WAVE FIELD WITH THE SPECTRUM OF MASSES

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Linear equations with higher derivatives of an unlimited order are studied. Limitations for the operators of such equations are established. The case of a scalar field is especially examined.

1. The field equations and quantization

Recent investigations of cosmic rays give evidence for the existence of large diversity of mesotron masses. The idea naturally arises to consider particles with various masses as different states of a single particle capable of interference, just as are the states of a particle of a definite mass but different energies.

The present investigation should be considered merely as a preliminary analysis of the possibilities in this direction.

We start from the most general type of linear equations for the components of the wave function ψ_α ($\alpha=1, 2, 3, \ldots, n$); these equations may possess some covariance property (scalar, vector, spinor, etc.). If x ($x_1, x_2, x_3, x_4=ix_0$) designates the coordinates of the world point these equations may be written in the form

$$\sum_\beta \int L_{\alpha\beta}(x-x')\,\psi_\beta(x')\,dx' \qquad (1)$$

and the quantization rules will be

$$[\psi_\alpha(x'), \psi_\beta(x'')] = D_{\alpha\beta}(x'-x''), \qquad (2)$$

where $[a, b]=ab-ba$, $D_{\alpha\beta}(x)$ is a certain function the general form of which will be established later. $L_{\alpha\beta}(x-x')$ is the matrix element of a linear operation which reduces to differentiation only in particular cases. From (1) and (2) it follows

$$\sum_\beta \int L_{\alpha\beta}(x-x')\,D_{\beta\gamma}(x'-x'')\,dx' = 0. \qquad (3)$$

Introducing the Fourier coefficients instead of these functions according to the formula

$$ig(x) = \frac{1}{(2\pi)^4}\int g(k)\,e^{ikx}\,dk, \qquad (4)$$

where k is the four-dimensional wave vector ($k_1, k_2, k_3, k_4=ik_0$) and the Fourier coefficients are denoted by the same symbol as the function itself, we obtain instead of (1) and (3)

$$\sum_\beta L_{\alpha\beta}(k)\,\psi_\beta(k)=0, \quad \sum_\beta L_{\alpha\beta}(k)\,D_{\beta\gamma}(k)=0. \qquad (5)$$

Let $\Delta(k^2)$ be the determinant $\|L_{\alpha\beta}^{(k)}\|$. If $D_{\beta\gamma}(k)\neq 0$ in the region $k^2>0$, then $\Delta(k^2)=0$ in the same region and, therefore, a non-trivial solution exists ψ_β for $k^2>0$. Such solutions, however, would correspond to particles with an imaginary mass μ. Therefore, for $k^2>0$ (space region) $\Delta(k^2)\neq 0$ and ψ_β and $D_{\beta\gamma}\equiv 0$. In the region $k^2<0$ (time region) the determinant $\Delta(k^2)$ may equal zero. For $\Delta(k^2)=0$ a non-zero solution of the wave equation exists. As to $D_{\beta\gamma}$ it may or may not equal zero depending on whether ψ_β are considered to commute or not.

The lines (hypersurfaces) and shaded bands (four-dimensional region) where $\Delta(k^2)=0$ are shown in Fig. 1. Each line

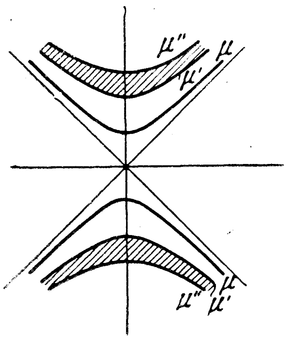

Fig. 1

corresponds to particles with a definite mass μ while the bands correspond to a continuous mass spectrum (say, from $\mu=\mu'$ to $\mu=\mu''$). Thus the most general equation (1) describes a field representing a particle capable of possessing various rest masses (particle with a mass spectrum).

2. Scalar field

The case of a scalar field ($n=1$) is naturally the most simple. On the other hand, this problem clearly illustrates the peculiarities of equations of type (1). For this reason we shall in the future direct our attention to just this case.

All the matrix elements $L_{\alpha\beta}$ equal zero except L_{11}, which is a function of k^2. Denoting it by $L(k^2)$ we obtain instead of (5)

$$L(k^2)\,\psi(k)=0, \quad L(k^2)\,D(k)=0, \tag{6}$$

$$iL(x)=\frac{1}{(2\pi)^4}\int L(k^2)\,e^{ikx}\,dk. \tag{7}$$

Besides this $\Delta(k^2)=L(k^2)$. According to the foregoing $L(k^2)$ can equal zero only on the lines or within the bands shown in Fig. 1. The integration in (7) may be extended over all the k-space if one subtracts the surplus part due to integration over the bands. One then obtains

$$L(x)=L_1(-\Box^2)\,\delta(x)-\frac{1}{(2\pi)^4}\int L_1(k^2)\,e^{ikx}\,d\mathbf{k}\,dk_0, \tag{8}$$

where $\Box^2=\sum\frac{\partial^2}{\partial x_\mu^2}$, $\delta(x)$ is the product of the δ-functions of x_1, x_2, x_3, x_4, and $L_1(k^2)$ is the operator corresponding to the discrete mass spectrum. The last integral should be extended over the shaded bands. Introducing instead of k_0 the square of the mass $\mu^2=k_0^2-\mathbf{k}^2$ after simple transformations one obtains

$$L(x)=L_1(-\Box^2)\,\delta(x)-\frac{1}{2\pi}\int L_1(-\mu^2)\,d\mu^2\,\Delta_s(x,\mu), \tag{8'}$$

where

$$\Delta_s(x,\mu)=\frac{1}{(2\pi)^3}\int e^{i\mathbf{kx}}\frac{\cos x_0\sqrt{\mathbf{k}^2+\mu^2}}{\sqrt{\mathbf{k}^2+\mu^2}}\,d\mathbf{k}, \quad \Delta_a(x,\mu)=\frac{1}{(2\pi)^3}\int e^{i\mathbf{kx}}\frac{\sin x_0\sqrt{\mathbf{k}^2+\mu^2}}{\sqrt{\mathbf{k}^2+\mu^2}}\,d\mathbf{k}. \tag{9}$$

The function $\Delta_a(x,\mu)$ enters $D(x)$*. Thus applying similar transformations to $D(x)$ one gets

$$D(x)=D_a(x)+D_s(x)=\int D_a(-\mu^2)\,\Delta_a(x,\mu)\,d\mu^2+\int D_s(-\mu^2)\,\Delta_s(x,\mu)\,d\mu^2, \tag{10}$$

* Explicit expressions for Δ_a and Δ_s in terms of Bessel functions are given in Pauli's paper (1).

where $D_a(-\mu^2)$ and $D_s(-\mu^2)$ differ from zero only within the bands or on the lines [where $L(-\mu^2)=0$]. In the latter case they have the form of δ-functions. The equation with the operator $L(x)$ (8) and commutator $D(x)$ (10) is the most general type of a linear equation for the scalar field and the most general rule for its quantization. The physical difference between this field and a set of fields with particles of various masses is that in the considered case the states with various masses are capable of interference.

The function $\Delta_a(x, \mu)$ and together with it also $D_a(x)$ disappear at $x_0=0$ and hence over the entire space region. Quantization with the aid of $D_a(x)$ means that the measurement of the field at the point x' perturbes the field at the point x'' only if these points can be connected by a signal propagating with the velocity of light or with a smaller velocity. The function $D_s(x)$, on the contrary, permits a mutual perturbation of the measurements carried out in points which may be connected only by signals propagating with a velocity greater than that of light. Quantization with the aid of D_s contradicts Hamilton's method (namely it turns out that $ih\dot{\psi} \neq [H, \psi]$, where H is the Hamilton function*).

3. Discrete mass spectrum

Let us examine in greater detail the case of a scalar field for which the particle possesses a discrete mass spectrum. Let the operator of the equation of such a field be denoted by $L(\Box^2)$. According to the preceding discussion concerning the roots of such an operator the latter should equal the product (finite or infinite) **

* If application of this method [cf. (²)] is not considered to be obligatory, then the function D_s will not seem to be so paradoxical, since the perturbations which propagate with a velocity greater than that of light will be confined to a small region of distances $|\mathbf{x}'-\mathbf{x}''| \sim 1/\mu$ and small time intervals $|t'-t''| \sim 1/\mu c$ (as $\Delta_s \to 0$ like $e^{-\mu\sigma}/\sqrt{\mu\sigma}$ for $\sigma \to \infty$, $\sigma^2 = |\mathbf{x}'-\mathbf{x}''|^2 - c^2|t'-t''|^2 > 0$). *Cf.* our paper (³) in which the same problem is examined from the different point of view.

** An example of such an operator is

$$L(\Box^2) = \mu^2 \frac{\sin\sqrt{\mu^2\Box^2}}{\sqrt{\mu^2\Box^2}} \left(\gamma = \frac{\pi^2 s^2}{\mu^2},\ s = 1, 2, 3, \ldots\right)$$

and many other transcendental operators too.

$$L(\Box^2) = \frac{1}{\gamma_1} \prod_s (1-\gamma_s\Box^2), \qquad (11)$$

where $\gamma_s = 1/\mu_s^2$. The general solution of the equation $L\Psi = 0$ is

$$\Psi = \sum_s \psi_s, \qquad (12)$$

where ψ_s satisfy the equations

$$\Box^2\psi_s - \frac{1}{\gamma_s}\psi_s = 0. \qquad (13)$$

The simplest energy-momentum tensor with a positive-definite energy density τ_{44} may be obtained in the following manner. From considerations of the necessary covariance of the tensor components one may put

$$\tau_{\mu\nu} = -\sum_s \frac{\partial\Psi_s}{\partial x_\mu}\frac{\partial\Psi_s}{\partial x_\nu} + \frac{1}{2}\delta_{\mu\nu}\left\{\sum_s\sum_\mu \left(\frac{\partial\Psi_s'}{\partial x_\mu}\right)^2 + \sum_s (\Psi_s'')^2\right\}, \qquad (14)$$

where Ψ_s, Ψ_s', Ψ_s'' are obtained from Ψ by means of the scalar operations: $\Psi_s = Q_s\Psi$, $\Psi_s' = Q_s'\Psi$, $\Psi_s'' = Q_s''\Psi$. In order that the divergence of this tensor would disappear for all Ψ it is necessary to put $Q_s' = Q_s$ and $Q_s'' = \beta_s Q_s$, then

$$\mathrm{div}_\nu\tau = -\sum_s \frac{\partial\Psi_s}{\partial x_\nu}(\Box^2\Psi_s - \beta_s^2\Psi_s). \qquad (15)$$

This expression equals zero identically, if

$$Q_s = a_s L(\Box^2)(\Box^2 - \beta_s^2)^{-1}, \qquad (16)$$

where a_s is a certain number*. If $(\Box^2 - \beta_s^2)$ is not a divisor of L, then $Q_s\Psi \equiv 0$. Hence $\beta_s^2 = \mu_s^2$. Assuming further

$$a_s = -\frac{\gamma_1}{\gamma_s}\frac{1}{\prod_{n\neq s}(1-\gamma_n/\gamma_s)}, \qquad (17)$$

one obtains from equations (11) and (13)

$$Q_s\Psi = \psi_s \qquad (18)$$

so that

$$\tau_\mu = \sum_s \tau_{\mu\nu}^{(s)}, \qquad (19)$$

* To be more general the a_s could be considered as operators not possessing any common zeros with the operator L. This possibility will not be considered here.

$$\tau_{\mu\nu}^{(s)} = -\frac{\partial\psi_s}{\partial x_\mu}\frac{\partial\psi_s}{\partial x_\nu} + $$
$$+\frac{1}{2}\delta_{\mu\nu}\left\{\sum_\mu\left(\frac{\partial\psi_s}{\partial x_\mu}\right)^2 + \mu_s^2\psi_s^2\right\}. \quad (20)$$

If desired, the tensor $\tau_{\mu\nu}$ may be represented explicitly as a quadratic form of Ψ by means of the operators (18).

Let us now examine an inhomogeneous equation of the form

$$L(\Box^2)\Psi = -4\pi g\rho, \quad (21)$$

where g is a certain constant and ρ is a scalar function of the coordinates, $g\rho$ is the "meson" charge. The solution of equation (21) equals the sum of the solutions of the homogeneous and inhomogeneous equations. The solution of the latter may be written in the form

$$\Psi(x) = -4\pi g L^{-1}(\Box^2)\rho(x). \quad (22)$$

This equality may be interpreted as follows. Let $\rho(x)$ depend on the time x_0 harmonically (otherwise one could consider a superposition)

$$\rho(x) = \rho_0(\mathbf{x})e^{i\omega x_0} =$$
$$= \frac{1}{(2\pi)^4}\int \rho_0(\mathbf{k})\,\delta(k_0-\omega)\,e^{ikx}dk_0 d\mathbf{k}. \quad (23)$$

Inserting this in (22)one obtains

$$\Psi(x) = e^{-i\omega x_0}\int \rho_0(\mathbf{x}')\,G(\mathbf{x}-\mathbf{x}',\omega)\,d\mathbf{x}', \quad (24)$$

where

$$G(\mathbf{x}-\mathbf{x}',\omega) =$$
$$= -\frac{4\pi g}{(2\pi)^3}\int L^{-1}(\omega^2-\mathbf{k}^2)\,e^{i(\mathbf{k},\mathbf{x}-\mathbf{x}')}d\mathbf{k}. \quad (25)$$

In order to obtain a retarded field it is necessary in integral (25) to choose in a certain manner the integration contour taken around the singularities of the expression $L^{-1}(\omega^2-\mathbf{k}^2)$. After simple calculations one gets

$$G(\mathbf{x}-\mathbf{x}',\omega) = \sum_s g_s\frac{e^{i\sqrt{\omega^2-\mu_s^2}\,r}}{r}, \quad g_s = -a_s g, \quad (26)$$

where a_s is given by formula (17). It can be seen from (26) that equation (21) has a general solution of the type (12) if the function ψ_s satisfy the inhomogeneous equations

$$\Box^2\psi_s - \frac{1}{\gamma_s}\psi_s = -4\pi g_s\rho, \quad (27)$$

i. e. each particular wave has its own charge $g_s\rho$.

From the form of the coefficient a_s it follows that the sign of the charge of the particular waves ψ_s is different and equal to $(-)^{s+1}$ (if the waves are arranged in the order of ascending masses).

It can also be seen from (26) that if the mass of the particle $\mu_s > \omega$ (the mass and frequency are measured in units of reciprocal lengths), then the corresponding field ψ_s will co-oscillate around the source and will differ from zero in the wave zone only for $\omega > \mu_s$. As the frequency ω increases, particles of ever increasing mass will be emitted.

The energy-momentum tensor (19) now cannot be expressed in any simple way in terms of the total field Ψ [as the operator $(\Box^2 - 1/\gamma_s)$ does not commute with the charge ρ].

Due to the presence of sources the divergence of the tensor $\tau_{\mu\nu}$ differs from zero and equals

$$\mathrm{div}_\nu\,\tau = 4\pi\rho\sum_s g_s\frac{\partial\psi_s}{\partial x_\nu}. \quad (28)$$

As the right-hand side of this equality cannot be expressed in terms of the total field, due to the difference in the charges of the particular waves, it is evident that the force which the field Ψ exerts on the sources cannot be expressed through the field Ψ but depends on its particular states ψ_s *.

* In general one could express the force in terms of the total field Ψ by taking instead of the tensor $\tau_{\mu\nu}$ the tensor of the form $\tau'_{\mu\nu} = \sum_s c_s\tau_{\mu\nu}^{(s)}$. But then one would obtain after quantization that the proper energy of the field equals $\int\tau_{44}\,d\mathbf{x} = \sum_s c_s N_s h\omega_s$, where N_s is an integer. Thus, generally speaking, a fractional number of photons $(|c_s|N_s)$ would be obtained; moreover, the energy of these photons would be either positive or negative according to s. This follows from the fact that if the force in the right-hand side of equation (29) is written in the form $-8\pi\Phi\sum_s b_s\psi_s$, then the condition of disappearance of the divergence of the tensor $T_{\mu\nu}$ would be $c_s g_s + b_s = 0$; if $b_s = b$, so that the force depends only on Ψ, then oefficients $c_s = -b/g_s$ and change their signs in the same way as g_s do. In particular for an equation of the fourth order $\left[L_2 = -\frac{1}{\gamma_1}(1-\gamma_1\Box^2)(1-\gamma_2\Box^2)\right]$ the requirement that the force would depend on Ψ leads to $c_1 = +1$, $c_2 = -1$ [*cf.* (4)]. An equation of the sixth order (L_3) already leads to fractional values of the coefficients c_1, c_2, c_3.

If, as an example, one assumes that the source of the field Ψ is the scalar field Φ of particles with a mass m so that $\rho = \Phi^2$, then the equation for Φ must be written in the form

$$\Box^2\Phi - m^2\Phi = -8\pi\Phi \sum_s g_s\psi_s. \qquad (29)$$

The complete field tensor for Φ and Ψ taken together will be

$$T_{\mu\nu} = \tau_{\mu\nu} + t_{\mu\nu} - 4\pi\delta_{\mu\nu}\Phi^2 \sum_s g_s\psi_s, \qquad (30)$$

where $t_{\mu\nu}$ is a tensor of the form (20) for the field Φ alone [one should put in (20) $\psi_s = \Phi$, $\mu_s = m$]. Because of (27) and (29) the divergence of tensor (30) equals zero and the work, the field carries out on the sources turns out to be equal to

$$\mathrm{div}_\nu t = 8\pi \frac{\partial \Phi}{\partial x_\nu} \Phi \sum g_s\psi_s. \qquad (31)$$

Thus although equation (21) permits one to find a unique field Ψ which is coherently composed of particular states ψ_s belonging to various masses μ_s the force which the field Ψ exerts on the sources Φ^2 of this field may be expressed only in terms of the particular states ψ_s taken separately. In a similar manner the tensor $\tau_{\mu\nu}$ may be represented only as the sum of tensors, each of which depends only on ψ_s.

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