{r}, \mathbf{r}, t)\, d\mathbf{r}\,.$$

The *weak-localization* correction to the conductance can be written as [1]

$$\Delta\sigma = -s\frac{e^2 D}{\pi\hbar\Omega}\int_0^\infty P_{int}(t)\left[\exp(-\gamma t) - \exp\left(\frac{-t}{\tau_e}\right)\right] dt\,, \tag{3}$$

Ω is the volume and s is the spin degeneracy. The contribution of the return probability is integrated between τ_e, the smallest time for diffusion, and the phase coherence time $\tau_\phi = 1/\gamma$. Similarly, *the variance of the conductance fluctuations* at $T = 0$ K is given by

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$$\langle \delta\sigma^2 \rangle = \frac{3}{\beta}\left(s\frac{e^2 D}{\pi\hbar\Omega}\right)^2 \times \int_0^\infty [P_{\rm cl}(t) + P_{\rm int}(t)]\left[\exp(-\gamma t) - \exp\left(\frac{-t}{\tau_{\rm e}}\right)\right] t\,{\rm d}t\,, \tag{4}$$

$\beta = 1$ in the absence of a magnetic field; $\beta = 2$ if the field completely breaks the time reversal symmetry. The interaction contribution to the average magnetization of a disordered electron gas can also be written as a function of field-dependent part of the return probability. For a screened interaction $U(\mathbf{r}-\mathbf{r}') = U\delta(\mathbf{r}-\mathbf{r}')$, one has [2, 3]

$$\langle M_{\rm ee}\rangle = -\frac{\lambda_0\hbar}{\pi}\frac{\partial}{\partial B} \times \int_0^\infty P_{\rm int}(t,B)\left[\exp(-\gamma t) - \exp\left(\frac{-t}{\tau_{\rm e}}\right)\right]\frac{{\rm d}t}{t^2}\,, \tag{5}$$

where $\lambda_0 = U\rho_0$ is the interaction parameter. Considering higher corrections in the Cooper channel leads to a ladder summation [4, 5], so that λ_0 should be replaced by $\lambda(t) = \lambda_0/[1+\lambda_0\ln(\epsilon_{\rm F}t)] = 1/\ln(T_0 t)$ where T_0 is defined as $T_0 = \epsilon_{\rm F}\exp(1/\lambda_0)$.

Finally the typical magnetization $M_{\rm typ}$, defined as $M^2_{\rm typ} = \langle M^2\rangle - \langle M\rangle^2$, is given by (neglecting the interaction) [2, 7]

$$M^2_{\rm typ} = \frac{\hbar^2}{2\pi^2} \times \int_0^{+\infty}[P''_{\rm int}(t,B) - P''_{\rm cl}(t,0)]\left[\exp(-\gamma t) - \exp\left(\frac{-t}{\tau_{\rm e}}\right)\right]\frac{{\rm d}t}{t^3}\,, \tag{6}$$

where $P''(t,B) = \partial^2 P(t,B)/\partial B^2$. In the geometry of a quasi-1D ring, the magnetization M is proportional to the *persistent current* $M = IS$ where S is the area of the ring. Eqns (5,6) can thus be used to get the persistent current of a diffusive ring from the known expression of $P(t)$ is such a geometry [3].

2. Diffusion on graphs

We wish to calculate the above quantities in any structure made of quasi-1D wires. To this end, we reformulate the different results. We first note that the quantities of interest have all the same structure. They are proportional to

$$\int_0^\infty t^\alpha P(t)\exp(-\gamma t)\,{\rm d}t\,, \tag{7}$$

where $P(t) = \sum_n \exp(-E_n t)$ and E_n are the eigenvalues of the diffusion equations (1) or (2). The time integral of $P(t)$ can be straightforwardly written in terms of a quantity called *the spectral determinant* $S_{\rm d}(\gamma)$:

$$\mathcal{P} \equiv \int_0^\infty {\rm d}t P(t) = \sum_n \frac{1}{E_n+\gamma} = \frac{\partial}{\partial\gamma}\ln\mathcal{S}_{\rm d}(\gamma)\,, \tag{8}$$

where $S_{\rm d}(\gamma)$ is, within a multiplicative constant independent of γ,

$$\mathcal{S}_{\rm d}(\gamma) = \prod_n(\gamma + E_n)\,. \tag{9}$$

Using standard properties of Laplace transforms, the above time integrals can be rewritten in terms of the spectral determinant, so that the physical quantities described above read [8]:

$$\Delta\sigma = -s\,\frac{e^2 D}{\pi\hbar\Omega}\frac{\partial}{\partial\gamma}\ln S_{\rm d}(\gamma)\,, \tag{10}$$

$$\langle\delta\sigma^2\rangle = -\frac{3}{\beta}\left(s\frac{{\rm e}^2 D}{\pi\hbar\Omega}\right)^2\frac{\partial^2}{\partial\gamma^2}\ln S_{\rm d}(\gamma)\,, \tag{11}$$

$$M^2_{\rm typ} = \frac{\hbar^2}{2\pi^2}\int_\gamma^{+\infty}{\rm d}\gamma_1(\gamma-\gamma_1)\frac{\partial^2}{\partial B^2}\ln\mathcal{S}_{\rm d}(\gamma_1)\big|_0^B\,, \tag{12}$$

$$\langle M_{\rm ee}\rangle = \frac{\lambda_0\hbar}{\pi}\int_\gamma^{+\infty}{\rm d}\gamma_1\frac{\partial}{\partial B}\ln S_{\rm d}(\gamma_1)\,. \tag{13}$$

These expressions are quite general, strictly equivalent to expressions (3)–(6). On a graph made of quasi-1D diffusive wires, the spectral determinant can be calculated explicitly. By solving the diffusion equation on each link, and then imposing Kirchhoff-type conditions at the nodes of the graph, the problem can be reduced to the solution of a system of N linear equations relating the eigenvalues at the N nodes. Let us introduce the $N\times N$ matrix M [9]:

$$M_{\alpha\alpha} = \sum_\beta\coth\left(\frac{l_{\alpha\beta}}{L_\phi}\right),\quad M_{\alpha\beta} = -\frac{\exp({\rm i}\theta_{\alpha\beta})}{\sinh\left(l_{\alpha\beta}/L_\phi\right)}\,. \tag{14}$$

The sum $\sum_\beta$ extends to all the nodes β connected to the node α; $l_{\alpha\beta}$ is the length of the link between α and β. The off-diagonal coefficient $M_{\alpha\beta}$ is non zero only if there is a link connecting the nodes α and β; $\theta_{\alpha\beta} = (4\pi/\phi_0)\int_\alpha^\beta \mathbf{A}\cdot{\rm d}\mathbf{l}$ is the circulation of the vector potential between α and β; $N_{\rm B}$ is the number of links in the graph. It can then be shown that for an isolated graph the spectral determinant $\mathcal{S}_{\rm d}$ is given by [8]

$$\mathcal{S}_{\rm d} = \left(\frac{L_\phi}{L_0}\right)^{N_{\rm B}-N}\prod_{(\alpha\beta)}\sinh\left(\frac{l_{\alpha\beta}}{L_\phi}\right)\det M\,. \tag{15}$$

L_0 is an arbitrary length independent of L_ϕ. We have thus transformed the spectral determinant which is an infinite product into a finite product related to $\det M$. The properties of this spectral determinant have been studied in details in Ref. [10], in particular, the role of the boundary conditions has been considered and a trace expansion of the spectral determinant has been obtained. As a physical important result, it has been found that when isolated rings are connected into an array of rings, the average magnetization per ring is reduced by a factor r given by [8]

$$r = \prod_i\left(\frac{2}{z_i}\right)\,, \tag{16}$$

where z_i are the coordinates of the nodes belonging to each ring.

3. Transmission through ballistic quantum graphs

We now consider a graph made of 1D ballistic wires. There are N nodes connected to $N_{\rm in}$ input channels and to $N_{\rm out}$ output channels. The total transmission coefficient T from the left to the right reservoirs is given by

$$T = \sum_{i,j}|t_{ij}|^2\,, \tag{17}$$

where $i \in [1, N_{\rm in}]$ denote the ith input channel and $j \in [N - N_{\rm out} + 1, N]$ denote the jth output channel. The transmission coefficient (17) is the sum of each individual transmission coefficient obtained by injecting a wave packet in the ith channel. It is assumed that there is no phase relationship between electrons in the different channels. Solving Schrödinger equation for the wave function ψ on each bond of the network, and writing current conservation at the nodes, one gets

$$M_{\alpha\alpha}\psi_\alpha + \sum_\beta M_{\alpha\beta}\psi_\beta = 0\,, \tag{18}$$

where $\sum_\beta$ extends to the nodes β connected to the node α. Current conservation at the input node i writes

$$M_{ii}\psi_i + \sum_\beta M_{i\beta}\psi_\beta = i(1 - r_{ii})\,, \tag{19}$$

and for an output node j, one has

$$M_{jj}\psi_j + \sum_\beta M_{j\beta}\psi_\beta = -{\rm i}t_{ij}\,, \tag{20}$$

with $\psi_i = 1 + r_{ii}$ and $\psi_j = t_{ij}$; M is an $N \times N$ matrix whose elements are

$$M_{\alpha\alpha} = \sum_\beta \cot(kl_{\alpha\beta})\,, \quad M_{\alpha\beta} = -\frac{\exp {\rm i}\theta_{\alpha\beta}}{\sin kl_{\alpha\beta}}\,. \tag{21}$$

Here k is the wave vector of the incident electron; $\theta_{\alpha\beta} = (2\pi/\phi_0)\int_\alpha^\beta \mathbf{A}\cdot{\rm d}\mathbf{l}$ is the circulation of the vector potential between α and β. The equations (18)–(20) constitute a linear system of N equations from which the t_{ij} can be calculated. The total transmission coefficient $T(k)$ is finally obtained from Eqn (17) by considering all the input channels. This formalism has been used recently to calculate the transmission coefficient of regular networks, in particular of the so-called $\mathcal{T}_3$ network which exhibits the Aharonov–Bohm cage effect, i.e. the absence of transmission for half flux quantum per plaquette [11].

4. Landau diamagnetism and magnetization of an interacting electron gas

Finally, we wish to emphasize an interesting correspondence between the magnetization of a phase coherent *interacting diffusive* system and the grand canonical magnetization M_0 of the corresponding *noninteracting clean* system. The latter can also be written in term of a spectral determinant. The grand canonical magnetization M_0 is given quite generally by

$$M_0 = -\frac{\partial\Omega}{\partial B} = \frac{\partial}{\partial B}\int_{-\epsilon_{\rm F}}^0 {\rm d}\epsilon N(\epsilon)\,, \tag{22}$$

where the integrated DOS can be rewritten as

$$N(\epsilon) = -\frac{1}{\pi}{\rm Im}\sum_{\epsilon_\mu}\ln(\epsilon_\mu - \epsilon_+) = -\frac{1}{\pi}{\rm Im}\ln\mathcal{S}(\epsilon_+)\,. \tag{23}$$

Here $\epsilon_+ = \epsilon + {\rm i}0$, $\mathcal{S}(\epsilon) = \prod_{\epsilon_\mu}(\epsilon_\mu - \epsilon) \propto \mathcal{S}_{\rm d}(\gamma = -\epsilon/\hbar)$, where ϵ_μ are the eigenvalues of the Schrödinger equation. For a clean system these eigenvalues are the same as those of the diffusion equation, with the substitutions $D \to \hbar/(2m)$ and $2e \to e$. Comparing Eqns (22), (23) with Eqn (13), we can now formally relate M_0 and the HF magnetization $\langle M_{\rm ee}\rangle$ of the same diffusive system:

$$M_0 \mathrel{\hat{=}} -\lim_{\lambda_0\to 0}\frac{1}{\lambda_0}{\rm Im}\Big[\langle M_{\rm ee}\rangle\Big(\gamma = -\frac{\epsilon_{\rm F}}{\hbar} - {\rm i}0\Big)\Big]\,, \tag{24}$$

where the sign $\hat{=}$ means that the two quantities are equal, provided the substitutions $D \to \hbar/(2m)$ and $2e \to e$ are made. This relation can be inverted to obtain

$$\langle M_{\rm ee}\rangle \mathrel{\hat{=}} -\frac{\lambda_0}{\pi}\int_0^\infty \frac{M_0(\epsilon)}{\epsilon + \hbar\gamma}\,{\rm d}\epsilon\,. \tag{25}$$

Taking into account the renormalization of the interaction parameter in the Cooper channel,

$$\lambda_0 \to \lambda(\epsilon) = \frac{\lambda_0}{1 + \lambda_0\ln(\epsilon_{\rm F}/\epsilon)} = \frac{1}{\ln(T_0/\epsilon)}\,, \tag{26}$$

where T_0 is defined as $T_0 = \epsilon_{\rm F}\exp(1/\lambda_0)$, the energy dependence of the parameter $\lambda(\epsilon)$ can be incorporated exactly in the integral so that:

$$\langle M_{\rm ee}\rangle \mathrel{\hat{=}} -\frac{1}{\pi}\int_0^\infty \lambda(\epsilon)\,\frac{M_0(\epsilon)}{\epsilon + \hbar\gamma}\,{\rm d}\epsilon\,. \tag{27}$$

As a simple example, consider the Landau spinless susceptibility in 2D, given by $\chi_0 = -e^2/24\pi m = \chi_L/2$. From Eqn (27), one gets the phase coherent contribution of the electron–electron interaction to the susceptibility [13]:

$$\chi_{\rm ee} \simeq 4|\chi_L|\frac{\epsilon_{\rm F}\tau_{\rm e}}{\hbar}\ln\frac{\ln(T_0\tau_\varphi/\hbar)}{\ln(T_0\tau_e/\hbar)}\,, \tag{28}$$

where χ_L is the Landau susceptibility. An ultraviolet cutoff $1/\tau_{\rm e}$ has been added to cure the divergence at large energy.

5. Conclusions

In conclusion, we have shown how to relate phase coherent transport and thermodynamic properties to the return probability for a diffusive particle. It is then possible to calculate straightforwardly these quantities by simple integrals of this return probability in simple geometries. For networks made of diffusive wires, we have developed a formalism which relates *directly* the persistent current, and the transport properties to the determinant of a matrix M describing the connectivity of the graph. From a loop expansion of this determinant, simple predictions for the persistent current in any geometry can now be compared with forthcoming experiments on connected and disconnected rings. Then we have shown that the transmission coefficient of a network of one-dimensional ballistic wires can be written in terms of a connectivity matrix very similar to M with the substitution $kL \to {\rm i}L/L_\phi$. Finally, we have found a correspondence between the phase coherent contribution to the orbital magnetism of a disordered interacting system and the orbital response of the corresponding clean noninteracting system.

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References

1. Khmel'nitskiĭ D E *Physica B* **126** 235 (1984); Bergmann G *Phys. Rep.* **107** 1 (1984); Chakravarty S, Schmid A *Phys. Rep.* **140** 193 (1986)
2. Argaman N, Imry Y *Phys. Scripta Vol. T* **T49A** 333 (1993)
3. Montambaux G *J. Phys. I* (Paris) **6** 1 (1996)
4. Eckern U *Z. Phys. B* **82** 393 (1991)
5. Ullmo D et al. *Physica E* **1** 268 (1997)
6. Argaman N, Imry Y, Smilansky U *Phys. Rev. B* **47** 4440 (1993)
7. Montambaux G, in *Quantum Fluctuations. Proc. Les Houches Summer School, Session LXIII* (Ed. E Giacobino et al.) (Amsterdam: Elsevier, 1996)
8. Pascaud M, Montambaux G *Phys. Rev. Lett.* **82** 4512 (1999)
9. Alexander S *Phys. Rev. B* **27** 1541 (1983); Douçot B, Rammal R *J. Phys.* (Paris) **47** 973 (1986)
10. Akkermans E et al. *Ann. Phys.* (New York) **284** 10 (2000)
11. Vidal J, Montambaux G, Douçot B *Phys. Rev. B* **62** R16294 (2000)
12. Montambaux G *Phys. Rev. Lett.* **86** 4640 (2001)
13. Al'tshuler B L, Aronov A G, Zyuzin A Yu *Zh. Eksp. Teor. Fiz.* **84** 1525 (1983) [*Sov. Phys. JETP* **57** 889 (1983)]