ELECTROMAGNETIC FIELD OF MULTIPOLES

By V. BERESTETZKY

Physical Technical Institute, Academy of Sciences of the USSR and *Leningrad State University*

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A simple derivation of the electric and magnetic fields of multipoles is given.

The expressions for the potentials and the fields of electric and magnetic multipoles are given in a paper by Heitler (1). However, Heitler's method is somewhat cumbersome and artificial; the gauge of Heitler's potentials is inconvenient in one of the most important applications of the multipole potentials — the calculation of the coefficients of internal conversion of γ-rays (2). Therefore, it seems reasonable to propose a simple derivation of the multipole potentials.

1. Consider firstly the scalar potential $\Phi(\mathbf{r})$ satisfying the wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\mathbf{r}) = 0.$$

The general solution of this equation is

$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \Phi(\mathbf{k})\, e^{i\mathbf{kr}}(d\mathbf{k}) + \text{c.c.} \quad (1)$$

Here in $\Phi(\mathbf{k})$ is involved the time factor e^{-ickt}, $(d\mathbf{k}) = k^2 dk\, d\omega$ is the volume element of the wave vector in $\mathbf{k}$-space; c. c. stands for "complex conjugate expression". We are interested in the angle dependent part of $\Phi(\mathbf{k})$. If we represent $\Phi(\mathbf{k})$ in the form:

$$\Phi(\mathbf{k}) = \sum_{l=0}^{\infty}\Phi(\mathbf{k}, l), \quad (2)$$

$$\Phi(\mathbf{k}, l) = \sum_{m=-l}^{l}(-1)^l c_l^{-m}(k)\, Y_l^m(\vartheta, \varphi), \quad (3)$$

where ϑ and φ are the spherical angles in $\mathbf{k}$-space, $Y_l^m(\vartheta, \varphi)$ — spherical harmonics, which we consider to be normalized to unity, quantities $\Phi(\mathbf{k}, l)$ and $\Phi(\mathbf{k})$ must be scalars and the coefficients c_l^m transform under rotation of coordinate system as Y_l^m, *i. e.* according to the $(2l+1)$-dimensional representation of the rotation group. For brevity we shall call in the sequel such a set of quantities an l-vector. Potential determined by the l-vector c_l^m we call "potential of the 2^l-pole". According to the expansion (2), (3) we write

$$\Phi(\mathbf{r}) = \int k^2 dk \sum_l \Phi(\mathbf{r}, k, l). \quad (4)$$

Using the familiar expansion

$$e^{i\mathbf{kr}} = 4\pi\sum_l i^l f_l(kr)\sum_m Y_l^m(\vartheta, \varphi)\, Y_l^m(\theta, \hat{\Phi}), \quad (5)$$

where θ and $\hat{\Phi}$ are the spherical angles in the space and

$$f_l(kr) = \sqrt{\frac{\pi}{2}}\,\frac{J_{l+1/2}(kr)}{\sqrt{kr}} \quad (6)$$

(J — the Bessel function), we obtain after integration with respect to k:

$$\Phi(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}}\, i^l\sum_m(-1)^m c_l^{-m}(k)\, Y_l^m(\theta, \hat{\Phi})\, f_l(kr) + \text{c. c.} \quad (7)$$

2. For the vector potential we write similarly to (1)

$$\mathbf{A}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \mathbf{A}(\mathbf{k})\, e^{i\mathbf{kr}} (d\mathbf{k}) + \text{c. c.} \qquad (8)$$

In the expansion of $\mathbf{A}(\mathbf{k})$ there arises the problem of construction of a vector from l-vectors of multipole coefficients and spherical harmonics. We can consider vector $\mathbf{A}$ as a 1-vector:

$$A^0 = A_z; \quad A^{\pm 1} = \pm \frac{1}{\sqrt{2}} (A_x \pm i A_y) \qquad (9)$$

(the lower index of 1-vector is dropped).

Such a construction can be realized simply with the help of the Clebsch-Gordon formula for the reduction of the product of two representations of a rotation group into irreducible representations. If two l-vectors $U_{l_1}^{m_1}$ and $V_{l_2}^{m_2}$ are given, one can build an l-vector W_l^m:

$$W_l^m = \sum_{m_1+m_2=m} \left(\begin{matrix} l_1 & l_2 \\ m_1 & m_2 \end{matrix} \middle| \begin{matrix} l \\ m \end{matrix} \right) U_{l_1}^{m_1} V_{l_2}^{m_2}, \qquad (10)$$

if $l = l_1 + l_2,\ l_1 + l_2 - 1, \ldots, |l_1 - l_2|$. The brackets in (10) denote the coefficients which can be found in the books on the group theory [3].

It must be noted that l-vectors can differ (for the same value of l) by the character of transformation under reflection. Similarly to the usual vector terminology we call "the polar l-vector" l-vector which gets by reflection a factor $(-1)^l$ and "the axial" or "the pseudo-l-vector" one which gets the factor $(-1)^{l+1}$. The spherical harmonic and the potential are polar l-vectors. l-vectors of multipoles can be of both types; polar multipole vectors are usually named electric and axial ones — magnetic.

3. From the form of formula (3), which is a particular case of (10), it is seen that it is impossible to construct the scalar potential from pseudo-l-vector and spherical harmonics (since Φ is a scalar but not a pseudoscalar). Hence for a magnetic multipole

$$\Phi = 0. \qquad (11)$$

For the vector potential of a magnetic multipole we obtain using (10), the following expression in which h_l^m stands for the multipole coefficients

$$A^\mu(\mathbf{k}) = \sum_l A^\mu(\mathbf{k}, l),$$
$$A^\mu(\mathbf{k}, l) = \sum_m (-1)^m h_l^{-m}(k)\, \gamma_{m\mu}^l\, Y_l^{m+\mu}(\vartheta, \varphi), \qquad (12)$$

quantities $\gamma_{m\mu}^l$ are the particular form of the coefficients in (10)

$$(-1)^m \gamma_{m\mu}^l = \left(\begin{matrix} l & l \\ -m & m+\mu \end{matrix} \middle| \begin{matrix} 1 \\ \mu \end{matrix} \right)$$

determined with an accuracy up to a factor depending upon l.

With certain normalization

$$\gamma_{m0}^l = \frac{m}{\sqrt{l(l+1)}}; \qquad \gamma_{m,\pm 1}^l = \pm \sqrt{\frac{(l \pm m + 2)(l \pm m + 1)}{2l(l+1)}}. \qquad (13)$$

If we put similarly to (4)

$$A^\mu(\mathbf{r}) = \int k^2 dk \sum_l A^\mu(\mathbf{r}, k, l), \qquad (14)$$

we get from (12) and (8) using expression (6)

$$A^\mu(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}}\, i^l \sum_m (-1)^m h_l^{-m}(k)\, \gamma_{m\mu}^l\, Y_l^{m+\mu}(\theta, \Phi)\, f_l(kr) + \text{c. c.} \qquad (15)$$

Electric field of the magnetic multipole is given by

$$\mathbf{E}(\mathbf{k}) = ik\, \mathbf{A}(\mathbf{k}).$$

Hence, according to (12)

$$E^\mu(\mathbf{k}, l) = ik \sum_m (-1)^m h_l^{-m}(k)\, \gamma_{m\mu}^l\, Y_l^{m+\mu}(\vartheta, \varphi) \qquad (16)$$

and

$$E^{\mu}(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}}\, i^{l+1} k \sum_{m} (-1)^m h_l^{-m}(k)\, \gamma_{m\mu}^{l} Y_l^{m+\mu}(\theta, \hat{\Phi})\, f_l(kr) + \text{c. c.} \tag{17}$$

For the determination of the magnetic field we shall use the expression:

$$\mathbf{H}(\mathbf{k}) = i\,[\mathbf{k}\mathbf{A}(\mathbf{k})]$$

or

$$H^{\mu}(\mathbf{k}) = -\sqrt{2}\, k \sum_{\sigma} \gamma'_{\mu\sigma} (-1)^{\sigma} n^{-\sigma} A^{\mu+\sigma}(\mathbf{k}),$$

where $\mathbf{k}$ is represented in the form of an l-vector

$$k^{\mu} = k n^{\mu}; \quad n^{0} = \cos\vartheta; \quad n^{\pm 1} = \pm \frac{1}{\sqrt{2}} \sin\vartheta e^{\pm i\varphi}, \tag{18}$$

and the vector product is also written in l-vector form. Using the known relations:

$$n^{\mu} Y_l^{m} = \alpha_{m\mu}^{l} Y_{l+1}^{m+\mu} + (-1)^{\mu} \beta_{m\mu}^{l} Y_{l-1}^{m+\mu}; \tag{19}$$

where

$$\alpha_{m0}^{l} = \sqrt{\frac{(l+1-m)(l+1+m)}{(2l+1)(2l+3)}}; \quad \alpha_{m,\pm 1}^{l} = \sqrt{\frac{(l\pm m+1)(l\pm m+2)}{2(2l+1)(2l+4)}};$$
$$\beta_{m0}^{l} = \sqrt{\frac{(l-m)(l+m)}{(2l+1)(2l-1)}}; \quad \beta_{m,\pm 1}^{l} = \sqrt{\frac{(l\mp m)(l\mp m-1)}{2(2l+1)(2l-1)}}. \tag{20}$$

Hence we obtain for $\mathbf{H}$

$$H^{\mu}(\mathbf{k}, l) = k \sum_{m} (-1)^m h_l^{-m}(k) \left[\sqrt{\frac{l}{l+1}}\, \alpha_{m\mu}^{l} Y_{l+1}^{m+\mu}(\vartheta, \varphi) - \sqrt{\frac{l+1}{l}}\, (-1)^{\mu} \beta_{m\mu}^{l} Y_{l-1}^{m+\mu}(\vartheta, \varphi) \right]; \tag{21}$$

$$H^{\mu}(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}}\, i^{l+1} k \sum_{m} (-1)^m h_l^{-m}(k) \Big[\sqrt{\frac{l}{l+1}}\, \alpha_{m\mu}^{l} Y_{l+1}^{m+\mu}(\theta, \hat{\Phi})\, f_{l+1}(kr) +$$
$$+ \sqrt{\frac{l+1}{l}}\, \beta_{m\mu}^{l} (-1)^{\mu} Y_{l-1}^{m+\mu}(\theta, \hat{\Phi})\, f_{l-1}(kr) \Big] + \text{c. c.} \tag{22}$$

4. For the vector potential of an electric multipole formula (10) permits us to construct the following general expression

$$A^{\mu}(\mathbf{k}, l) = \sum_{m} (-1)^m a_l^{-m}(k)\, \alpha_{m\mu}^{l} Y_{l+1}^{m+\mu}(\vartheta, \varphi) + \sum_{m} (-1)^m b_l^{-m}(k)\, (-1)^{\mu} \beta_{m\mu}^{l} Y_{l-1}^{m+\mu}(\vartheta, \varphi). \tag{23}$$

Here a_l^m and b_l^m are two arbitrary independent l-vectors; coefficients $\alpha_{m\mu}^{l} = (-1)^m \times \times \begin{pmatrix} l & l+1 & 1 \\ -m & m+\mu & \mu \end{pmatrix}$ and $\beta_{m\mu}^{l} = (-1)^m \begin{pmatrix} l & l-1 & 1 \\ -m & m+\mu & \mu \end{pmatrix}$ can be made by a certain normalization identical with the coefficients in (19).

For scalar potential we have the expression (3) in which the Lorentz condition:

$$\Phi(\mathbf{k}) = \mathbf{n}\mathbf{A}(\mathbf{k})$$

or

$$\Phi(\mathbf{k}, l) = \sum_{\mu} (-1)^{\mu} n^{-\mu} A^{\mu}(k, l),$$

leads [with the help of (19)] to the following relation between a_l^m, b_l^m and c_l^m:

$$(2l+1)\, c_l^m = (l+1)\, a_l^m + l b_l^m. \tag{24}$$

The electric field is

$$\mathbf{E}(\mathbf{k}) = ik\,[\mathbf{A}(\mathbf{k}) - \mathbf{n}\,\Phi(\mathbf{k})],$$

$$E^{\mu}(\mathbf{k}, l) = ik \sum_m (-1)^m d_l^{-m}(k) \left[\sqrt{\frac{l}{l+1}} \alpha^l_{m\mu} Y^{m+\mu}_{l+1}(\vartheta, \varphi) - \sqrt{\frac{l+1}{l}} (-1)^{\mu} \beta^l_{m\mu} Y^{m+\mu}_{l-1}(\vartheta, \varphi) \right], \quad (25)$$

where

$$d_l^m = \frac{\sqrt{l(l+1)}}{2l+1} (a_l^m - b_l^m). \quad (26)$$

The magnetic field is readily determined by means of the relation

$$\mathbf{H}(\mathbf{k}) = [\mathbf{n}\mathbf{E}(\mathbf{k})], \quad \mathbf{E}(\mathbf{k}) = -[\mathbf{n}\mathbf{H}(\mathbf{k})].$$

Noting that (25) has the form of (21), we conclude that $H^{\mu}(\mathbf{k}, l)$ must have the form of (16). The comparison gives

$$H^{\mu}(\mathbf{k}, l) = k \sum_m (-1)^m d_l^{-m}(k) \gamma^l_{m\mu} Y_l^{m+\mu}(\vartheta, \varphi). \quad (27)$$

Again, by the transformation of the type (3)—(7) we find

$$A^{\mu}(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}} i^{l+1} \sum_m (-1)^m [a_l^{-m}(k) \alpha^l_{m\mu} Y^{m+\mu}_{l+1}(\theta, \Phi) f_{l+1}(kr) -$$
$$- b_l^{-m}(k) (-1)^{\mu} \beta^l_{m\mu} Y^{m+\mu}_{l-1}(\theta, \Phi) f_{l-1}(kr)] + \text{c. c;} \quad (28)$$

$$E^{\mu}(\mathbf{r}, k, l) = -\sqrt{\frac{2}{\pi}} i^l k \sum_m (-1)^m d_l^{-m}(k) \left[\sqrt{\frac{l}{l+1}} \alpha^l_{m\mu} Y^{m+\mu}_{l+1}(\theta, \Phi) f_{l+1}(kr) + \right.$$
$$\left. + \sqrt{\frac{l+1}{l}} \beta^l_{m\mu} (-1)^{\mu} Y^{m+\mu}_{l-1}(\theta, \Phi) f_{l-1}(kr) \right] + \text{c. c.;} \quad (29)$$

$$H^{\mu}(\mathbf{r}, k, l) = \sqrt{\frac{2}{\pi}} i^l k \sum_m (-1)^m d^{-m}(k) \gamma^l_{m\mu} Y_l^{m+\mu}(\theta, \Phi) f_l(kr) + \text{c. c.} \quad (30)$$

The expressions for the field vectors $\mathbf{E}$ and $\mathbf{H}$ involve one l-vector d_l^m determined according to (26). l-vectors a_l^m and b_l^m, entering the expression for the potentials, separately are arbitrary. This is in accordance with the gauge invariance of the field vectors. We may put in particular $c_l^m = 0$ (Heitler's potential). Then from (24)

$$a_l^m = \sqrt{\frac{l}{l+1}} d_l^m;$$
$$b_l^m = -\sqrt{\frac{l+1}{l}} d_l^m \quad (c_l^m = 0). \quad (31)$$

If we put $a_l^m = 0$, then

$$b_l^m = -\frac{2l+1}{\sqrt{l(l+1)}} d_l^m;$$
$$c_l^m = -\sqrt{\frac{l}{l+1}} d_l^m \quad (a_l^m = 0). \quad (32)$$

5. The quantized field amplitudes one can obtain from the energy expression

$$W = \frac{1}{2\pi} \int |\mathbf{E}(\mathbf{k})|^2 (d\mathbf{k}) = \frac{1}{2\pi} \int |\mathbf{H}(\mathbf{k})|^2 (d\mathbf{k}). \quad (33)$$

Expanding $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$ into series of multipole fields we get

$$W = \sum_l \sum_m \sum_k \{W_{\text{el}}(k, l, m) +$$
$$+ W_{\text{magn}}(k, l, m)\}, \quad (34)$$

where

$$W_{\text{el}}(k, l, m) = \frac{1}{2\pi} \int_{\Delta k} k^2 |d_l^m(k)|^2 k^2 dk =$$
$$= \frac{k^4}{2\pi} |d_l^m(k)|^2 \Delta k,$$
$$W_{\text{magn}}(k, l, m) = \frac{k^4}{2\pi} |h_l^m(k)|^2 \Delta k. \quad (35)$$

On the other hand, putting

$$W_{\substack{\text{el}\\ \text{magn}}}(k, l, m) = \hbar ck\,\xi^*\xi, \tag{36}$$

where ξ are the usual oscillator operators, satisfying the commutation rule

$$\xi\xi^* - \xi^*\xi = 1,$$

we find:

$$\sqrt{\frac{2}{\pi}}\, k^2\Delta k\, d_l^m(k) = 2\sqrt{\hbar ck\,\Delta k}\,\xi; \tag{37}$$

and similarly for $h_l^m(k)$. In the left-hand side of (37) we include the factor which always enters together with d_l^m (and h_l^m, correspondingly) into the expressions for potentials and fields (8), (15), etc.

Let us write also the expression for the probability of the emission of a given quantum by any non-relativistic system, *i. e.* under the interaction law

$$V = -\frac{e}{2Mc}(\mathbf{Ap} + \mathbf{pA}) + e\Phi \tag{38}$$

($\mathbf{p}$ — operator of the momentum). The dimensions of the system are supposed to be small as compared with the wave length of radiation

$$kr \ll 1. \tag{39}$$

Such a probability gives the estimation of the life-time of the excited states of nuclei.

Using, *e. g.* the potentials (28) and the condition (31), we get* the following expression for the emission probability

$$N_{\text{el}}^{(l)} = \frac{2\pi}{\hbar}\left(\frac{e^2}{2Mc}\right)^2 4k\,\frac{l+1}{l}\left[\frac{2^{l-1}(l-1)!}{(2l-1)!}\right]^2 k^{2l-2}\,\frac{2l-1}{4\pi}\left|\sum_{\mu}\beta_{m\mu}^l\,(p^{-\mu}Q_{l-1}^{m+\mu} + Q_{l-1}^{m+\mu}p^{-\mu})\right|^2. \tag{40}$$

Here eQ_l^m is the electric 2^l-pole moment of the system, given by

$$Q_l^m = \sqrt{\frac{4\pi}{2l+1}}\, r^l Y_l^m(\theta, \hat{\Phi}). \tag{41}$$

The normalization in (41) is such, that

$$\sum_m |Q_l^m|^2 = r^{2l}.$$

[In the brackets in (40) is meant the matrix element of the corresponding operator.]

It can be shown that similarly to the relation

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}$$

it holds

$$\frac{dQ_l^m}{dt} = ick\,Q_l^m = \frac{\sqrt{(2l+1)(2l-1)}}{2Mc}\sum_{\mu}\beta_{m\mu}^l\,(p^{-\mu}Q_{l-1}^{m+\mu} + Q_{l-1}^{m+\mu}p^{-\mu}), \tag{42}$$

which allows to transform (40) into the form

$$N_{\text{el}}^{(l)} = \frac{2}{2l+1}\,\frac{l+1}{l}\left[\frac{2^{l-1}(l-1)!}{(2l-1)!}\right]^2 \frac{k^{2l+1}}{\hbar}\,|eQ_l^m|^2. \tag{43}$$

Using the potentials (15) we obtain the expression of the emission probability for the magnetic 2^l-pole

$$N_{\text{magn}}^{(l)} = \frac{8}{(2l+1)(l+1)\,l}\left[\frac{2^l l!}{(2l)!}\right]^2 \frac{k^{2l+1}}{\hbar}\,|\mathrm{M}_l^m|^2, \tag{44}$$

* Owing to (39) it is possible to disregard the first term in (28) and to put

$$f_l(kr) = \frac{2^l l!}{(2l+1)!}\,(kr)^l.$$

where the magnetic multipole moment of the system is determined as

$$\mathbf{M}_l^m = \frac{e}{2Mc}\left\{-i\sqrt{l(l+1)}\sum_{\mu}(-1)^{\mu}\gamma_{m\mu}^{l}Q_l^{m+\mu}p^{-\mu} + \frac{(l+1)\sqrt{(2l+1)(2l-1)}}{2}\sum_{\mu}\beta_{m\mu}^{l}Q_{l-1}^{m+\mu}\sigma^{-\mu}\right\} \quad (45)$$

($\vec{\sigma}$ is Pauli's spin operator).

6. In constructing the expressions for the potentials of multipole field it is not necessary to start from the solution (1) and (8), using further the expansion in the k-space. It is possible to solve the wave equation in spherical harmonics immediately. In this case the transformation properties determine the character of the separation of variables and we arrive at the expressions (7), (15) and (28), where we must consider $f_l(kr)$ as a solution of the corresponding equation for radial function. If we choose the radial functions (6) without singularities in the origin we obtain the solution with the energy flux vanishing at infinity (standing wave). If it is necessary to obtain the solution with a non-vanishing energy flux at infinity (radiation of classical multipole) we have to choose:

$$f_l(kr) = c\,\frac{H_{l+1/2}(kr)}{\sqrt{kr}}, \quad (46)$$

where $H_{l+1/2}(kr)$ is the Hankel function of the first kind. Such solutions have the poles in the origin. In particular, the vector-potential of the electric field (28) will have the pole of $(l+1)$th order [the first term in (28)]. However, if we choose the coefficients according to (32) only the pole of $(l-1)$th order remains. These are the only potentials adequate for the calculation of the internal conversion coefficients.

If we put in (46) $\sqrt{2/\pi}\cdot c = 1$, the energy flux of the multipole corresponds to the emission probability

$$N_{\mathrm{el}}^{(l)} = \frac{|d_l^m|^2}{\pi^2\hbar k}; \quad N_{\mathrm{magn}}^{(l)} = \frac{|h_l^m|^2}{\pi^2\hbar k}. \quad (47)$$

Comparison of (47) with (43) and (44) gives the connection between the coefficients d_l^m and h_l^m and the electric and magnetic 2^l-pole momenta of the radiating system.

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