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ON THE INFLUENCE OF THE TERRESTRIAL MAGNETIC FIELD ON THE REFLECTION OF RADIO WAVES FROM THE IONOSPHERE *

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The question of the influence of the terrestrial magnetic field on the reflection of radio waves and signals from an inhomogeneous ionized layer (Heaviside layer) is considered. In particular the propagation of waves at a small angle to the direction of the magnetic field is investigated, and it is shown that in this case a very peculiar splitting of the reflected signal into three pulses, and not into two, as observed in other cases, must take place.

Introduction

As is well known, the magnetic field of the Earth causes a double refraction of radio waves, which are propagated in the ionosphere. If therefore a radio signal is sent upwards from the Earth, then after reflection from one of the Heaviside layers, generally speaking, two signals are returned. A qualitative understanding of the corresponding phenomena can be obtained very easily from a consideration of the influence of the magnetic field on an homogeneous medium, containing free electrons. An investigation of this problem has been carried out by Appleton, Lassen and others**.

As far as I am aware, only in the work of Försterling and Lassen (2) is found an attempt to account in detail for the fact that in reality the wave or the signal are propagated in an inhomogeneous magnetized medium. Such a consideration is, of course, absolutely necessary, since the total reflection of the signal from the ionized layer is itself due to the arrival of this signal at the region of the layer where the refraction index vanishes (we are speaking here of vertical incidence). The above mentioned authors (2) attempted to describe the propagation of waves in an inhomogeneous medium by replacing the latter by a large number of homogeneous layers and by considering the reflection and refraction at the boundaries of these auxiliary layers. There is, however, no reason for using such a method, since, as is shown below (§ 2), the corresponding results can easily be obtained from general considerations.

The main problem of the present paper consists, however, in the investigation of peculiarities arising, when a radio wave, which is propagated upwards vertically, makes a small angle with the direction of the magnetic field, which takes place in the vicinity of the magnetic poles of the Earth. In this case, which, so far as I know, has not been especially noted by anybody, a very interesting and peculiar partial reflection of the radio waves from a certain region of the Heaviside layer ought to be observed: as a result the reflected radio signal consists, under certain conditions, not of two but of three pulses. This phenomenon can be understood already on the example of an homogeneous medium, which is discussed in § 1. It is considered in more detail and more precisely in § 3, where formulae are obtained, allowing to estimate the conditions, under which this partial reflection of the wave takes place. In § 4 the question of the reflection from the ionosphere not of a monochromatic wave, but

* Translated by S. Frenkel.

** The bibliography of the question can be found in the review of Mimno (1).

of a radio signal, *i. e.* of a quasi-monochromatic wave group is briefly discussed. The latter question is touched upon in other paragraphs of this paper also.

1. Fundamental relations. Propagation in a homogeneous medium

The propagation of electromagnetic waves in an anisotropic medium, in particular in a medium placed in a magnetic field, is described phenomenologically by the following equations:

$$\left.\begin{aligned} &\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \qquad \operatorname{div}\mathbf{D}=0,\\ &\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{H}=0,\\ &D_i=\sum_{k=1}^{3}\varepsilon_{ik}E_k, \end{aligned}\right\} \quad (1)$$

where ε_{ik} is the dielectric constant tensor, no macroscopic charges being present and the magnetic permeability of the medium μ being equal to unity; all the vectors in (1) refer to the wave field. In the case of an inhomogeneous medium the same equations hold, the components ε_{ik} being, however, functions of the coordinates.

By applying to these equations (1), which contain time derivatives, the operation rot the following equations are easily obtained:

$$\left.\begin{aligned} &\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\Delta\mathbf{E}-\operatorname{grad}\operatorname{div}\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\\ &=\Delta\mathbf{E}+4\pi\operatorname{grad}\operatorname{div}\mathbf{P}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t}=0,\\ &\Delta\mathbf{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}+\frac{4\pi}{c}\frac{\partial}{\partial t}\operatorname{rot}\mathbf{P}=0, \end{aligned}\right\} \quad (2)$$

where $\mathbf{P}$ denotes the electric polarization, determined by the relations (α_{ik} is the polarizability tensor and δ_{ik} — the unit tensor):

$$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}, \qquad P_i=\sum_{k=1}^{3}\alpha_{ik}E_k, \qquad \varepsilon_{ik}=\delta_{ik}+4\pi\alpha_{ik}. \quad (3)$$

In the presence of dispersion ε_{ik} and α_{ik} depend on the angular frequency of the wave ω. The last of equations (1) refers therefore to the case of monochromatic solutions only, which we shall assume to be proportional to the factor $e^{i\omega t}$. In this case the operation $\partial/\partial t$ in (1) and (2) is replaced by a multiplication by $i\omega$ and, for instance, the first of equations (2) assumes the following form (the factor $e^{i\omega t}$ is omitted):

$$\operatorname{rot}\operatorname{rot}\mathbf{E}-\frac{\omega^2}{c^2}\mathbf{D}= \\ =\Delta\mathbf{E}-\operatorname{grad}\operatorname{div}\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{D}=0. \quad (4)$$

In the case we are interested in the anisotropic medium represents an electronic gas, placed in a magnetic field. If the field is absent this gas is isotropic and its dielectric constant is equal to

$$\varepsilon_0=n_0^2=1-\frac{4\pi e^2N}{m\omega^2}, \quad (5)$$

where e and m are the charge and the mass of an electron, respectively, and N — the concentration of the electrons, *i. e.* their number in 1 cm³. We shall assume that in the ionosphere N varies with the height only, *i. e.* depends on the coordinate x, which is directed vertically upwards from the surface of the Earth.

In order to find the tensor ε_{ik} in the presence of the field, we shall consider the motion of an electron in an external magnetic field, which is constant with respect to time and homogeneous in space. The equation of the motion of the electron in this field and simultaneously in the field of the wave has the following form:

$$m\ddot{\mathbf{r}}=-e\mathbf{E}-\frac{e}{c}[\dot{\mathbf{r}}\mathbf{H}^t] \quad (6)$$

where $\mathbf{E}$ denotes the electric field of the wave and $-e$ — the charge of the electron ($e>0$). Since the electric polarization may be in our case put equal to $\mathbf{P}=-eN\mathbf{r}$ we at once obtain an equation for $\mathbf{P}$, which will be written down for a harmonic process:

$$-\omega^2\mathbf{P}=\frac{e^2N}{m}\mathbf{E}-\frac{i\omega e}{mc}[\mathbf{P}\mathbf{H}^t]. \quad (7)$$

Hence if the direction of the field $\mathbf{H}^t$ is taken for the z axis, we get

$$\left.\begin{aligned} D_z &= E_z + 4\pi P_z = \left(1 - \frac{\omega_0^2}{\omega^2}\right) E_z, \\ D_x + iD_y &= \left(1 - \frac{\omega_0^2}{\omega(\omega + \omega_H)}\right)(E_x + iE_y), \\ D_x - iD_y &= \left(1 - \frac{\omega_0^2}{\omega(\omega - \omega_H)}\right) E_x - iE_y, \end{aligned}\right\} \quad (8)$$

where

$$\omega_0^2 = \frac{4\pi e^2 N}{m}, \quad \omega_H = \frac{e\mathbf{H}^t}{mc}. \quad (9)$$

The explicit expressions for α_{ik} and ε_{ik} can be obtained from (8) directly and we shall not write them down here. Equation (1) [or (2)] and (8) give a complete description of the propagation of radio waves in an inhomogeneous, magnetized medium. Their solution for an inhomogeneous medium is, however, generally speaking, extremely complicated (see § 2).

If the medium is homogeneous, *i. e.* if $N = \text{const}$, then plane harmonic waves can be propagated in it. A solution is obtained in this case, by assuming the field vectors to be proportional to $e^{\pm i\mathbf{kr}}$. As a result equation (4) is reduced to the form:

$$[\mathbf{k}[\mathbf{kE}]] + \frac{\omega^2}{c^2}\mathbf{D} = 0. \quad (10)$$

Thus we get for the determination of the field, taking into account (8), three homogeneous linear equations for the components $E_x, E_y; E_z$. The condition of the existence of a non-trivial solution of these equations determines the dependence of the magnitude of the wave vector $\mathbf{k}$ on its orientation with respect to the field $\mathbf{H}^t$ and on the parameters ω, ω_0 and ω_H. The problem stated can, however, be solved much easily with a different choice of the axes. Namely we choose for the x axis the direction of the vector $\mathbf{k}$ and direct the y axis normally both to the wave vector and to the direction of the field $\mathbf{H}^t$. The components of the field $\mathbf{H}^t$ along the axes x and z will be denoted by H_L^t and H_T^t respectively, where $H_L^t = H^t\cos\alpha$ and $H_T^t = H^t\sin\alpha$, α being the angle between $\mathbf{k}$ and $\mathbf{H}^t$. The advantages of such a choice of axes are preserved in the case of an inhomogeneous medium also, the properties of which depend on the cordinate x only; in this case the propagation of plane waves, with the field vectors depending on the coordinates x only, is also possible; equations (4) assume in this case the form:

$$\left.\begin{aligned} \frac{d^2E_y}{dx^2} + \frac{\omega^2}{c^2} D_y &= 0, \\ \frac{d^2E_z}{dx^2} + \frac{\omega^2}{c^2} D_z &= 0, \\ D_x = E_x + 4\pi P_x &= 0. \end{aligned}\right\} \quad (11)$$

Using (7) we obtain further the relation between $\mathbf{D}$ and $\mathbf{E}$:

$$\left.\begin{aligned} D_y &= E_y + 4\pi P_y = \left\{1 + \frac{\omega_0^2(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)(\omega_L^2 - \omega^2) + \omega^2\omega_T^2}\right\} E_y + \frac{i\omega_L - \omega_0^2(\omega^2\omega_0^2)}{\omega[(\omega^2 - \omega_0^2)(\omega_L^2 - \omega^2) + \omega^2\omega_T^2]} E_z, \\ D_z &= E_z + 4\pi P_z = \frac{-i\omega_L\omega_0^2(\omega^2 - \omega_0^2)}{\omega[(\omega^2 - \omega_0^2)(\omega_L^2 - \omega^2) + \omega^2\omega_T^2]} E_y + \left\{1 + \frac{\omega_0^2(\omega^2 - \omega_0^2 - \omega_T^2)}{(\omega^2 - \omega_0^2)(\omega_L^2 - \omega^2) + \omega^2\omega_T^2}\right\} E_z, \\ P_x &= i\frac{\omega\omega_T}{\omega^2 - \omega_0^2} P_y, \quad E_x = -4\pi P_x, \end{aligned}\right\} \quad (12)$$

where

$$\omega_L = \frac{eH_L^t}{mc} = \omega_H\cos\alpha, \quad \omega_T = \frac{eH_T^t}{mc} = \omega_H\sin\alpha, \quad \omega_L^2 + \omega_T^2 = \omega_H^2. \quad (13)$$

Substituting (12) into (11) we obtain finally the following system of equations, which describe the propagation of the wave in the case under consideration*

* This case is especially important for the investigation of the ionosphere, since it corresponds to the reflection of radio signals, which are sent vertically upwards.

$$\left.\begin{aligned}\frac{d^2E_y}{dx^2}+\frac{\omega^2}{c^2}(AE_y+iCE_z)=0,\\ \frac{d^2E_z}{dx^2}+\frac{\omega^2}{c^2}(-iCE_y+BE_z)=0,\end{aligned}\right\} \quad (14)$$

where

$$\left.\begin{aligned}A&=\frac{u-(1-v)^2-uv\cos^2\alpha}{u-(1-v)-uv\cos^2\alpha},\\ B&=\frac{u(1-v)-(1-v)^2}{u(1-v)-uv\cos^2\alpha},\\ C&=\frac{\sqrt{u}\cos\alpha\cdot v(1-v)}{u-(1-v)-uv\cos^2\alpha}\end{aligned}\right\} \quad (15)$$

and

$$u=\frac{\omega_H^2}{\omega^2}, \quad \sqrt{u}=+\frac{\omega_H}{\omega};$$
$$v=\frac{\omega_0^2}{\omega^2}=\frac{4\pi e^2N(x)}{m\omega^2}. \quad (16)$$

In the case of an inhomogeneous medium, the coefficients A, B and C depend on x, through $v(x)$. If the medium is homogeneous, the equations (14) may be solved without difficulty by putting $E_{y,z}=E^0_{y,z}e^{\pm i\frac{\omega}{c}nx}$ where n is the refractive index. The conditions of the existence of a non-trivial solution of the system of algebraical equations obtained in this way lead to a quadratic equation for the determination of n^2; in an implicit form this equation is

$$\begin{vmatrix} A-n^2 & +iC \\ -iC & B-n^2\end{vmatrix}=0,$$

its solution being given by*

$$n^2_{1,2}=\frac{-2(1-v)^2+2u-vu(1+\cos^2\alpha)\mp v\sqrt{u^2\sin^4\alpha+4u(1-v)^2\cos^2\alpha}}{2[u-(1-v)-vu\cos^2\alpha]}=$$
$$=1-\frac{2v(1-v)}{2(1-v)-u\sin^2\alpha\pm\sqrt{u^2\sin^4\alpha+4u(1-v)^2\cos^2\alpha}}. \quad (17)$$

If in (17) the upper sign is taken, then the square of the refractive index n_2^2 of the «ordinary» wave is obtained; the lower sign corresponds to the «extraordinary» wave $(n^2=n_1^2)$. The refractive index is assumed to be equal to $n_{1,2}=+\sqrt{n^2_{1,2}}$. The solutions $n_{1,2}=-\sqrt{n^2_{1,2}}$ correspond to waves, propagated in the opposite direction, which will be taken into account below in the expression for the phase of the wave. If the magnetic field $\mathbf{H}^t=0$, $u=0$, and, consequently,

$$n^2_{1,2}=n_0^2=1-v \quad (5')$$

as it ought to be [see (5), (16)].

The refractive index n_2, as well as n_0, vanishes when

$$v=v_{20}=1. \quad (18)$$

The refractive index n_1 vanishes, when*

$$v=v^{\pm}_{10}=1\pm\sqrt{u}=1\pm\frac{\omega_H}{\omega}, \quad (19)$$

and becomes infinite, when

$$v=v_{1\infty}\frac{1-u}{1-u\cos^2\alpha}. \quad (20)$$

The character of the vibrations in both waves, propagated in an homogeneous medium, is determined from (14):

$$\frac{E_y^{(1,2)}}{E_z^{(1,2)}}=\frac{-iC}{A-n^2_{1,2}}=\frac{B-n^2_{1,2}}{iC}=$$
$$=+i\frac{u\sin^2\alpha\mp\sqrt{u^2\sin^4\alpha+4u(1-v)^2\cos^2\alpha}}{2\sqrt{u}(1-v)\cos\alpha}, \quad (21)$$

where, as before, the upper sign of the radical refers to the wave 2 $(E_y^{(2)}, E_z^{(2)}, n_2^{(2)})$ and the lower sign—to the wave 1; we obviously have $E_y^{(1)}/E_z^{(1)}=E_z^{(2)}/E_y^{(2)}$. The polarization of both waves is, in general, an elliptical one, the axes of the ellipses being parallel to the axes y and z.

In order to get an idea of the propagation of waves in an inhomogeneous medium the dependence of $n^2_{1,2}$ on $N(x)$ is usually considered(1,2). If the function $N(x)$ is monotonic and, in particular, linear with respect to x, the corresponding graph of the dependence of $n^2_{1,2}$ on N describes simultaneously the dependence of the refractive index of both waves on the height. If $n^2_{1,2}$ varies with changing

* We shall limit ourselves below to the most interesting case only, when $u<1$, and assume that $\alpha\neq 0$ (see below).

* The expressions for n^2 could also be obtained with a smaller number of transformations (1). But we prefered to give the above derivation having in view to use in the sequel the expressions (14), (15)

of x slowly enough, then, generally speaking, quasi-stationary behaviour must take place, *i. e.* in the vicinity of a given point the field must be nearly the same, as in the case of an homogeneous medium with the corresponding values of $n^2_{1,2}$. A foundation of this intuitively clear assertion can be reached by way of a transition to the approximation of geometrical optics (see § 2). Under the condition that this approximation is valid, a knowledge of $n^2_{1,2}(x)$ is actually sufficient for a qualitative description of the propagation of waves in an inhomogeneous medium. In particular, the total inner reflection of the waves must take place in the vicinity of the points where $n^2_{1,2}=0$ (the reflection from the region in the neighbourhood of the point $n_1 \to \infty$ will not be considered for the present).

Besides $N(x)$ or the dimensionless parameter $v=\frac{4\pi e^2 N(x)}{m\omega^2}$, the value of $n^2_{1,2}$ depends on u and α. In particular cases, when $\alpha=0$ (propagation along the field) and $\alpha=\pi/2$ (propagation in a direction perpendicular to the field) the expression for $n^2_{1,2}$ is greatly simplified:

$$\alpha=0: \quad n^2_{1,2}=1-\frac{4\pi e^2 N}{m\omega(\omega\pm\omega_H)}=1-\frac{v}{1\pm\sqrt{u}}, \tag{22}$$

$$\alpha=\pi/2: \quad n^2_{1,2}=1-\frac{v(1-v)}{1-u-v}, \quad n^2_2=n^2_0=1-v. \tag{23}$$

When $\alpha=0$ both waves are circularly polarized; if $\alpha=\pi/2$ they are linearly polarized, the electric vector of the wave 2 being, parallel to the magnetic field, *i. e.* to the z axis and $E_y=E_x=0$; for the wave 1 in this case $E_z=0$, the components E_x and E_y being different from zero*. It is important to note that for any angle $\alpha=0$ between the wave vector and the magnetic field n^2_1 vanishes at two points (19) and n^2_2 at one point (18), the positions of these points being wholly independent of α. If, however, $\alpha=0$, then, as can easily be seen from (22) or directly from (17), both n^2_1 and n^2_2 vanish at one point only, the position of the root of the equation $n^2_2=0$ jumping, in the case $\alpha\to 0$, from $v=1$ to $v=1+\sqrt{u}$. The dependence of $n^2_{1,2}$ on v, and the «jumping» of the roots of n^2_2 is illustrated in Fig. 1. For the sake of definiteness it was here assumed that $u=0.25$,

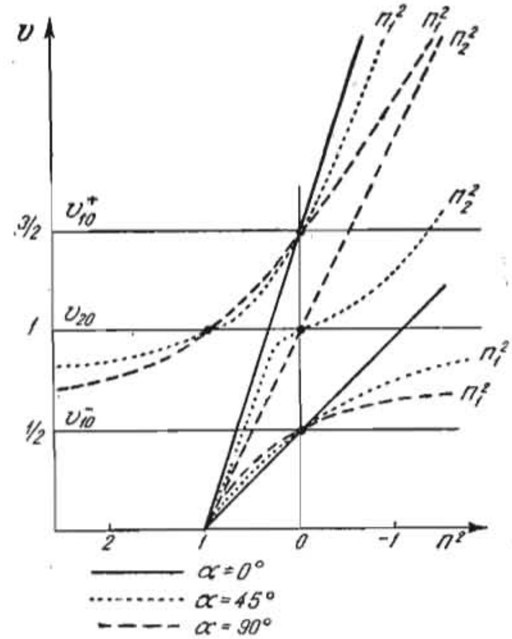

Fig. 1

i. e. that $\omega=2\omega_H$*. Thus for very small angles α a peculiarity takes place, which becomes clear from a consideration of the curves $n^2(v)$ for small values of α (Fig. 2). We see that, formally speaking, there exists no continuous transition to the case when $\alpha=0$. It is clear, however, from the physical point of view that such a transition must exist and that the following phenomenon must be observed: when $v<1$ the wave 1 («extraordinary» wave) is propagated in all cases to the point v_{10} only where it suffers a total reflection (Fig. 1). So long as α is large enough, the wave 2 («ordinary»

* The opposite assertion (that $E_x=0$) leads also to a wrong expression for n^2_1 (3).

* Since $H' \approx 0.5$ gauss, $\omega_H=\frac{eH}{mc}\approx\frac{e}{2mc}\approx 8.82\cdot 10^6$ and

$$\lambda_H=\frac{2\pi c}{\omega_H}\approx 214 \text{ m}.$$

wave) is completely reflected at point v_{20} (more exactly in a region near this point). With decrease of v the properties of the wave 2 in the region $v < 1$ become more and more like those of the wave of the type 1, which can be propagated when $v > 1$. In fact, when α is small and the value of v is but slightly less than unity, n_2^2 approaches the value of n_1^2 corresponding to a value of v slightly exceeding unity; further the character of the oscillations of the waves 2 and 1 in the points $v_{\rm I}$ and $v_{\rm II}$ (Fig. 2) becomes

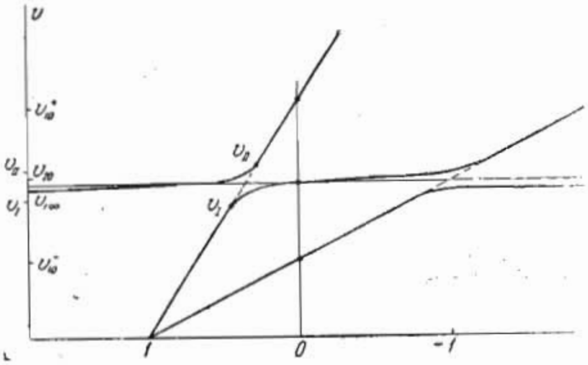

Fig. 2. The full curve for $\alpha \cong 10°$, the dotted—for $\alpha = 0°$

more and more similar; this is connected with the fact that, as follows from (21), the ratio $E_y^{(1,2)} / E_z^{(1,2)}$ changes its sign at the transition of v through unity. In virtue of this

$$\left(\frac{E_y^{(2)}}{E_z^{(2)}}\right)_{v_{\rm I}} \approx \left(\frac{E_y^{(1)}}{E_z^{(1)}}\right)_{v_{\rm II}}.$$

Hence it is clear that it suffices for the wave 2 to penetrate to the point $v_{\rm II}$, what from the point of view of the wave theory always takes place to a certain extent, to ensure its propagation upwards to the point v_{10}^{+}, where it will be finally reflected (it is assumed here of course that Fig. 1 and 2 describe the dependence of n^2 on the height, *i. e.* for instance $v = ax$). In the vicinity of the point v_{20} a partial reflection of the wave 2 and its partial penetration into the region with $v > 1$ must take place. The reflection coefficient depends very sharply (exponentially) on α, as is usually the case in problems of such a type. A partial reflection (see § 3) therefore takes place practically in the case only if α becomes smaller than a certain angle α_0, which has to be calculated. If $\alpha > \alpha_0$ the reflection will be complete, and when $\alpha = 0$ it must vanish. All these considerations remain, generally speaking, valid for a radio signal, which is propagated upwards, *i. e.* for a quasi-monochromatic wave group. In the case of sufficiently small angles between the direction of the terrestrial magnetic field and the direction of the normal to the wave front of the signal (*i. e.* the direction of the vertical), the latter* must be reflected from the Heaviside layer not at two points (v_{10} and v_{20}), in the case $\alpha > \alpha_0$, but at three (v_{10}^{-}, v_{20}, and v_{10}^{+}); the intensity of the third reflection must increase with decrease of α at the cost of the intensity of the refection from the point v_{20}, until for $\alpha = 0$ the latter completely vanishes. The difference of the retardations of the two signals will be in this case considerably larger than when $\alpha > \alpha_0$.

2. Propagation in an inhomogeneous medium

The propagation of electromagnetic waves in an inhomogeneous ionized medium in the presence of an external magnetic field is described, as has been mentioned above, by equations (3) and (7) or (12). In the case we are interested in, one can assume that the properties of the medium (*i. e.* N, n or v) depend on the x coordinate only. This circumstance leads, however, to an essential simplification of the equations only in the case of propagation of plane waves with a normal parallel to the x axis; in this case the differential equations have a rather simple form (14); the solution of even these equations is, however, not at all easy, even for functions $v(x)$ of the simplest form. In the case of oblique incidence of the waves on the Heaviside layer a refraction takes place, whence it follows, that in this case the waves cannot be plane. It is thus clear that it is very important to ascertain the applicability to equations

* The effect of «trebling» of the signal must take place only in the case when the signal is propagated vertically upwards since in the opposite case its reflection takes place when $n_{1,2}^2 < 0$ and the anomalous region in the neighbourhood of the point $v \doteq 1$ is never reached. The angle α may therefore be small and the effect of the «trebling» can be observed in the vicinity of the magnetic poles of the Earth only.

(2) and (14) of approximate methods of solution and, in the first place, of the method of geometrical optics. The latter consists, as is well known, in the expansion of the solution into a series of powers of a small parameter $\frac{c}{\omega}=\frac{\lambda}{2\pi}$; more exactly the solution of equation (4), where **D** is expressed through **E**, will be found accordingly in the following form:

$$\mathbf{E}=\left(\mathbf{E}^{(0)}_{(\mathbf{r})}+\frac{c}{\omega}\mathbf{E}^{(1)}_{(\mathbf{r})}+\frac{c^2}{\omega^2}\mathbf{E}^{(2)}_{(\mathbf{r})}+\dots\right)e^{-i\frac{\omega}{c}\Psi(\mathbf{r})}. \quad (24)$$

Substituting (24) into (4) and equalling to zero the factors of the different powers of c/ω we get

$$[\nabla\Psi[\nabla\Psi\mathbf{E}^{(0)}]]+\mathbf{D}^{(0)}=0, \quad (25,0)$$

$$[\nabla\Psi[\nabla\Psi\mathbf{E}^{(1)}]]+\mathbf{D}^{(1)}= = -i\{\mathrm{rot}[\nabla\Psi\mathbf{E}^{(0)}]+[\nabla\Psi\,\mathrm{rot}\,\mathbf{E}^{(0)}]\}, \quad (25,1)$$

and so on.

The equations of the zero approximation (25,0) coincide with (10) if **k** is replaced by $\frac{\omega}{c}\nabla\Psi$. The condition of the existence of a non-trivial solution of these equations gives an equation for the determination of Ψ, *i. e.* the so-called equation of the eikonal. The condition of the existence of a solution of the system (25,1) leads to an equation determining $\mathbf{E}^{(0)}$, and so on. If $\mathbf{H}^t=0$, *i. e.* for an isotropic medium a degeneracy takes place: in this case the eikonal equation has multiple solutions in agreement with the fact that waves with different polarizations have the same propagation velocity. In this case the solution of the zero approximation must be described as a linear combination of two different particular solutions; an investigation of this question has been carried out by Rytov([4]).

In so far as we are interested in the propagation of a wave directed vertically upwards (see § 1), the further discussion of the approximation of geometrical optics will be carried out for this case; the results obtained in this way will remain valid for a more general problem.

The substitution of (24) in (14) leads to the equations:

$$\{A-(\Psi')^2\}E_y^{(0)}+iCE_z^{(0)}=0, \qquad -iCE_y^{(0)}+\{B-(\Psi')^2\}E_z^{(0)}=0; \quad (26,0)$$

$$\begin{aligned}&\{A-(\Psi')^2\}E_y^{(1)}+iCE_z^{(1)}=i\{\Psi''E_y^{(0)}+2\Psi'E_y'^{(0)}\},\\ &-iCE_y^{(1)}+\{B-(\Psi')^2\}E_z^{(1)}=i\{\Psi''E_z^{(0)}+2\Psi'E_z'^{(0)}\},\end{aligned} \quad (26,1)$$

$$\begin{aligned}&\{A-(\Psi')^2\}E_y^{(2)}+iCE_z^{(2)}=i\{\Psi''E_y^{(1)}+2\Psi'E_y'^{(1)}\}-E_y''^{(0)},\\ &-iCE_z^{(2)}+\{B-(\Psi')^2\}E_z^{(2)}=i\{\Psi''E_z^{(1)}+2\Psi'E_z'^{(1)}\}-E_z''^{(0)},\end{aligned} \quad (26,2)$$

etc., where $\Psi'=\frac{d\Psi}{dx}$, $\Psi''=\frac{d^2\Psi}{dx^2}$, and so on.

As is clear from § 1, the equation for $(\Psi')^2$ obtained from (26,0) coincides with the equation for n^2, which is used in the case of an inhomogeneous medium. It follows therefore that there exist two solutions for $(\Psi')^2$, which are equal to

$$(\Psi')^2_{1,2}=n^2_{1,2} \quad (27)$$

where $n^2_{1,2}$ are determined according to (17). We have, further

$$\Psi_{1,2}=\int_a^x n_{1,2}\,dx. \quad (28)$$

Solutions, which differ from each other with respect to sign, will not be distinguished here, since the solution $\Psi_{1,2}= =+\int_a^x n_{1,2}\,dx$ corresponds to propagation in the direction of the positive x axis (according to the condition of § 1, $n_{1,2}=+\sqrt{n^2_{1,2}}$) while the solution $\Psi_{1,2}=-\int_a^x n_{1,2}\,dx$ corresponds to a propagation in the opposite direction x. We speak here therefore about two solutions for Ψ, whereas there actually exist four solutions.

In the region, where geometrical optics is applicable, these solutions are wholly independent, so that, for instance, the propagation of a wave of the type 1 upwards does not lead to the appearance of a wave 1, propagated downwards, nor of upwards or downward propagated waves of the type 2. This result holds, of course, also for an isotropic medium with respect to propagated upwards and downwards waves (5); since, however, in this case $n_1^2 = n_2^2$ the waves 1 and 2, which correspond to two possible polarizations, are not independent and under certain conditions a rotation of the plane of polarization takes place (4).

From (26,0) it follows that

$$\frac{E_z^{(0)}}{E_y^{(0)}} = \frac{iC}{B-(\Psi'_{1,2})^2} = \frac{A-(\Psi'_{1,2})^2}{-iC} = R_{1,2}. \quad (29)$$

[see also (21)].

The condition for the existence of a non-trivial solution for system (26,1) leads to an equation which yields $E_y^{(0)}$:

$$E_y^{\prime(0)} + \left(\frac{\Psi''}{\Psi'} - \frac{RR'}{1-R^2}\right) E_y^{(0)} = 0.$$

Hence

$$E_y^{(0)} = \frac{\text{const}}{\sqrt{\Psi'(1-R^2)}} \equiv \frac{\text{const}}{\sqrt{\mu}}. \quad (30)$$

Furthermore, equation (26,1) gives directly the relation between $E_z^{(1)}$ and $E_y^{(1)}$:

$$E_z^{(1)} = RE_y^{(1)} + \frac{1}{C}(\Psi''E_y^{(0)} + 2\Psi' E_y^{\prime(0)}) =$$

$$= RE_y^{(1)} + \frac{2RR'}{1-R^2}\Psi' E_y^{(0)}.$$

The condition for the existence of a solution for system (26,2) leads to the following equation determining $E_y^{(1)}$:

$$E_y^{\prime(1)} + \left(\frac{\Psi''}{2\Psi'} - \frac{RR'}{1-R^2}\right) E_y^{(1)} = f(x),$$

where

$$f(x) = \frac{1}{2i\Psi'(1-R^2)}\left\{(1-R^2)E_y^{\prime\prime(0)} - 2RR'E_y^{\prime(0)} - RR''E_y^{(0)} + \right.$$
$$\left. + \frac{i\Psi''R}{C}\left(\frac{2RR'}{1-R^2}\Psi' E_y^{(0)}\right) + 2i\Psi' R\frac{d}{dx}\left(\frac{2RR'}{C(1-R^2)}\Psi' E_y^{(0)}\right)\right\}.$$

The geometrical optics approximation is valid if

$$\frac{c}{\omega}|E^{(1)}| \equiv \frac{\lambda_0}{2\pi}|E^{(1)}| = \frac{\lambda_0}{2\pi}\left|\frac{1}{\sqrt{\mu}}\int_0^x f(x)\sqrt{\mu}\,dx\right| \ll |E_0| = \frac{\text{const}}{\sqrt{\mu}}. \quad (31)$$

Before investigating this inequality we shall ascertain what form the condition of validity of geometrical optics will assume in the simplest case of the scalar wave equation investigated by G a n s (5). We give the relevant expressions without further explanations:

$$E'' + \frac{\omega^2}{c^2}n^2(x)E = 0,$$

$$E = \left(E^{(0)} + \frac{c}{\omega}E^{(1)} + \frac{c^2}{\omega^2}E^{(2)} + \ldots\right)e^{-i\frac{\omega}{c}\Psi},$$

$$(-(\Psi')^2 + n^2)E^{(0)} = 0, \quad E'^{(0)} + \frac{\Psi''}{2\Psi'}E^{(0)} = 0,$$

$$E'^{(1)} + \frac{\Psi''}{2\Psi'}E^{(1)} = \frac{E''^{(0)}}{2i\Psi'}; \text{ etc.}$$

Hence

$$(\Psi')^2 = n^2,$$

$$E^{(0)} = \frac{\text{const}}{\sqrt{\Psi'}},$$

$$E^{(1)} = \frac{\lambda_0}{2\pi}\cdot\frac{1}{\sqrt{\Psi'}}\int_0^x \frac{E_0''\sqrt{\Psi'}}{2i\Psi'}\,dx.$$

The condition of applicability of geometrical optics

$$\frac{\lambda_0}{2\pi}|E^{(1)}| \ll |E^{(0)}| \quad (32,1)$$

can be substituted for more convenient and more rigid conditions. Namely, let

$$\frac{\lambda_0}{2\pi}\cdot\frac{E_0''}{2i\Psi'} \ll \frac{d}{dx}(\ln\Psi')E^{(0)}. \quad (32,2)$$

Then $\frac{\lambda_0}{2\pi}\frac{|E^{(1)}|}{E^{(0)}} \ll |\ln \Psi'(x)|$ and if

$$|\ln n| \leqslant 1 \qquad (32,3)$$

condition (32,1) will be fulfilled. It will easily be seen that inequality (32,2) will certainly be observed if the following inequalities are fulfilled:

$$\left|\frac{\lambda_0 n'}{2\pi n^2}\right| \ll 1, \quad \left|\frac{\lambda_0 n''}{2\pi n' n}\right| \ll 1. \qquad (32,4)$$

Inequalities (32,4) can be obtained directly if we require, following Gans' example (5), that the term $E''^{(0)}$ is negligible as compared to the terms $\Psi' E'^{(0)}$ and $\Psi'' E^{(0)}$ when a solution of the form $E = E^{(0)} e^{-i\frac{\omega}{c}\Psi}$ is substituted into the initial wave equation $E'' + \frac{\omega^2}{c^2} n^2 E = 0$. It will be obvious from the preceding that only sufficient and somewhat too rigid conditions are obtained by this method. However, these conditions are very convenient and simple. We shall therefore apply a similar method to inequality (31)*. On the condition that

$$|\ln \mu| \leqslant 1 \qquad (33,1)$$

[we must remember that $\mu = \Psi'(1 - R^2) = n(1 - R^2)$], inequality (31) will follow from

$$\frac{\lambda}{2\pi} f(x) \ll \frac{d}{dx}(\ln \mu) \cdot \frac{1}{\sqrt{\mu}}. \qquad (33,2)$$

The latter, in its turn, will be fulfilled if

$$\left.\begin{aligned} &\left|\frac{\lambda \mu'}{2\pi n \mu}\right| \ll 1, \quad \left|\frac{\lambda}{2\pi}\cdot\frac{\mu''}{n\mu'}\right| \ll 1, \\ &\left|\frac{\lambda}{2\pi}\frac{RR'}{\mu}\right| \ll 1, \quad \left|\frac{\lambda}{2\pi}\frac{RR''}{\mu'}\right| \ll 1, \\ &\left|\frac{\lambda}{2\pi}\left(\frac{R}{C}\right)\frac{nn'}{\mu'}\frac{RR'}{(1-R^2)}\right| \ll 1, \\ &\left|\frac{\lambda}{2\pi}\frac{n\sqrt{\mu}R}{\mu'}\frac{d}{dx}\left(\frac{RR'n}{C(1-R^2)\sqrt{\mu}}\right)\right| \ll 1. \end{aligned}\right\} \qquad (33,3)$$

The last of these inequalities as well as several others can be transformed into simpler relations; we shall not examine this point in detail. Application of the properties of functions R, C and n [see (21), (15), (17), (29)] readily shows* that in the case of a smooth monotonic dependence of v on x the conditions of the applicability of geometrical optics are practically reduced to a single condition:

$$\left.\begin{aligned} &\left|\frac{\lambda_0}{2\pi}\frac{n'_{(1,2)}}{n_{1,2}}\right| \ll 1, \\ &|\ln n_{1,2}| < 1. \end{aligned}\right\} \qquad (33)$$

Moreover, geometrical optics is invalid for problems connected with polarization when $v \to 0$, as in this case there is a tendency to degeneracy (*i. e.* to isotropy). This case is of no special interest; the principal side of the problem for the case of slight anisotropy has been examined by the author in a paper «Investigation of Stresses by Optical Methods»**

It can be seen from (33) that when $n_{1,2} \to 0$ the approximation of geometrical optics is not applicable, which is quite obvious since in this case $\lambda = \lambda_0/n \to \infty$. In the region lying close to the points $n_1 = 0$ and $n_2 = 0$ the different solutions obtained in the approximation of the geometrical optics are therefore not independent. As a result, in the isotropic case, in the region near the point $n = 0$ a reflection takes place, *i. e.* the wave $\sim e^{-i\frac{\omega}{c}\Psi}$ generates the wave $\sim e^{+i\frac{\omega}{c}\Psi}$. If the region, where $n^2 < 0$ extends far enough, the reflection is complete and, although the geometrical optics is, strictly speaking, inapplicable in the vicinity of the points $n = 0$, its corresponding solution, allowing for the existence of total reflection, is qualitatively correct. This refers, in the case of sufficiently smooth variation of n with x, both to the calculation of the phase of the reflected wave (6) and to the value of the amplitude of the wave (32) near the point $n = 0$ (7). There is no reason to doubt that in the case of an anisotropic medium the situation will be quite similar. Leaving this question aside, let us look for consequences of the existence of

* The investigation of the region of applicability of geometrical optics was incorrectly carried out in the Russian publication of this paper (Journ. Exper. a. Theor. Phys., **13**, 149 (1943)) and formulae (30)—(33) refer to the isotropic case exclusively. This error is of no importance to the consequent results.

* We can show, for example, that C and C' tend to infinity only when $v \to v_{1\infty}$ and $R_2 \to \infty$, and $R'_2 \to \infty$ when $v \to 1$. R_1 never tends to ∞. For $v \to 1$, $R_1 \to 0$. Indices 1 and 2 are referred here to the extraordinary and ordinary waves.

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two types of waves. From the aforesaid it is clear that in the region where $n_1 \to 0$ geometrical optics is, generally speaking, applicable to the wave 2 and, on the contrary, in the region $n_2 \to 0$ it is applicable to the wave 1. Hence it follows directly that the reflection of the wave 1 «from the point» v_{10}^- (where $n_1 = 0$) can not be accompanied by the appearance of a wave of the type 2. The non-applicability of geometrical optics leads only to the coupling of the waves of the type 1 propagated in different directions; the same refers to the reflection of the wave 2. The same conclusion has been reached by Försterling and Lassen [2] by considering the reflection and refraction of the waves at the boundaries of the homogeneous layers by which they replaced the inhomogeneous medium. Apart from the fact that such a derivation is much less conclusive, it is also much more complicated than the preceding considerations.

It was assumed above that in the region where one of the waves cannot be dealt with by the method of geometrical optics this method can be applied to the other wave. There exists, however, a case when the two waves are not independent. This takes place under just those conditions, when a «trebling» of signals should be observed, *i. e.* for very small values of α. As can easily be seen from Fig. 2 in the vicinity of the point v_{20}, *i. e.* near the value $v=1$, $n_2 \to 0$ and $\frac{dn_1}{dx} \to \infty$. The approximation of geometrical optics is thus in this case (when $v \to 1$) actually not applicable to the waves of both types [see (33)].

The solution of the problem of partial reflection from the point v_{20} when $\alpha \to 0$ must be based on equations (14). The difficulties arising here are extremely great. A certain simplification may be reached, if equations (14) are solved for the region v between $v_{\rm I}$ and $v_{\rm II}$ only (Fig. 2), where the «leaking» and the partial reflection come just into play. In the regions, where $v > v_{\rm II}$ and $v < v_{\rm I}$, the medium can either be treated as homogeneous, or the approximation of geometrical optics can be applied; at the boundaries of the region, *i. e.* in the points $v_{\rm I}$ and $v_{\rm II}$, the solutions must, of course, be adequately connected with each other. Further, if the region $v_{\rm II} - v_{\rm I}$ is small enough, as is the case when $\alpha \to 0$, one can put $(1-v) = -ax$, where x is reckoned from the point $v=1$, and neglect in expressions (15) the term $(1-v)^2$ in comparison with the others [under the condition that the inequalities $(1-v_{\rm I}) \ll 1$ and $(1-v_{\rm II}) \ll 1$ hold]. As a result equations (14) are reduced to the form:

$$\left.\begin{aligned} &\{a(1-u)\,x + u\sin^2\alpha\}\frac{d^2E_y}{dx^2} + \\ &+\frac{\omega^2}{c^2}\{(u\sin^2\alpha - aux)\,E_y - \\ &\qquad - ia\sqrt{u}xE_z\} = 0, \\ &\{a(1-u)\,x + u\sin^2\alpha\}\frac{d^2E_z}{dx^2} + \\ &+\frac{\omega^2}{c^2}\{ia\sqrt{u}xE_y - auxE_z\} = 0, \end{aligned}\right\} \qquad (34)$$

where it is taken into account that $\cos\alpha \approx 1$.

With the help of Laplace's transformation this system of equations of the second order* can be reduced to a system of two linear equations, which is equivalent to one linear equation of the second order with variable coefficients. Hence it is clear that an exact solution of the system (34) can probably be obtained and discussed in detail without extreme difficulty. In view of the approximate character of the system (34) and, especially, of the possibility of an extremely simple approximate solution of the problem by another method, equations (14) and (15) have not been further investigated. Before passing to the just mentioned method of solution, we shall pause to consider one general question. In an anisotropic medium one must distinguish, as is well known, between the direction of the normal to the wave and the direction of the rays, *i. e.* the direction of the energy flux. If the problem of the propagation of the wave front in the approximation of geometrical optics is solved [*i. e.* equations (25,0) are solved] one can determine the paths of the rays. This can also be done, however, in principle, directly, using the Fermat principle [see in more detail in [8]]. If the wave is propagated vertically upwards, the wave normal, as we have seen, preserves its upward direction; the Pointing vector has in this case also a different direction as follows from the fact that in this case $E_x \neq 0$ [see (11), (12)]. On reflection both E_y and E_z change their sign, *i. e.* a standing wave is formed. As can be

* Equations of the type (14) are also obtained, for example in the problem of two coupled oscillators, if the coupling depends on the time explicitly; in this case E_y and E_z are replaced by the coordinates of the vibrating points and x by the time t.

seen from (12) the sign of the component E_x is here also changed. If therefore a certain portion of a plane wave is sent urwards, it will return after reflection to the same place on the Earth (Fig. 3). An actual emitter sends upwards not a portion of a plane wave, but a more complicated wave field; inasmuch, however, the inhomogeneous layer lies far away from the reflector, this case is reduced to the preceding one. It is clear from the aforesaid that the non-coincidence of the direction of the beam with the x axis does not lead to any essential changes in the process of propagation, considered above apart from the fact that the signal directed downwards is reflected, not from the region of the Heaviside layer situated directly above the observations point, but from a somewhat shifted region; for the waves of the type 1 and 2 these regions are different (Fig. 3). It must be mentioned, by the way, that when $\alpha=0$, the beam is directed along the x axis, and for small α the deviation of the beam is smaller than in other cases.

3. Calculation of the transmission coefficient

The calculation of the transmission coefficient, *i. e.* of the ratio D of the intensity of the wave of type 1 in the region $v>1$ to the intensity of the incident wave of the type 2 in the region $v<1$, will be carried out in the following way. With the help of the solutions of geometrical optics we shall build up solutions for the incident and reflected waves of the type 2 and the transmitted wave of the type 1, which should correspond as closely as possible to the actual conditions. Further using the method of Ritz the combination of the above mentioned solutions can be determined, which is the best approximation to an exact solution of the problem. In order to apply the Ritz method, it is necessary to replace equations (14) by an equivalent variational problem. The corresponding Lagrange function is easily seen to be

$$L=\left(E_y''+\frac{\omega^2}{c^2}\left[AE_y+iCE_z\right]\right)\frac{E_y^*}{2}+ \\ +\left(E_z''+\frac{\omega^2}{c^2}\left[BE_z-iCE_y\right]\right)\frac{E_z^*}{2}+\text{c. c.} \quad (35)$$

where c. c. denotes the conjugate complex expression, the same is indicated by a star. The variation of the function (35) with respect to E_y and E_z leads to equations (14)

which thus reduce to zero the first variation of the integral $\int L\,dx$. The variation of (35) with respect to E_y^* and E_z^* leads to equations, which are conjugate to the system (14). At first sight, it seems to be more natural to choose as a Lagrange function the expression which differs from (35) by the total

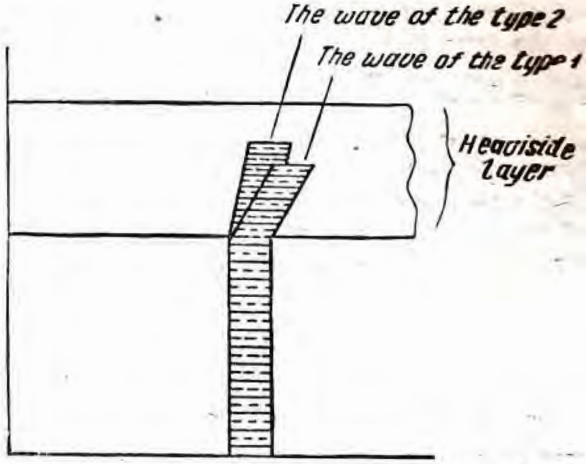

Fig. 3

derivative $\frac{1}{2}\frac{d}{dx}\left(E_y'E_y^*+E_yE_y^{*\prime}+E_z'E_z^*+E_zE_z^{*\prime}\right)$ and which does not contain the second derivatives of E_y and E_z. It is necessary, however, to choose just the function (35), since we are interested in a variational problem with a free variation at the boundary. In fact, the amplitude of the incident wave outside the medium for large negative values of x (conventionally, for $x=-\infty$) is assumed to be equal to unity, while the amplitude of the reflected wave is unknown, and is to be determined, the variation at the boundary, $x=\pm\infty$, is, generally speaking, different from zero and one must choose such a Lagrange function that an arbitrary combination of harmonic plane waves should reduce the variation to zero* at the boundaries. Here waves of the type $e^{\pm ikx}$ are meant since just such a solution holds outside the medium. The variation $\int L\,dx$ at the boundary [see (9), p. 198] is

* I am for this remark indebted to Prof. Ig. Tamm whom I wish also to thank for the discussion of the whole variational method of the solution of the problem.

$$\delta I = \frac{\partial L(E, E')}{\partial E_y'^*}\delta E_y^* + \frac{\partial L(E, E')}{\partial E_z'^*}\delta E_z^*$$

or, according to (35):

$$\delta I = -\frac{E_y^*}{2}\delta E_y^* + \frac{E_y}{2}\delta E_y'^* - \frac{E_z'}{2}\delta E_z^* + \frac{E_z}{2}\delta E_z'^*;$$

if $E_{y,z} = e^{-ikx} + ae^{+ikx}$ then $\delta I = 0$ for any value of a, as it should be.

We shall start with the following initial solutions. In the region $v < 1$ for the wave of the type 2, which is propagated upwards $E^{II}_{y,z} = F^{II}_{y,z} e^{-i\frac{\omega}{c}\Psi_2(x)}$ and for the reflected wave of the type 2, $E^{III}_{y,z} = F^{III}_{y,z} e^{+i\frac{\omega}{c}\Psi_2(x)}$ $[F^{III} = F^{II},]$; in the region $v > 1$ the waves of the type 2 must be damped with increase of the values of v (or x) and, consequently, for these waves the expression

$$E^{II}_{y,z} = E^{III}_{y,z} = F^{II}_{y,z} e^{+i\frac{\omega}{c}\Psi_2(x)}$$

must be taken.

In fact in the region $v > 1$, $n_2^2 < 0$, $n_2 = +\sqrt{n_2^2} = +i\sqrt{-n_2^2}$, $\Psi_2 = i\int\sqrt{-n_2^2}\,dx$ and $\mathrm{Re}\left\{\frac{i\omega}{c}\Psi_2\right\} < 0$. It is clear that, if we wish to have for $E^{II}_{y,z}$ a single analytical expression and leave aside the existence of a branching point for $v = 1$, the field $E^{II}_{y,z}$ must be assumed to be equal to $E^{II}_{y,z} = F^{II}_{y,z} e^{-i\frac{\omega}{c}\Psi_2^*}$; for the reflected wave the expression $E^{III}_{y,z} = F^{II}_{y,z} e^{i\frac{\omega}{c}\Psi_2}$ is preserved within all the range of variation of v. For the process under consideration only one wave of the type 1, which is directed upwards and which is damped in the region $v < v_{1\infty}$ is of importance [see (10) and Fig. 1]. Considerations, which are similar to the above mentioned, lead to the choice for $E^{I}_{y,z}$ from the expression $E^{I}_{y,z} = F^{I}_{y,z} e^{-i\frac{\omega}{c}\Psi_1}$. The general solution of equations (14), in the approximation used, is thus as follows (the amplitude of the incident wave of the type 2 is assumed to be equal to unity):

$$E_{y,z} = F^{II}_{y,z} e^{-i\frac{\omega}{c}\Psi_2^*} + aF^{II}_{y,z} e^{+i\frac{\omega}{c}\Psi_2} + bF^{I}_{y,z} e^{-i\frac{\omega}{c}\Psi_1}, \tag{36}$$

where the constants a and b must be determined from the condition of minimum of the integral $\int L\,dx$, in which the solution (36) [(9), p. 163] is substituted. Substituting (36) in (35) and integrating L with respect to x from the origin of the layer (conventionally $x = -\infty$) to its end ($x = +\infty$) we obtain a bilinear form with respect to a, a^*, b and b^*. The condition of minimum of this form I is equivalent to the equations $\frac{\partial I}{\partial a^*} = \frac{\partial I}{\partial b^*} = 0$, or in an explicit form [see (29)—(32)]

$$P_2 a + Rb = S, \quad R^* a + P_1 b = T, \tag{37}$$

where $F^{I}_{y} \equiv F_{yI}$ and so on:

$$\left.\begin{aligned}
P_1 &= \int_{-\infty}^{+\infty} (F''_{yI}F^*_{yI} + F''_{zI}F^*_{zI} + \text{c. c.})\, e^{i\frac{\omega}{c}(\Psi_1^* - \Psi_1)}\,dx, \\
P_2 &= \int_{-\infty}^{+\infty} (F''_{yII}F^*_{yII} + F''_{zII}F^*_{zII} + \text{c. c.})\, e^{i\frac{\omega}{c}(\Psi_2 - \Psi_2^*)}\,dx, \\
R &= \int_{-\infty}^{+\infty} (F''_{yI}F_{yII} + F^{*\prime\prime}_{yII}F_{yI} + F''_{zI}F_{zII} + F^{*\prime\prime}_{zII}F_{zI})\, e^{-i\frac{\omega}{c}(\Psi_1 + \Psi_2^*)}\,dx, \\
S &= -\int_{-\infty}^{+\infty} (F''_{yII}F^*_{yII} + F''_{zII}F^*_{zII} + \text{c. c.})\, e^{-2i\frac{\omega}{c}\Psi_2^*}\,dx, \\
T &= -\int_{-\infty}^{+\infty} (F''_{yII}F^*_{yI} + F^{*\prime\prime}_{yI}F_{yII} + F''_{zII}F^*_{zI} + F^{*\prime\prime}_{zI}F_{zII})\, e^{i\frac{\omega}{c}(\Psi_1^* - \Psi_2^*)}\,dx.
\end{aligned}\right\} \tag{38}$$

The transmission coefficient D is obviously equal to

$$D=|b|^2=\left|\frac{TP_2-SR^*}{P_2P_1-RR^*}\right|^2. \qquad (39)$$

The factor $e^{-i\frac{\omega}{c}(\Psi_1+\Psi_2^*)}$ is for all values of x rapidly oscillating, therefore R is much smaller than P_1 and P_2. For $v>1$ the factor $e^{i\frac{\omega}{c}(\Psi_2-\Psi_2^*)}$ is exponentially damped, if $v<1$ it is equal to unity; the same refers to the factor $e^{i\frac{\omega}{c}(\Psi_1^*-\Psi_1)}$ in the regions $v<v_{1\infty}$ and $v>v_{1\infty}$, respectively. The factor $e^{-2i\frac{\omega}{c}\Psi_2}$ appearing in S is, on the contrary, rapidly oscillating when $v<1$ and exponentially damped if $v>1$. Hence it is clear that S is essentially smaller than the quantities P_1 and P_2 which exceed the integral T also. The value of the latter determines the behaviour and the magnitude of the coefficient D. If the angle α is large the factor $e^{i\frac{\omega}{c}(\Psi_1^*-\Psi_2^*)}$ rapidly oscillates everywhere, the value of T is very small and the coefficient D practically vanishes. When $\alpha\to 0$ in the region near the point $v=1$, $e^{i\frac{\omega}{c}(\Psi_1^*-\Psi_2^*)}$ is oscillating the slower the smaller is α; the quantity D is here therefore different from zero. When $T\gg S$ we thus have with a good degree of accuracy

$$D\approx\left|\frac{T}{P_1}\right|^2. \qquad (40)$$

The integral T satisfies those conditions, which refer to the approximate calculation of integrals by the saddle point method [see for instance (9)]. The saddle point which is determined from the condition $\frac{\partial}{\partial x}(\Psi_1^*-\Psi_2^*)=0$ or the equivalent condition $n_1^*=n_2^*$ is given by

$$v_n=1-i\frac{\sqrt{u}\sin^2\alpha}{2\cos\alpha}=1-iw_n. \qquad (41)$$

If we limit ourselves to the basic exponential factor, the value of $|T|^2$ and, consequently, with the same accuracy, that of D is as follows:

$$D\sim e^{-2\frac{\omega}{c}\operatorname{Im}\{\Psi_1^*(v_n)-\Psi_2^*(v_n)\}}, \qquad (42)$$

where, as is clear from the notation, the values Ψ are taken at the saddle point and Im denotes the imaginary part. A more exact calculation of D, according to equation (40) with account of the preexponential factors, is, apparently, possible approximately without much difficulty. Without a further investigation it is, however, not clear, whether in the formula for D obtained in this way the accuracy of the initial approximation will not be exceeded. Expression (42) reproduces doubtless correctly the mean features of the dependence of D on the parameters of the problem *. We shall limit ourselves to the calculation of expression (42) which requires only the determination of the value of:

$$\operatorname{Im}\{\Psi_1^*(v_n)-\Psi_2^*(v_n)\}=$$
$$=-\operatorname{Im}\left\{\int\limits_{x(v_a)}^{x(v_n)} n_1\,dx-\int\limits_{x(v_b)}^{x(v_n)} n_2\,dx\right\}, \qquad (43)$$

where v_a and v_b are points on the curves n_1 and n_2, respectively, where n_1 and n_2 are real. The choice of these points does not evidently influence the imaginary part of the integrals $\int n_{1,2}\,dx$. We shall therefore choose these points as follows: $v_a=v_b=1$. In so far as, according to (41), $\operatorname{Re}(v_n)=1$, the integration in (43) must be carried out along the imaginary x axis. Taking further as an independent variable v instead of x and taking into account that along the path of integration $v=1-iw$ we get

$$\operatorname{Im}\{\Psi_1^*(v_n)-\Psi_2^*(v_n)\}=$$
$$=-\operatorname{Im}\left\{\int\limits_1^{v_n} n_1(v)\frac{dx}{dv}\,dv-\int\limits_1^{v_n} n_2(v)\frac{dx}{dv}\,dv\right\}=$$
$$=\operatorname{Re}\left\{\int\limits_0^{w_n} n_1(w)\frac{dx}{dv}\,dw-\int\limits_0^{w_n} n_2(w)\frac{dx}{dv}\,dw\right\}. \qquad (44)$$

* The problem of a partial reflection and leaking of radio waves is akin to some problems of the quantum theory of collisions. In order to solve these problems L a n d a u [10] has proposed a methods, which differs from that presented here, but also leading to a formula of the type (42). I am very obliged to Prof. L. Landau who pointed out that his formula can be applied to the problem in question also.

We shall assume for the sake of definiteness that in the neighbourhood of the point $v=1$ the quantity $\frac{dv}{dx}=\frac{4\pi e^2}{m\omega^2}\frac{dN(x)}{dx}$ does not depend on x, *i. e.* that in this region $(1-v)=-ax$, where $\frac{dv}{dx}=a=\text{const}$. Further, since $v=\frac{4\pi e^2 N(x)}{m\omega^2}$ [see (16)], $\left(\frac{dv}{dx}\right)_{v=1}=\left(\frac{dN/dx}{N}\right)_{v=1}$. If $\alpha\to 0$ one can take for $n_{1,2}$ along the path of integration in (44) the expression [see (17)]:

$$n_{1,2}=\sqrt{\frac{-u\sin^2\alpha\pm\sqrt{u^2\sin^4\alpha-4uw^2}}{2iw-u\sin^2\alpha\pm\sqrt{u^2\sin^4\alpha-4uw^2}}}.$$

An exact calculation of the integral $\int_0^{w_n}\text{Re}\,(n_{1,2})dx$ (the quantity $\frac{dx}{dv}$ is taken outside the integral sign) cannot be carried out. From the fact that

$$\text{Re}\,n_1(w=0)=n_1(w=0)=1,$$

$$\text{Re}\,n_1(w_n)=\text{Re}\,n_2(w_n)=\sqrt{\frac{\sqrt{u}(\sqrt{u}+\sqrt{1+u})}{2(1+u)}},$$

$$\text{Re}\,n_2(w=0)=n_2(w=0)=0,$$

and that the functions $n_{1,2}$ vary monotonically, it is, however, clear that

$$\text{Im}\,\{\Psi_1^*(v_n)-\Psi_2^*(v_n)\}=\beta w_n\left(\frac{dx}{dv}\right)_{v=1}=$$

$$=\beta\left(\frac{N}{\frac{dN}{dx}}\right)_{v=1}\frac{\sqrt{u}}{2}\sin^2\alpha, \qquad (45)$$

where $\beta\leqslant 1$ (it should be remembered that in the practically interesting frequency range $u<1$ also) and $\cos\alpha$ is put equal to unity.

In the example considered above, corresponding to $u=0.25$ the numerical integration leads to the value $\beta=0.6$. We thus get, taking into account (45), finally:

$$D\sim e^{-\frac{2\pi}{\lambda_0}\beta\sqrt{u}\left(N/\frac{dN}{dx}\right)_{v=1}\sin^2\alpha}=e^{-\gamma}. \qquad (46)$$

It is clear from (46) that D is the larger the larger are $\frac{dN}{dx}$ and λ_0 and the smaller are N and $\sin^2\alpha$.

In (46) the expression γ in the exponent can be rewritten in the following form:

$$\gamma=\frac{m\omega^3}{4\pi e^2 c}\beta\sqrt{u}\frac{\sin^2\alpha}{\left(\frac{dN}{dx}\right)_{v=1}}=$$

$$=1.06\cdot 10^{-20}\,\omega^3\beta\sqrt{u}\frac{\sin^2\alpha}{\left(\frac{dN}{dx}\right)_{v=1}}. \qquad (47)$$

If $u=0.25$ this expression is equal to $(\omega=2\omega_H=2\cdot 8.82\cdot 10^6)$:

$$\gamma=17.5\frac{\sin^2\alpha}{\left(\frac{dN}{dx}\right)_{v=1}}. \qquad (47')$$

The value of $\left(\frac{dN}{dx}\right)_{v=1}$ can be but estimated: taking into account, for instance, that in the E layer the concentration of the electrons, if the properties of this layer are determined by the electrons *, reaches a value of $\sim 10^5$ at a distance of ~ 10 km, we find $\frac{dN}{dx}\sim 0.1$; the same value is obtained by a similar estimate for the F layer. With this value of $\frac{dN}{dx}$ one can easily see, using (47′), that $\gamma=2$, *i. e.* according to (46) D is equal to 0.135 if $\alpha=6°$. As is clear from the aforesaid, owing to the preexponential factor, D can, in the case $\alpha=6°$, differ very markedly from this value. The conclusion that D falls exponentially with increase of α remains, however, valid. In order to determine the absolute value of $\left(\frac{dN}{dx}\right)_{v=1}$ from the measurement of the coefficient D, one must carry out a more exact calculation based on the solution of equations (14) or at least (34); sufficient accuracy will perhaps here be obtained by the solution of (40).

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* The question, whether the reflection of the radio waves from the E layer is due to electrons or to ions, remains, in spite of all its importance open (1,3). The part of the refraction index, which is due to the ions, is practically not influenced by the magnetic field. The existence of a double refraction proves, therefore, the electronic structure of the Heaviside layer. This is the case with respect to the F layer. The observation of the splitting of the signals in the E layer is considerably more complicated than in the F layer, owing to its tenfold smaller thickness and has so far not given an unambiguous result.

In certain cases (on various days and hours, for different values of ω) the quantity $\frac{dN}{dx}$ can apparently be much larger than 0.1.

The investigation of the «trebling» of the signals can serve as a fine probe of the Heaviside layer, which leads to a determination or to an estimate of $\frac{dN}{dx}$. This phenomenon can be observed in regions, lying relatively close to the magnetic poles of the Earth, *i. e.*, for instance, in the case considered above in regions, which are removed from the poles by less than 6°. Regions of this kind are at present quite accessible for carrying out the necessary observations.

In conclusion we shall make two remarks.

When the values of α are so small that $D \rightarrow 1$, formulae (40), (42) and (46) are no longer applicable, as well as the method of calculation itself. This region lies quite close to the magnetic poles, and is therefore less interesting than the preceding one. The intensity of the wave reflected from the point $v = 1$ is here small and the fundamental rôle is played by the reflection from the points $v^{\pm}_{10}$.

Further, as is clear from Fig. 2, the quantity $\frac{dn_{1,2}}{dN}$ tends to infinity when $\alpha \rightarrow 0$ and $v \rightarrow 1$. Therefore for small values of α (in the region of «trebling») the local inhomogeneities in the ionosphere (*i. e.* macroscopic fluctuations of the concentration N) will lead to a strong scattering of radio waves. This possibility must be taken into account.

4. Reflection of the signals

The most widespread method of investigating the ionosphere consists in sending upwards radio signals and in observing the time which is needed for their return. The theory must be able to calculate this time and the change of the shape of the reflected signal in comparison with the incident one. If the reflection of a monochromatic wave from the layer, *i. e.* if the intensity and the phase shift with respect to the incident wave is known, the problem is reduced to the investigation of Fourier integrals, representing the incident and the reflected signals. In the case, when the magnetic field of the Earth can be left out of account*, the question has been studied by the author earlier ([11]). The results of this work can be extended to the case when the presence of the magnetic field plays an essential rôle. It must be here, of course, taken into account, from the very beginning, that each wave, constitutes a signal and, consequently, the signal as a whole, is split up accordingly into two waves or two signals, which differ from each other with respect to the state of polarization.

If the «trebling» and polarization of the reflected signals are left aside (see below) and one of the two signals only is considered, then the only complication, introduced by the magnetic field, is connected with the difficulty of evaluating the integrals $\int n_{1,2}x\,(dx)$ and $\int \frac{dx}{n_{1,2}(x)}$ **. The general conclusions obtained in ([11]) remain valid under the conditions specified there. The specific influence of the field refers to the polarization of the reflected waves; the signals returning to the Earth consist: one of waves of the type 1 and the other of waves of the type 2, and have, consequently, on the whole, different polarizations. Moreover each of the waves, which are present in the expansion of the signal into a Fourier integral, possesses its own polarization. For small values of all the waves of the types 1 and 2 can be regarded as circularly polarized [see (21) and ([2])]. Both signals or both waves of the types 1 and 2 are coherent and if they are not completely separated on the Earth surface a rather complicated interference pattern arises ([12]). In studying the effect of «trebling» it seems therefore expedient to send upwards a signal, consisting of waves circularly polarized in that sense, which corresponds to the wave 2 (in the vicinity of the North Pole the electric vector in this wave has an anti-clockwise rotation if the clock is looked at from above; because the magnetic field is there

* Strictly speaking, if the properties of the E layer are determined by the electrons, the influence of the magnetic field can never be neglected. Approximately this can be done sometimes in the determination of the retardation time, but the details of the shape of the reflected signal, in the case of its splitting into two are just the same, different from the case when such splitting does not take place.

** Only in the case $\alpha = 0$, the function $n_{1,2}$, as is clear from (5) and (22), is not more complicated than n_0.

directed downwards). The absence of the waves **of the** type 1 in the region, where no «trebling» **exists**, will lead to the appearance of a single **reflected** signal. The «trebling» will in this case be revealed by a doubling of this signal. **In so** far as the transmission coeffecient D depends sharply on the frequency, decreasing with its increase [see (46), (47)], a signal reflected from the point $v=1$ is richer with respect to high frequencies, while the signal, reflected from the point $v=v_{10}^{+}$ is richer with respect to lower frequencies. A more detailed spectral analysis of both signals is of no great practical interest at the present time.

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